SOLUTION of DIFFERENTIAL EQUATION in SERIES by FROBENIUS METHOD

Roll:1-32

Assigned By

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Power Series

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Power Series: A Power Series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \ldots\ldots\ldots$$

where "$\mathbf{a_n}$" represents the co-efficient of the "$\mathbf{n^{th}}$" term and "**c**" is a constant.

Analytic Function: A function f(x) defined on interval containing the point $x=x_0$ is called analytic at x_0 if its Taylor Series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

exists and converges to f(x) for all x in the interval of the convergence

Hence, we find that all polynomial functions, e^x, sin x, cos x, sinh x and cosh x are analytic everywhere. A rational function is analytic except all those values of x at which its denominator is zero, for example, the rational function defined by $x/(x^2-3x+2)$ is analytic everywhere except at x =1 and x = 2.

Ordinary and Singular Points

Definition:

A point $x=x_0$ is called an ordinary point of the equation.

$$y''+P(x)y'+Q(x)y=0$$

if both the functions P(x) and Q(x) are analytic at $x=x_0$

If the point $x=x_0$ is not an ordinary point of the differential equation, then it is called a **singular point of the differential equation**.

There are two types of singular points:

- Regular singular points
- Irregular singular points

A singular point $x=x_0$ of the differential equation is called a **regular singular point of the differential equation** if both $(x=x_0)P(x)$ and $(x=x_0)^2Q(x)$ are analytic at $x=x_0$

A singular point, which is not regular is called an **irregular singular point of the differential equation**.

Related Mathematical Problems:

<u>Problem 01:</u> Determine whether x=0 is an ordinary point or a regular singular point of the differential equation $2x^2(d^2v/dx^2)+7x(x+1)(dv/dx)-3y=0$

<u>Solution:</u>

Dividing by $2x^2$, the given equation becomes

$$\frac{d^2y}{dx^2}+\frac{7(x+1)}{2x}\frac{dy}{dx}-\frac{3}{2x^2}y=0 \;\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Comparing (1) with standard equation **y" +P(x)y'+ Q(x)y=0**, we have

P(x)=[7(x+1)]/2x

Q(x)= -3/(2x2)

Since both P(x) and Q(x) are undefined at x=0, so both P(x)and Q(x) are not analytic at x=0. Thus x=0 is not an ordinary point and so x=0 is a singular point.

Also, (x-0)P(x)=7(x+1)/2 and $(x-0)^2Q(x)=-3/2$, showing that both (x-0)P(x) and $(x-0)^2Q(x)$ are analytic at x=0. Therefore x=0 is a regular singular point. **(showed)**

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<u>Problem 02:</u> Show that x=0 is an ordinary point of $(x^2-1)y''+xy'-y=0$, but x=1 is a regular singular point.

<u>Solution:</u>

Dividing by (x^2-1), the given equation becomes

$$\frac{d^2y}{dx^2}+\frac{x}{(x-1)(x+1)}\frac{dy}{dx}-\frac{1}{(x-1)(x+1)}y=0 \;\ldots\ldots\ldots\ldots(1)$$

Comparing (1) with standard equation **y"+P(x)y'+Q(x)y=0**, we get

P(x)=x/[(x-1)(x+1)]

Q(x)=-1/{(x-1)(x+1)}

Since both P(x) and Q(x) are analytic at x=0, so x=0 is an ordinary point of the given equation (1)

Since both P(x) and Q(x) are undefined at x=1, so they are not analytic at x=1. Thus x=1 is not an ordinary point and so x=1 is a singular point.

Also $(x-1)P(x)=x/(x+1)$ and $(x-1)^2Q(x)=-(x-1)/(x+1)$, showing that both (x-1)P(x) and $(x-1)^2Q(x)$ are analytic at x=1.

Therefore x=1 is a regular singular point. **(showed)**

<u>Problem 03:</u> Verify that x=0 is a regular singular point of $2x^2y''+y'-(x+1)y=0$ equation.

<u>Solution:</u>

Given that,

$$2x^2y''+y'-(x+1)y=0$$

$$\text{Or, } y'' + \frac{1}{2x}y' - \frac{x+1}{2x^2}y = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Comparing (1) with standard equation,

$$\mathbf{y'' + P(x)y' + Q(x)y = 0}$$

we get,

$P=1/2x$

$Q=-(x+1)/2x^2$

Since both P and Q are not analytic ($P=\infty$, $Q=\infty$) at x=0, therefore x=0 is a singular point

Now,

$P_1=(x-x_0)P(x)$
$\quad =(x-0)\times(1/2x)$
$\quad =1/2$

and,

$Q_1=(x-x_0)^2Q(x)$
$\quad =(x-0)^2[-(x+1)/2x^2]$
$\quad =-(x+1)/2$

Here both P_1 and Q_1 are analytic ($P_1\neq\infty$; $Q_1\neq\infty$) at x=0.

Hence x=0 is a regular singular point of equation (1) **(showed)**

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<u>Problem 04:</u> Show that x=0 and x=1 both are regular singular point for the equation x(x-1)y"+(3x-1)y'+y=0

<u>Solution:</u>

Given that,

$$x(x-1)y''+(3x-1)y'+y=0$$

Or, $y'' + \frac{3x-1}{x(x-1)}y' + \frac{1}{x(x-1)}y = 0$……………(1)

Comparing (1) with standard equation,

$$y''+P(x)y'+Q(x)y=0$$

we get,

$P=\frac{3x-1}{x(x-1)}$

$Q=\frac{1}{x(x-1)}$

Since both P and Q are not analytic (P=∞, Q=∞) at x=0 and x=1, hence both these points are singular point

At x=0,

$$P_1=(x-x_0)P(x)=(x-0)\times\frac{3x-1}{x(x-1)}=\frac{3x-1}{x-1}$$

$$Q_1=(x-x_0)^2Q(x)=(x-0)^2\times\frac{1}{x(x-1)}=\frac{x}{x-1}$$

Since P_1 and Q_1 are analytic ($P_1\neq\infty$; $Q_1\neq\infty$) at x=0. Hence x=0 is a regular singular point.

At x=1,

$$P_1=(x-x_0)P(x)=(x-1)\times\frac{3x-1}{x(x-1)}=\frac{3x-1}{x}$$

$$Q_1=(x-x_0)^2Q(x)=(x-1)^2\times\frac{1}{x(x-1)}=\frac{x-1}{x}$$

Since P_1 and Q_1 are analytic ($P_1\neq\infty$; $Q_1\neq\infty$) at x=1. Hence x=1 is a regular singular point.

(showed)

<u>Problem 05:</u> Find regular singular points of the differential equation.
$\mathbf{x^2(x-2)^2y'' + 2(x-2)y' + (x+3)y = 0}$

<u>Solution:</u>

Given that,

$$x^2(x-2)^2y'' + 2(x-2)y' + (x+3)y = 0$$

or, $y'' + \frac{2}{x^2(x-2)} y' + \frac{x+3}{x^2(x-2)^2} y = 0$

Comparing (1) with standard equation,

$$y'' + P(x)y' + Q(x)y = 0$$

we get,

$$P = \frac{2}{x^2(x-2)}$$
$$Q = \frac{x+3}{x^2(x-2)^2}$$

P and Q are not analytic ($P=\infty$, $Q=\infty$) at x = 0 and x = 2.

Hence both these points are singular points of equation (1).

(i) At x = 0

$P_1 = xP(x) = \frac{2}{x(x-2)}$

$Q_1 = x^2Q(x) = x^2 . \frac{(x+3)}{x^2(x-2)^2} = \frac{(x+3)}{(x-2)^2}$

Since P_1 is not analytic ($P_1 = \infty$) at x = 0, so x = 0 is an irregular singular point.

(ii) At x = 2

$P_1 = (x-2)P(x) = (x-2) . \frac{2(x-2)}{x^2(x-2)^2} = \frac{2}{x^2}$

$Q_1 = (x-2)^2Q(x) = (x-2)^2 . \frac{(x+3)}{x^2(x-2)^2} = \frac{(x+3)}{x^2}$

Since both P_1 and Q_1 are analytic ($P_1 \neq \infty$, $Q_1 \neq \infty$) at x = 2, so x = 2 is a regular singular Point **(showed)**

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Problem 06: **$x^2y'' + xy' + (x^2-4)y = 0$; Prove that x=0 is a singular point**

Solution:

Given that,

$$x^2y'' + xy' + (x^2-4)y = 0$$

Or, $y'' + \frac{1}{x}y' + \frac{x^2-4}{x^2}y = 0$ …………….(1)

Comparing (1) with standard equation,

$$y'' + P(x)y' + Q(x)y = 0$$

$P = \frac{1}{x}$

$Q = \frac{x^2 - 4}{x^2}$

Since both P and Q are not analytic (P=∞, Q=∞) at x=0, therefore x=0 is a singular point for equation (1). **(showed)**

<u>Problem 07:</u> Show that x = 0 is a regular singular point of $\mathbf{x^2y'' + xy' + \frac{(x^2-1)}{4}y = 0}$

<u>Solution:</u>

Dividing by x^2, the given equation becomes,

$y'' + \frac{1}{x}y' + \frac{(x^2-1)}{4x^2}y = 0$…………(1)

Comparing (1) with standard equation,

$$y'' + P(x)y' + Q(x)y = 0 \text{…………(1)}$$

Here, $P(x) = \frac{1}{x}$ and $Q(x) = \frac{x^2-1}{4x^2}$

Since both P(x) and Q(x) are undefined

At x = 0, so they are not analytic at x = 0
So, x = 0 is a singular point

Now, (x-0) P(x)= $x.\frac{1}{x}$ and $(x-0)^2.Q(x) = \frac{x^2-1}{4}$
Both (x-0) P(x) and $(x-0)^2.Q(x)$ are analytic at x = 0

∴ x = 0 is a regular singular point **(showed)**

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<u>Problem 08:</u> Determine whether x = 0 is an ordinary point or singular point of $\mathbf{(2x^2+1)y'' - xy' - y = 0}$

<u>Solution:</u>

Dividing the given equation by $2x^2 + 1$ and we get,

$$y'' - \frac{x}{(2x^2+1)}y' - \frac{1}{(2x^2+1)}y = 0 \ldots\ldots\ldots(1)$$

Comparing (1) with standard equation,

$y'' + P(x)y' + Q(x)y = 0$

We have ,

$$P(x) = \frac{x}{2x^2 + 1}; Q(x) = -\frac{1}{2x^2 + 1}$$

Since, P(x) and Q(x) are analytic at $x = 0$. so, $x = 0$ is an ordinary point at equation (1)

Problem 09: **Determine whether $x = 0$ is an ordinary point or singular point of $\left(1 - x^2\right)y'' - xy' + 4y = 0$**

Solution:

Dividing the given equation by $(1 - x^2)$ and we get,

$$y'' - \frac{x}{(1 - x^2)}y' + \frac{4}{(1 - x^2)}y = 0 \ldots\ldots\ldots.(1)$$

Comparing (1) with standard equation,

$$y'' - P(x)y' + Q(x)y = 0$$

We have,

$$P(x) = -\frac{x}{1-x^2};\ Q(x) = \frac{4}{1-x^2}$$

Since P(x) and Q(x) are analytic at $x = 0$. So, $x = 0$ is an ordinary point at equation (1)

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Problem 10: **Find the ordinary point or singular point of differential equation.**

$$\mathbf{(x^2 + 2)y'' + xy' - (1 + x)y = 0}$$

Solution:

Given that,

$$(x^2 + 2)y'' + xy' - (1 + x)y = 0$$

Dividing the equation with $(x^2 + 2)$ we get,

$$y'' + \frac{x}{x^2+2}y' - \frac{1+x}{x^2+2}y = 0\ldots\ldots\ldots\ldots\ldots(1)$$

Comparing (1) with standard equation,

$$y'' + P_1y' - P_2y = 0$$

We get,

$$P_1 = \frac{x}{x^2 + 2}; \qquad P_2 = \frac{1 + x}{x^2 + 2}$$

Here, P_1 and P_2 are both defined at $x = 0$ $(P_1 \neq \infty, P_2 \neq \infty)$. So, $x = 0$ is an ordinary point at equation (1).

Problem 11: Find the ordinary point or singular point of differential equation.

$$\mathbf{y'' + xy' + x^2y = 0}$$

Solution:

Here,

$$P_1 = x; \qquad P_2 = x^2$$

$y'' + P_1y' + P_2y = 0$

Since both P_1 and P_2 are defined at $x = 0$ $(P_1 \neq \infty, P_2 \neq \infty)$. Here $x = 0$ is an ordinary point of $y'' + xy' + x^2y = 0$

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Problem 12: Show that x = 0 and x = -1 are singular points of $x^2 (x+1)^2 y'' + (x^2-1) y' + 2y = 0$ Where the first is irregular and the other is regular.

Solution:

Dividing by $x^2 (x+1)^2$ the given equation becomes

$$\frac{d^2y}{dx^2} + \frac{x-1}{x^2(x+1)}\frac{dy}{dx} + \frac{2}{x^2(x+1)^2}y = 0$$

Comparing (1) with standard equation

$$\mathbf{y'' + P(x)y' + Qy = 0}$$

we get

$P(x) = (x-1) / [x^2(x+1)]$

$Q(x) = 2 / [x^2(x+1)^2]$

Since both P(x) and Q(x) are undefined at x =0 and x = -1 , so they are not analytic at x = 0 and x = -1 .

Hence x = 0 and x = -1 are both singular points.

