Laplace Equation

❖ Laplace equation in two dimensional Cartesian coordinate (x,y) is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

❖ Laplace equation in two dimensional Polar coordinate (r,θ) is $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0.$

❖ Laplace equation in three dimensional Cartesian coordinate (x,y,z) is $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$

❖ Laplace equation in three dimensional cylindrical coordinate (r,θ,z) is $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0.$

❖ Laplace equation in three dimensional Spherical coordinate (r,θ,φ) is

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}+\frac{\cot\varphi}{r^2}\frac{\partial u}{\partial \varphi}+\frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2}=0.$$

❖ The operator ∇^2 is called the Laplacian operator and is defined as $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$.

❖ Harmonic function: A function u which satisfies Laplace's equation is called harmonic function.

Question 1: Solve the Laplace equation in two dimensional Cartesian coordinate (x,y).

Solution: We know the Laplace equation in two dimensional Cartesian coordinate (x,y) is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \qquad \text{.........................(i)}$$

In order to apply the method of separation of variables assume that $u(x,y)=X(x).Y(y)$ is a solution of (i).

$$\therefore \frac{\partial u}{\partial x}=X'Y, \qquad or, \frac{\partial^2 u}{\partial x^2}=X''Y$$

$$\therefore \frac{\partial u}{\partial y}=XY', \qquad or, \frac{\partial^2 u}{\partial y^2}=XY''$$

Putting these values in equation (i) we get,

$$X''Y+XY''=0$$

or, $X''Y=-XY''$

or, $\frac{X''}{X}=-\frac{Y''}{Y}$ ....(ii)

Since the left hand side of (ii) is a function of x only and the right hand side of (ii) is a function of y only then the equation (ii) is true only if each side is equal to the same constant.

So, Let, $\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda^2$

$$or, \frac{X''}{X}=-\lambda^2 \qquad \text{and} \qquad -\frac{Y''}{Y}=-\lambda^2$$

$$or, X''+\lambda^2 X=0, \qquad \text{or, } Y''-\lambda^2 Y=0$$

Which are second order homogeneous differential equations.

Hence, the solutions are, $X = A\cos\lambda x + B\sin\lambda x$

and $\quad Y = Ce^{\lambda y} + De^{-\lambda y}$

Therefore the required solution of equation (i) is $u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y})$. Ans.

Here, A, B, C, D are arbitrary constants.

Question 2: Solve the Laplace equation in two dimensional Polar coordinate (r,θ).

Solution: We know the Laplace equation in two dimensional Polar coordinate (r,θ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \text{........................(i)}$$

In order to apply the method of separation of variables assume that $u(r,\theta) = R(r).F(\theta)$ is a solution of (i).

$$\therefore \frac{\partial u}{\partial r} = R'F, \qquad or, \frac{\partial^2 u}{\partial r^2} = R''F$$

$$\therefore \frac{\partial u}{\partial \theta} = RF', \qquad or, \frac{\partial^2 u}{\partial \theta^2} = RF''$$

Putting these values in equation (i) we get,

$$R''F + \frac{1}{r}R'F + \frac{1}{r^2}RF'' = 0$$

or, $R''F + \frac{1}{r}R'F = -\frac{1}{r^2}RF''$

or, $(R''r^2 + R'r)F = -RF''$

or, $\frac{1}{R}(R''r^2 + R'r) = -\frac{F''}{F}$ ....(ii)

Since the left hand side of (ii) is a function of r only and the right hand side is a function of θ only then the equation (ii) is true only if each side is equal to the same constant.

So, Let, $\frac{1}{R}(R''r^2 + R'r) = -\frac{F''}{F} = n^2$

$or, r^2R'' + rR' - n^2R = 0$ ....(iii)

and $\quad F'' + n^2F = 0$ ....(iv)

Equation (iii) is a linear homogeneous differential equation.

Let, $r = e^z$

or, $z = Log\, r$

$$\therefore \frac{dz}{dr} = \frac{1}{r}$$

Now, $\frac{dR}{dr} = \frac{dR}{dz}.\frac{dz}{dr} = \frac{1}{r}\frac{dR}{dz}$ ....(v) or, $r\dfrac{dR}{dr}=\dfrac{dR}{dz}$

