Fourier Series

Introduction: French Mathematician Joseph B. J. Fourier (1768-1830) at first uses his trigonometric series (Fourier series) for heat transfer in physics. According to his name this trigonometric series is called Fourier series. Now Fourier series is applied to solve the problem in various branches of physics.

Periodic functions: A function $f(x)$ of real variable x is said to be periodic if there exists a non zero number T independent of x, such that $f(x+T)=f(x)$.

Examples: $f(x)=\sin x$ is periodic with period 2π.

Odd or Even functions: A function $f(x)$ is said to be odd if it changes sign with the change of sign of the variable x that is if $f(-x)=-f(x)$,

and A function $f(x)$ is said to be even if it does not changes sign with the change of sign of the variable x that is if $f(-x)=f(x)$.

Examples: (i) $f(x)=x^3$ is odd function and (ii) $f(x)=x^2$ is even function.

Fourier series: Under certain condition of the function $f(x)$ in a period $-\pi \le x \le \pi$ can be represent a trigonometric series $\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$ which is known as Fourier series with the Fourier constants or coefficients a_0, a_n, b_n.

Determination of the Fourier constants in a period $-\pi \le x \le \pi$ or 2π :

We consider that the Fourier series in a period $-\pi \le x \le \pi$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)\text{....................(i)}$$

Now integrating (i) in $(-\pi,\pi)$ we get,

$$\int_{-\pi}^{\pi} f(x)dx=\frac{a_0}{2}\int_{-\pi}^{\pi}dx+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nxdx+b_n\int_{-\pi}^{\pi}\sin nxdx\right)$$

$$\text{or, } \int_{-\pi}^{\pi} f(x)dx=\frac{a_0}{2}[x]_{-\pi}^{\pi}+0+0$$

$$\text{or, } \int_{-\pi}^{\pi} f(x)dx=\frac{a_0}{2}\times 2\pi$$

$$\therefore a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx\text{.....................(ii)}$$

Multiplying (i) by $\cos nx$ and integrating in $(-\pi,\pi)$ we get,

$$\int_{-\pi}^{\pi} f(x)\cos nxdx=\frac{a_0}{2}\int_{-\pi}^{\pi}\cos nxdx+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos^2 nxdx+b_n\int_{-\pi}^{\pi}\sin nx\cos nxdx\right)$$

$$\text{or, } \int_{-\pi}^{\pi} f(x)\cos nxdx=0+a_n\pi+0$$

$$\therefore a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx\text{....................(iii)}$$

Similarly multiplying (i) by $\sin nx$ and integrating in $(-\pi,\pi)$ we get,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \text{ ....(iv)}$$

Therefore the Fourier constants are: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

Note: i) If $f(x)$ is an even function then we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx \neq 0,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \neq 0,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$$

ii) If $f(x)$ is an odd function then we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx = 0,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = 0,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \neq 0$$

Determination of the Fourier constants in a period $2L$:

In a addition to the Fourier series with period 2π we like to develop a Fourier series with period $2L$. Let us consider the function $\phi(y)$ in $(-L,\ L)$ which is integrable.

Let, $y = \frac{Lx}{\pi}$ then $\phi\left(\frac{Lx}{\pi}\right)$ is the function of period 2π

and $f(x) = \phi\left(\frac{Lx}{\pi}\right) = \phi(y)$

In that case the Fourier series for $f(x)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$ will be converted to a Fourier series for $\phi(y)$ as

$$\phi(y) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi y}{L} + b_n \sin\frac{n\pi y}{L}\right)$$

Hence the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) dx = \frac{1}{L}\int_{-L}^{L}\phi(y) dy,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{L}\int_{-L}^{L}\phi(y)\cos\frac{n\pi y}{L} dy,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{L}\int_{-L}^{L}\phi(y)\sin\frac{n\pi y}{L} dy.$$

Theorem: State and Prove Parseval's theorem.

Statement: If the Fourier series of a function $f(x)$ is converses uniformly in the interval (-L, L) then

$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$ where a_0, a_n, b_n are Fourier constants.

Proof: We have the Fourier series of a function $f(x)$ in the interval $(-L,\ L)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)\text{.....................(i)}$$

Multiplying both sides of (i) by $f(x)$ and integrating with respect to x in the interval $(-L,\ L)$

we get, $\int_{-L}^{L}\{f(x)\}^2dx=\frac{a_0}{2}\int_{-L}^{L}f(x)dx+\sum_{n=1}^{\infty}\left(a_n\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx+b_n\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx\right)$...............(ii)

But, we know by Fourier coefficients

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx,\qquad \text{or, }\int_{-L}^{L}f(x)dx=La_0,$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx,\ \text{or, }\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx=La_n,$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx.\ \text{or, }\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx=Lb_n$$

Using these values in (ii) we get,

$$\int_{-L}^{L}\{f(x)\}^2dx=\frac{a_0^2}{2}.L+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)L$$

or, $\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2dx=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)$ which is the Parseval's theorem (Proved).

Question 1: What is half range cosine series? Expand the half range cosine series.

Or, What is cosine series? Expand the cosine series.

Fisrt Part: The part which contains only the cosine term in Fourier series is called half range cosine series. Its range is $(0,\pi)$ which is half range of $(-\pi,\pi)$ of Fourier series. In the case of even function cosine term remain.

Expansion: The cosine series in the range $(0,\pi)$ is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$ ....(i)

Now integrating both sides with respect to x in the limit 0 to π we obtain

$$\int_0^{\pi}f(x)dx=\frac{a_0}{2}\int_0^{\pi}dx+\sum_{n=1}^{\infty}a_n\int_0^{\pi}\cos nx dx$$

$$=\frac{a_0}{2}[x]_0^{\pi}+0$$

$$=\frac{a_0}{2}\times\pi$$

$$\therefore a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx\text{ ....(ii)}$$

Multiplying (i) by $\cos nx$ and integrating in $(0,\pi)$ we get,

$$\int_0^{\pi}f(x)\cos nx dx=\frac{a_0}{2}\int_0^{\pi}\cos nx dx+\sum_{n=1}^{\infty}a_n\int_0^{\pi}\cos^2 nx dx$$

$$=0+\frac{a_n}{2}\int_0^{\pi}2\cos^2 nx dx$$

$$=0+\frac{a_n}{2}\int_0^{\pi}(1+\cos 2nx)dx$$

$$=0+\frac{a_n}{2}\times\pi$$

$$\therefore a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx\,\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Hence the required cosine series $f(x)=\frac{1}{\pi}\int_0^{\pi}f(x)dx+\frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi}f(x)\cos^2 nxdx$ [Using (ii) & (iii) in (i)]

Question 2: What is half range sine series? Expand the half range sine series.