Also $(x-0) P(x) = (x-1) / [x(x+1)]$ and $(x-0)^2 Q(x) = 2 / (x+1)^2$,

Showing that P(x) is not analytic at x = 0 and so x=0 is an irregular singular point .

Again,

$(x+1) P(x) = (x-1) / x^2$ and $(x+1)^2 Q(x) = 2/x^2$,

Showing that both $(x+1) P(x)$ and $(x+1)^2 Q(x)$ are analytic at x = -1 and hence x = -1 is a regular singular points.

Problem 13: **Find regular singular points of the differential equation**

$$2x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+(x^2-4)y=0$$

Solution:

$$\frac{d^2y}{dx^2}+\frac{3}{2x}\frac{dy}{dx}+\frac{(x^2-4)}{2x^2}y=0$$

$$P_1=\frac{3}{2x}\ \text{ and }\ P_2=\frac{x^2-4}{2x^2}$$

$$Q_1=x.P_1=x\left(\frac{3}{2x}\right)=\frac{3}{2},\quad Q_2=x^2.P_2=x^2.\frac{x^2-4}{2x^2}=\frac{1}{2}(x^2-4)$$

Since both P_1 and P_2 are not analytic ($P_1=\infty$, $P_2=\infty$) at x = 0 therefore x = 0 is a singular point of (1) . Moreover both Q_1 and Q_2 are analytic ($Q_1\neq\infty$, $Q_2\neq\infty$) at x = 0 . Hence x = 0 is a regular singular point of (1).

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Problem 14: **Find regular singular points of differential equation**

$$x^2(x-4)^2\frac{d^2y}{dx^2}+(x-4)\frac{dy}{dx}+(x+2)y=0$$

Solution:

Given that

$$x^2(x-4)^2\frac{d^2y}{dx^2}+(x-4)\frac{dy}{dx}+(x+2)y=0 \quad \ldots\ldots\ldots\ldots(1)$$

Now, $P_1=\frac{(x-4)}{x^2(x-4)^2}$ $\quad P_2=\frac{(x+2)}{x^2(x-4)^2}$

P_1 and P_2 are not analytic ($P_1=\infty$, $P_2=\infty$) at x=0 and x=4

Hence both these points are singular points of (1)

At x=0

$Q_1=x\,P_1=x.\frac{(x-4)}{x^2(x-4)^2}=\frac{1}{x(x-4)}$

$Q_2=x^2\,P_2=x^2\,\frac{(x+2)}{x^2(x-4)^2}=\frac{(x+2)}{(x-4)^2}$

Since Q_1 is not analytic ($Q_1=\infty$) at x=0, so x=0 is an irregular singular point

At x=4

$Q_1 = (x-4).P_1 = (x-4).\frac{(x-4)}{x^2(x-4)^2} = \frac{1}{x^2}$

$Q_2 = (x-4)^2 P_2 = (x-4)^2 \frac{(x+2)}{x^2(x-4)^2} = \frac{(x+2)}{x^2}$

Since both Q_1 and Q_2 are analytic at x=4, so x= 4 is a regular singular point

Therefore x=0 is an irregular singular point

x=4 is a regular singular point

Problem 15: Show that x=0 is a regular singular point of

$$x^4\frac{d^2y}{dx^2} + x^3\frac{dy}{dx} + x^2y = 0$$

Solution:

Given that, $x^4\frac{d^2y}{dx^2} + x^3\frac{dy}{dx} + x^2y = 0$

Dividing by x^4 the given equations becomes
$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}y = 0$ .... (1)

Now, $P_1 = \frac{1}{x}$ $P_2 = \frac{1}{x^2}$

P_1 and P_2 are not analytic ($P_1=\infty$, $P_2 = \infty$) at x=0

Hence x=0 singular points of (1)
$Q_1 = x P_1 = x.\frac{1}{x} = 1$
$Q_2 = x^2 P_2 = x^2 \frac{1}{x^2} = 1$

Since both Q_1 and Q_2 are analytic at x=0, so x= 0 is a regular singular point
(showed)

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Problem 16: Determine whether x=0 is on ordinary point or singular point of $(x^2+1)y^4+xy^2+xy=0$

Solution:

Dividing the given eq^n by (x^2+1)

$y'' + \frac{x}{x^2+1}y' - \frac{x}{x^2+1}y = 0$ .....(1)

Comparing (1) with standard eq^n

y”+p(x)y’+ Q(x)y=0

We have
$P(x) = \frac{x}{x^2+1}$ and $Q(x) = -\frac{x}{x^2+1}$

since P(x) and Q(x) are analytic at x=0 , so x=0 is an ordinary point of equation (1)

<u>Problem 17:</u> Show that, x=0 is a regular singular point and x=1 is an irregular singular point of

$$x(x-1)^3y''+ 2(x-1)^3y'+3y=0$$

<u>Solution:</u>

dividing by $x(x-1)^3$, the given equation becomes

$y''+\frac{2}{x}y'+\frac{3}{x(x-1)^3}y=0$ ……(1)

comparing (1) with standard equation

$$y''+P(x)y'+\theta(x)y=0$$

we get $p(x)=\frac{2}{x}$, $Q(x)=\frac{3}{x(x-1)^3}$

P(x) and θ(x) both are undefined at x=0.

So they are not analytic. So x=0 is a singular point

$(x-0)p(x) = 2$ and $(x-0)^2Q(x) = \frac{3}{x(x-1)^3}$

Both $(x-0)p(x)$ and $(x-0)^2Q(x)$ are analytic at x=0

$x = 0$ is a regular singular point .

Q(x) is not defined st x=1 so Q(x) is not analytic at x=1.

So,x=1 is a singular point .

$(x-1)P(x) = \frac{2(x-1)}{x}$, $(x-1)^2Q(x) = \frac{3}{x(x-1)^3}$

$(x-1)^2Q(x)$ is undefined at x=1.

So $(x-1)^2Q(x)$ is not analytic at x=1..

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<u>Problem 18:</u> Show that $x = 0$ is an ordinary point of $y'' - xy' + 2y = 0$

<u>Solution:</u>

Given that,

$y'' - xy' + 2y = 0$ …(1)

Comparing (1) with standard equation,
$y' + P_1(x)\, y' + P_2(x)\, y = 0$

We get,
$P_1 = -1$ $P_2 = 2$

Since both P_1 and P_2 are defined at $x = 0$ ($P_1 \neq \infty$, $P_2 \neq \infty$) therefore $x = 0$ is an ordinary point of (1).

<u>Problem 19:</u> Prove that , $x = 0$ is a regular singular point for $2x^2 y'' - xy' + (x - 5) = 0$

<u>Solution:</u>

Given that ,

$2x^2 y'' - xy' + (x - 5) = 0$
$y'' - 1/2x + (x - 5)/2x^2 = 0$ …(1)

Comparing (1) with the standard equation ,

$y'' + P_1(x)\, y' + P_2(x)\, y = 0$

We get ,
$P_1 = -1/2x$; $P_2 = (x - 5)/2x^2$

Since both P_1 and P_2 are not analytic ($P_1 = \infty$, $P_2 = \infty$) ; therefore $x = 0$ is a singular point .

$Q_1 = (x - x_O)\, P_1(x)$
$= (x - 0)(-1/2x)$
$= -1/2$

$Q_2 = (x - x_O)^2\, P_2(x)$
$= (x - 0)^2 . (x - 5)/2x^2$
$= (x - 5)/2$

Here , both Q_1 and Q_2 are analylic ($Q_1 \neq \infty$, $Q_2 \neq \infty$) at $x = 0$; therefore $x = 0$ is a regular singular point of (1).

<u>Problem 20:</u> Prove that , $x = 0$ is a regular singular point for $3xy'' + 2y' + y = 0$

<u>Solution:</u>

Given that ,

$3xy'' + 2y' + y = 0$

Dividing by 3x we get ,

$$y'' + \frac{2}{3x}\,y' + \frac{y}{3x} = 0 \qquad \text{... (1)}$$

Comparing with standard equation we get
$y'' + P_1\, y' + P_2\, y = 0$

We get ,
$P_1 = 2/3x$; $P_2 = 1/3x$

Since, P_1 and P_2 both are undefined at $x = 0$ ($P_1 = \infty$ $P_2 = \infty$)
So , $x = 0$ is a singular point of equation (1).

Again ,

$Q_1 = P_1(x - 0)$	$Q_2 = P_2(x - 0)$
$= 2/3x\,(x - 0)$	$= 1/3x\,(x - 0)$
$= 2/3$	$= 1/3$

Q_1 and Q_2 both are defined at $x = 0$

So , $x = 0$ is a regular singular point of equation (1)

FROBENIUS METHOD

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If $x = 0$ is a regular singularity of the equation,

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \ldots\ldots\ldots.(1) \quad [p(0) = 0]$$

Then the series solution is $y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) = \sum_{k=0}^{\infty} a_k x^{m+k}$

The value of m will be determined by substituting the expressions for $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1), we get the identity.

On equating the coefficient of lowest power of x in the identity to zero, a quadratic equation in m (indicial equation) is obtained.
Thus, we will get two values of m. The series solution of (1) will depend on the nature of the roots of the indicial equation.

1. Case 1: When roots m_1, m_2 are distinct and not differing by an integer $m_1 - m_2 \neq 0$ or a positive integer.

e.g., $m_1 = \frac{1}{2}, m_2 = 2$

The complete solution is $y = c_1(y)_{m_1} = c_2(y)_{m_2}$

2. Case 2: When roots m_1, m_2 are equal i.e., $m_1 = m_2$

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

3. Case 3: When roots m_1, m_2 are distinct and differ by an integer $(m_1 < m_2)$

e.g., $m_1 = \frac{3}{2}, m_2 = \frac{5}{2}$ or $m_1 = 2, m_2 = 4$

If some of the coefficients of y series become infinite when $m = m_1$, to overcome this difficulty, replace a_0 by b_0 $(m - m_1)$. We get a solution which is only a constant multiple of the first solution.

Complete solution is $y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$

4. Case 4: Roots are distinct and differing by an integer, making some coefficient indeterminate

Complete solution is $y = c_1(y)_{m_1} + c_2(y)_{m_2}$

If the coefficient does not become infinite when $m_1 = m_2$

Case1: when the roots are distinct and differing by an integer.

Related Examples

Example 1: Find solution in generalized series form about x = 0 of the differential equation

$$3x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$$

Solution: $3x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0 \dots\dots\dots\dots..(1)$

Since x=0 is a regular singular point, we assume the solution in the form

$y=\sum_{k=0}^{\infty} a_k x^{m+k}$

Such that, $\frac{dy}{dx}=\sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1}, \frac{d^2y}{dx^2}=\sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}$

Substituting for $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in the given equation (1) we get

$$3\Sigma a_k(m+k)(m+k-1)x^{m+k-1}+2\Sigma a_k(m+k)x^{m+k-1}+\Sigma a_k x^{m+k}=0$$
$$\Sigma a_k[3(m+k)(m+k-1)+2(m+k)]x^{m+k-1}+\Sigma a_k x^{m+k}=0 \dots\dots..(2)$$

The co efficient of the lowest degree term x^{n-1} in the identity (2) is obtained by putting $k=0$ in first summation only and equating it to zero. Then the indicial equation is

$a_0[3(m-1)m+2m]=0 \; or \; a_0[3m^2-m]=0 \; or \; a_0 m(3m-1)=0$

Since $a_0 \neq 0, m=0 \; or \; \frac{1}{3}$

The coefficient of next lowest degree term x^m in the identity (2) is obtained by putting $k=1$ in first summation and $k=0$ in the second summation and equating it to zero.

$$a_1[3(m+1)m+2(m+1)]+a_0=0$$
$$or \; a_1[3m^2+5m+2]+a_0=0 \text{ or } a_1(3m+2)(m+1)+a_0=0$$
$$a_1=-\frac{1}{(3m+2)(m+1)}a_0$$

Equating to zero the coefficient of x^{m+k}, the recurrence relation is given by

$$a_{k+1}[3(m+k+1)(m+k)+2(m+k+1)]+a_k=0$$
$$a_{k+1}(m+k+1)(3m+3k+2)+a_k=0 \text{ or } a_{k+1}=\frac{-1}{(m+k+1)(3m+3k+2)}a_k$$

This gives

For $k=0,\; a_1=\frac{-1}{(m+1)(3m+2)}a_0$

For $k = 1$, $a_2 = \frac{-1}{(m+2)(3m+5)} a_1 = \frac{1}{(m+1)(m+2)(3m+2)(3m+5)} a_0$

For $k = 2$, $a_3 = \frac{-1}{(m+3)(3m+8)} a_2 = \frac{-1}{(m+1)(m+2)(m+3)(3m+2)(3m+5)(3m+8)} a_0$

For m = 0,

$a_1 = -\frac{1}{2} a_0$ $\quad a_2 = \frac{1}{20} a_0$ $\quad a_3 = -\frac{1}{480} a_0$

Hence for $m = 0$, $y_1 = a_0\left(1 - \frac{1}{2}x + \frac{1}{20}x^2 - \frac{1}{480}x^3 + \cdots\right)$

For $m = \frac{1}{3}$

$a_1 = -\frac{1}{4} a_0$ $\quad a_2 = \frac{1}{56} a_0$ $\quad a_3 = -\frac{1}{1680} a_0$

Hence for $m = \frac{1}{3}$, $y_2 = a_0(x^{\frac{1}{3}} - \frac{1}{4}x^{\frac{4}{3}} + \frac{1}{56}x^{\frac{7}{3}} - \frac{1}{1680}x^{\frac{10}{3}} + \cdots)$

