Non Linear Partial Differential Equation

Theorem If $F(x,y,z,p,q)=0$ be a nonlinear partial differential equation then the Charpit's auxiliary equations are:

$$\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}}=\frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dx}{-\frac{\partial F}{\partial p}}=\frac{dy}{-\frac{\partial F}{\partial q}}$$

or, $$\frac{dp}{F_x+pF_z}=\frac{dq}{F_y+qF_z}=\frac{dz}{-pF_p-qF_q}=\frac{dx}{-F_p}=\frac{dy}{-F_q}$$

Problem 1: Find the complete integral of the given partial differential equation by Charpit's method

$$p^2-y^2q=y^2-x^2$$

Solution: Given that, $p^2-y^2q=y^2-x^2$

Let, $F(x,y,z,p,q)=p^2-y^2q-y^2+x^2=0$ ....(i)

We know the Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}}=\frac{dq}{\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}}=\frac{dz}{-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}}=\frac{dx}{-\frac{\partial F}{\partial p}}=\frac{dy}{-\frac{\partial F}{\partial q}}$$

or, $$\frac{dp}{2x+0}=\frac{dq}{-2yq-2y+0}=\frac{dz}{-2p^2+y^2q}=\frac{dx}{-2p}=\frac{dy}{y^2}$$

From 1st and 4th ratio, we get,

$$\frac{dp}{2x}=\frac{dx}{-2p}$$

$or,\ pdp+xdx=0$

or, $p^2+x^2=c_1$ ....(ii) [By integrating]

Now, solving (i) and (ii) we get, $c_1-y^2q-y^2=0$

or, $q=\frac{c_1-y^2}{y^2}=\frac{c_1}{y^2}-1$

and $p=\sqrt{c_1-x^2}$

We know, $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

or, $dz=pdx+qdy$

or, $dz=\sqrt{c_1-x^2}\,dx+\left(\frac{c_1}{y^2}-1\right)dy$

or, $\int dz=\int\sqrt{c_1-x^2}\,dx+\int\left(\frac{c_1}{y^2}-1\right)dy$

or, $z=\frac{\sqrt{c_1-x^2}}{2c_1}+\frac{c_1}{2}\sin^{-1}\left(\frac{x}{\sqrt{c_1}}\right)-\frac{c_1}{y}-y+k$ [By integrating]

which is the required complete integral/ Solution of (i).

Problem 2: Solve $z^2(p^2z^2+q^2)=1$ by Charpit's method

Solution: Given that, $z^2(p^2z^2+q^2)=1$

Let, $F(x,y,z,p,q)=p^2z^4+q^2z^2-1=0$ ....(i)

We know, the Charpit's auxiliary equations are

$$\frac{dp}{F_x+pF_z}=\frac{dq}{F_y+qF_z}=\frac{dz}{-pF_p-qF_q}=\frac{dx}{-F_p}=\frac{dy}{-F_q}$$

$$\text{or, } \frac{dp}{0+p(4p^2z^3+2zq^2)}=\frac{dq}{0+q(4p^2z^3+2zq^2)}=\frac{dz}{-p.2pz^4-q.2qz^2}=\frac{dx}{-2pz^4}=\frac{dy}{-2qz^2}$$

From 1st and 2nd ratio, we get,

$$\frac{dp}{p}=\frac{dq}{q}$$

$$or, Logp = Logq + Logc \text{ [By integrating]}$$

or, $p=qc$ ....(ii)

Now, solving (i) and (ii) we get, $c^2q^2z^4+z^2q^2-1=0$

$$\text{or, } q=\frac{1}{z\sqrt{c^2z^2+1}} \quad \text{and } p=\frac{c}{z\sqrt{c^2z^2+1}}$$

We know, $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

or, $dz=pdx+qdy$

$$\text{or, } dz=\frac{c}{z\sqrt{c^2z^2+1}}dx+\frac{1}{z\sqrt{c^2z^2+1}}dy$$

$$\text{or, } \int z\sqrt{c^2z^2+1}dz=\int c\,dx+\int 1\,dy$$

or, $\frac{1}{3c^2}(c^2z^2+1)^{\frac{3}{2}}=cx+y+k$ [By integrating] which is the required complete Solution.

