Application of Partial Differential Equations

 # Problem: A rectangular plate with insulated surfaces is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge $y = 0$ is given by

$$u(x,0) = \begin{cases} 20x, & 0 < x \le 5 \\ 20\,(10 - x), & 5 < x < 10 \end{cases}$$

while the two long edges $x = 0$ and $x = 10$ as well as the other short edges are kept at 0^0C. Find the steady state temperature at any point (x, y) of the plate.

Solution: In the steady state, the temperature $u(x, y)$ at any point $p(x, y)$ satisfy the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

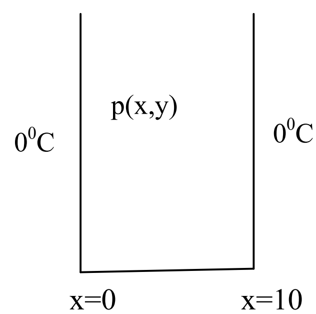
The boundary conditions are

$$u(0, y) = 0 \text{ for all values of } y \qquad (2)$$

$$u(10, y) = 0 \text{ for all values of } y \qquad (3)$$

$$u(x, \infty) = 0 \text{ for all values of } x \qquad (4)$$

$$u(x,0) = \begin{cases} 20x, & 0 < x \le 5 \\ 20\,(10 - x), & 5 < x < 10 \end{cases} \qquad (5)$$



Now three possible solutions of (1) are

$$u(x, y) = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \qquad (6)$$

$$u(x, y) = (C_5 \cos px + C_6 \sin px)(C_7 e^{py} + C_8 e^{-py}) \qquad (7)$$

$$u(x, y) = (C_9 x + C_{10})(C_{11} y + C_{12}) \qquad (8)$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solutions (6) and (8) cannot satisfy the condition (2), (3) and (4). Thus, only possible solution is (7) i.e., of the form.

$$u(x, y) = (C_5 \cos px + C_6 \sin px)(C_7 e^{py} + C_8 e^{-py}) \qquad (9)$$

By (2), $u(0, y) = C_5 (C_7 e^{py} + C_8 e^{-py}) = 0$ for all values of y

$\therefore \; C_5 = 0$

$\therefore$ Eq. (9) reduces to $u(x, y) = C_6 \sin px (C_7 e^{py} + C_8 e^{-py})$ (10)

By (3), $u(10, y) = C_6 \sin 10p (C_7 e^{py} + C_8 e^{-py}) = 0,$

$$\because \; C_6 \ne 0, \quad \therefore \; \sin 10p = 0 = \sin n\pi$$
$$10p = n\pi$$
$$\Rightarrow p = \frac{n\pi}{10}$$

Also to satisfy the condition (4), i.e., $u = 0$ as $y \to \infty$, $\therefore \; C_7 = 0$.

Hence (10) takes the form $u(x,y) = C_6 C_8 \sin px e^{-py}$

According to the conditions (2)-(4), the general solution is $u(x,y) = \sum_{n=1}^{\infty} b_n \sin px e^{-py}$ (11)

Putting $y = 0$, $u(x,y) = \sum_{n=1}^{\infty} b_n \sin px$, where $p = \dfrac{n\pi}{10}$ and $b_n = C_6 C_8$

Using Fourier series, $b_n = \dfrac{2}{10}\int_0^5 20x \sin px dx + \dfrac{2}{10}\int_5^{10} 20(10-x)\sin px dx$

$$b_n = 4\int_0^5 x \sin px dx + 4\int_5^{10} (10-x)\sin px dx$$

$$= 4\left[x\left(\frac{-\cos px}{p}\right) - (1)\left(\frac{-\sin px}{p^2}\right)\right]_0^5 + 4\left[(10-x)\left(\frac{-\cos px}{p}\right) - (-1)\left(\frac{-\sin px}{p^2}\right)\right]_5^{10}$$