Thus, the complete solution is

$$y = Ay_1 + By_2$$

$$y = a_0(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \cdots) + b_0 x^{\frac{1}{3}}(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \cdots)$$ **Ans.**

1800012

Example 2: 4xy" + 2y' + y = 0

Consider 4xy" + 2y' + y = 0--------------(1)

So, x=0 is a regular singular point with $p(x) = \frac{1}{2}$ and $q(x) = \frac{x}{4}$. The power series in y_1 and y_2 will converge for $|x| < \infty$ since p and q have convergent power series in this interval. The indicial equation is

$$r(r-1) + \frac{1}{2}r = 0 \Rightarrow r^2 - \frac{1}{2}r = 0\text{-----------------(2)}$$

so $r_1 = \frac{1}{2}$ and $r_2 = 0$ (Note: $p_0 = \frac{1}{2}$, $q_0 = 0$). Substituting $y = x^r \sum_{k=0}^{\infty} a_k x^k$ into (1) and shifting the indices of the first two series so all terms are of form x^{k+r} we get

$$4\sum_{k=-1}^{\infty}(k+r+1)(k+r)a_k + 1x^{k+r} + 2\sum_{k=-1}^{\infty}(k+r+1)a_{k+1}x^{k+r} + X\sum_{k=0}^{\infty} a_k x^{k+r} = 0\text{---------------(3)}$$

All coefficients of powers x^{k+r} must equate to zero to obtain a solution. The lowest power is x^{r-1} for $k = -1$ and this yield

$$4r(r-1) + 2r = 0 \Rightarrow r^2 - \frac{1}{2}r = 0\text{--------------------(4)}$$

which is just the indicial equation as expected. For $k \geq 0$, we obtain

$$4(k+r+1)(k+r)a_k + 1 + 2(k+r+1)a_k + 1 + a_k = 0\text{------------------(5)}$$

Corresponding to the recurrence relation

$a_{k+1} = \frac{-a_k}{(2k+2r+2)(2k+2r+1)}$,   k = 0, 1, 2...--------------------(6)

First Solution: To find y_1 apply (6) with $r = r_1 = \frac{1}{2}$ to get the recurrence relation

$a_{k+1} = \frac{-a_k}{(2k+3)(2k+2)}$,   k = 0, 1, 2...------------------(7)

Then

$a_1 = -\frac{a_0}{3.2}$, $a_2 = -\frac{a_1}{5.4}$, $a_3 = -\frac{a_2}{7.6}, \ldots$------------------------- (8)

so

$a_1 = -\frac{a_0}{3!}$, $a_2 = \frac{a_0}{5!}$, $a_3 = -\frac{a_0}{7!}, \ldots$-------------------------- (9)

Since a_0 is arbitrary, let $a_0 = 1$ so

$a_k(r_1) = \frac{(-1)^k}{(2k+1)!}$, k = 0, 1, 2...-----------------------(10)

and

$y_1(x) = x^{1/2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^k$.-----------------------(11)

Second Solution: To find y_2, just apply (6) with $r = r_2 = 0$ to get the recurrence relation

$b_{k+1} = -\frac{b_k}{(2k+2)(2k+1)}$.-----------------------------(12)

Letting the arbitrary constant $b_0 = 1$, then

$b_k(r_2) = \frac{(-1)^k}{(2k)!}$ ------------------(13)

so

$y_2(x) = \sum_{K=0}^{\infty}\frac{(-1)^K}{(2k)!}x^k$.----------------(14)

1800013

Example 3: xy” + 2y’ − xy = 0

Given that xy” + 2y’ − xy = 0

Since x=0 is the regular singular point of the given equation-

$$y = \sum_{n=0}^{\alpha} C_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} C_n (n+1)_x^{n+r-2}$$

Substituting y,y',y" is the given differential equation, We have-

$$\sum_{n=0}^{\infty} C_n (n+r)(n+r-1)x^{n+r-1} + \sum_{n=0}^{\infty} 2C_n (n+r)x^{n+r-1} - \sum_{n=0}^{\infty} C_n x^{n+r-1} = 0$$

OR, $C_0 r(r-1)x^{r-1} + C_1(r+1)rx^r + \sum_{n=2}^{\infty} C_n(n+r)(n+r-1)x^{n+r-1} + 2C_0 r x^{r-1} +$
$2C_1(r+1)x^r + \sum_{n=2}^{\infty} 2C_n\,(n+r)(n+r-1)X^{n+r-1} - \sum_{n=2}^{\infty} C_{n-2}x^{n+r-1} = 0$

$$C_0 r(r+1)x^{r-1} + c_1(r+1)(r+2)x^r + \sum_{n=2}^{\infty} [C_r(n+r)(n+r-1) - C_{n-2]}x^{n+r-1} = 0$$

$C_0 r(r+1) = 0$; $so\, r = 0, -1$

$C_1(r+1)(r+2) = 0$
$C_n(n+r)(n+R-1)C_{n-2} = 0$
$r = 0{,}3C_1 = 0, C_l = 0$
$C_n n(n+1) - c_{n-2} = 0$
SO, $C_n = \frac{C_{n-2}}{n(n+1)}; n = 1,2,3 \ldots\ldots\ldots$
PUTTING $n = 2$

$C_3 = \frac{Co}{6}$
PUTTING $n = 3$

$C_3 = \frac{C_2}{12} = 0$
PUTTNG $n = 4$
$C_1 = \frac{C_0}{120}$
$y_1 = x^r(C_0 + C_1 x + C_{2x}^2 + \cdots \ldots\ldots\ldots..)$
=1$(c_0 + \frac{C_0}{6}x^2 \frac{C_0}{6}x^2 + \frac{C_0}{120}x^4 + \cdots \ldots\ldots\ldots..)$
$r = -1 \quad so\ C_1 = 0$
$C_n(n-1)n - C_{n-2} = 0$; $C_n = \frac{c_n - 2}{n(n-1)}$; $n = 2,3 \ldots\ldots$
PUTTING $n = 2;\ C_2 = \frac{C_0}{2}$
$n = 0; C_3 = \frac{C_1}{6} = 0$
$\eta = 4; C_1 = \frac{C_2}{12} = \frac{Co}{24}$
$y_2 = x^{-1}C_0(1 + \frac{1}{x}x_+^2 \frac{1}{24}x^4 + \cdots \ldots\ldots)$
SOLUTION- $y = y_1 + y_2$

$$y_1 = C_0\left(1 + \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \cdots \ldots \ldots\right) = C_0 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2^n}$$

$$y_2 = C_1 x^{-1} \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$

SO $y = y_1 + y_2$

$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2^n} + \boldsymbol{C_1 x_{n=0}^{-I_{\Sigma}^{\infty}} \frac{1}{(2n)}!\, x^{2^n}}$ **(ANS)**

1800014

<u>Example 04:</u> Find the solution in generalized series from about x=0 of the differential equation

$$2x(1\text{-}x)\frac{d^2y}{dx^2} + (5 - 7x)\frac{dy}{dx} - 3y = 0$$

<u>Solution:</u> Given that,

$$2x(1-x)\frac{d^2}{dx^2} + (5-7x)\frac{dy}{dx} - 3y + 0 \text{-------------------}(1)$$

Since x=0 is a regular singular point, we assume the solution in the form

$y = \sum_{k=0}^{\infty} a_k x^{m+k}$

Such that,

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1}$$

$$\frac{d^2}{dx^2} = \sum_{k=0}^{\infty} a^k (m+k)(m+k-1) x^{m+k-2}$$

Substituting for y, $\frac{dy}{dx}$ and $\frac{d^2}{dx^2}$ in the equation (1) we get,

$$(2x - 2x^2)\sum a_k (m+k)(m+k-1)x^{m+k-2} + (5-7x)\sum a_k (m+k) x^{m+k-1} - 3\sum a_k x^{m+k} = 0$$

=> $\sum 2a_k (m+k)(m+k-1)x^{m+k-1} - \sum 2a_k (m+k)(m+k-1)x^{m+k} + \sum 5a_k (m+k)x^{m+k-1} - \sum 7a_n (m+k)x^{m+k} - 3\sum a_k x^{m+k} = 0$---------------------(2)

The coefficient of the lowest degree term x^{m-1} in the identity (2) is obtained by putting k=0 and equating it to zero. Then the indicial equation is,

$$2a_0m(m-1)+5a_0m=0$$

=> $a_0(2m^2-2m+5m)=0$

∵a_0≠0 ∴$2m^2+3m=0$ [Indicial equation]

∴ m=0, $-\frac{3}{2}$

Equating to the zero the co-efficient of x^{m+k}, the recurrence relation is given by

$$2a_{k+1}(m+k+1)(m+k)-2a_k(m+k)(m+k-1)+5a_{k+1}(m+k+1)-7a_k(m+k)-3a_k=0$$

∴$a_{k+1}=\frac{2m+2k+3}{2m+2k+5}a_k$

This gives,

For k=0, $a_1=\frac{2m+3}{2m+5}a_0$

For k=1, $a_2=\frac{2m+5}{2m+7}a_1=\frac{2m+3}{2m+7}a_0$

Now,

For m=0, $a_1=\frac{3a_0}{5}$ and $a_2=\frac{3a_0}{7}$

For $m=-\frac{3}{2}$, a_1=0 and a_2=0

Here, $y=x^m[a_0+a_1x+a_2x^2+\cdots]$

For m=0, $y_1=a_0\left[1+\frac{3}{5}x+\frac{3}{7}x^2+\cdots\right]$

For $m=-\frac{3}{2}$, $y_2=a_0x^{-3/2}[1+0+0+\cdots]=a_0x^{-3/2}$

Hence the complete solution is,

$$y=Ay_1+By_2$$

$$y=Aa_0\left[a+\frac{3}{5}x+\frac{3}{7}x^2+\cdots\right]+Ba_0x^{-\frac{3}{2}}$$

1800015

#<u>Example 05</u>: $2x^2y''+7x(x+1)y'-3y=0$

<u>Solution:</u>

The given equation,

$2x^2y''+7x(x+1)y'-3y=0$------------(1)

Since x=0 is a regular singular point, we assume the solution in the form,

$$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

$$\therefore y' = \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1}$$

And,

$$y'' = \sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2}$$

Putting the values of y, y', y" in equation (1) we get,

$$2x^2 \sum a_k (m+k)(m+k-1) x^{m+k-2} + 7x(x+1) \sum a_k (m+k) x^{m+k-1} - 3 \sum a_k x^{m+k} = 0$$

=> $2 \sum a_k (m+k)(m+k-1) x^{m+k} + 7 \sum a_k (m+k) x^{m+k+1} + 7 \sum a_k (m+k) x^{m+k} - 3 \sum a_k x^{m+k} = 0$---------(2)

The co-efficient of the lowest degree term x^m in the equation (2) is obtained by putting k=0 in the 1st , 3rd and 4th summation and equating it to zero. Then the indicial equation is,

$$2a_0 m(m-1) + 7a_0 m - 3a_0 = 0$$

$$\Rightarrow a_0(2m^2 + 5m - 3) = 0$$

$$\Rightarrow a_0(2m-1)(m+3) = 0$$

Since $a_0 \neq 0$, $\therefore m = \frac{1}{2}, -3$

The co-efficient of next lowest degree x^{m+1} in the equation (2) is obtained by putting k=1 in the 1st , 3rd and 4th summation and k=0 in the 2nd summation,

$$2a_1 m(m+1) + 7a_0 m + 7a_1(m+1) - 3a_1 = 0$$

$$\Rightarrow a_1(2m^2 + 2m + 7m + 7 - 3) + 7a_0 m = 0$$

$$\Rightarrow a_1 (2m+1)(m+4) + 7a_0 m = 0$$

$$\Rightarrow a_1 = -\frac{7ma_0}{(2m+1)(m+4)}$$

Equating to zero the co-efficient x^{m+k}, the relation is given by,

$$[2(m+k)(m+k+1) + 7(m+k) - 3]a_k + 7(m+k-1)a_{k-1} = 0$$

$$\therefore a_k = \frac{-7(m+k-1)a_{k-1}}{[2(m+k)-1](m+k+3)}$$

For, $m = \frac{1}{2}$

$$a_k = \frac{-7\left(\frac{1}{2}+k-1\right)a_{k-1}}{2\left(\frac{1}{2}+k-1\right]\left(\frac{1}{2}+k+3\right)} = \frac{-7(2k-1)a_{k-1}}{2k(2k+7)} \quad [k \geq 1]$$

Now, for k=1, $a_1 = -\frac{7a_0}{18}$

For k=2, $a_2 = -\frac{21a_1}{44} = \frac{147}{792}a$

$$\therefore y_1 = a_0 x^{\frac{1}{2}}(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \cdots)$$----------------(3)

For,

m=-3

$$a_k = \frac{-7(-3+k-1)a_{k-1}}{[2(-3+k)-1](-3+K+3)} = \frac{-7(k-4)a_{k-1}}{k(2k-7)} \quad [k\geq 1]$$

Now,

For k=1, $a_1 = -\frac{21}{5}a_0$

For k=2, $a_2 = -\frac{7}{3}a_1 = \frac{49}{5}a_0$

For k=3, $a_3 = -\frac{7a_2}{3} = -\frac{343}{15}a_0$

$$\therefore y_2 = a_0 x^{-3}(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3 + \cdots)$$-------------------(4)

So, the general solution is given by,

$$y = Ax^{\frac{1}{2}}\left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \cdots\right) + Bx^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3 + \cdots\right)$$

(Ans)

