

Partial Differential Equation

❖ **Partial Differential Equation (PDE):** An equation involving one or more partial derivatives is called a partial differential equation.

Or, A differential equation involving derivatives of one or more dependent variables with respect to more than one independent variables is called partial differential equation.

Example:(i) $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$ is a partial differential equation.

(ii) $\frac{\partial u}{\partial s} + \frac{\partial v}{\partial t} = v$ is a Partial Differential Equation.

❖ **Classification:** Partial differential equation are two kinds such as :

A) Linear partial differential equation

B) Nonlinear partial differential equation

❖ **Linear Partial Differential Equation:** A partial differential equation is said to be linear if it is of the 1st degree in the partial derivatives.

Example: $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$ is a linear partial differential equation.

❖ **Order of the Differential Equation:** The order of the differential equation is the highest ordered derivatives involved in that equation.

Example: $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$ is a 1st Order Differential Equation.

❖ **Degree of the Differential Equation:** The degree of the differential equation is the power of the highest ordered derivatives.

Example: i) $\frac{d^3 y}{dx^3} + 3 \left(\frac{dy}{dx} \right)^4 = e^x$ is a 1st degree Differential Equation.

ii) $\left(\frac{d^4 y}{dx^4} \right)^3 + 5 \frac{d^2 y}{dx^2} + 3y = 0$ is a 3rd degree Differential Equation.

❖ **Theorem:** If $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be any two independent solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ where

P, Q, R are function of x, y, z then prove that $\phi(u, v) = 0$ or, $v = \phi(u)$ is a general solution of the Lagrange's linear equation $Pp + Qq = R$, where ϕ is an arbitrary function.

Proof: Let u and v be any two function of the variables x, y, z connected by the relation $\phi(u, v) = 0$ (i)

Partially differentiating equation (i) with respect to x and y respectively we get,

$$\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} \right) = 0 \dots\dots\dots(ii)$$

and $\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial y} \right) = 0 \dots\dots\dots(iii)$

Now eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (ii) and (iii) we get,

$$\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial y} \end{vmatrix} = 0$$

or, $\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \end{vmatrix} = 0$ where, $\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q$

or, $\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) - \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right) \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) = 0$

or, $p \left(\frac{\partial u}{\partial z} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial z}\right) + q \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial x}\right) = \left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial y}\right) \dots\dots\dots(iv)$

Let, $\lambda P = \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial z}; \quad \lambda Q = \frac{\partial u}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial x}; \quad \lambda R = \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial y}$

Hence the equation (iv) becomes

$$p\lambda P + q\lambda Q = \lambda R$$

or, $Pp + Qq = R \dots\dots\dots(v)$ this is Lagrange's linear partial differential equation.

Again let, $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two solution of equation (v).

Differentiating these equation with respect to x, y, z respectively we get,

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \dots\dots\dots(vi)$$

and $\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0 \dots\dots\dots(vii)$

Solving equation (vi) and (vii) we get,

$$\frac{dx}{\frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}}$$

or, $\frac{dx}{\lambda P} = \frac{dy}{\lambda Q} = \frac{dz}{\lambda R}$

or, $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ which are the required Lagrange's auxiliary equation.

Example 1: Find the partial differential equation of $\phi\left(\frac{z}{x^2}, x^2 - y^2 + z\right) = 0$.

Solution: Given, $\phi\left(\frac{z}{x^2}, x^2 - y^2 + z\right) = 0$ (i)

Let, $u = \frac{z}{x^2}$, $v = x^2 - y^2 + z$ then (i) becomes $\phi(u, v) = 0$ (ii)

Partially differentiating equation (ii) with respect to x and y respectively we get,

$$\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} \right) = 0$$

$$\text{and } \frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial y} \right) = 0$$

$$\text{or, } \frac{\partial \phi}{\partial u} \left(\frac{-2z}{x^3} + \frac{1}{x^2} p \right) + \frac{\partial \phi}{\partial v} (2x + 1.p) = 0 \text{(iii)}$$

$$\text{and } \frac{\partial \phi}{\partial u} \left(0 + \frac{1}{x^2} q \right) + \frac{\partial \phi}{\partial v} (-2y + 1.q) = 0 \text{(iv)}$$

Now eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (iii) and (iv) we get,

$$\begin{vmatrix} \frac{-2z}{x^3} + \frac{p}{x^2} & 2x + p \\ \frac{q}{x^2} & -2y + q \end{vmatrix} = 0$$

$$\text{or, } \left(\frac{-2z}{x^3} + \frac{p}{x^2} \right) (-2y + q) - (2x + p) \left(\frac{q}{x^2} \right) = 0$$

$$\text{or, } \frac{4yz}{x^3} - \frac{2zq}{x^3} - \frac{2yp}{x^2} + \frac{pq}{x^2} - \frac{2q}{x} - \frac{pq}{x^2} = 0$$

$$\text{or, } 4yz - 2zq - 2xyp - 2x^2q = 0$$

or, $xyp + q(x^2 + z) = 2yz$ which is the required PDE.

