Question 4: Solve the Laplace equation in three dimensional cylindrical coordinates (r,θ,z).

Solution: We know the Laplace equation in three dimensional cylindrical coordinate (r,θ,z) is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0. \text{.........................(i)}$$

In order to apply the method of separation of variables assume that $u(r,\theta,z)=R(r).F(\theta).Z(z)$ is a solution of (i).

$$\therefore \frac{\partial u}{\partial r}=R'FZ, \quad or, \frac{\partial^2 u}{\partial r^2}=R''FZ$$

$$\therefore \frac{\partial u}{\partial \theta}=RF'Z, \quad or, \frac{\partial^2 u}{\partial \theta^2}=RF''Z$$

$$\therefore \frac{\partial u}{\partial z}=RFZ', \quad or, \frac{\partial^2 u}{\partial z^2}=RFZ''$$

Putting these values in equation (i) we get,

$$R''FZ+\frac{1}{r}R'FZ+\frac{1}{r^2}RF''Z+RFZ''=0$$

$$\text{or, } R''FZ+\frac{1}{r}R'FZ+\frac{1}{r^2}RF''Z=-RFZ''$$

$$\text{or, } \frac{R''}{R}+\frac{R'}{rR}+\frac{F''}{r^2F}=-\frac{Z''}{Z} \text{ ....(ii)}$$

Since the left hand side of (ii) is a function of r and θ while the right hand side is a function of z only then the equation (ii) is true only if each side is equal to the same constant.

$$\text{So, Let, } \frac{R''}{R}+\frac{R'}{rR}+\frac{F''}{r^2F}=-\frac{Z''}{Z}=-\lambda^2$$

$$\therefore Z''-\lambda^2 Z=0 \text{ ....(iii)}$$

$$\text{and} \quad \frac{R''}{R}+\frac{R'}{rR}+\frac{F''}{r^2F}=-\lambda^2$$

$$\text{or, } \frac{R''}{R}+\frac{R'}{rR}+\lambda^2=-\frac{F''}{r^2F}$$

$$\text{or, } r^2\left(\frac{R''}{R}+\frac{R'}{rR}+\lambda^2\right)=-\frac{F''}{F} \text{ ....(iv)}$$

Since the left hand side of (iv) is a function of r only and the right hand side is a function of θ only then the equation (iv) is true only if each side is equal to the same constant.

$$\text{So, Let, } r^2\left(\frac{R''}{R}+\frac{R'}{rR}+\lambda^2\right)=-\frac{F''}{F}=\mu^2$$

$$\therefore F''+\mu^2 F=0 \text{ ....(v)}$$

$$\text{and} \quad r^2R''+rR'+\lambda^2r^2R-\mu^2R=0$$

$$\text{or, } r^2R''+rR'+\left(\lambda^2r^2-\mu^2\right)R=0 \text{ ....(vi)}$$

Equation (vi) is a Bessel's differential equation and it's general solutions are:

$R(r)=A_1 J_\mu(\lambda r)+B_1 J_{-\mu}(\lambda r)$ for fractional μ.

and $\quad R(r) = A_1 J_\mu(\lambda r) + B_1 Y_\mu(\lambda r)$ for integer μ.

Also the solution of equations (iii) and (v) are

$$Z(z) = A_2 e^{\lambda z} + B_2 e^{-\lambda z}$$

and $\quad F(\theta) = A_3 \cos\mu\theta + B_3 \sin\mu\theta$

Therefore the required solution of equation (i) are

$$u(r,\theta,z) = (A_1 J_\mu(\lambda r) + B_1 J_{-\mu}(\lambda r))(A_3 \cos\mu\theta + B_3 \sin\mu\theta)(A_2 e^{\lambda z} + B_2 e^{-\lambda z}).$$

and $\quad u(r,\theta,z) = (A_1 J_\mu(\lambda r) + B_1 Y_\mu(\lambda r))(A_3 \cos\mu\theta + B_3 \sin\mu\theta)(A_2 e^{\lambda z} + B_2 e^{-\lambda z})$ Ans.

Here $A_1, A_2, A_3, B_1, B_2, B_3$ are arbitrary constants.

Question 5: Solve the Laplace equation in three dimensional Spherical polar coordinates (r,θ,φ).