1800016

#Example 06: Solve the differential equation: $2x(1-x)y'' = (1-x)y' + 3y=0$

Solution: $2x(1-x)y'' = (1-x)y' + 3y = 0$ . . . . (1)

Since, x = 0 is a regular singular point, we assume the solution in the form:

$$y = \sum_{n=0}^{\infty} a_k \cdot x^{m+k}$$

Such that, $y' = \sum_{k=0}^{\infty} a_k(m+k) \cdot x^{m+k-1}$

$$y'' = \sum_{k=o}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting for y, y' and y'' in the given equation (1), we get:

$$2x(1-x)[\sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k-2}] +$$

$$(1-x)[\sum_{k=0}^{\infty} a_k (m+k)x^{m+k-1}] + 3\sum_{k=0}^{\infty} a_k\, x^{m+k} = 0$$

Or, $\sum_{k=0}^{\infty} a_k 2(m+k)(m+k-1)x^{m+k-1} - \sum_{k=0}^{\infty} a_k\, 2(m+k)(m+n-1)x^{m+k} +$
$\sum_{k=0}^{\infty} a_k\, (m+k)x^{m+k-1} - \sum_{k=0}^{\infty} a_k (m+k)x^{m+k} - \sum_{k=0}^{\infty} a_k\, 3x^{m+k} = 0$

Or, $\sum_{k=0}^{\infty} a_k\, 2[(m+k)(m+k-1) + (m+k)]x^{m+k-1} -$
$\sum_{k=0}^{\infty} a_k [2(m+k)(m+k-\ 1) + (m+k) - 3]\, x^{m+k} = 0$ Or,
$\sum_{k=0}^{\infty} a_k\, (m+k)[2(m+k-1)+1]x^{m+k-1} - \sum_{k=0}^{\infty} a_k (m+k)[(2m+2k-1) -$
$3]x^{m+k} = 0$

Or, $\sum_{k=0}^{\infty} a_k [(m+k)(2m+2k-1)]x^{m+k-1} - \sum_{k=0}^{\infty} a_k [(m+k)(2m+2k-1) - 3]x^{m+k} = 0. \ldots\ldots (2)$

The coefficient of the lowest degree term $\mathbf{x^{m-1}}$ in the identity (2) is obtained by putting k=0 in the first summation only and equating it to zero. Then the Indicial equation is,

$a_0(m+0)(2m+0-1) = 0$

$a_0 \cdot m(2m-1) = 0$

$(2m-1) \cdot m = 0$ **because** $a_0 \neq 0$

$\therefore 2m - 1 = 0$ **and** $m = 0$ or, $m = \frac{1}{2}$

$\therefore \quad m = \frac{1}{2}\,, 0$

Now $m_1 \neq m_2$ and $m_1 - m_2 = \frac{1}{2} - 0 = \frac{1}{2} \neq$ Integer. [Case – 1]

The coefficient of next lowest degree term x^m in the identity (2) is obtained by putting k = 1 in the first summation and k = 0 in the second summation and equating it to zero.

$a_1(m+1)(2m+2\cdot 1-1) - a_0[(m+0)(2m+0-1) - 3] = 0$

$a_1(m+1)(2m+1) - a_0[m(2m-1) - 3] = 0$

$a_1(m+1)(2m+1) - a_0(2m^2 - m - 3) = 0$

$a_1(m+1)(2m+1) - a_0(m+1)(2m-3) = 0$

$$a_1 = \frac{(m+1)(2m-3)}{(m+1)(2m+1)} a_0$$

$$a_1 = \frac{(2m-3)}{(2m+1)}a_0$$

Equating to zero the coefficient of x^{m+k}, the recurrence relation is given by,

$$a_{k+1}(m+k+1)(2m+2k+1) - a_k[(2m+2k-1)(m+n)-3] = 0$$

$$a_{k+1}(m+k+1)(2m+2k+1) = a_k[\{2(m+k)-1\}(m+n)-3$$

$$a_{k+1}(m+k+1)(2m+2k+1) = a_k[2(m+k)^2-(m+k)-3]$$

$$a_{k+1}(m+k+1)(2m+2k+1) = a_k[2(m+k)^2-3(m+k)+2(m+k)-3]$$

$$a_{k+1}(m+k+1)(2m+2k+1) = a_k[(m+k)\{2(m+k)-3\}+1\{2(m+k)-3)$$

$$a_{k+1}(m+k+1)(2m+2k+1) = a_k[(2m+2k-3)\cdot(m+k+1)]$$

$$a_{k+1} = \frac{(2m+2k-3)(m+k+1)}{(2m+2k+1)(m+k+1)}a_k$$

$$a_{k+1} = \frac{(2m+2k-3)}{(2m+2k+1)}a_k$$

This gives,

For k = 0, $a_1 = \frac{(2m-3)}{(2m+1)}a_0$

For k=1 $a_2 = \frac{(2m-1)}{(2m+3)}a_1 = \frac{(2m-1)(2m-3)}{(2m+3)(2m+1)}a_0$

For k = 2, $a_2 = \frac{(2m+1)}{(2m+3)}a_1 = \frac{(2m-1)(2m-3)}{(2m+5)(2m+3)}a_0$

Now we know that the complete solution of case - 2 is given by,

$$y = C_1\,(y)_{m_1} + C_2\,(y)_{m2}$$

Trial Solution: $y = \sum_{k=0}^{\infty} a_k x^{m+k}$

Or, $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \ldots.. +\infty$

Now putting the values of a_1, a_2, a_3 in the trial solution, we get:

$$y = a_0x^m + \frac{(2m-3)}{(2m+1)}a_0\,x^{m+1} + \left[\frac{(2m-1)(2m-3)}{(2m+3)(2m+1)}\right]a_0x^{m+2} + [\frac{(2m-1)(2m-3)}{(2m+5)(2m+3)}]a_0x^{m+3} + \ldots\ldots\ldots +\infty$$

Now at m = ½,

$$(\boldsymbol{y})_{\boldsymbol{m_1}} = a_0 x^{\frac{1}{2}} - a_0 x^{\frac{3}{2}} + 0 + 0 + 0 + \ldots\ldots..$$

$$(y)_{m1} = a_0(x^{\frac{1}{2}} - x^{\frac{3}{2}})$$

At m = 0,

$$(\boldsymbol{y})_{\boldsymbol{m_2}} = a_0(1 - 3x + x^2 + \frac{x^3}{5} + \ldots\ldots\ldots..)$$

Thus, the complete solution:

$$y = C_1\ (y)_{m_1} + C_2\ (y)_{m2}$$

$$\boldsymbol{y = C_1[a_0(x^{\frac{1}{2}} - x^{\frac{3}{2}})] + C_2[a_0(1 - 3x + x^2 + \frac{x^3}{5} + \ldots\ldots.]}$$

$$\boldsymbol{y = A[(x^{\frac{1}{2}} - x^{\frac{3}{2}})] + B[1 - 3x + x^2 + \frac{x^3}{5} + \ldots\ldots.]}$$

1800017

#Example 07: Find a fundamental set of Frobenius solution of

$$\boldsymbol{x^2(3 + x)y'' + 5x(1 + x)y' - (1 - 4x)y = 0}$$

Give explicit formulas for the coefficients in the solutions.

Solution: For this equation, the polynomials defined in Theorem 7.5.2 are

$$p_0(r) = 3r(r-1) + 5r - 1 = (3r-1)(r+1),$$
$$p_1(r) = r(r-1) + 5r + 4 = (r+2)^2,$$
$$p_2(r) = 0.$$

The zeros of the indicial polynomial p_0 are $r_1 = \frac{1}{3}$ and $r_2 = -1$, so $r_1 - r_2 = \frac{4}{3}$. Therefore theorem 7.5.3 implies that

$$y_1 = x^{\frac{1}{3}} \sum_{n=0}^{\infty} a_n\left(\frac{1}{3}\right) x^n$$

and

$$y_2 = x^{-1} \sum_{n=0}^{\infty} a_n(-1) x^n$$

form a fundamental set of Frobenius solutions of (7.5.21). To find the coefficients in these series, we use the recurrence relations (7.5.20); thus,

$$a_0(r) = 1,$$

$$a_n(r) = -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r)$$

$$= -\frac{(n+r+1)^2}{(3n+3r-1)(n+r+1)}a_{n-1}(r) \qquad (7.5.22)$$

$$= -\frac{n+r+1}{3n+3r-1}a_{n-1}(r), \;\; n \geq 1.$$

Setting $r = \frac{1}{3}$ in (7.5.22) yields

$$a_0\left(\frac{1}{3}\right) = 1,$$

$$a_n\left(\frac{1}{3}\right) = -\frac{3n+4}{9n}a_{n-1}\left(\frac{1}{3}\right), \quad n \geq 1.$$

By using the product notation introduced in section 7.2 and proceeding as we did in the examples in that section yields

$$a_n\left(\frac{1}{3}\right) = \frac{(-1)^n \prod_{j=1}^{n}(3j+4)}{9^n n!}, \qquad n \geq 0.$$

Therefore

$$y_1 = x^{\frac{1}{3}}\sum_{n=0}^{\infty}\frac{(-1)^n \prod_{j=1}^{n}(3j+4)}{9^n n!}x^n$$

is a Frobenius solution of (7.5.21)

Setting $r = -1$ in (7.5.22) yields

$$a_0(-1) = 1,$$

$$a_n(-1) = -\frac{n}{3n-4}a_{n-1}(-1), \quad n \geq 1,$$

So

$$a_n(-1) = \frac{(-1)^n n!}{\prod_{j=1}^{n}(3j-4)}$$

Therefore

$$y_2 = x^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n n!}{\prod_{j=1}^{n}(3j-4)}x^n$$

is a Frobenius of solution of (7.5.21), and $\{y_1, y_2\}$ is fundamental set of solutions.
We now consider equations of the form

$$x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y = 0 \qquad (7.5.23)$$

with $\alpha_0 \neq 0$. For this equation, $\alpha_1 = \beta_1 = \gamma_1 = 0$, so $p_1 \equiv 0$ and the recurrence relations in Theorem 7.5.2 simplify to

$$a_0(r) = 1,$$
$$a_1(r) = 0,$$
$$a_n(r) = -\frac{p_2(n+r-2)}{p_0(n+r)} a_{n-2}(r), \quad n \geq 2.$$

Since $a_1(r) = 0$, the last equation implies that $a_n(r) = 0$ if n is odd, so the Frobenius solutions are of the form

$$y(x,r) = x^r \sum_{m=0}^{\infty} a_{2m}(r) x^{2m},$$

where

$$a_0(r) = 1,$$
$$a_{2m}(r) = -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r), \ \ m \geq 1.$$

1800018

#Example 08: Find the general solution to $x^2y'' + xy' + (x-2)y = 0$

Solution:

In standard form this ODE has

$$p(x) = \frac{1}{x} \text{ and } q(x) = \frac{x-2}{x^2},$$

Neither of which is analytic at x=0. However, both

$$xp(x) = 1 \text{ and } x^2q(x) = x - 2$$

are analytic at $x = 0$, so we have a regular singularity with

$$p_0 = \lim_{x\to 0} xp(x) = 1 \text{ and } q_0 = \lim_{x\to 0} x^2q(x) = -2.$$

The indicial equation is

$$r^2 + (1-1)r - 2 = 0$$
$$r = \pm\sqrt{2}$$

Applying the method of Frobenius, we set

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n x^{n+r} \qquad (a_0 \neq 0)$$

and substitute into the ODE, obtaining

$$(r^2-2)a_0x^r + \sum_{n=1}^{\infty}\left(((n+r)^2-2)a_n + a_{n-1}\right)x^{n+r} = 0.$$

Here we must have $r^2 - 2 = 0$ (which we already knew) and

$$a_n = \frac{-a_{n-1}}{(n+r)^2-2}$$
$$= \frac{-a_{n-1}}{n(n+2r)} \quad for\; n \geq 1.$$

Taking $a_0 = 1$ one readily sees that

$$a_n = \frac{(-1)^n}{n!\,(1+2r)(2+2r)(3+2r)\ldots(n+2r)}$$

Since the difference of the roots is $\sqrt{2} - (-\sqrt{2}) = 2\sqrt{2} \notin \mathbb{Z}$, the two R-values give independent solutions:

$$y_1 = x^{\sqrt{2}}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!\,(1+2\sqrt{2})(2+2\sqrt{2})(3+2\sqrt{2})\ldots(n+2\sqrt{2})},$$

$$y_2 = x^{-\sqrt{2}}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!\,(1-2\sqrt{2})(2-2\sqrt{2})(3-2\sqrt{2})\ldots(n-2\sqrt{2})}$$

And the general solution (for $x > 0$) is

$$y = c_1y_1 + c_2y_2.$$

1800019

#Example 09: Solve the equation in series $9x(1-x)y'' - 12y' + 4y = 0$

Solution: Given Equation,

$$9x(1-x)y'' - 12y' + 4y = 0 \qquad \ldots\ldots (1)$$

Comparing the equation with $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$, we get

$$P_0(x) = 9x(1-x)$$

Putting x = 0,

$P_0(0) = 0$ $\therefore x = 0$ is a singular point.