Example 2: Find the partial differential equation arising from $\phi(x + y + z, x^2 + y^2 - z^2) = 0$.

Solution: Given, $\phi(x + y + z, x^2 + y^2 - z^2) = 0$ (i)

Let, $u = x + y + z$, $v = x^2 + y^2 - z^2$ then (i) becomes $\phi(u, v) = 0$ (ii)

Partially differentiating equation (ii) with respect to x and y respectively we get,

$$\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} \right) = 0$$

$$\text{and } \frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial y} \right) = 0$$

$$\text{or, } \frac{\partial \phi}{\partial u} (1 + 1.p) + \frac{\partial \phi}{\partial v} (2x - 2z.p) = 0 \text{(iii)}$$

and $\frac{\partial \phi}{\partial u}(1+1.q) + \frac{\partial \phi}{\partial v}(2y-2z.q) = 0 \dots\dots\dots(iv)$

Now eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (iii) and (iv) we get,

$$\begin{vmatrix} 1+p & 2x-2zp \\ 1+q & 2y-2zq \end{vmatrix} = 0$$

or, $(1+p)(2y-2zq) - (2x-2zp)(1+q) = 0$

or, $2y-2zq+2yp-2zpq-2x-2xq+2zp+2zpq = 0$

or, $p(y+z) - q(x+z) = x-y$ which is the required PDE.

Example 3: Find the solution of the partial differential equation $(3x + y - z)p + (x + y - z)q = 2(z - y)$.

Solution: Given that, $(3x + y - z)p + (x + y - z)q = 2(z - y)$

We know the Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or, $\frac{dx}{3x+y-z} = \frac{dy}{x+y-z} = \frac{dz}{2(z-y)}$

Choosing 1,-3,-1; 1,1,-1; 1,-1,1 as multiplier we get,

$$\frac{dx}{3x+y-z} = \frac{dy}{x+y-z} = \frac{dz}{2(z-y)} = \frac{dx-3dy-dz}{0} = \frac{dx+dy-dz}{4(x+y-z)} = \frac{dx-dy+dz}{2(x-y+z)}$$

From 4th ratio we get, $dx - 3dy - dz = 0$

or, $x - 3y - z = c_1$ [By integrating]

Again from 5th and 6th ratio we get, $\frac{dx + dy - dz}{2(x + y - z)} = \frac{dx - dy + dz}{(x - y + z)}$

or, $\frac{1}{2} \text{Log}(x + y - z) = \text{Log}(x - y + z) + \text{Log}c_2$ [By integrating]

or, $\text{Log} \frac{\sqrt{(x + y - z)}}{(x - y + z)} = \text{Log}c_2$

or, $\frac{\sqrt{x + y - z}}{x - y + z} = c_2$

Hence the general solution is $\phi \left(x - 3y - z, \frac{\sqrt{x + y - z}}{x - y + z} \right) = 0$, where ϕ is an arbitrary function.

Example 4: Solve: $(y + zx)p - (x + yz)q = x^2 - y^2$.

Solution: Given that, $(y + zx)p - (x + yz)q = x^2 - y^2$

We know the Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{or, } \frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2}$$

Choosing $y, x, 1$ and $x, y, -z$ as multiplier we get,

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2} = \frac{ydx + xdy + dz}{0} = \frac{xdx + ydy - zdz}{0}$$

From 4th ratio we get, $ydx + xdy + dz = 0$

$$\text{or, } d(xy) + dz = 0$$

$$\text{or, } xy + z = c_1 \quad [\text{By integrating}]$$

From 5th ratio we get, $xdx + ydy - zdz = 0$

$$\text{or, } x^2 + y^2 - z^2 = c_2 \quad [\text{By integrating}]$$

Hence the general solution is $\phi(xy + z, x^2 + y^2 - z^2) = 0$, where ϕ is an arbitrary function.

H.W. Example 5: Find the general solution of the partial differential equation

(i) $y^2zp - x^2zq = x^2y$. [Hints, Taking 1st and 2nd ratio; Taking 2nd and 3rd ratio; then $\phi(x^3 + y^3, y^2 + z^2) = 0$ Ans]

(ii) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

(iii) $(y - z)p + (x - y)q = z - x$. [Hints 1,1,1 and x, z, y as multiplier then $\phi(x + y + z, x^2 + 2yz) = 0$ Ans]

(iv) $x(2y^4 - z^4)p + y(z^4 - 2x^4)q = z(x^4 - y^4)$ [Hints x^3, y^3, z^3 as multiplier and $x, y, \frac{z}{2}$ as divider then

$$\phi(x^4 + y^4 + z^4, xyz^2) = 0 \text{ Ans.}]$$