Solution: We know the Laplace equation in three dimensional Spherical coordinate (r,θ,φ) is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\cot\varphi}{r^2}\frac{\partial u}{\partial \varphi} + \frac{1}{r^2 \sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2} = 0. \text{.........................(i)}$$

In order to apply the method of separation of variables assume that $u(r,\theta,\varphi) = R(r).F(\theta).\Phi(\varphi)$ is a solution of (i).

$$\therefore \frac{\partial u}{\partial r} = R'F\Phi, \quad or, \frac{\partial^2 u}{\partial r^2} = R''F\Phi$$

$$\therefore \frac{\partial u}{\partial \theta} = RF'\Phi, \quad or, \frac{\partial^2 u}{\partial \theta^2} = RF''\Phi$$

$$\therefore \frac{\partial u}{\partial \varphi} = RF\Phi', \quad or, \frac{\partial^2 u}{\partial \varphi^2} = RF\Phi''$$

Putting these values in equation (i) we get,

$$R''F\Phi + \frac{2}{r}R'F\Phi + \frac{1}{r^2}RF\Phi'' + \frac{\cot\varphi}{r^2}RF\Phi' + \frac{1}{r^2\sin^2\varphi}RF''\Phi = 0$$

or, $$R''F\Phi + \frac{2}{r}R'F\Phi + \frac{1}{r^2}RF\Phi'' + \frac{\cot\varphi}{r^2}RF\Phi' = -\frac{1}{r^2\sin^2\varphi}RF''\Phi$$

or, $$\left\{\frac{1}{R}\left(R'' + \frac{2}{r}R'\right) + \frac{1}{r^2\Phi}(\Phi'' + \cot\varphi\Phi')\right\}r^2\sin^2\varphi = -\frac{F''}{F} \text{ ....(ii)}$$

Since the left hand side of (ii) is a function of r and φ while the right hand side is a function of θ only then the equation (ii) is true only if each side is equal to the same constant.

So, Let, $$\left\{\frac{1}{R}\left(R'' + \frac{2}{r}R'\right) + \frac{1}{r^2\Phi}(\Phi'' + \cot\varphi\Phi')\right\}r^2\sin^2\varphi = -\frac{F''}{F} = m^2$$

$$\therefore F'' + m^2 F = 0 \text{ ....(iii)}$$

and $$\left\{\frac{1}{R}\left(R'' + \frac{2}{r}R'\right) + \frac{1}{r^2\Phi}(\Phi'' + \cot\varphi\Phi')\right\}r^2 = \frac{m^2}{\sin^2\varphi}$$

or, $$\frac{1}{R}\left(r^2R'' + 2rR'\right) + \frac{1}{\Phi}(\Phi'' + \cot\varphi\Phi') = \frac{m^2}{\sin^2\varphi}$$

$$\text{or, } \frac{1}{\Phi}\left(\Phi'' + \cot\varphi\Phi'\right) - \frac{m^2}{\sin^2\varphi} = -\frac{1}{R}\left(r^2R'' + 2rR'\right) \text{.......................(iv)}$$

Since the left hand side of (iv) is a function of φ only and the right hand side is a function of r only then the equation (iv) is true only if each side is equal to the same constant.

$$\text{So, Let, } \frac{1}{\Phi}\left(\Phi'' + \cot\varphi\Phi'\right) - \frac{m^2}{\sin^2\varphi} = -\frac{1}{R}\left(r^2R'' + 2rR'\right) = -n(n+1)$$

$$\therefore r^2R'' + 2rR' - n(n+1)R = 0 \text{ ....(v)}$$

$$\text{and } \quad \frac{1}{\Phi}\left(\Phi'' + \cot\varphi\Phi'\right) - \frac{m^2}{\sin^2\varphi} = -n(n+1)$$

$$\text{or, } \Phi'' + \cot\varphi\Phi' + \left\{n(n+1) - \frac{m^2}{\sin^2\varphi}\right\}\Phi = 0 \text{ ....(vi)}$$

Let, $\cos\varphi = \mu$ then $\sin^2\varphi = 1 - \cos^2\varphi = 1 - \mu^2$

$$\text{Now, } \Phi' = \frac{d\Phi}{d\varphi} = \frac{d\Phi}{d\mu}.\frac{d\mu}{d\varphi} = -\sin\varphi\frac{d\Phi}{d\mu}$$

$$\therefore \cot\varphi\Phi' = \frac{\cos\varphi}{\sin\varphi}\frac{d\Phi}{d\varphi} = \frac{\cos\varphi}{\sin\varphi}.\left(-\sin\varphi\frac{d\Phi}{d\mu}\right) = -\cos\varphi\frac{d\Phi}{d\mu} = -\mu\frac{d\Phi}{d\mu}$$

$$\Phi'' = \frac{d}{d\varphi}\left(\Phi'\right) = \frac{d}{d\varphi}\left(-\sin\varphi\frac{d\Phi}{d\mu}\right)$$

$$= -\cos\varphi\frac{d\Phi}{d\mu} - \sin\varphi.\frac{d}{d\varphi}\left(\frac{d\Phi}{d\mu}\right)$$

$$= -\cos\varphi\frac{d\Phi}{d\mu} - \sin\varphi.\frac{d}{d\mu}\left(\frac{d\Phi}{d\mu}\right)\frac{d\mu}{d\varphi}$$

$$= -\cos\varphi\frac{d\Phi}{d\mu} - \sin\varphi.\frac{d^2\Phi}{d\mu^2}\left(-\sin\varphi\right)$$