$\therefore$ The solution of equation (1) be,

$$y = \sum_{n=0}^{\infty} a_n x^{m+n} = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots$$

$$= x^m(a_0 + a_1x + a_2x^2 + \cdots)$$

$$\therefore y' = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \cdots$$

$\therefore y" = m(m-1)a_0x^{m-2} + m(m+1)a_1x^{m-1} + (m+1)(m+2)a_2x^m + \cdots$

Substituting $y, y', y"$ in equation (1),

$9x(1-x)[m(m-1)a_0x^{m-2} + m(m+1)a_1 \cdot x^{m-1} + (m+1)(m+2)a_2x^n \ldots]$

$-12[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} \ldots] + 4x^m(a_0 + a_1x + a_2x^2 + \cdots)$

$= [(9m(m-1)a_0x^{m-1} + 9(m+1)ma_1x^m + 9(m+2)(m+1)a_2x^{m+1} \ldots$

$-9m(m-1)a_0x^m + 9(m+1)\text{m}a_1x^{m+1} + 9(m+2)(m+1)a_2x^{m+2} + \cdots]$

$+[-12ma_0x^{m-1} - 12(m+1)a_1x^m - 12(m+2)a_2x^{m+1} \ldots]$
$+ [4a_0x^m + 4a_1x^{m+1} + 4a_2x^{m+2} + \cdots] = 0$

The lowest power of x is $(m-1)$. Hence equating its coefficient to zero,

$9m(m-1)a_0 - 12ma_0 = 0$

or $a_0(9m^2 - 9m - 12m) = 0$

or $9m^2 - 9m - 12m = 0$ $[a_0 \neq 0]$

or $9m^2 - 21m = 0$

or $3m(3m-7) = 0$

$\therefore m = 0, \frac{7}{3}$

i.e. roots are distinct but do not differ by an integer.

Equating the coefficient of x^m to zero,

$9(m+1)ma_1 - 9m(m-1)a_0 - 12(m+1)a_1 + 4a_0 = 0$

or $a_1(9m^2 + 9m - 12m - 12) = a_0(9m^2 - 9m - 4)$

or $3a_1(3m^2 + 3m - 4m - 4) = a_0(9m^2 + 3m - 12m - 4)$

or $3a_1(m+1)(3m-4) = a_0(3m+1)(3m-4)$

$\therefore a_1 = \frac{3m+1}{3(m+1)}a_0$

Equating the co-efficient of x^{m+1} to zero,

$9(m+2)(m+1)a_2 - 9(m+1)ma_1 - 12(m+2)a_2 + 4a_1 = 0$

or $a_2[9(m+2)(m+1) - 12(m+2)] = a_1(9m^2 + 9m - 4)$

or $(m+2)a_2(9m + 9 - 12) = a_1(9m^2 - 3m + 12m - 4)$

or $(m+2)a_2(9m - 9) = a_1[3m(3m-1) + 4(3m-1)]$

or $3a_2(m+2)(3m-1) = a_1(3m-1)(3m+4)$

or $\quad a_2 = \frac{3m+4}{3(m+2)} a_1 = \frac{(3m+4)(3m+1)}{3^2(m+2)(m+1)} a_0$ and so on

∴ we have,

$$a_1 = \frac{3m+1}{3(m+1)} a_0 \ ; \qquad a_2 = \frac{(3m+4)(3m+1)}{3^2(m+2)(m+1)}$$

When $m = 0$,

$$a_1 = \frac{1}{3} a_0 \quad ; \quad a_2 = \frac{1\times4}{9\times2\times1} a_0 = \frac{1}{3}\times\frac{4}{6} a_0 \qquad \text{and so on}$$

$$\therefore y_1 = (y)_{m=0} = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + \frac{1}{3} x a_0 + \frac{1}{3}\times\frac{4}{6}\cdot a_0 x^2 + \cdots$$

$$= a_0[1 + \frac{1}{3} + \frac{1}{3}\times\frac{4}{6} + \cdots$$

when $m = \frac{7}{3}$,

$$a_1 = \frac{3\cdot\frac{7}{3}+1}{3\left(\frac{7}{3}+1\right)} a_0 = \frac{8}{10} a_0$$

$$a_2 = \frac{\left(3\cdot\frac{7}{3}+1\right)\left(3\cdot\frac{7}{3}+4\right)}{3^2\left(\frac{7}{3}+1\right)\left(\frac{7}{3}+1\right)} a_0 = \frac{8}{10}\times\frac{11}{13} a_0 \quad \text{and so on}$$

$$\therefore y_2 = (y)_{m=\frac{7}{3}} = x^{\frac{7}{3}}[a_0 + a_1 x + a_2 x^2 + \cdots]$$

$$= x^{\frac{7}{3}}\left[a_0 + \frac{8}{10} a_0 x + \frac{8}{10}\times\frac{11}{13} a_0 x^2 + \cdots\right]$$

$$= a_0 x^{\frac{7}{3}}\left[1 + \frac{8}{10} x + \frac{8}{10}\times\frac{11}{13} x^2 + \cdots\right]$$

Hence the complete solution is

$$y = c_1 (y)_{m_1} + c_2 (y)_{m_2}$$

$$= c_1 a_0\left[1 + \frac{1}{3} x + \frac{1}{3}\times\frac{4}{6} x^2 + \cdots\right] + c_2 a_0 x^{\frac{7}{3}}\left[1 + \frac{8}{10} x + \frac{8}{10}\times\frac{11}{13} x^2 + \cdots\right]$$

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Example 10: $xy'' + y' + 2xy = 0$

Solution: $x = 0$ *is* a regular point, we assume the solution in the form,

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

And $y' = \sum_{n=0}^{\infty} c_n (n + r) x^{n+r-1}$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)\, x^{n+r-2}$$

We get, $xy'' + y' + 2xy = 0$

$$\to x \sum_{n=0}^{\infty} C_n (n+r)(n+r-1) x^{n+r-2} + \sum_{n=0}^{\infty} C_n (n+r) x^{n+r-1}$$
$$+2x \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\to \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+n-1} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$
$$+ \sum_{n=0}^{\alpha} 2c_n x^{n+r+1} = 0$$

Let, k=n+1, n=k+1

If n=0, then k=1

And k=n+1, n=k-1

If n=0, k=1

Now,

$$\sum_{k=1}^{\infty} c_{k+1}(k+r+1)(k+r) x^{k+r} + \sum_{k=1}^{\infty} c_{k+1}(k+1+r) x^{k+r} + \sum_{k=1}^{\infty} 2c_{k-1} x^{k+r} = 0$$

Evaluate at,

For first series, k=-1, k=0

$$c_0(r)(r-1)x^{r-1} + c_1(r+1)rx^r + \sum_{k=1}^{\infty} c_{k+1}(k+1+r)(k+r)x^{k+r}$$

For 2nd and 3rd Series, k=-1, k=0

$$c_0(r)x^{r-1} + c_1(r+1)x^r + \sum_{k=1}^{\infty} c_{k+1}(k+1+r)x^{k+r} + \sum_{k=1}^{\infty} 2c_{k-1}x^{k+r} = 0$$

$\to c_0 x^{r-1}(r(r-1)+r) + c_1 x^r (r(r+1) + (r+1)) + \sum_{k=1}^{\infty} [c_{k+1}(k+1+r)(k+r) + c_{k+1}(k+1+r) + 2c_{k-1}] x^{k+r} = 0$

Linear independence will imply,

$c_0(r(r-1)+r) = 0, c_0 r^2 = 0, r = 0$ $\quad$ $[c_0 = 0]$

$c_1(r+1)(r+1) = 0; c_1(r+1)^2 = 0, c_1 = 0$ $\quad$ $[r = 0]$

$c_{k+1}(k+1)k + c_{k+1}(k+1) + 2c_{k-1} = 0, \quad k \geq 1$

$\rightarrow c_{k+1}(k+1)k + c_{k+1}(k+1) + 2c_{k-1} = 0 \quad [r = 0]$

$\therefore c_{k+1} = \frac{-2c_{k-1}}{(k+1)^2}; k \geq 1$

$k = 1, c_2 = \frac{-2c_0}{2^2} = \frac{-2c_0}{4} = -\frac{c_0}{2}$

$k = 2, c_3 = \frac{-2c_1}{3^2} = 0 \qquad [c_1 = 0]$

$k = 3, c_4 = \frac{-2c_2}{4^2} = -\frac{2}{16}\left(-\frac{c_0}{2}\right) = \frac{c_0}{16}$

$k = 5, c_6 = -\frac{2c_4}{6^2} = -\frac{2}{36}\left(\frac{c_0}{16}\right) = -\frac{c_0}{288}$

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \cdots \qquad [r = 0]$$

As, $c_1, c_2, c_3 = 0$

$$y = c_0 - \frac{c_0}{2}x^2 + \frac{c_0}{16}x^4 - \frac{c_0}{288}x^6 + \cdots$$

1800021

Case 2: When roots m_1, m_2 are equal i.e., $m_1 = m_2$

Related Math

Example 01: $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$

Solution:

THE GIVEN DE is: $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \ldots (1)$

Put, $y=x^m$

$\Rightarrow \frac{dy}{dx} = mx^{m-1} \Rightarrow \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$

From eq^n. (1)

$$x.m(m-1)x^{m-2}+mx^{m-1}+\text{x.xm}=0$$
$$\Rightarrow m(m-1)x^{m-1}+mx^{m-1}+x^{m+1}=0$$

x=0 Is a regular singular point.
Let, sol^n of the eq^n. is

$$y=\sum_{k=0}^{\infty} a_k\, x^{m+2k}$$

$$\frac{dy}{dx}=\sum_{k=0}^{\infty} a_k\,(m+2k)x^{m+2k-1}$$

$$\frac{d^2y}{dx^2}=\sum_{k=0}^{\infty} a_k\,(m+2k)(m+2k-1)x^{m+2k-2}$$

Substituting for $y,\frac{dy}{dx},\frac{d^2y}{dx^2}$ $in\ equation\ 1$

$$x.\sum_{k=0}^{\infty} a_k\,(m+2k)(m+2k-1)x^{m+2k-2}+\sum_{k=0}^{\infty} a_k\,(m+2k)x^{m+2k-1}+x.\sum_{k=0}^{\infty} a_k\,x^{m+2k}$$

$$\Rightarrow\left[\left\{\sum_{k=0}^{\infty} a_k\,(m+2k)x^{m+2k-1}\right\}(m+2k-1+1)\right]+x.\sum_{k=0}^{\infty} a_k\,x^{m+2k}=0\ \ldots\ldots\ldots\ldots..(2)$$

THE Coefficient Of lowest degree x^{m-1}

In equation 2 is obtained by putting k=0 in the 1st summation only of 2 and equating it to 0

$$a_o(m+0)^2=0$$

$$m=0{,}0\ as\ a_o\neq 0$$

The coefficient of 2nd lowest degree x^m

In equation 2 is obtained by putting k=0 in 2nd summation and k=1 in 1st summation of only 2 we get,

$$\sum_{k=0}^{\infty} a_k\,(m+2k)x^{m+2k-1}+\sum_{k=0}^{\infty} a_{k-1}\,x^{m+2k-1}=0$$

$$\Rightarrow\sum_{k=0}^{\infty}[a_k(m+2k)^2+a_{k-1}]x^{m+2k-1}=0$$

$$\Rightarrow a_k(m+2k)^2+a_{k-1}$$

$$\Rightarrow a_k=\frac{-a_{k-1}}{(m+2k)^2}$$

Put, k=1

$$\Rightarrow a_1 = \frac{-a_0}{(m+2)^2}$$

K=2

$$\Rightarrow a_2 = \frac{-a_1}{(m+4)^2}$$

$$a_2 = \frac{a_0}{(m+2)^2(m+4)^2}$$

we know

y=$x^m[\,a_o + \; a_1x^2 + a_2x^4 + \cdots]$

$$\Rightarrow y = x^m \left[a_o - \frac{a_0}{(m+2)^2}x^2 + \frac{a_0}{(m+2)^2(m+4)^2}x^4 + \cdots\right]$$

$$\Rightarrow y = a_o x^m \left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} + \cdots\right]$$

$$y = Ay_{m=0} + B\frac{\partial y}{\partial m_{m=0}}$$

$y_{m=0} = a_o\ [1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} + \cdots]$ …. (3)

$$\frac{\partial y}{\partial m} = a_o x^m logx\left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} + \cdots\right] + a_o x^m\left[0 + \frac{2x^2}{(m+2)^3} - \frac{3x^4}{(m+2)^2.(m+4)^2} \ldots\right]$$

$\frac{\partial y}{\partial m_{m=0}} = a_o \text{logx}\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \cdots\right] + a_o\left[\frac{x^2}{2^2} - \frac{3x^4}{2^2.4^2} + \cdots\right]$…………………. (4)

General solution=A $\left(a_o[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} + \cdots]\right)$+B$\left(a_o \text{logx}\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \cdots\right] + a_o\left[\frac{x^2}{2^2} - \frac{3x^4}{2^2.4^2} + \cdots\right]\right)$

1800022

Example 02: $x\frac{d^2y}{dx^2} + (1 + x)\frac{dy}{dx} + 2y = 0$

Solution:

The Given DE is: $x\frac{d^2y}{dx^2} + (1 + x)\frac{dy}{dx} + 2y = 0$ ………………. (1)

Here x=0 is a regular singular point of the given DE.