Or, What is sine series? Expand the sine series.

Fisrt Part: The part which contains only the sine term in Fourier series is called half range sine series. Its range is $(0,\pi)$ which is half range of $(-\pi,\pi)$ of Fourier series. In the case of odd function sine term remain.

Expansion: The sine series in the range $(0,\pi)$ is $f(x)=\sum_{n=1}^{\infty}b_n\sin nx\,\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$

Multiplying (i) by $\sin nx$ and integrating in $(0,\pi)$ we get,

$$\int_0^{\pi}f(x)\sin nxdx=\sum_{n=1}^{\infty}b_n\int_0^{\pi}\sin^2 nxdx$$

$$=\frac{b_n}{2}\int_0^{\pi}2\sin^2 nxdx$$

$$=\frac{b_n}{2}\int_0^{\pi}(1-\cos 2nx)dx$$

$$=\frac{b_n}{2}\times\pi$$

$$\therefore b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nxdx\,\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Hence the required sine series $f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi}f(x)\sin^2 nxdx$ [Using (ii) in (i)]

Dirichlet's conditions: Let any function $f(x)$

i) $f(x)$ is defined in $(-L,L)$ except some finite points.

ii) $f(x)$ is with period $2L$ outside $(-L,L)$ and

iii) in $(-L,L)$ the function $f(x)$ and $f'(x)$ are continuous.

Then the series $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$

where, $a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx,\quad (n=0)$

$$a_n==\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx,\qquad (n=0,1,2,\cdots\cdots\cdots)$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx\qquad (n=0,1,2,\cdots\cdots\cdots)$$

a) $f(x)$ is convergent to $f(x)$ if $f(x)$ is continuous at x.

b) $f(x)$ is convergent to $\dfrac{f(x+0)+f(x-0)}{2}$ if $f(x)$ is discontinuous at x.

Problem-1: Expand the Fourier series for $f(x)=x^2$ within $-\pi<x<\pi$,

Hence show that $\dfrac{\pi^2}{6}=\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots\cdots\cdots$

Solution: Given that $f(x)=x^2$ within $-\pi<x<\pi$,

Since, $f(-x)=(-x)^2=x^2=f(x)$, So, the function $f(x)$ is even and in Fourier series $b_n=0$.

In this case the Fourier series is $f(x)=\dfrac{a_0}{2}+\sum\limits_{n=1}^{\infty}a_n\cos nx$ ....(i)

Here, $a_0=\dfrac{2}{\pi}\int\limits_0^{\pi}f(x)dx$

$$=\frac{2}{\pi}\int_0^{\pi}x^2dx=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^2\cos nxdx=\frac{2}{\pi}\left[x^2\sin nx\right]_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}2x\times\frac{\sin nx}{n}dx$$

$$=0-\frac{4}{n\pi}\left[-\frac{x\cos nx}{n}\right]_0^{\pi}+\frac{4}{n\pi}\int_0^{\pi}\frac{-\cos nx}{n}dx$$

$$=\frac{4}{n^2\pi}(\pi\cos n\pi-0)-\frac{4}{n^2\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$=\frac{4\cos n\pi}{n^2}-0$$

$$=\frac{4(-1)^n}{n^2}$$

Now putting these values in (i) we get the expanded Fourier series

$$x^2=\frac{1}{2}\times\frac{2\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \text{ Ans.}$$

$$\text{or, } x^2=\frac{\pi^2}{3}+4\left(-\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}-\frac{\cos 3x}{3^2}+\cdots\cdots\cdots\right)$$

$$\text{or, } x^2=\frac{\pi^2}{3}-4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\cdots\cdots\cdots\right) \text{ Ans.}$$

Putting $x=\pi$ in above equation we get,

$$\pi^2=\frac{\pi^2}{3}-4\left(\frac{\cos\pi}{1^2}-\frac{\cos 2\pi}{2^2}+\frac{\cos 3\pi}{3^2}-\cdots\cdots\cdots\right)$$

$$\text{or, } \frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots\cdots \quad \text{(showed).}$$

Problem-2: Expand the Fourier series for $f(x)=\begin{cases}-k, & -\pi<x<0\\ k, & 0<x<\pi\end{cases}$

Hence show that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\cdots\cdots$

Solution: Given that $f(x)=\begin{cases}-k, & -\pi<x<0\\ k, & 0<x<\pi\end{cases}$

We know that the Fourier series in $-\pi<x<\pi$ is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$............(i)

Here, $$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}f(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-k)dx+\frac{1}{\pi}\int_{0}^{\pi}k\,dx$$

$$=\frac{-k}{\pi}[x]_{-\pi}^{0}+\frac{k}{\pi}[x]_{0}^{\pi}$$

$$=\frac{-k}{\pi}(0+\pi)+\frac{k}{\pi}(\pi-0)$$

$$=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}-k\cos nxdx+\frac{1}{\pi}\int_{0}^{\pi}k\cos nxdx$$

$$=\frac{-k}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}+\frac{k}{\pi}\left[\frac{\sin nx}{n}\right]_{0}^{\pi}$$

$$=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}-k\sin nxdx+\frac{1}{\pi}\int_{0}^{\pi}k\sin nxdx$$

$$=\frac{-k}{\pi}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0}+\frac{k}{\pi}\left[\frac{-\cos nx}{n}\right]_{0}^{\pi}$$

$$=\frac{k}{n\pi}(1-\cos n\pi)-\frac{k}{n\pi}(\cos n\pi-1)$$

$$=\frac{2k}{n\pi}(1-\cos n\pi)$$

$$=\frac{2k}{n\pi}(1-(-1)^n)$$

Putting the value of a_0, a_n, b_n in (i), $f(x)=0+\sum_{n=1}^{\infty}\left(0\times\cos nx+\frac{2k}{n\pi}\{1-(-1)^n\}\sin nx\right)$

$$=\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\{1-(-1)^n\}\sin nx$$

$$\text{or, } f(x)=\frac{2k}{\pi}\left\{\frac{2\sin x}{1}+0+\frac{2\sin 3x}{3}+0+\frac{2\sin 5x}{5}+0+\frac{2\sin 7x}{7}+\cdots\cdots\right\}\text{...........(ii)}$$

Now putting $x=\frac{\pi}{2}$ in (ii) we get,

$$f(\pi/2)=\frac{4k}{\pi}\left\{\frac{\sin\pi/2}{1}+\frac{\sin 3\pi/2}{3}+\frac{\sin 5\pi/2}{5}+\frac{\sin 7\pi/2}{7}+\cdots\cdots\right\}$$

$$\text{or, } k=\frac{4k}{\pi}\left\{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\cdots\right\}$$

$$\text{or, } \frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\cdots \qquad \text{(Proved)}$$

Problem-3: If $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in the range 0 to 2π then show that $f(x)=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$.