Problem 3: Solve $pxy+pq+qy=yz$ by Charpit's method

Solution: Given that, $pxy+pq+qy=yz$

Let, $F(x,y,z,p,q)=pxy+pq+qy-yz=0$ ....(i)

We know, the Charpit's auxiliary equations are

$$\frac{dp}{F_x+pF_z}=\frac{dq}{F_y+qF_z}=\frac{dz}{-pF_p-qF_q}=\frac{dx}{-F_p}=\frac{dy}{-F_q}$$

$$\text{or, } \frac{dp}{py+p(-y)}=\frac{dq}{q+px-qy-z}=\frac{dz}{-pxy-pq-pq-qy}=\frac{dx}{-xy-q}=\frac{dy}{-(p+y)}$$

From 1st ratio, we get,

$$dp=0$$

or, $p=c$ ....(ii) [By integrating]

Now, solving (i) and (ii) we get, $cxy+cq+qy-yz=0$

$$\text{or, } q = \frac{yz - cxy}{c+y} \quad \text{and } p = c$$

We know, $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

or, $dz = pdx + qdy$

or, $dz = cdx + \frac{yz - cxy}{c+y}dy$

or, $dz - cdx = \frac{y(z-cx)}{c+y}dy$

or, $\frac{dz - cdx}{z - cx} = \frac{y}{c+y}dy$

or, $\frac{dz - cdx}{z - cx} = \left(1 - \frac{c}{c+y}\right)dy$

or, $Log(z - cx) = y - cLog(c+y) + Logk$ [By integrating]

or, $z - cx = e^{-y}k(c+y)^{-c}$

or, $z = cx + e^{-y}k(c+y)^{-c}$ which is the required complete Solution.

Problem 4: Solve $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$ by Charpit's method and identifying the surface.

Solution: Given that, $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$

Let, $F(x, y, z, p, q) = 16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$ ....(i)

We know, the Charpit's auxiliary equations are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\text{or, } \frac{dp}{32p^3z + 18pq^2z + 8pz} = \frac{dq}{32p^2qz + 18q^3z + 8qz} = \frac{dz}{-32p^2z^2 - 18q^2z^2} = \frac{dx}{-32pz^2} = \frac{dy}{-18qz^2}$$

Choosing $4z, 0, 4p, 1, 0$ as multiplier,

$$\frac{dp}{32p^3z + 18pq^2z + 8pz} = \frac{dq}{32p^2qz + 18q^3z + 8qz} = \frac{dz}{-32p^2z^2 - 18q^2z^2} = \frac{dx}{-32pz^2} = \frac{dy}{-18qz^2} = \frac{dx + 4pdz + 4zdp}{0}$$

i.e, $dx + 4pdz + 4zdp = 0$

or, $dx + 4d(pz) = 0$

or, $x + 4pz = a$ [By integrating]

or, $p = \frac{a - x}{4z}$ ....(ii)

Now, solving (i) and (ii) we get, $16\frac{(a-x)^2}{16z^2}.z^2 + 9q^2z^2 + 4z^2 - 4 = 0$

or, $(a-x)^2 + 9q^2z^2 + 4z^2 - 4 = 0$

or, $9q^2z^2 = 4 - 4z^2 - (a-x)^2$

$$\text{or, } q^2 = \frac{4}{9z^2}\left\{1 - z^2 - \frac{1}{4}(a-x)^2\right\}$$

$$\text{or, } q = \frac{2}{3z}\sqrt{1 - z^2 - \frac{1}{4}(a-x)^2}$$

We know, $dz = pdx + qdy$

$$\text{or, } dz = \frac{a-x}{4z}dx + \frac{2}{3z}\sqrt{1 - z^2 - \frac{1}{4}(a-x)^2}\,dy$$

$$\text{or, } zdz = \frac{a-x}{4}dx + \frac{2}{3}\sqrt{1 - z^2 - \frac{1}{4}(a-x)^2}\,dy$$