Let, the solution of the given DE be,

$y=\sum_{n=0}^{\infty} a_n\, x^{m+n}$; $a_o \neq 0$

Then, $\frac{dy}{dx} = \sum_{n=0}^{\infty}(m + n)\, a_n x^{m+n-1}$

$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(m + n)(m + n - 1)a_n x^{m+n-2}$

Substituting these in equation (1),

$x\sum_{n=0}^{\infty}(m + n)(m + n - 1)a_n x^{m+n-2} + (1 + x)\sum_{n=0}^{\infty}(m + n)a_n\, x^{m+n-1} + 2\sum_{n=0}^{\infty} a_n x^{m+n} = 0$

$\sum_{n=0}^{\infty}(m + n)(m + n - 1)a_n x^{m+n-1} + \sum_{n=0}^{\infty}(m + n)a_n x^{m+n-1} + \sum_{n=0}^{\infty}(m + n)a_n x^{m+n} +$
$2\sum_{n=0}^{\infty} a_n\, x^{m+n} = 0$

$\sum_{n=0}^{\infty}(m + n)(m + n - 1 + 1)a_n x^{m+n-1} + \sum_{n=0}^{\infty}(m + n + 2)\, a_n x^{m+n} = 0$ ....(2)

Equating coefficient of x^{m-1} to zero,

$m^2a_o=0$

$m=0,0$

Now, equating coefficient of x^{m+n-1} to zero in eq^n (2),

$(m+n)^2a_n + (m+n-1+2)\, a_{n-1} = 0$

$a_n = -\frac{m+n+1}{(m+n)^2}a_{n-1}$ …………………… (3)

Putting n=1,2,3,4,…………. In eq^n (3)

$a_1=-\frac{m+2}{(m+1)^2}a_o$

$a_2=-\frac{m+3}{(m+2)^2}a_1 = \frac{(m+3)(m+2)}{(m+2)^2(m+1)^2}a_o = \frac{(m+3)}{(m+1)^2(m+2)}a_o$

$a_3=-\frac{m+4}{(m+3)^2}a_2 = -\frac{m+4}{(m+1)^2(m+2)(m+3)}a_o$

Thus,

$y = a_o x^m [1-\frac{m+2}{(m+1)^2}x+\frac{(m+3)}{(m+1)^2(m+2)}x^2 - \frac{m+4}{(m+1)^2(m+2)(m+3)}x^3 + \cdots \ldots \ldots \ldots]$

$\frac{\partial y}{\partial m} = a_o x^m \log x\ [1-\frac{m+2}{(m+1)^2}x+\frac{(m+3)}{(m+1)^2(m+2)}x^2 - \frac{m+4}{(m+1)^2(m+2)(m+3)}x^3 + \cdots \ldots \ldots \ldots]\ + a_o x^m[\{-\frac{1}{(m+1)^2} + \frac{2(m+2)}{(m+1)^3}\}x + [\frac{1}{(m+1)^2(m+2)} + \frac{(m+3)}{(m+1)^2}\left\{\frac{-1}{(m+2)}\right\} + \frac{m+3}{m+2}\{\frac{-2}{(m+1)^3}\}]x^2 + \cdots \ldots \ldots \ldots$

Hence, the complete solution of eq^n (1) is,

$Y=C_1(y)_{m=0}+C_2\left(\frac{\partial y}{\partial m}\right)_m = C_1a_o(1-2x+\frac{3x^2}{2} - \frac{4x^3}{6} + \cdots \ldots \ldots \ldots) + \{3x + (\frac{1}{2} - \frac{6}{2} - \frac{3}{4})x^2\} + \cdots \ldots \ldots \ldots$

i.e. $y = (A + B\log x)(1 - 2x + \frac{3x^2}{2} - \frac{4x^3}{6} + \cdots \ldots \ldots \ldots) + B\ \{3x + (\frac{1}{2} - \frac{6}{2} - \frac{3}{4})x^2\} + \cdots \ldots$

where, $A=C_1a_o$ and $B=C_2a_o$ are arbitrary constant.

(Ans)

1800023

Example 03: $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

Solution:

THE GIVEN DE is : $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0 \ldots . (1)$

x=0 Is a regular singular point. We assume the solution In the form that,

$$y = \sum_{k=0}^{\infty} a_k\, x^{m+k}$$

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k\, (m+k)x^{m+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k\, (m+k)(m+k-1)x^{m+k-2}$$

Substituting for $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ *in equation* 1

$$\sum x\, a_k(m+k)(m+\ k-\ 1)x^{m+k-2} + \sum a_k\, (m+k)x^{m+k-1} - \sum a_k\, x^{m+k} = 0$$

Or $\sum a_k[\ x^{m+k-1}\{(m+k)(m+k-1) + (m+k)\}] - \sum a_k\, x^{m+k} = 0$

$\sum a_k \{x^{m+k-1}(m+k)^2\} - \sum a_k x^{m+k} = 0 \ldots\ldots (2)$

THE Coefficient Of lowest degree x^{m-1}

in equation 2 is obtained by putting k=0 in the 1st summation only of 2 and equating it to 0

$$a_o m^2 = 0$$

$$m = 0 \; as \; a_o \neq 0$$

The coefficient of 2nd lowest degree x^m

In equation 2 is obtained by putting k=0 in 2nd summation and k=1 in 1st summation of only 2 we get,

$a_1 (m+1)^2 - a_o = 0$

$a_1 = \frac{a_o}{(m+1)^2}$

equating the coefficient of x^{m+k} to zero

$a_{k+1}(m+k+1)^2 - a_k = 0$

k=0

$a_1 = \frac{a_o}{(m+1)^2}$

k =1

$a_2 = \frac{a_o}{(m+1)^2(m+2)^2}$

k =2

$a_3 = \frac{a_o}{(m+1)^2 \; (m+2)^2 \; (m+3)^2}$

we know

$y = x^m[\, a_o + \; a_1 x^1 + a_2 . x^2 + \cdots]$

Or,$y = x^m \; [a_o + \frac{a_o}{(m+1)^2} x + \frac{a_o}{(m+1)^2(m+2)^2(m+3)^2} x^2 + \cdots]$

$$or, y = x^m \, a_o \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2(m+2)^2} + \frac{x^3}{(m+1)^2(m+2)^2(m+3)^2} + \cdots \right]$$

$(y)_{m=0} = a_o[\, 1 + \frac{x}{1^2} + \frac{x^2}{1^2 2^2} + \frac{x^3}{1^2 2^2 3^2} + \cdots \ldots] \ldots . (3)$

$(\frac{\partial y}{\partial m})=a_o x^m\left[-\frac{2x}{(m+1)^3}-\frac{2x^2}{(m+1)^2(m+2)^3}-\frac{2x^2}{(m+1)^3(m+2)^2}-\cdots\ldots\right]+a_o x^m logx[1+\frac{x}{(m+1)2}+\frac{x^2}{(m+1)^2(m+2)^2}+\cdots..]$

$(\frac{\partial y}{\partial m})_{m=0}$=[$a_o\, x^0\{-\frac{2x}{1^3}-\frac{2x^2}{1^2 2^3}-\frac{2x^2}{1^3 2^2}-.\ldots\ldots\}]+[\{1+\frac{x}{1^2}+\frac{x^2}{1^2 2^2}+\cdots.\}a_o x^0 logx]$

=-2$a_o\left(\frac{x}{1^3}+\frac{x^2}{1^2 2^3}+\frac{x^2}{1^3 2^2}+\cdots.\right)+a_o logx\,\left(1+\frac{x}{1^2}+\frac{x^2}{1^2 2^2}+\cdots\right)$

General solution,

Y=Aa_o (1+$\frac{x}{1^2}+\frac{x^2}{1^2 2^2}+\cdots)+B[-2a_o\left\{\frac{x}{1^3}+\frac{x^2}{1^2 2^3}+\frac{x^2}{1^3 2^2}+\cdots..\right\}+\{1+\frac{x}{1^2}+\frac{x^2}{1^2 2^2}+\cdots..\}]$

1800024

Example 04: $x^2y'' - 5xy' + (9-x)y = 0$

Solution:

Given the differential equation $x^2y'' - 5xy' + (9-x)y = 0$

Let y = $\sum_{n=0}^{\infty} a_n\, x^{n+r}$

Y'=$\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$

Y" = $\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

Substituting y, y' & y" in given DE: $x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - 5x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + (9-x)\sum_{n=0}^{\infty} a_n\, x^{n+r}$= 0

Expand, $\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - 5\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + 9\sum_{n=0}^{\infty} a_n\, x^{n+r} - \sum_{n=0}^{\infty} a_n\, x^{n+r+1} = 0$

Equate to smallest power, x^{n+r}

$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - 5\sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + 9\sum_{n=0}^{\infty} a_n\, x^{n+r} - \sum_{n=1}^{\infty} a_n\, x^{n+r} = 0$

Indicial equation (n=0)

[r(r-1)-5r+9]a_0=0

$r^2 - r - 5r + 9 = 0\ \ since\ a_0 \neq 0$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

r = 3 (one root => case 2)

Recurrence relation:

$(n+r-3)^2 a_n - a_{n-1} = 0$

$$a_n = \frac{1}{(n+r-3)^2} a_{n-1}$$

When n=1, $a_1 = \frac{1}{(r-2)^2} a_0$

n = 2, $a_2 = \frac{1}{(r-1)^2} a_1 = \frac{1}{(r-1)^2 (r-2)^2} a_0$

y =$\sum_{n=0}^{\infty} a_n \, x^{n+r}$

$= x^r [a_0 + a_1 x + a_2 x^2 + \cdots]$

$= x^r a_0 [1 + \frac{x}{(r-2)^2} + \frac{x^2}{(r-1)^2 (r-2)^2} + \cdots]$

$$y_1 = y_{(r=3)} = x^3 a_0 [1 + \frac{1}{1^2} x + \frac{1}{2^2 1^2} x^2 + \cdots]$$

$$y_1 = a_0 [x^3 + x^4 + \frac{x^5}{4} + \cdots]$$

$$\frac{\partial y}{\partial x} = y lnx + a_0 x^r [\frac{1}{(r-2)^2}\left(\frac{-2}{r-2}\right) x + \frac{1}{(r-1)^2 (r-2)^2}\left(\frac{-2}{r-1} - \frac{2}{r-2}\right) x^2 + \cdots]$$

$$y_2 = \frac{\partial y}{\partial x_{r=3}} = y_1 lnx + a_0 x^3 [\frac{1}{(1)^2}\left(\frac{-2}{1}\right) x + \frac{1}{(2)^2 (1)^2}\left(\frac{-2}{2} - \frac{2}{1}\right) x^2 + \cdots]$$

$$y_2 = y_1 lnx + a_0 \left[-2x^4 - \frac{3}{4} x^5 + \cdots\right]$$

General equation,

$$y = A a_0 \left[x^3 + x^4 + \frac{x^5}{4} + \cdots\right] + B [y_1 lnx + a_0 \left(-2x^4 - \frac{3}{4} x^5 + \cdots\right)]$$

[answer]

1800025

Example 05: $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + x^2y = 0$

Solution:

THE GIVEN DE is : $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0 \ldots.(1)$

x=0 is a regular singular point.

By Frobenius method,

Let the trial sol^n, y = $\sum_{n=0}^{\infty} a_n x^{m+n}$

$\frac{dy}{dx} = \sum_{n=0}^{\infty}(m+n)a_n x^{(m+n-1)}$

$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{(m+n-2)}$

Now put the value of y, $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in given D.E. ……………… (1)

=>$x[\ \sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{(m+n-2)}\ ] + [\ \sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1}\ ] + x^2[\sum_{n=0}^{\infty} a_n x^{m+n}\ ] = 0$

=> $\sum_{n=0}^{\infty}(m+n)(m+n-1)a_n x^{(m+n-1)} + \sum_{n=0}^{\infty}(m+n)a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0$

=> $\sum_{n=0}^{\infty}[(m+n)(m+n-1) + (m+n)\ ]\ a_n x^{(m+n-1)} + \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0$

=> $\sum_{n=0}^{\infty}[(m+n)(m+n-1+1)]\ a_n x^{(m+n-1)} + \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0$

=> $\sum_{n=0}^{\infty}(m+n)^2\ a_n x^{(m+n-1)} + \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0$

Now by equating lead power of x ,

$i.e. x^{m-1}$ equal to zero :

$(M)^2 a_0 = 0$

=> $m^2 = 0$ $\quad [as, a_0 \neq 0]$

Which is the required Indicial equation,

M = 0, 0

So, $m_1 = 0$ & $m_2 = 0$

So, root of indicial equation are same

$m_1 = m_2$ [Case (2)]

From equation (2),

$\sum_{n=0}^{\infty}(m+n+1)^2\ a_{n+1} x^{(m+n)} + \sum_{n=0}^{\infty} a_{n-2} x^{m+n} = 0$

$\sum_{n=0}^{\infty}[(m+n+1)^2\, a_{n+1} + \ a_{n-2}]\, x^{(m+n)} = 0$……………(3)

$Now\ put\ the$ co-efficient of $x^{(m+n)}$ equal to zero,

$(m+n+1)^2 a_{n+1} + \ a_{n-2} = 0$

$(m+n+1)^2 a_{n+1} = -\ a_{n-2}$

$a_{n+1} = \frac{-1}{(m+n+1)^2} a_{n-2}$……………(4)