Solution: Given that $f(x)=\left(\frac{\pi-x}{2}\right)^2=\frac{\pi^2}{4}-\frac{\pi x}{2}+\frac{x^2}{4}$

We know that the Fourier series is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$............(i)

Here, $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx$

$$=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi^2}{4}-\frac{\pi x}{2}+\frac{x^2}{4}\right)dx$$

$$=\frac{1}{\pi}\left[\frac{\pi^2 x}{4}-\frac{\pi x^2}{4}+\frac{x^3}{12}\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[\frac{\pi^2.2\pi}{4}-\frac{\pi.4\pi^2}{4}+\frac{8\pi^3}{12}\right]$$

$$=\frac{\pi^2}{6}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nxdx$$

$$=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi^2}{4}-\frac{\pi x}{2}+\frac{x^2}{4}\right)\cos nxdx$$

$$=\frac{\pi}{4}\int_0^{2\pi}\cos nxdx-\frac{1}{2}\int_0^{2\pi}x\cos nxdx+\frac{1}{4\pi}\int_0^{2\pi}x^2\cos nxdx$$

$$=\frac{\pi}{4}\left[\frac{\sin nx}{n}\right]_0^{2\pi}-\frac{1}{2}\left[\frac{x\sin nx}{n}\right]_0^{2\pi}+\frac{1}{2}\left[\frac{-\cos nx}{n^2}\right]_0^{2\pi}+\frac{1}{4\pi}\left[\frac{x^2\sin nx}{n}\right]_0^{2\pi}-\frac{1}{4\pi}\int_0^{2\pi}2x\frac{\sin nx}{n}dx$$

$$=0-0-\frac{1}{2n^2}(\cos 2n\pi-1)+0-\frac{1}{2n\pi}\left[\frac{-x\cos nx}{n}\right]_0^{2\pi}+\frac{1}{2n\pi}\left[\frac{-\sin nx}{n^2}\right]_0^{2\pi}$$

$$=-0+\frac{1}{2n^2\pi}(2\pi\cos 2n\pi-0)+0$$

$$= \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi^2}{4} - \frac{\pi x}{2} + \frac{x^2}{4}\right)\sin nx\,dx$$

$$= \frac{\pi}{4}\int_0^{2\pi}\sin nx\,dx - \frac{1}{2}\int_0^{2\pi}x\sin nx\,dx + \frac{1}{4\pi}\int_0^{2\pi}x^2\sin nx\,dx$$

$$= \frac{\pi}{4}\left[\frac{-\cos nx}{n}\right]_0^{2\pi} - \frac{1}{2}\left[\frac{-x\cos nx}{n}\right]_0^{2\pi} + \frac{1}{2}\left[\frac{-\sin nx}{n^2}\right]_0^{2\pi} + \frac{1}{4\pi}\left[\frac{-x^2\cos nx}{n}\right]_0^{2\pi} - \frac{1}{4\pi}\int_0^{2\pi}2x\frac{(-\cos nx)}{n}dx$$

$$= 0 + \frac{1}{2n}(2\pi\cos 2n\pi - 0) - 0 - \frac{1}{4n\pi}(4\pi^2\cos 2n\pi - 0) + \frac{1}{2n\pi}\left[\frac{-x\sin nx}{n}\right]_0^{2\pi} - \frac{1}{2n\pi}\left[\frac{-\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{\pi}{n} - \frac{\pi}{n} - 0 + \frac{1}{2n^3\pi}(\cos 2n\pi - 1)$$

$$= 0$$

Putting the values of a_0, a_n, b_n in (i), $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ (Showed)

Problem-4: Find the Fourier series in the expansion of a function represented by $f(x) = \begin{cases} 0, if\ -\pi < x < 0 \\ 2x^3, if\ 0 < x < \pi. \end{cases}$

Solution: Given that $f(x) = \begin{cases} 0, if\ -\pi < x < 0 \\ 2x^3, if\ 0 < x < \pi. \end{cases}$

We know that the Fourier series in $-\pi < x < \pi$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)$...........(i)

Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0dx + \frac{1}{\pi}\int_0^{\pi} 2x^3dx$$

$$= 0 + \frac{2}{\pi}\left[\frac{x^4}{4}\right]_0^{\pi}$$

$$= \frac{\pi^3}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\times\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} 2x^3\cos nx\,dx$$

$$= 0 + \frac{2}{\pi}\left[\frac{x^3\sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} 3x^2\frac{\sin nx}{n}dx$$

$$= 0 - \frac{6}{n\pi}\left[\frac{-x^2\cos nx}{n}\right]_0^{\pi} + \frac{6}{n\pi}\int_0^{\pi} 2x\frac{-\cos nx}{n}dx$$

$$= \frac{6}{n\pi}\times\frac{\pi^2(-1)^n}{n} - \frac{12}{n^2\pi}\left[\frac{x\sin nx}{n}\right]_0^{\pi} + \frac{12}{n^2\pi}\left[\frac{-\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{6\pi(-1)^n}{n^2} - 0 - \frac{12}{n^4\pi}\{(-1)^n - 1\}$$

$$= \frac{6\pi(-1)^n}{n^2} - \frac{12}{n^4\pi}\{(-1)^n - 1\}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\times\sin nx dx + \frac{1}{\pi}\int_0^{\pi} 2x^3\sin nx dx$$

$$= 0 + \frac{2}{\pi}\left[\frac{x^3(-\cos nx)}{n}\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} 3x^2\frac{(-\cos nx)}{n}dx$$

$$= -\frac{2}{n\pi}(\pi^3\cos n\pi - 0) + \frac{6}{n\pi}\left[\frac{x^2\sin nx}{n}\right]_0^{\pi} - \frac{6}{n\pi}\int_0^{\pi} 2x\frac{\sin nx}{n}dx$$

$$= -\frac{2\pi^2(-1)^n}{n} + \frac{6}{n^2\pi}(0) - \frac{12}{n^2\pi}\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{12}{n^2\pi}\left[\frac{-\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2\pi^2(-1)^{n+1}}{n} + \frac{12}{n^3\pi}\{\pi(-1)^n - 0\} - 0$$

$$= \frac{2\pi^2(-1)^{n+1}}{n} + \frac{12(-1)^n}{n^3}$$

Putting the value of a_0, a_n, b_n in (i),

$$f(x) = \frac{\pi^3}{4} + \sum_{n=1}^{\infty}\left[\left(\frac{6\pi(-1)^n}{n^2} - \frac{12}{n^4\pi}((-1)^n - 1)\right)\cos nx + \left\{\frac{2\pi^2(-1)^{n+1}}{n} + \frac{12(-1)^n}{n^3}\right\}\sin nx\right] \text{ Ans.}$$