$$= -\cos\varphi\frac{d\Phi}{d\mu} + \sin^2\varphi.\frac{d^2\Phi}{d\mu^2}$$

$$= -\mu\frac{d\Phi}{d\mu} + (1-\mu^2)\frac{d^2\Phi}{d\mu^2}$$

Putting the values of Φ'' and $\cot\varphi\Phi'$ in (vi) we get,

$$-\mu\frac{d\Phi}{d\mu} + (1-\mu^2)\frac{d^2\Phi}{d\mu^2} - \mu\frac{d\Phi}{d\mu} + \left\{n(n+1) - \frac{m^2}{\sin^2\varphi}\right\}\Phi = 0$$

$$\text{or, } (1-\mu^2)\frac{d^2\Phi}{d\mu^2} - 2\mu\frac{d\Phi}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Phi = 0 \text{ ....(vii)}$$

Equation (vii) is associated with Legendre's differential equation and it's general solution is:

$$\Phi(\varphi) = AP_n^m(\mu) + BQ_n^m(\mu)$$

$\therefore \Phi(\varphi) = AP_n^m(\cos\varphi) + BQ_n^m(\cos\varphi)$ ....(viii) $[\because \mu = \cos\varphi]$

and the solution of equation (iii) is

$F(\theta) = C\cos m\theta + D\sin m\theta$ ....(ix)

Also we have to solve the equation (v): $r^2R'' + 2rR' - n(n+1)R = 0$ ....(x)

Let, $r = e^z$

or, $z = Log\, r$

$\therefore \frac{dz}{dr} = \frac{1}{r}, \quad or, \frac{d^2z}{dr^2} = -\frac{1}{r^2}$

Now, $R' = \frac{dR}{dr} = \frac{dR}{dz}.\frac{dz}{dr} = \frac{1}{r}\frac{dR}{dz}$ ....(v)

or, $rR' = \frac{dR}{dz}$

Differentiating Eq. (v) w, r, to r we get,

$$R'' = \frac{d}{dr}\left(\frac{dR}{dr}\right) = \frac{d}{dr}\left(\frac{1}{r}\frac{dR}{dz}\right)$$

$$\text{or, } R'' = \frac{d}{dz}\left(\frac{1}{r}\frac{dR}{dz}\right)\frac{dz}{dr}$$

$$= \left(\frac{1}{r}\frac{d^2R}{dz^2} - \frac{1}{r^2}\frac{dr}{dz}\frac{dR}{dz}\right)\frac{1}{r}$$

$$= \frac{1}{r^2}\frac{d^2R}{dz^2} - \frac{1}{r^2}.r.\frac{1}{r}\frac{dR}{dz}$$

$$= \frac{1}{r^2}\left(\frac{d^2R}{dz^2} - \frac{dR}{dz}\right)$$

$$\text{or, } r^2R'' = \frac{d^2R}{dz^2} - \frac{dR}{dz}$$

Putting these values in (x) we get,

$$\frac{d^2R}{dz^2} - \frac{dR}{dz} + 2\frac{dR}{dz} - n(n+1)R = 0$$

or, $\frac{d^2R}{dz^2} + \frac{dR}{dz} - n(n+1)R = 0$ which is linear homogeneous differential equation.

So, the auxiliary equation is $m^2 + m - n(n+1) = 0$

or, $m^2 - n^2 + m - n = 0$

or, $(m-n)(m+n+1) = 0$

$\therefore m = n, \quad m = -(n+1)$

Hence the solution is $R(r) = Ee^{nz} + Fe^{-(n+1)z}$

$$= E\left(e^z\right)^n + F\left(e^z\right)^{-(n+1)}$$

$$= E r^n + F r^{-(n+1)} \quad \text{.............(xi)} \qquad [\because r = e^z]$$

Therefore the required solution of equation (i) is

$$u(r,\theta,\varphi) = (E r^n + F r^{-(n+1)})(C\cos m\theta + D\sin m\theta)\left\{A\, P_n^m(\cos\varphi) + B\, Q_n^m(\cos\varphi)\right\} \text{ Ans.}$$

Here, A, B, C, D, E, F are arbitrary constants.

[N.B: $$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\cot\varphi}{r^2}\frac{\partial u}{\partial \varphi} + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

or, $$\frac{1}{r^2}\left[r^2\frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \varphi^2} + \frac{\cos\varphi}{\sin\varphi}\frac{\partial u}{\partial \varphi} + \frac{1}{\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2}\right] = 0.$$

or, $$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\varphi}\left(\sin\varphi\frac{\partial^2 u}{\partial \varphi^2} + \cos\varphi\frac{\partial u}{\partial \varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2}\right] = 0.$$

or, $$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial u}{\partial \varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2}\right] = 0.$$ which is also the Laplace equation in three dimensional Spherical polar coordinates (r,θ,φ). [Another form]]