Differentiating Eq. (v) with respect to r we get,

$$\frac{d}{dr}\left(\frac{dR}{dr}\right)=\frac{d}{dr}\left(\frac{1}{r}\frac{dR}{dz}\right)$$

$$\text{or, } \frac{d^2R}{dr^2}=\frac{d}{dz}\left(\frac{1}{r}\frac{dR}{dz}\right)\frac{dz}{dr}$$

$$=\left(\frac{1}{r}\frac{d^2R}{dz^2}-\frac{1}{r^2}\frac{dr}{dz}\frac{dR}{dz}\right)\frac{1}{r}$$

$$=\frac{1}{r^2}\frac{d^2R}{dz^2}-\frac{1}{r^2}.r.\frac{1}{r}\frac{dR}{dz}$$

$$=\frac{1}{r^2}\left(\frac{d^2R}{dz^2}-\frac{dR}{dz}\right)$$

$$\text{or, } r^2\frac{d^2R}{dr^2}=\frac{d^2R}{dz^2}-\frac{dR}{dz}$$

Using these values in equation (iii) we get,

$$\frac{d^2R}{dz^2}-\frac{dR}{dz}+\frac{dR}{dz}-n^2R=0$$

$$\text{or, } \frac{d^2R}{dz^2}-n^2R=0$$

Which is second order homogeneous differential equation.

Hence, the solutions is, $R(r)=Ae^{nz}+Be^{-nz}$

or, $R(r)=A(e^{z})^{n}+B(e^{z})^{-n}$

or, $R(r)=Ar^{n}+Br^{-n}$

Also the solution of equation (iv) is $F(\theta)=C\cos n\theta+D\sin n\theta$

Therefore the required solution of equation (i) is $u(r,\theta)=(Ar^{n}+Br^{-n})(C\cos n\theta+D\sin n\theta)$. Ans.

Here A,B,C,D are arbitrary constants.

Question 3: Solve the Laplace equation in three dimensional Cartesian coordinate (x,y,z).

Solution: We know the Laplace equation in three dimensional Cartesian coordinate (x,y,z) is

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}=0.........................\text{(i)}$$

In order to apply the method of separation of variables assume that $u(x,y,z)=X(x).Y(y).Z(z)$ is a solution of(i).

$$\therefore \frac{\partial u}{\partial x}=X'YZ\,, \quad or, \frac{\partial^2u}{\partial x^2}=X''YZ$$

$$\therefore \frac{\partial u}{\partial y}=XY'Z\,, \quad or, \frac{\partial^2u}{\partial y^2}=XY''Z$$

$\therefore \dfrac{\partial u}{\partial z} = X\,Y\,Z'$, $\quad or, \dfrac{\partial^2 u}{\partial z^2} = X\,Y\,Z''$

Putting these values in equation (i) we get,

$$X''\,Y\,Z + X\,Y''\,Z + X\,Y\,Z'' = 0$$

or, $X''\,Y\,Z + X\,Y''\,Z = -X\,Y\,Z''$

or, $\dfrac{X''}{X} + \dfrac{Y''}{Y} = -\dfrac{Z''}{Z}$ ....(ii)

Since x, y *and* z are independent variables then the equation (ii) is true only if each term on each side is equal to a constant. So, the following cases are arises:

Case-I: When each term is zero,

Thus we get, $X'' = 0$, $\quad Y'' = 0$, $\quad Z'' = 0$.

Whose solutions are $X = Ax + B$, $\quad Y = C\,y + D$ and $Z = Ez + F$ respectively.

Hence the general solution of (i) is $u(x, y, z) = (Ax + B)(Cy + D)(Ez + F)$ Ans.

Here, A, B, C, D, E, F are arbitrary constants.

Case-II: Let, $\dfrac{X''}{X} = -n^2$, $\dfrac{Y''}{Y} = -m^2$ and $\dfrac{Z''}{Z} = p^2$ such that $m^2 + n^2 = p^2$

Then, $X'' + n^2 X = 0$ and its solution, $X = A\cos nx + B\sin nx$

$Y'' + m^2 Y = 0$ and its solution, $Y = C\cos my + D\sin my$

$Z'' - p^2 Z = 0$ and its solution, $Z = E\,e^{pz} + F\,e^{-pz}$

Hence the general solution of (i) is $u(x, y, z) = (A\cos nx + B\sin nx)(C\cos my + D\sin my)(E\,e^{pz} + F\,e^{-pz})$ Ans.

Here, A, B, C, D, E, F are arbitrary constants.

Case-III: Let, $\dfrac{X''}{X} = n^2$, $\dfrac{Y''}{Y} = m^2$ and $\dfrac{Z''}{Z} = -p^2$ such that $m^2 + n^2 = p^2$

Then, $X'' - n^2 X = 0$ and its solution, $X = A\,e^{nx} + B\,e^{-nx}$

$Y'' - m^2 Y = 0$ and its solution, $Y = C\,e^{my} + D\,e^{-my}$

$Z'' + p^2 Z = 0$ and its solution, $Z = E\cos pz + F\sin pz$

Hence the general solution of (i) is $u(x, y, z) = (A\,e^{nx} + B\,e^{-nx})(C\,e^{my} + D\,e^{-my})(E\cos pz + F\sin pz)$ Ans.

Here, A, B, C, D, E, F are arbitrary constants.