Problem-5: Expand the Fourier series in $-2 < x < 2$ for $f(x) = \begin{cases} 2; & -2 < x < 0 \\ x; & 0 < x < 2 \end{cases}$

Solution: Given that $f(x) = \begin{cases} 2; & -2 < x < 0 \\ x; & 0 < x < 2 \end{cases}$

We know that the Fourier series in $-2 < x < 2$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{2} + b_n\sin\frac{n\pi x}{2}\right)$.....(i)

Here, $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx$

$$= \frac{1}{2}\int_{-2}^{0} f(x)dx + \frac{1}{2}\int_0^{2} f(x)dx$$

$$= \frac{1}{2}\int_{-2}^{0} 2dx + \frac{1}{2}\int_0^{2} x\,dx$$

$$= [x]_{-2}^{0} + \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2$$

$$= (0+2) + \frac{1}{2}(2-0)$$

$$= 3$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx$$

$$= \left[\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_{-2}^{0} + \frac{1}{2}\left[\frac{x\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2 - \frac{1}{2}\left[\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]_0^2$$

$$= 0 + 0 + \frac{2}{n^2\pi^2}(\cos n\pi - 1)$$

$$= \frac{2}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \frac{1}{2}\int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx$$

$$= \left[\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_{-2}^{0} + \frac{1}{2}\left[\frac{-x\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2 - \frac{1}{2}\left[\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right]_0^2$$

$$= -\frac{2}{n\pi}(1-\cos n\pi) - \frac{1}{n\pi}(2\cos n\pi - 0) + 0$$

$$= -\frac{2}{n\pi}$$

Putting the values of a_0, a_n, b_n in (i), $f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}\left\{(-1)^n - 1\right\}\cos\frac{n\pi x}{2} - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]$ Ans.

Home Works

Problem-6: Expand the Fourier series in $-\pi < x < \pi$ for $f(x) = x$

Problem-7: Expand the Fourier series in $-\pi < x < \pi$ for $f(x) = \begin{cases} -\frac{\pi}{4}; & -\pi < x < 0 \\ \frac{\pi}{4}; & 0 < x < \pi \end{cases}$

Problem-8: Expand the Fourier series in $-\pi < x < \pi$ for $f(x) = \begin{cases} -1; & -\pi < x < 0 \\ 0; & x = 0 \\ 1; & 0 < x < \pi \end{cases}$

Problem-9: Expand the Fourier series in $-\pi < x < \pi$ for $f(x)=\begin{cases} -x; & -\pi < x < 0 \\ x; & 0 < x < \pi \end{cases}$

Problem-10: Expand the Fourier series for $f(x)=\begin{cases} 0; & -2 < x < 0 \\ 1; & 0 < x < 2 \end{cases}$

Theorem: State and prove Fourier integral theorem.

Statement: It states that $f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\,dt\,du$.

Proof: We know that the Fourier series of a function $f(x)$ in the interval $(-L,\ L)$ is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)\text{.....................(i)}$$

where the Fourier coefficients are $a_0=\frac{1}{L}\int_{-L}^{L} f(t)dt,\quad a_n=\frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt,\ b_n=\frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt$.

Substituting the values of a_0, a_n, b_n in (i) we get,

$$f(x)=\frac{1}{2L}\int_{-L}^{L} f(t)\,dt+\sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L}dt+\frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}dt\right)$$

$$\text{or, } f(x)=\frac{1}{2L}\int_{-L}^{L} f(t)\,dt+\sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(t)\left[\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L}+\sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}\right]dt\right)$$

$$\text{or, } f(x)=\frac{1}{2L}\int_{-L}^{L} f(t)\,dt+\sum_{n=1}^{\infty}\left(\frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}(t-x)dt\right)$$

$$\text{or, } f(x)=\frac{1}{2L}\int_{-L}^{L} f(t)\left[1+2\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}(t-x)\right]dt\text{ ....(ii)}$$

Since cosine functions are even, i.e; $\cos(-\theta)=\cos\theta$.

So, the expression $1+2\sum_{n=1}^{\infty}\cos\frac{n\pi}{L}(t-x)=\sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{L}(t-x)$.

Then the equation (ii) becomes $f(x)=\frac{1}{2L}\int_{-L}^{L} f(t)\left[\sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{L}(t-x)\right]dt$

$$\text{or, } f(x)=\frac{1}{2\pi}\int_{-L}^{L} f(t)\left[\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{L}(t-x)\right]dt\text{ ....(iii)}$$

Let us assume that L increases indefinitely, so that we may write $\frac{n\pi}{L}=u$ and $\frac{\pi}{L}=du$

This assumption gives, $\lim_{L\to\infty}\left[\frac{\pi}{L}\sum_{n=-\infty}^{\infty}\cos\frac{n\pi}{L}(t-x)\right]=\int_{-\infty}^{\infty}\cos u(t-x)du=2\int_0^{\infty}\cos u(t-x)du$ ....(iv)

Using (iv) in (iii) we get, $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t).2\int_0^{\infty}\cos u(t-x)\,du\,dt$

$$\text{or, } f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\,dt\,du \text{ (proved).}$$

Fourier sine and cosine integrals:

Fourier sine integral is $f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin ut \sin ux \, du\, dt$

and Fourier cosine integral is $f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\cos ut \cos ux \, du\, dt$

Fourier integral transform: We know, the Fourier complex integral is $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{iut} e^{-iux} dt\, du$.