Which is the required recurrence relation,

Now put, n=2 $\quad a_3 = \frac{-1}{(m+3)^2} a_0$

Now by equating co-efficient of x^m both series in equation (2),

$(M+1)^2 a_1 = 0$

$a_1 = 0$

$Now\ by\ equating$ co-efiicient of x^{m+1} both sides in equation (2)

$$(m+2)^2 a_2 = 0$$

$$a_2 = 0$$

Now put n=3 in equation (4)

$$a = -\frac{1}{(m+4)^2} \quad a_1 = -\frac{1}{(m+4)^2}(0)$$

$$a_4 = 0$$

Put n=4 in equation (4)

$$a_5 = -\frac{1}{(m+5)^2} \quad a_2 = -\frac{1}{(m+5)^2}(0)$$

$$a_5 = 0$$

Put n=5 in equation (4)

$$a_6 = -\frac{1}{(m+6)^2}\, a_3 = -\frac{1}{(m+6)}\left[-\frac{1}{(M+3)^2} a_0\right]$$

$$a_6 = \frac{1}{(m+6)^2(m+3)^2} a_0$$

Now put the value of a_1, a_2, a_3, a_4, a_5, a_6

Trial solution

$$y = \sum_{n=0}^{\infty} a_n x^{m+n} = a_6 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + a_4 x^{m+4} + a_5 x^{m+5} + a_6 x^{m+6} + \cdots$$

$$= x^m[a_0 + (0)x^2 + \left[-\frac{1a_0}{(m+3)^2}\right]x^2 + (0)x^4 + (0)x^5 + [\frac{1}{(M+6)^2 + (m+3)^2}a_0x^6 + \cdots]$$

$$= x^m[a_0 - \frac{1}{(m+3)^2}a_0x^3 + \frac{1}{(m+6)^2(m+3)^2}a_6x^6 + \cdots]$$

$$a_0x^m[1 - \frac{1}{(m+3)^2}x^3 + \frac{1}{(m+3)(m+6)^2}x^6 + \cdots]$$

Now complete solution of case II is given as

$$y = C_1(y)_{m_1=0} + C_2\left(\frac{dy}{dm}\right)_{m_1=0} y_{m=0}$$

Now $y_{m=0} = a_0x^0\left[1 - \frac{x^3}{3^2} + \frac{x^6}{(3)^2(6)^2} + \cdots \dots \dots \dots \infty\right]$

Now partially Differentiate eq (5) w.r.t m, we get

$\frac{\partial y}{\partial m} = a_o(x^m logx)\left[1 - \frac{1}{(m+3)^2}x^3 + \frac{1}{(m+3)^2(m+6)^2}x^6 + \cdots \dots \dots \infty\right] + a_ox^m[0 + \frac{2}{(m+3)^3}x^3 +$ $\{-2(m+3)^{-3}(m+6)^{-2} + (m+3)^{-2}(-2)(m+6)^{-3}\}x^6 + \cdots \dots \dots$[By Product Rule]

$=a_o(x^m logx)\left[1 - \frac{1}{(m+3)^2}x^3 + \frac{1}{(m+3)^2(m+6)^2}x^6 + \cdots \dots \dots \infty\right] + a_ox^m[\frac{2x^3}{(m+2)^3} - \frac{2x^6}{(m+3)^3(m+6)^2} - \frac{2x^6}{(m+3)^2(m+6)^3} + \cdots \dots \dots \dots \infty]$ $(\frac{\partial y}{\partial m})_{m=0} = a_o(1)logx\left[1 - \frac{x^3}{3^2} + \frac{x^6}{(3)^2(6)^2} + \cdots \dots \dots \dots \infty\right] + a_o(1)[\frac{2x^3}{3^3} - \frac{2x^6}{(3)^3(6)^2} - \frac{2x^6}{(3)^2(6)^3} + \cdots \dots \dots \dots \infty]$

Now put $(y)_{m+0}$ and $(\frac{\partial y}{\partial m})_{m=0}$ *in eq.* (6) *we get,*

$y = C_1[a_o\left\{1 - \frac{x^3}{3^2} + \frac{x^6}{(3)^2(6)^2} + \cdots \dots \dots \dots \infty\right] + C_2[a_o logx\left\{1 - \frac{x^3}{3^2} + \frac{x^6}{(3)^2(6)^2} + \cdots \dots \dots \dots \infty\right\} + a_o\{\frac{2x^3}{3^3} - \frac{2x^6}{(3)^3(6)^2} - \frac{2x^6}{(3)^2(6)^3} + \cdots \dots \dots \dots \infty\}]$

Y=A (1-$\frac{x^3}{3^2}$ +$\frac{x^6}{(3)^2(6)^2}$ ……… ∞) + B $\left[\log(1 - \frac{x^3}{3^2} + \frac{x^6}{(3)^2(6)^2} \dots \dots \dots \infty) + (\frac{2x^3}{3^3} - \frac{2x^6}{(3)^3(6)^2} - \frac{2x^6}{(3)^2(6)^3})\right]$

Y = (A + B $logx$) (1-$\frac{x^3}{3^2} + \frac{x^6}{(3)^2(6)^2} \dots \dots \dots \infty) + 2B\left[\frac{x^3}{3^3} - \frac{1}{(3)^2(6)^2}\left(\frac{1}{3} + \frac{1}{6}\right)x^6 + \cdots \dots \dots \infty\right]$

Which is the required complete solution of given D.E. (Answer)

1800026

Example 06: Solve x (x – 1) y” + (3x – 1) y’ + y = 0

Solution.

x (x – 1) y” + (3x – 1) y’ + y = 0 …… (1)

Since, x = 0 is a regular singular point, we assume the solution in the form

$$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

Such that

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k\,(m+k)x^{m+k-1}$$

$$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting the expressions for y, $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1) we have

$$\sum x(x-1)\,a_k(m+k)(m+k-1)x^{m+k-2} +$$

$$(3x-1)\sum a_k(m+k)^{x^{m+k-1}} + \sum a_k x^{m+k} = 0$$

or $\quad \sum a_k(m+k)(m+k-1)x^{m+k} - \sum a_k(m+k)(m+k-1)x^{m+k-1} +$

$$3\sum a_k(m+k)^{x^{m+k}} - \sum a_k(m+k)x^{m+k-1} + \sum a_k x^{m+k} = 0$$

or $\quad \sum a_k[(m+k)(m+k-1)+3(m+k)+1]\,x^{m+k} -$

$$\sum a_k[(m+k)(m+k-1)+(m+k)]\,x^{m+k-1} = 0$$

or $\quad \sum a_k[(m+k)(m+k+2)+1]\,x^{m+k} - \sum a_k(m+k)^2\,x^{m+k-1} = 0 \dots\dots (2)$

The coefficient of lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the second summation only of (2) and equating it to zero. Then the indicial equation is

$$a_0(m+0)^2 = 0 \quad \Rightarrow m = 0,\ 0\ as\ a_0 \neq 0$$

The coefficient of the next lowest degree term x^m in (2) is obtained by putting k = 0 in the first summation and k = 1 in the second summation only of (2) and equating it to zero, we get

$$a_1[(m+0)(m+2)+1] - a_0(m+0)^2 = 0$$

$$a_1 - a_0 = 0 \qquad \Rightarrow \quad a_1 = a_0 \text{ (as m = 0)}$$

Equating the coefficient of x^{m+k} to zero, the recurrence relation is given by

$$a_k[(m+k)(m+k+2)+1] - a_{k+1}(m+k+1)^2 = 0$$

$$a_k[(m+k+1)^2] - a_{k+1}(m+k+1)^2 = 0$$

Hence $\quad a_{k+1} = a_k$

$$y = x^m[a_0 + a_1x + a_2x^2 + \cdots]$$

$$y = a_0x^m[1 + x + x^2 + x^3 \dots]$$

When m = 0, this gives only one solution instead of two.

Second solution is given by

$\left(\frac{\partial y}{\partial m}\right)_{m=0}$ and $y_1 = a_0(1 + x + x^2 + x^3)$

$$\frac{\partial y}{\partial m} = a_0 x^m \log x[1 + x + x^2 + x^3 + \cdots]$$

$$y_2 = a_0 \log x[1 + x + x^2 + x^3 + \cdots] \qquad \text{m = 0}$$

$$y_1 = a_0[1 + x + x^2 + x^3 + \cdots] \qquad \text{m = 0}$$

$$y = Ay_1 + By_2$$

$y = A[1 + x + x^2 + x^3 + \cdots] + B \log x[1 + x + x^2 + x^3 + \cdots]$ **Ans.**

1800028

Example 07: Using extended power series method find one solution of the differential equation $xy" + y' + x^2y = 0$. Indicate the form of a second solution which is linearly independent of the first obtained above.

Solution. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + x^2y = 0$

Let $y = \sum a_k x^{m+k}$ …… (1)

$\frac{dy}{dx} = \sum a_k(m + k)x^{m+k-1}$, $\frac{d^2y}{dx^2} = \sum a_k(m + k)(m + k - 1)x^{m+k-2}$

Substituting the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$x\sum a_k(m + k)(m + k - 1)x^{m+k-2} + \sum a_k(m + k)x^{m+k-1} + x^2\sum a_k x^{m+k} = 0$$

or $\sum a_k(m + k)(m + k - 1)x^{m+k-1} + \sum a_k(m + k)x^{m+k-1} + \sum a_k x^{m+k+2} = 0$

or $\sum a_k[(m + k)(m + k - 1) + (m + k)]x^{m+k-1} + \sum a_k x^{m+k+2} = 0$

or $\sum a_k(m + k)^2x^{m+k-1} + \sum a_k x^{m+k+2} = 0$

The coefficient of lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in first summation of (2) only and equating it to zero. Then the indicial equation is

$$a_0m^2 = 0 \qquad \Rightarrow \qquad m^2 = 0 \text{ or } m = 0, 0$$

The coefficient of the next lowest degree term x^m in (2) is obtained by putting k = 1 in first summation only and equating it to zero.

$$a_1(m+1)^2 = 0 \qquad \text{or} \qquad a_1 = 0$$

Equating the coefficient of x^{m+1} for $k = 2$ we get

$$a_2(m+2)^2 = 0 \qquad \Rightarrow \qquad a_2 = 0$$

Equating the coefficient of x^{m+k+2} to zero, we have

$$a_{k+3}(m+k+3)^2 + a_k = 0$$

$$a_{k+3} = -\frac{a_k}{(m+k+3)^2}$$

$k = 0$, $a_3 = -\frac{1}{(m+3)^2} a_0$

$k = 1$, $a_4 = -\frac{1}{(m+4)^2} a_1 = 0$, $a_7 = 0, a_{10} = 0$

$k = 2$, $a_5 = -\frac{1}{(m+5)^2} a_2 = 0$, $a_8 = 0, a_{11} = 0$

$k = 3$, $a_6 = -\frac{1}{(m+6)^2} a_3 = \frac{1}{(m+3)^2(m+6)^2} a_0$

$$a_9 = -\frac{1}{(m+9)^2} a_6 = -\frac{1}{(m+3)^2(m+6)^2(m+9)^2} a_0$$

$$y = x^m a_0 \left[1 - \frac{x^3}{(m+3)^2} + \frac{x^6}{(m+3)^2(m+6)^2} - \frac{x^9}{(m+3)^2(m+6)^2(m+9)^2} + \cdots\right] \qquad \ldots\ldots (3)$$

To get the first solution, let $m = 0$ in (3), then

$$y_1 = a_0 \left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2 \times 6^2} - \frac{x^9}{3^2 \times 6^2 \times 9^2} + \cdots\right] \qquad \ldots\ldots (4)$$

To get the second independent solution, differentiate (3) w.r.t. m. Then

$$\frac{\partial y}{\partial m} = (x^m \log x)\, a_0 \left[1 - \frac{x^3}{(m+3)^2} + \frac{x^6}{(m+3)^2(m+6)^2} - \frac{x^9}{(m+3)^2(m+6)^2(m+9)^2} + \cdots\right] + x^m a_0 \left[\frac{2x^3}{(m+3)^3} - \frac{2x^6}{(m+3)^3(m+6)^2} - \frac{2x^6}{(m+3)^2(m+6)^3} + \frac{2x^9}{(m+3)^3(m+6)^2(m+9)^2} + \frac{2x^9}{(m+3)^2(m+6)^3(m+9)^2} + \frac{2x^9}{(m+3)^2(m+6)^2(m+9)^3} + \cdots\right] \qquad \ldots\ldots (5)$$

Putting m = 0 in (5), we get

$$y_2 = (\log x) a_0 \left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2 \times 6^2} - \frac{x^9}{3^2 \times 6^2 \times 9^2} + \cdots\right] + a_0 \left[\frac{2x^3}{3^3} - \frac{2x^6}{3^3 \times 6^2} - \frac{2x^6}{3^2 \times 6^3} + \frac{2x^9}{3^3 \times 6^2 \times 9^2} + \frac{2x^9}{3^2 \times 6^3 \times 9^2} + \frac{2x^9}{3^2 \times 6^2 \times 9^3} + \cdots\right] \qquad \ldots\ldots (6)$$

Hence the general solution is given by (4) and (6)

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 a_0 \left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2 \times 6^2} - \frac{x^9}{3^2 \times 6^2 \times 9^2} + \cdots\right] + c_2 (\log x) a_0 \left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2 \times 6^2} - \frac{x^9}{3^2 \times 6^2 \times 9^2} + \cdots\right] + \left[\frac{2x^3}{3^3} - \frac{2x^6}{3^2 \times 6^6}\left(\frac{1}{3} + \frac{1}{6}\right) + \frac{2x^9}{3^2 \times 6^2 \times 9^2}\left(\frac{1}{3} + \frac{1}{6} + \frac{1}{9}\right) + \cdots\right]$$

$$= (c_1 + c_2 \log x)a_0\left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2\times6^2} - \frac{x^9}{3^2\times6^2\times9^2} + \cdots\right] + 2c_2a_0\left[\frac{x^3}{3^3} - \frac{x^6}{3^5\times2^2}\left(1+\frac{1}{2}\right) + \frac{2x^9}{3^9\times2^2}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \cdots\right]$$ **Ans.**

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Case III: When m_1 and m_2 are distinct and differing by an integer

Then, $y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_2}$ $\left[\begin{matrix} If\ coefficient = \infty \\ when\ m = m_2 \end{matrix}\right]$

#Example 01: Find the solution of $xy'' + (x-1)y' - y = 0$ using Frobenius method.