$$\text{or, } zdz - \frac{1}{4}(a-x)dx = \frac{2}{3}\sqrt{1 - z^2 - \frac{1}{4}(a-x)^2}\,dy$$

$$\text{or, } \frac{3}{2}\frac{zdz - \frac{1}{4}(a-x)dx}{\sqrt{1 - z^2 - \frac{1}{4}(a-x)^2}} = dy$$

or, $-\frac{3}{2}\sqrt{1 - z^2 - \frac{1}{4}(a-x)^2} = y + c$ [By integrating] which is the required Solution

$$\text{or, } \frac{9}{4}\left(1 - z^2 - \frac{1}{4}(a-x)^2\right) = (y+c)^2$$

$$\text{or, } 1 - z^2 - \frac{1}{4}(a-x)^2 = \frac{4}{9}(y+c)^2$$

or, $\frac{4}{9}(y+c)^2 + \frac{1}{4}(a-x)^2 + z^2 = 1$ it is represent an ellipsoid.

Problem 5: Solve $p^2 + q^2 = py - qx$ by Charpit's method

Solution: Given that, $p^2 + q^2 = py - qx$

Let, $F(x, y, z, p, q) = p^2 + q^2 - py + qx = 0$ ....(i)

We know, the Charpit's auxiliary equations are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\text{or, } \frac{dp}{q} = \frac{dq}{-p} = \frac{dz}{-2p^2 + yp - 2q^2 - qx} = \frac{dx}{-2p + y} = \frac{dy}{-2q - x}$$

From 1^{st} and 2^{nd} ratio, we get $\frac{dp}{q} = \frac{dq}{-p}$

or, $pdp + qdq = 0$

or, $p^2 + q^2 = a$ [By integrating].................(ii)

Now, solving (i) and (ii) we get, $a - q^2 + q^2 - \sqrt{a - q^2}\,y + qx = 0$

or, $a + qx = \sqrt{a - q^2}\,y$

$$\text{or, } a^2+2aqx+q^2x^2=ay^2-q^2y^2 \text{ [by Squaring]}$$

$$\text{or, } q^2(x^2+y^2)+2aqx+a^2-ay^2=0$$

$$\text{or, } q=\frac{-2ax\pm\sqrt{4a^2x^2-4(x^2+y^2)(a^2-ay^2)}}{2(x^2+y^2)}$$

$$\text{or, } q=\frac{-ax\pm\sqrt{ay^2(x^2+y^2-a)}}{(x^2+y^2)}$$

$$\text{and} \quad p=\left[a-\left\{\frac{-ax\pm\sqrt{ay^2(x^2+y^2-a)}}{(x^2+y^2)}\right\}^2\right]^{\frac{1}{2}}$$

We know, $dz=pdx+qdy$

$$\text{or, } dz=\left[a-\left\{\frac{-ax\pm\sqrt{ay^2(x^2+y^2-a)}}{(x^2+y^2)}\right\}^2\right]^{\frac{1}{2}}dx+\left\{\frac{-ax\pm\sqrt{ay^2(x^2+y^2-a)}}{(x^2+y^2)}\right\}dy$$

$$\text{or, } z=\int\left[a-\left\{\frac{-ax\pm\sqrt{ay^2(x^2+y^2-a)}}{(x^2+y^2)}\right\}^2\right]^{\frac{1}{2}}dx+\int\left\{\frac{-ax\pm\sqrt{ay^2(x^2+y^2-a)}}{(x^2+y^2)}\right\}dy+k$$

[By integrating] which is the required Solution

H.W. Example 6: Find the complete integral of the given partial differential equation by Charpit's method

(i) $2z+p^2+qy+2y^2=0$ [Hints,Taking 1st and 4th ratio and $2y^2z+y^2(x-a)^2+y^4=c$ Ans]

Problem 7: Find the complete and singular integral of $(p^2+q^2)y=qz$

Solution: Given that, $(p^2+q^2)y=qz$

Let, $F(x,y,z,p,q)=(p^2+q^2)y-qz=0$ ....(i)