If we put $F(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{iut} dt$ ....(i)

Then we get, $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(u) e^{-iux}\, du$ ....(ii)

If a pair of function is reciprocally related as in equation (i) and (ii) then they are called Fourier integral transform of one another.

Fourier sine and cosine transform:

We know, the Fourier sine integral is $f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\sin ut \sin ux \, du\, dt$ ....(i)

If we put $h(u) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin ut\, dt$ ....(ii)

Then we get from (i), $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty h(u)\sin ux\, du$ ....(iii)

If a pair of function is reciprocally related as in equation (ii) and (iii) then they are called Fourier sine transform of one another.

We know, the Fourier cosine integral is $f(x) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(t)\cos ut \cos ux \, du\, dt$ ....(iv)

If we put $g(u) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos ut\, dt$ ....(v)

Then we get from (iv), $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(u)\cos ux\, du$ ....(vi)

If a pair of function is reciprocally related as in equation (v) and (vi), then they are called Fourier cosine transform of one another.

Problem-1: Determine Fourier sine and cosine transform of $f(x) = e^{-ux}$.

Solution: We know Fourier sine transform $h(v) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin vx\, dx$

or, $h(v) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ux}\sin vx\, dx$

or, $h(v) = \sqrt{\frac{2}{\pi}} \times \frac{v}{u^2+v^2}$ [By Laplace integral transform]

The reciprocal relation is given by $f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty h(v)\sin vx\,dv$

$$\text{or, } f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\times\frac{v}{u^2+v^2}\sin vx\,dv$$

$$\text{or, } e^{-ux}=\frac{2}{\pi}\int_0^\infty \frac{v\,\sin vx}{u^2+v^2}\,dv$$

Also, we know Fourier cosine transform $g(v)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos vx\,dx$

$$\text{or, } g(v)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ux}\cos vx\,dx$$

$$\text{or, } g(v)=\sqrt{\frac{2}{\pi}}\times\frac{u}{u^2+v^2}\qquad\text{[By Laplace integral transform]}$$

The reciprocal relation is given by $f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty g(v)\cos vx\,dv$

$$\text{or, } f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\times\frac{u}{u^2+v^2}\cos vx\,dv$$

$$\text{or, } e^{-ux}=\frac{2}{\pi}\int_0^\infty \frac{u\,\cos vx}{u^2+v^2}\,dv$$

Problem-2: Determine Fourier transform of $e^{-x^2/2}$ Or, Determine Fourier cosine transform of $e^{-x^2/2}$.

Solution: We know Fourier cosine transform $g(v)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos vx\,dx$ ....(i)

$$\text{or, } g(v)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2/2}\cos vx\,dx\text{ ....(ii)}$$

Let, $I=\int_0^\infty e^{-x^2/2}\cos vx\,dx$ ....(iii)

$$\text{or, } \frac{\partial I}{\partial v}=-\int_0^\infty x\,e^{-x^2/2}\sin vx\,dx$$

$$=-\left[-\sin vx\,e^{-x^2/2}\right]_0^\infty-\int_0^\infty v\,\cos vx\,e^{-x^2/2}\,dx$$

$$=0-v\int_0^\infty \cos vx\,e^{-x^2/2}\,dx$$

$$=-vI$$

$$\therefore \frac{dI}{I}=-vdv$$

$$\text{or, } LogI=-\frac{v^2}{2}+Logc$$

$$\text{or, } I=c\,e^{-\frac{v^2}{2}}\text{ ....(iv)}$$

when $v=0$, then from (iii) we get,

$$I = \int_0^\infty e^{-x^2/2}\, dx$$

$$= \int_0^\infty e^{-z^2}\sqrt{2}\, dz \qquad \text{Put, } \frac{x}{\sqrt{2}} = z \quad \Rightarrow dx = \sqrt{2}\, dz$$

$$= \sqrt{2}\int_0^\infty e^{-z^2}\, dz$$

$$= \sqrt{2}\times\frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{2}} \text{ ....(v)}$$

Also, when $v = 0$, then from (iv) we get, $I = c$ ....(vi)

Hence from (v) and (vi) we get $c = \sqrt{\frac{\pi}{2}}$

and from (iv) we get, $I = \sqrt{\frac{\pi}{2}}e^{-\frac{v^2}{2}}$

Therefore we obtain from (ii), $g(v) = \sqrt{\frac{2}{\pi}}\times I = \sqrt{\frac{2}{\pi}}\times\sqrt{\frac{\pi}{2}}e^{-\frac{v^2}{2}} = e^{-\frac{v^2}{2}}$

The reciprocal relation is given by $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(v)\cos vx\, dv$

$$\text{or, } e^{-\frac{x^2}{2}} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-\frac{v^2}{2}}\cos vx\, dv \text{ Ans.}$$

Problem-3: Find the Fourier transformation of $F(x) = e^{-|x|}$ in $-\infty < x < \infty$.

Solution: We know Fourier integral transform $f(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(u)e^{-iu\omega}\, du$ .... (i)

$$f(\omega) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} F(u)e^{-iu\omega}\, du + \int_0^\infty F(u)e^{-iu\omega}\, du\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{u}\, e^{-iu\omega}\, du + \int_0^\infty e^{-u}\, e^{-iu\omega}\, du\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{(1-i\omega)u}\, du + \int_0^\infty e^{-(1+i\omega)u}\, du\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{(1-i\omega)u}}{1-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(1+i\omega)u}}{-(1+i\omega)}\right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{1-i\omega} + \frac{1}{1+i\omega}\right] \qquad \left[\because e^0 = 1, \quad e^\infty = 0\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{1+\omega^2}\right]$$

Also the reciprocal relation of (i) is given by

$$F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\omega)\, e^{i\omega x}\, d\omega$$

$$\text{or, } F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\left[\frac{2}{1+\omega^2}\right] e^{i\omega x}\, d\omega$$

$$\text{or, } F(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{1+\omega^2}d\omega \text{ Ans.}$$

Problem-4: Using Fourier transform solve the heat equation with boundary condition:

$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$; $x>0, t>0$; subject to the condition $u(0,t)=0,\ u(x,0)=\begin{cases}1, & 0<x<1\\ 0, & x>1.\end{cases}$ and $u(x,t)$ is bounded.