$$= 4\left[\frac{-5\cos 5p}{p} + \frac{\sin 5p}{p^2}\right] + 4\left[0 - \frac{\sin 10p}{p^2} + \frac{5\cos 5p}{p} + \frac{\sin 5p}{p^2}\right]$$

$$= 4\left[\frac{2\sin 5p}{p^2} - \frac{\sin 10p}{p^2}\right] = \frac{4}{p^2}\left[2\sin 5p - \sin 10p\right]$$

$$= \frac{400}{n^2\pi^2}\left[2\sin\frac{n\pi}{2} - \sin n\pi\right] = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Putting the value of b_n in (11) the temperature at any point (x,y) is given by

$$u(x,y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}\, e^{-n\pi y/10}$$

Problem: The diameter of a semi-circular plate of radius a is kept at 0^0C and the temperature at the semi-circular boundary is T^0C. Find the steady state temperature at the plate.

Solution: Let the center O of the semi-circular plate be the pole and the bounding diameter be as the initial line. Let $u(r,\theta)$ be the steady state temperature at any point $p(r,\theta)$ and u satisfies the equation

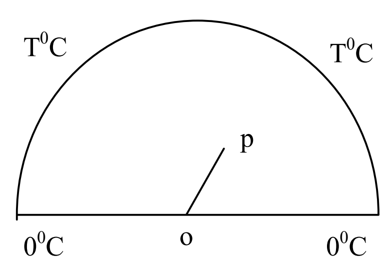
$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad (1)$$

The boundary conditions are

$u(r,0) = 0$, $0 \le r \le a$ (2)

$u(r,\pi) = 0$, $0 \le r \le a$ (3)

$u(a,\theta) = T$. (4)



From condition (3) and (4), we have $u \to 0$ as $r \to 0$.

Hence the solution of (1) is

$$u(x,y)=(C_1r^p+C_2r^{-p})(C_3\cos p\theta+C_4\sin p\theta) \tag{5}$$

Putting, $u(r,0)=0$

$$0=(C_1r^p+C_2r^{-p})C_3$$

$$\therefore C_3=0$$

Eq. (5) becomes

$$u(x,y)=(C_1r^p+C_2r^{-p})C_4\sin p\theta \tag{6}$$

Putting, $u(r,\pi)=0$

$$0=(C_1r^p+C_2r^{-p})C_4\sin p\pi$$

$$\therefore \sin p\pi=0$$
$$\Rightarrow p\pi=n\pi$$
$$\therefore p=n$$

Eq. (6) becomes

$$u(x,y)=(C_1r^n+C_2r^{-n})C_4\sin n\theta \tag{7}$$

Since $u=0$ when $r=0$, $\therefore C_2=0$

Eq. (7) becomes

$$u(x,y)=C_1C_4r^n\sin n\theta$$

The general solution of (1) is $u(r,\theta)=\sum_{n=1}^{\infty}b_nr^n\sin n\theta$ (8)

putting $r=a$ and $u=T$ in (8) we get

$$T=\sum_{n=1}^{\infty}b_na^n\sin n\theta$$

By Fourier half range series, we get

$$\begin{aligned}
b_na^n &= \frac{2}{\pi}\int_0^{\pi}T\sin n\theta\,d\theta\\
&= \frac{2}{\pi}T\left(\frac{-\cos n\theta}{n}\right)_0^{\pi}\\
&= \frac{2T}{n\pi}[-(-1)^n+1]
\end{aligned}$$

$$\therefore b_na^n = 0 \quad \textit{when n is even}$$
$$= \frac{4T}{n\pi} \quad \textit{when n is odd}$$

$$\therefore b_n=\frac{4T}{n\pi\,a^n} \quad \textit{when n is odd}$$

Hence Eq. (8) becomes $u(r,\theta)=\sum_{n=1}^{\infty}\frac{4T}{n\pi a^n}r^n\sin n\theta$ when n is odd. Ans.