We know, the Charpit's auxiliary equations are

$$\frac{dp}{F_x+pF_z}=\frac{dq}{F_y+qF_z}=\frac{dz}{-pF_p-qF_q}=\frac{dx}{-F_p}=\frac{dy}{-F_q}$$

$$\text{or, } \frac{dp}{-pq}=\frac{dq}{p^2+q^2-q^2}=\frac{dz}{-2p^2y-2q^2y+qz}=\frac{dx}{-2py}=\frac{dy}{-2qy+z}$$

From first and 2nd ratio, we get $\frac{dp}{-q}=\frac{dq}{p}$

$$\text{or, } pdp+qdq=0$$

or, $p^2+q^2=a$ [By integrating].................(ii)

Now, solving (i) and (ii) we get, $ay=qz$

$$\text{or, } q=\frac{ay}{z}$$

and $\quad p=\dfrac{\sqrt{az^2-a^2y^2}}{z}$

We know, $dz = pdx + qdy$

or, $dz=\dfrac{\sqrt{az^2-a^2y^2}}{z}dx+\dfrac{ay}{z}dy$

or, $\dfrac{azdz-a^2ydy}{\sqrt{az^2-a^2y^2}}=dx$

or, $\sqrt{az^2-a^2y^2}=ax+b$ [By integrating] which is the complete Solution

or, $az^2-a^2y^2=(ax+b)^2$ ....(iii)

Differentiating equation (iii) with respect to a and b respectively

$z^2-2ay^2=2x(ax+b)$ ....(iv)

and $0=2(ax+b)\quad or, ax+b=0$ ....(v)

From (iv), $z^2-2ay^2=0\quad or,\quad a=\dfrac{z^2}{2y^2}$

and from (v) , $\quad b=\dfrac{-xz^2}{2y^2}$

Putting these values of a and b in (iii) we get, $\dfrac{z^4}{2y^2}-\dfrac{z^4}{4y^4}y^2=\left(\dfrac{z^2x}{2y^2}-\dfrac{z^2x}{2y^2}\right)^2$

or, $z^4=0$ which is the required singular solution.

Problem 8: Find the complete and singular integral of $2xz-px^2-2qxy+pq=0$

Solution: Given that, $2xz-px^2-2qxy+pq=0$

Let, $F(x,y,z,p,q)=2xz-px^2-2qxy+pq=0$ ....(i)

We know, the Charpit's auxiliary equations are

$$\frac{dp}{F_x+pF_z}=\frac{dq}{F_y+qF_z}=\frac{dz}{-pF_p-qF_q}=\frac{dx}{-F_p}=\frac{dy}{-F_q}$$

or, $\dfrac{dp}{2z-2px-2qx+2px}=\dfrac{dq}{-2qx+2qx}=\dfrac{dz}{2qxy-pq+2x^2p-pq}=\dfrac{dx}{x^2+q}=\dfrac{dy}{2xy-p}$

From 2nd ratio, we get $dq=0$

or, $q=a$ [By integrating]................(ii)

Now, solving (i) and (ii) we get, $2xz-px^2-2axy+pa=0$

or, $p(a-x^2)=2axy-2xz$

or, $p=\dfrac{2axy-2xz}{a-x^2}$

We know, $dz = pdx + qdy$

or, $dz = \dfrac{2axy - 2xz}{a - x^2} dx + ady$

or, $dz - ady = \dfrac{2x(z - ay)}{x^2 - a} dx$

or, $\dfrac{dz - ady}{z - ay} = \dfrac{2x}{x^2 - a} dx$

or, $Log(z - ay) = Log(x^2 - a) + Logb$ [By integrating] which is the complete Solution

or, $z - ay = b(x^2 - a)$

or, $z = ay + b(x^2 - a)$ ....(iii)

Differentiating equation (iii) with respect to a and b respectively

$0 = y - b$ or, $b = y$ ....(iv)

and $0 = x^2 - a$ $or, a = x^2$ ....(v)

Putting these values of a and b in (iii) we get, $z = x^2 y + y(x^2 - x^2)$

or, $z = x^2 y$ which is the required singular solution.