Solution: Given, $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$

Taking Fourier sine transform on both sides we get,

$$\int_0^{\infty}\frac{\partial u}{\partial t}\sin nx dx=\int_0^{\infty}\frac{\partial^2 u}{\partial x^2}\sin nx dx$$

$$\text{or, } \frac{\partial}{\partial t}\int_0^{\infty} u\sin nx dx=\left[\sin nx\frac{\partial u}{\partial x}\right]_0^{\infty}-n\int_0^{\infty}\cos nx\frac{\partial u}{\partial x}dx$$

$$=0-n[\cos nx\times u]_0^{\infty}-n^2\int_0^{\infty}\sin nx\times u dx \qquad \left[\because \frac{\partial u}{\partial x}\to 0 \text{ as } x\to\infty\right]$$

$$=nu(0,t)-n^2\int_0^{\infty}\sin nx\times u dx \qquad [\because u\to 0 \text{ as } x\to\infty]$$

$$\text{or, } \frac{\partial}{\partial t}\int_0^{\infty} u\sin nx dx=-n^2\int_0^{\infty} u\sin nx dx \text{ ....(i)}$$

$$\text{Let, } F(n,t)=\int_0^{\infty} u(x,t)\sin nx dx \text{ ....(ii)}$$

then from (i) we get,

$$\text{or, } \frac{\partial F(n,t)}{\partial t}=-n^2F(n.t)$$

$$\text{or, } \frac{\partial F(n,t)}{F(n.t)}=-n^2\partial t$$

$$\text{or, } LogF(n,t)=-n^2t+LogA \quad \text{[By integrating]}$$

$$\text{or, } F(n,t)=Ae^{-n^2t} \quad \text{..........................(iii)}$$

$$\text{or, } F(n,0)=A \quad \text{.........................(iv)}$$

$$\text{Again from (ii) we get, } F(n,0)=\int_0^{\infty} u(x,0)\sin nx dx$$

$$=\int_0^1 \sin nx dx \qquad [\because u(x,0)=1,\ 0<x<1]$$

$$=\left[\frac{-\cos nx}{n}\right]_0^1=\frac{1-\cos n}{n}$$

$$\therefore A=\frac{1-\cos n}{n} \qquad \text{[Using (iv)]}$$

Hence from (iii) we get, $F(n,t)=\frac{1-\cos n}{n}e^{-n^2t}$

Now, the reciprocal relation of (ii) is given by $u(x,t)=\frac{2}{\pi}\int_0^\infty F(n,t)\sin nxdn$

$$\text{or, } u(x,t)=\frac{2}{\pi}\int_0^\infty \frac{1-\cos n}{n}e^{-n^2t}\sin nx\,dn$$

Which is the required solution.

Problem-5: Using Fourier transform solve the heat equation with boundary condition: $\frac{\partial F}{\partial t}=\frac{\partial^2 F}{\partial x^2}$; $-\infty<x<\infty, t>0$; subject to the condition: $F(x,t)=f(x)\ when\ t=0,\ \frac{\partial F}{\partial x}=0\ when\ x=\pm\infty$.

Solution: Given, $\frac{\partial F}{\partial t}=\frac{\partial^2 F}{\partial x^2}$

Taking Fourier sine transform on both sides we get,

$$\int_{-\infty}^{\infty}\frac{\partial F}{\partial t}\sin nxdx=\int_{-\infty}^{\infty}\frac{\partial^2 F}{\partial x^2}\sin nxdx$$

$$\text{or, } \frac{\partial}{\partial t}\int_{-\infty}^{\infty}F\sin nxdx=\left[\sin nx\frac{\partial F}{\partial x}\right]_{-\infty}^{\infty}-n\int_{-\infty}^{\infty}\cos nx\frac{\partial F}{\partial x}dx$$

$$=0-n[\cos nx\times F]_{-\infty}^{\infty}-n^2\int_{-\infty}^{\infty}\sin nx\times Fdx \qquad \left[\because \frac{\partial u}{\partial x}\to 0\ as\ x\to\infty\right]$$

$$=-n^2\int_{-\infty}^{\infty}F\sin nx\,dx$$

$$\text{or, } \frac{\partial}{\partial t}\int_{-\infty}^{\infty}F\sin nxdx=-n^2\int_{-\infty}^{\infty}F\sin nxdx\ \text{............(i)}$$

Let, $u(n,t)=\int_{-\infty}^{\infty}F(x,t)\sin nxdx$ ....(ii)

then from (i) we get,

$$\text{or, } \frac{\partial u(n,t)}{\partial t}=-n^2u(n.t)$$

$$\text{or, } \frac{\partial u(n,t)}{u(n.t)}=-n^2\partial t$$

or, $Log\ u(n,t)=-n^2t+LogA$ [By integrating]

or, $u(n,t)=Ae^{-n^2t}$ ....(iii)

or, $u(n,0)=A$ ....(iv)

Again from (ii) we get, $u(n,0)=\int_{-\infty}^{\infty}F(x,0)\sin nxdx$

$$\therefore A=\int_{-\infty}^{\infty}f(x)\sin nxdx \qquad \text{[Using (iv)]}$$

Hence from (iii) we get, $u(n,t)=\int\limits_{-\infty}^{\infty} f(x)\sin nx\, e^{-n^2t}dx$

Now, the reciprocal relation of (ii) is given by $F(x,t)=\dfrac{2}{\pi}\int\limits_{-\infty}^{\infty} u(n,t)\sin nxdn$

$$\text{or, } F(x,t)=\frac{2}{\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(x)\; e^{-n^2t}\sin^2 nx\; dx\; dn$$

Which is the required solution.

Problem-6: Prove that the solution of the boundary value problem $\dfrac{\partial U}{\partial t}=3\dfrac{\partial^2 U}{\partial x^2}$;

$U(0,t)=U(2,t)=0, t>0$; and $U(x,0)=x, \quad 0<x<2$ is $U(x,t)=\sum\limits_{n=1}^{\infty}\dfrac{4(-1)^{n+1}}{n\pi}\sin\left(\dfrac{n\pi x}{2}\right)e^{-\frac{3n^2\pi^2 t}{4}}$.

