

Partial Differential Equation

Problem 1: Find the equation of the integral surface satisfying $4yzp + q + 2y = 0$ and passing through $y^2 + z^2 = 1, x + z = 0$.

Solution: Given that, $4yzp + q + 2y = 0$

We know the Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$
$$\text{or, } \frac{dx}{4yz} = \frac{dy}{1} = \frac{dz}{-2y}$$

From 2nd and 3rd ratio, we get,

$$\frac{dy}{1} = \frac{dz}{-2y}$$
$$\text{or, } -2ydy - dz = 0$$
$$\text{or, } 2ydy + dz = 0$$
$$\text{or, } y^2 + z = c_1 \dots\dots\dots\text{(i) [By integrating]}$$

Again from 1st and 3rd ratio we get, $\frac{dx}{4yz} = \frac{dz}{-2y}$

$$\text{or, } dx + 2zdz = 0$$
$$\text{or, } x + z^2 = c_2 \dots\dots\dots\text{(ii) [By integrating]}$$

Adding (i) and (ii) we get, $y^2 + z + x + z^2 = c_2 + c_1$

$$\text{or, } 1 + 0 = c_2 + c_1$$
$$\text{or, } c_1 + c_2 = 1$$
$$\text{or, } y^2 + z + x + z^2 = 1 \text{ which is the required integral surface.}$$

Problem 2: Find the integral surface of the linear partial differential equation

$x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contain the line $x + y = 0, z = 1$.

Solution: Given that, $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$

We know the Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$
$$\text{or, } \frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{(x^2 - y^2)z}$$

Choosing $x, y, -1$ as multiplier and x, y, z as divider we get,

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{(x^2 - y^2)z} = \frac{xdx + ydy - dz}{0} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

From 4th ratio we get, $xdx + ydy - dz = 0$

$$\text{or, } x^2 + y^2 - 2z = c_1 \dots\dots\dots\text{(i) [By integrating]}$$

Again from 5th ratio we get, $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$

$$\text{or, } \text{Log}x + \text{Log}y + \text{Log}z = \text{Log}c_2 \text{ [By integrating]}$$

$$\text{or, } xyz = c_2 \dots\dots\dots\text{(ii)}$$

Since (i) and (ii) passes through, $x + y = 0, z = 1$.

$$\therefore xy = c_2 \text{ and } x^2 + y^2 - 2 = c_1$$

$$\text{or, } (x + y)^2 - 2xy - 2 = c_1$$

$$\text{or, } 0 - 2c_2 - 2 = c_1$$

$$\text{or, } c_1 + 2c_2 + 2 = 0$$

$$\text{or, } x^2 + y^2 - 2z + 2xyz + 2 = 0 \text{ which is the required integral surface.}$$

Problem 3: Find the integral surface of the linear partial differential equation

$$2y(z-3)p + (2x-z)q = y(2x-3) \text{ which passes through the circle } z = 0, x^2 + y^2 = 2x.$$

Solution: Given that, $2y(z-3)p + (2x-z)q = y(2x-3)$

We know the Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{or, } \frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}$$

Choosing 1, 2y, -2 as multiplier we get,

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} = \frac{dx + 2ydy - 2dz}{0}$$

From 4th ratio we get, $dx + 2ydy - 2dz = 0$

$$\text{or, } x + y^2 - 2z = c_1 \dots\dots\dots\text{(i) [By integrating]}$$

Again taking 1st and 3rd ratio we get, $\frac{dx}{2(z-3)} = \frac{dz}{(2x-3)}$

$$\text{or, } (2x-3)dx = 2(z-3)dz$$

$$\text{or, } x^2 - 3x - z^2 + 6z = c_2 \dots\dots\dots\text{(ii) [By integrating]}$$

Since (i) and (ii) passes through, $z = 0, x^2 + y^2 = 2x$.

Adding (i) and (ii) we get, $x^2 + y^2 - 2x - z^2 + 4z = c_1 + c_2$

$$\text{or, } 2x - 2x = c_1 + c_2$$

$$\text{or, } c_1 + c_2 = 0$$

$$\therefore x^2 + y^2 - 2x - z^2 + 4z = 0 \text{ which is the required integral surface.}$$

Problem 4: Find the integral surface of the linear partial differential equation

$$(x-y)y^2 p + (y-x)x^2 q = (x^2 + y^2)z \text{ which passes through } xz = a^3, y = 0.$$

Solution: Given that, $(x-y)y^2 p + (y-x)x^2 q = (x^2 + y^2)z$

We know, the Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{or, } \frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2 + y^2)z}$$

Taking 1st and 2nd ratio we get, $\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2}$

$$\text{or, } \frac{dx}{(x-y)y^2} = -\frac{dy}{(x-y)x^2}$$

$$\text{or, } x^2 dx + y^2 dy = 0$$

$$\text{or, } x^3 + y^3 = c_1 \dots\dots\dots\text{(i) [By integrating]}$$

Choosing 1, -1, 0 as multiplier we get,

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2 + y^2)z} = \frac{dx - dy}{(x-y)(x^2 + y^2)}$$

Again taking 3rd and 4th ratio we get, $\frac{dz}{z} = \frac{dx - dy}{(x-y)}$

$$\text{or, } \text{Log}z = \text{Log}(x-y) + \text{Log}c_2 \text{ [By integrating]}$$

$$\text{or, } \frac{z}{x-y} = c_2 \dots\dots\dots\text{(ii)}$$

Since (i) and (ii) passes through, $xz = a^3, y = 0$.

from (i) we get, $x^3 = c_1$

from (ii) we get, $\frac{xz}{x^2} = c_2$

$$\text{or, } \frac{a^3}{x^2} = c_2$$

$$\text{or, } \frac{a^9}{(x^3)^2} = c_2^3$$

$$\text{or, } \frac{a^9}{c_1^2} = c_2^3$$

$$\text{or, } c_1^2 c_2^3 = a^9$$

$$\text{or, } (x^3 + y^3)^2 \left(\frac{z}{x-y} \right)^3 = a^9$$

$$\text{or, } z^3 (x^3 + y^3)^2 = a^9 (x-y)^3 \text{ which is the required integral surface.}$$

H.W. Example 5: Find the integral surface of the linear partial differential equation

(i) $4yzp + q + 2y = 0$ which passes through $y^2 + z^2 = 1$, $x + z = 0$.

[Hints, Taking 1st and 3rd ratio; Taking 2nd and 3rd ratio; then adding and $y^2 + z^2 + x + z = 3$ Ans]

(ii) $(x - y)p + (y - x - z)q = z$ Which contain the circle $z = 1$, $x^2 + y^2 = 1$.

[Hints, 1, 1, 1 and 1, -1, 1 as multiplier, Taking 4th ratio; Taking 3rd and 5th ratio;

and $z^4(x + y + z)^2 - 2z^4(x + y + z) - 2z^2(x - y + z) + (x - y + x)^2 = 0$ Ans]