Proof: Given, $\dfrac{\partial U}{\partial t}=3\dfrac{\partial^2 U}{\partial x^2}$

Taking finite Fourier sine transform (with L=2) on both sides we get,

$$\int\limits_0^2 \frac{\partial U}{\partial t}\sin\left(\frac{n\pi x}{2}\right)dx=\int\limits_0^2 3\frac{\partial^2 U}{\partial x^2}\sin\left(\frac{n\pi x}{2}\right)dx$$

$$\text{or, } \frac{\partial}{\partial t}\int\limits_0^2 U\sin\left(\frac{n\pi x}{2}\right)dx=3\left[\sin\left(\frac{n\pi x}{2}\right)\frac{\partial U}{\partial x}\right]_0^2-\frac{3n\pi}{2}\int\limits_0^2\cos\left(\frac{n\pi x}{2}\right)\frac{\partial U}{\partial x}dx$$

$$=0-\frac{3n\pi}{2}\left[\cos\left(\frac{n\pi x}{2}\right)\times U(x,t)\right]_0^2-\frac{3n^2\pi^2}{4}\int\limits_0^2\sin\left(\frac{n\pi x}{2}\right)\times Udx$$

$$=0-\frac{3n^2\pi^2}{4}\int\limits_0^2 U\sin\left(\frac{n\pi x}{2}\right)dx \qquad [\because U(0,t)=U(2,t)=0]$$

$$\therefore \frac{\partial}{\partial t}\int\limits_0^2 U\sin\left(\frac{n\pi x}{2}\right)dx=-\frac{3n^2\pi^2}{4}\int\limits_0^2 U\sin\left(\frac{n\pi x}{2}\right)dx \text{............(i)}$$

Let, $F(n,t)=\int\limits_0^2 U(x,t)\sin\left(\dfrac{n\pi x}{2}\right)dx$ ....(ii)

then from (i) we get,

$$\text{or, } \frac{\partial F(n,t)}{\partial t}=-\frac{3n^2\pi^2}{4}F(n.t)$$

$$\text{or, } \frac{\partial F(n,t)}{F(n.t)}=-\frac{3n^2\pi^2}{4}\partial t$$

$$\text{or, } Log\ F(n,t)=-\frac{3n^2\pi^2 t}{4}+LogA \quad \text{[By integrating]}$$

$$\text{or, } F(n,t)=Ae^{-\frac{3n^2\pi^2 t}{4}} \quad \text{...........................(iii)}$$

$$\text{or, } F(n,0)=A \quad \text{..........................(iv)}$$

Again from (ii) we get, $F(n,0)=\int\limits_0^2 U(x,0)\sin\left(\dfrac{n\pi x}{2}\right)dx$

$$= \int_0^2 x \sin\left(\frac{n\pi x}{2}\right) dx \qquad [\because U(x,0)=x, \quad 0<x<2]$$

$$= \left[-\frac{2}{n\pi} x \cos\left(\frac{n\pi x}{2}\right) \right]_0^2 + \frac{2}{n\pi} \int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= -\frac{4}{n\pi} \cos n\pi + 0 + \frac{4}{n^2\pi^2} \left[\sin\left(\frac{n\pi x}{2}\right) \right]_0^2$$

$$= -\frac{4}{n\pi} \cos n\pi$$

$$\therefore A = -\frac{4}{n\pi} \cos n\pi \qquad \text{[Using (iv)]}$$

Hence from (iii) we get, $F(n,t) = -\frac{4}{n\pi} \cos(n\pi)\, e^{-\frac{3n^2\pi^2 t}{4}}$

Now, the reciprocal relation of (ii) is given by $U(x,t) = \frac{2}{2} \sum_{n=1}^{\infty} F(n,t) \sin\left(\frac{n\pi x}{2}\right)$

$$\text{or, } U(x,t) = \sum_{n=1}^{\infty} -\frac{4}{n\pi} \cos(n\pi)\, e^{-\frac{3n^2\pi^2 t}{4}} \sin\left(\frac{n\pi x}{2}\right)$$

$$\text{or, } U(x,t) = \sum_{n=1}^{\infty} -\frac{4}{n\pi} (-1)^n\, e^{-\frac{3n^2\pi^2 t}{4}} \sin\left(\frac{n\pi x}{2}\right)$$

$$\text{or, } U(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{3n^2\pi^2 t}{4}} \quad \text{(Proved)}$$

Problem-7: Solve the boundary value problem $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$; where $u(0,y) = 8e^{-3y}$ by the method of variables separation.

Solution: Given, $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ ....(i)

In order to apply the method of separation of variables let, $u(x,y) = X(x).Y(y)$ is a solution of (i).

$\therefore \frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$,

Putting these values in equation (i) we get,

$X'Y = 4XY'$

or, $\frac{X'}{4X} = \frac{Y'}{Y}$ ....(ii)

Since the left hand side of (ii) is a function of x only and the right hand side of (ii) is a function of y only then the equation (ii) is true only if each side is equal to the same constant.

So, Let, $\frac{X'}{4X} = \frac{Y'}{Y} = \lambda$

$or, X' - 4\lambda X = 0,$ or, $Y' - \lambda Y = 0$

The solutions of these equation are, $X = Ae^{4\lambda x}$ and $Y = Be^{\lambda y}$

Therefore the general solution of Eq. (i) is $u(x,y) = ABe^{4\lambda x}e^{\lambda y} = ke^{(4x+y)\lambda}$ Here, *k* are arbitrary constants.

$\therefore u(0,y)=ke^{\lambda y}$

or, $8e^{-3y}=ke^{\lambda y}$ this is possible only when $k=8 \quad and \quad \lambda=-3$

Hence the required solution is $u(x,y)=8e^{-3(4x+y)}$ Ans.

Problem-8: Find the solution of $\frac{\partial^2 u}{\partial x^2}=h^2\frac{\partial u}{\partial t}$ where $u(0,t)=u(l,t)=0$ and $u(x,0)=\sin\left(\frac{\pi x}{l}\right)$ by the method of variables separation.

Solution: Given that, $\frac{\partial^2 u}{\partial x^2}=h^2\frac{\partial u}{\partial t}$.............(i)

In order to apply the method of separation of variables let, $u(x,t)=X(x).T(t)$ is a solution of (i).

$\therefore \frac{\partial u}{\partial x}=X'T$ and $\frac{\partial u}{\partial t}=XT'$,

and $\frac{\partial^2 u}{\partial x^2}=X''T$

Putting these values in equation (i) we get,

$X''T=h^2XT'$

or, $\frac{X''}{h^2X}=\frac{T'}{T}$.............(ii)

Since the left hand side of (ii) is a function of x only and the right hand side of (ii) is a function of t only then the equation (ii) is true only if each side is equal to the same constant.

So, Let, $\frac{X''}{h^2X}=\frac{T'}{T}=-\lambda^2$

$or, X''+\lambda^2h^2X=0,$ or, $T'+\lambda^2T=0$

The solutions of these equation are, $X=A_1\cos\lambda hx+B_1\sin\lambda hx$ and $T=C_1e^{-\lambda^2 t}$

Therefore the general solution of Eq. (i) is $u(x,t)=\left(A_1\cos\lambda hx+B_1\sin\lambda hx\right)C_1e^{-\lambda^2 t}$

$=\left(A\cos\lambda hx+B\sin\lambda hx\right)e^{-\lambda^2 t}$......................(ii)

Here, $A_1C_1=A, \quad B_1C_1=B$ are arbitrary constants.

since $u(0,t)=0$

or, $Ae^{-\lambda^2 t}=0$ this is possible only when $A=0$

Then from Eq. (ii) $u(x,t)=Be^{-\lambda^2 t}\sin\lambda hx$.................(iii)

Again since $u(l,t)=0$

$\therefore Be^{-\lambda^2 t}\sin\lambda hl=0$

or, $\sin\lambda hl=0$ $\quad \left[\because B\neq 0, \quad e^{-\lambda^2 t}\neq 0\right]$

or, $\lambda hl=n\pi$

or, $\lambda=\frac{n\pi}{hl}$

Then from Eq. (iii) $u(x,t)=Be^{-\left(\frac{n\pi}{hl}\right)^2 t}\sin\frac{n\pi x}{l}$.................(iv)

Again since $u(x,0)=\sin\left(\frac{\pi x}{l}\right)$

$B\sin\frac{n\pi x}{l} = \sin\left(\frac{\pi x}{l}\right)$ this is possible only when $B = 1$ and $\frac{n\pi x}{l} = \frac{\pi x}{l} \Rightarrow n = 1$

Then from Eq. (iv), $u(x,t) = e^{-\left(\frac{\pi}{hl}\right)^2 t} \sin\frac{\pi x}{l}$ Ans.

Problem-9: Find the solution of $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$ which is bounded for $x > 0,\quad t = 0$ by the method of variables separation.

Solution: Given that, $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ ....(i)

In order to apply the method of separation of variables let, $u(x,t) = X(x).T(t)$ is a solution of (i).

$\therefore \frac{\partial u}{\partial x} = X'T$ and $\frac{\partial u}{\partial t} = XT'$,

Putting these values in equation (i) we get,

$X'T = 2XT' + XT$

or, $\frac{X'}{X} = 2\frac{T'}{T} + 1$ ....(ii)

Since the left hand side of (ii) is a function of x only and the right hand side of (ii) is a function of t only then the equation (ii) is true only if each side is equal to the same constant.

So, Let, $\frac{X'}{X} = 2\frac{T'}{T} + 1 = c$

$or, X' - cX = 0,$

and $2T' - (c-1)T = 0$

The solutions of these equation are, $X = Ae^{cx}$ and $T = Be^{\frac{(c-1)t}{2}}$

Therefore the general solution of Eq. (i) is $u(x,t) = Ae^{cx}\, Be^{\frac{(c-1)t}{2}}$

$= Ke^{cx + \frac{(c-1)t}{2}}$ ....(iii)

Here, $AB = K$ are arbitrary constants.

since $u(x,0) = 6e^{-3x}$

or, $Ke^{cx} = 6e^{-3x}$ then $K = 6$ and $c = -3$

Then from Eq. (iii) $u(x,t) = 6e^{-3x-2t}$ Ans.

Problem-10: A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = lx - x^2$ from which it is released at time $t = 0$. Find the displacement of any point on the string at a distance of x from one end at time t.

Solution: The equation of vibration of the string is given by $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ ....(i)

As the end points of the string are fixed for all time, so, $y(0,t) = 0$ ....(ii)

and $y(l,t) = 0$ ....(iii)

since the initial transverse velocity of any point of the string is zero, so, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ ....(iv)

and $y(x,0) = lx - x^2$ ....(v)

Solution of (i) is $y = (A_1\cos\lambda x + A_2\sin\lambda x)(A_3\cos\lambda ct + A_4\sin\lambda ct)$ ....(vi)

Using (ii), $y(0,t)=0$

or, $0 = A_1(A_3\cos\lambda ct + A_4\sin\lambda ct)$ so, $A_1 = 0$

Hence from Eq.(vi) becomes, $y = A_2\sin\lambda x(A_3\cos\lambda ct + A_4\sin\lambda ct)$ ....(vii)

$$\therefore \frac{\partial y}{\partial t} = A_2\sin\lambda x(-A_3\lambda c\sin\lambda ct + A_4\lambda c\cos\lambda ct)$$

Using (iv), $0 = A_2A_4\lambda c\sin\lambda x$ $\because A_2 \neq 0, \quad so,\ A_4 = 0.$

Hence Eq. (vii) is reduce to, $y = A_2A_3\sin\lambda x\cos\lambda ct$ ....(viii)

Using (iii), $y(l,t)=0$

or, $0 = A_2A_3\sin\lambda l\cos\lambda ct$

$\therefore \sin\lambda l = 0$

$\Rightarrow \lambda l = n\pi$

$\Rightarrow \lambda = \dfrac{n\pi}{l}$ where $n = 1,2,3,\cdots$

Hence Eq. (viii) becomes $y = b_n\sin\left(\dfrac{n\pi x}{l}\right)\cos\left(\dfrac{n\pi ct}{l}\right)$ where $b_n = A_2A_3$

So the complete solution is $y(x,t) = \sum\limits_{n=1}^{\infty} b_n\sin\left(\dfrac{n\pi x}{l}\right)\cos\left(\dfrac{n\pi ct}{l}\right)$.............(ix)

Using (v) $lx - x^2 = \sum\limits_{n=1}^{\infty} b_n\sin\left(\dfrac{n\pi x}{l}\right)$............................ (x)

Hence by Fourier transformation

$$b_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(-\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{l}\right) + \left(-2\frac{l^3}{n^3\pi^3}\right)\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[(-1)^{n+1}\frac{2l^3}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right] = \begin{cases} \dfrac{8l^2}{n^3\pi^3} & when\ n\ is\ odd \\ 0 & when\ n\ is\ even \end{cases}$$

Putting the value of b_n in (ix), $y(x,t) = \sum\limits_{n=1}^{\infty}\dfrac{8l^2}{n^3\pi^3}\sin\left(\dfrac{n\pi x}{l}\right)\cos\left(\dfrac{n\pi ct}{l}\right)$ when *n* is odd. Ans.