Solution:

$xy'' + (x-1)y' - y = 0$……….(i)

Here,

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$\therefore y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

Putting these values in (i) we get

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1) - (n-r)]a_n x^{n+r-1} + \sum_{n=0}^{\infty}[(n+r)-1]a_n x^{n+r} = 0$$

$eq^n(ii)$

Now putting $n = 0$ in the first summation of equation (ii) and equating it to zero, we get the coefficient of the lowest degree term x^{r-1} to be

$$a_0\{r(r-1) - r\} = 0$$

$$a_0\{r(r-2)\} = 0$$

$$\therefore r = 0\,,2 \quad [\because a_0 \neq 0]$$

The coefficient of next lowest degree x^r in $eq^n(ii)$ is obtained by putting $n = 1$ in the first summation and $n = 0$ in the second summation and equating it to zero,

$$a_1\{(r+1)r - (r+1)\} + a_0(r-1) = 0$$
$$a_1(r^2 + r - r - 1) + a_0(r-1) = 0$$
$$a_1(r^2 - 1) + a_0(r-1) = 0$$
$$a_1(r+1) + a_0 = 0$$
$$a_1 = -\frac{1}{r+1}a_0$$

Now equating the coefficient of x^{n+r} from $eq^n(ii)$

$$a_{n+1}\{(n+r+1)(n+r) - (n+r+1)\} + a_n(n+r-1) = 0$$

$$a_{n+1}(n+r+1)(n+r-1) + a_n(n+r-1) = 0$$

$$a_{n+1}(n+r+1) + a_n = 0$$
$$\therefore a_{n+1} = \frac{-1}{n+r+1}a_n$$

$eq^n(iii)$

For $n = 0, 1, 2, 3 \ldots .\, in\ eq^n(iii)$

$$a_1 = \frac{-1}{r+1}a_0$$
$$a_2 = \frac{-1}{r+2}a_1 = \frac{-1}{(r+1)(r+2)}a_0$$
$$a_3 = \frac{-1}{r+3}a_2 = \frac{-1}{(r+1)(r+2)(r+3)}a_0$$

and so on

For $r = 0$

$$a_1 = \frac{-1}{1}a_0$$
$$a_2 = \frac{-1}{2}a_0$$
$$a_3 = \frac{-1}{3.2}a_0$$

and so on

For $r = 2$

$$a_1 = \frac{-1}{3}a_0$$
$$a_2 = \frac{-1}{4.3}a_0$$
$$a_3 = \frac{-1}{5.4.3}a_0$$

and so on

$\therefore$ *General sol^n of eq^n(ii)is*

$$y = c_1(y)_{r=0} + c_2(y)_{r=2}$$

$$= c_1[x^0\left\{a_0 + \left(\frac{-1}{1}\right)a_0x + \left(\frac{-1}{2}\right)a_0x^2 + \left(\frac{-1}{3.2}\right)a_0x^3 + \cdots\right\}$$
$$+\ c_2\left[x^2\left\{a_0 + \left(\frac{-1}{3}\right)a_0x + \left(\frac{-1}{4.3}\right)a_0x^2 + \left(\frac{-1}{5.4.3}\right)a_0x^3 + \cdots\right\}\right]$$

$$= c_1a_0\left(1 - \frac{1}{1!}x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \cdots\right) + \ c_2a_0(x^2 - \frac{2}{3!}x^3 - \frac{2}{4!}x^4 - \frac{2}{5!}x^5 + \cdots)$$

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#Example 02: xy'' − (4 + x)y' + 2y = 0

Solve:

xy'' − (4 + x)y' + 2y = 0……(a)

Here, x=0 is a regular singular point.

Y=$\sum_{k=0}^{\infty} a_kx^{m+k}$…………… (1)

Y'=$\sum_{k=0}^{\infty}(m+k)a_kx^{m+k-1}$……………… (2)

Y''=$\sum_{k=0}^{\infty}(m+k)(m+k-1)a_kx^{m+k-2}$……………. (3)

Putting the values of eq^n (1), (2) and (3) in eq^n (a)we get,

x[$\sum_{k=0}^{\infty}(m+k)(m+k-1)a_kx^{m+k-2}$]-(4 + x) [$\sum_{k=0}^{\infty}(m+k)a_kx^{m+k-1}$]+2[$\sum_{k=0}^{\infty} a_kx^{m+k}$]=0

→$\sum_{k=0}^{\infty}(m+k)(m+k-1)a_kx^{m+k-1}$]- [$\sum_{k=0}^{\infty} 4(m+k)a_kx^{m+k-1}$]- [$\sum_{k=0}^{\infty}(m+k)a_kx^{m+k}$]+ [$\sum_{k=0}^{\infty} 2a_kx^{m+k}$]=0

→[$\sum_{k=0}^{\infty}(m+k)(m+k-1-4)a_kx^{m+k-1}$]- [$\sum_{k=0}^{\infty}(m+k-2)a_kx^{m+k}$]=0

→[$\sum_{k=0}^{\infty}(m+k)(m+k-5)a_kx^{m+k-1}$]- [$\sum_{k=0}^{\infty}(m+k-2)a_kx^{m+k}$]=0

→[$(m-5)ma_0x^{m-1}$] + [$\sum_{k=0}^{\infty}(m+k)(m+k-5)a_kx^{m+k-1}$]- [$\sum_{k=0}^{\infty}(m+k-3)a_{k-1}x^{m+k-1}$]=0

Equating the like power of x co-efficient

(m-5)ma_0=0; a_0≠0

→(m-5)m=0

→m=0 or, m-5=0

So, m=0 or, m=5

Hence, m_1=0, m_2=5

These two roots are district and differ by an integer

Now,

$(m+k-5)(m+k)a_k-(m+k-3)a_{k-1}=0$

$\rightarrow(m+k-5)(m+k)a_k = (m+k-3)a_{k-1}$

$\rightarrow a_k = \frac{(m+k-3)a_{k-1}}{(m+k-5)(m+k)}$

For m=0 (smaller root)

$\rightarrow a_k = \frac{(k-3)a_{k-1}}{(k-5)k}$ [for a_k is an arbitrary constant]

For k=1, $a_1=\frac{-2*a_0}{-4} = \frac{a_0}{2}$

K=2, $a_2=\frac{-1*a_1}{-3*2} = \frac{a_1}{6}$

K=3, $a_3=\frac{0*a_2}{-2*3} = 0$

K=4, $a_4=\frac{1*a_3}{-1*4} = \frac{a_3}{-4} = 0$

K=5, $a_5=\frac{2*a_4}{0*5} = undefined$

K=6, $a_6=\frac{3*a_5}{1*6} = \frac{a_5}{2}$

K=7, $a_7=\frac{4*a_6}{2*7} = \frac{2a_6}{7} = \frac{a_5}{7}$

General solution:

$Y=\sum_{k=0}^{\infty} a_k x^{m+k}$

$=\sum_{k=0}^{\infty} a_k x^k$

$=a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 + a_6x_6 + a_7x_7 + \cdots$

$=a_0 + \frac{a_0}{2}x + \frac{a_0}{12}x^2 + a_5x^5 + \frac{a_5}{2}x^6 + \frac{a_5}{7}x^7 + \cdots$

$=a_0\left[1 + \frac{x}{2} + \frac{x^2}{12}\right] + a_5[x^5 + \frac{x^6}{2} + \frac{x^7}{7} + \cdots]$

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#Example 03: $x^2 y'' + 2xy' + (x^2 - 6)y = 0$

Solution:

Let, $y = \sum a_k \cdot x^{(m+k)}$

$$\therefore \frac{dy}{dx} = \sum a_k (m+k) \cdot x^{(m+k-1)}$$

$$\therefore \frac{d^2y}{dx^2} = \sum a_k (m+k) \cdot (m+k-1) x^{(m+k-2)}$$

From given equation we can write by putting these values –

$$x^2 \sum a_k (m+k)(m+k-1) \cdot x^{(m+k-2)} + 2x \sum a_k (m+k) \cdot x^{(m+k-1)} + (x^2-6) \cdot \sum a_k \cdot x^{(m+k)} = 0$$

$$\Rightarrow \sum a_k [(m+k)(m+k-1) + 2(m+k) - 6] x^{(m+k)} + \sum a_k \cdot x^{(m+k+2)} = 0$$

$$\Rightarrow \sum a_k [m^2 + 2mk + k^2 + m + k - 6] x^{(m+k)} + \sum a_k \cdot x^{(m+k+2)} = 0$$

… … … … … … … … … … … … … (1)

The co-efficient of next lowest term x^m in (1) is obtained by putting k=0 in first summation only and equating it to zero. Then the identical equation is-

$$a_0 [m^2 + m - 6] = 0$$

$$\therefore m = -3, 2$$

The co-efficient of next lowest term $x^{(m+1)}$ in (1) is obtained by putting k=1 in first summation only and equating it to zero.

$\Rightarrow a_1 (m-1)(m+4) = 0 \qquad \Rightarrow a_1 = 0$

Equating the co-efficient of $x^{(m+k-2)}$

$$a_{(k+2)} \{(m+k)(m+k+5)\} + a_k = 0$$

or, $a_{(k+2)} = \dfrac{-a_k}{(m+k)(m+k+5)}$

$$\therefore a_1 = a_3 = a_5 = \cdots = 0$$

$$\therefore a_2 = \frac{-a_0}{m(m+5)}$$

$$\therefore a_4 = \frac{-a_2}{(m+2)(m+7)} = \frac{-a_0}{m(m+2)(m+5)(m+7)}$$

$$\therefore a_6 = \frac{-a_4}{(m+4)(m+9)} = \frac{-a_0}{m(m+2)(m+4)(m+5)(m+7)(n+9)}$$

Hence,

$$y = a_0 \cdot x^m \left[1 - \frac{x^2}{m(m+5)} + \frac{x^4}{m(m+2)(m+5)(m+7)} - \frac{x^6}{m(m+2)(m+4)(m+5)(m+7)(m+9)} + \cdots\right]$$

… … … … … … … … … … … … … (2)

Puting $m = 2$ in equation (2) we get,

$$y_1 = a_0 x^2 \left[1 - \frac{x^2}{2 \times 7} + \frac{x^4}{2 \times 4 \times 7 \times 9} - \frac{x^6}{2 \times 4 \times 6 \times 7 \times 9 \times 11} + \cdots\right]$$

… … … … … … … … … … … … … (3)

Co-efficient of x^4, x^6 etc in equation (2) becomes infinite on putting m=-2. To overcome this difficulty we put $a_0 = b_0\,(m+2)$ in equation (2) and we get,

$$y = b_0 \cdot x^m \left[(m+2) - \frac{(m+2)x^2}{m(m+5)} + \frac{x^4}{m(m+5)(m+7)} - \frac{x^6}{m(m+4)(m+5)(m+7)(m+9)} + \cdots\right]$$

… … … … … … … … … … … … … (4)

By differentiating (4) w.r.t 'm', we get

$$\frac{\partial y}{\partial m} = b_0(x^m \log x)\ [(m+2) - \frac{(m+2)x^2}{m(m+5)} + \frac{x^4}{m(m+5)(m+7)} - \frac{x^6}{m(m+4)(m+5)(m+7)(m+9)} + \cdots\Big]$$
$$+ b_0 x^m \left[1 - \frac{(m+2)x^2}{m(m+5)}\left\{\frac{1}{(m+2)} - \frac{1}{m} - \frac{1}{(m+5)}\right\} + \frac{x^4}{m(m+5)(m+7)}\left\{-\frac{1}{m} - \frac{1}{(m+5)} - \frac{1}{(m+7)}\right\} + \cdots\right]$$

Or replacing m by -2, we get

$$\left(\frac{\partial y}{\partial m}\right)_{m=-2} = (b_0 \cdot x^{-2} \cdot \log\ x)\left[0 - 0 + \frac{x^4}{(-2)\cdot(3)\cdot(5)} - \frac{x^6}{(-2)(2)(3)(5)(7)} + \cdots\right]$$
$$+ b_0 x^{-2}\Big[1 + \frac{x^2}{2 \times 3} + \frac{x^4}{2 \times 3 \times 5}\left(\frac{31}{30}\right) + \cdots$$

Or,

$$y_2 = b_0 x^2 \log x\left(-\frac{1}{2\times 3\times 5}+\frac{x^2}{2^2\times 3\times 5\times 7}-\frac{x^4}{2^3\times 3^2\times 5\times 7}+\cdots\right)$$
$$+b_0 x^{-2}\left(1+\frac{x^2}{2\times 3}+\frac{x^4}{2\times 3\times 5}\times\left(\frac{31}{30}\right)+\cdots\right)$$

General solution is $y = c_1 y_1 + c_2 y_2$

$$y = c_1 x^2\left(1-\frac{x^2}{2\times 7}+\frac{x^4}{2\times 4\times 7\times 9}-\frac{6}{2\times 4\times 6\times 7\times 9\times 11}+\cdots\right)$$
$$+c_2\left[x^2\log x\left(-\frac{1}{2\times 3\times 5}+\frac{x^2}{2^2\times 3\times 5\times 7}-\frac{x^4}{2^3\times 3^2\times 5\times 7}+\cdots\right)\right.$$
$$\left.+x^{-2}\left(1+\frac{x^2}{2\times 3}+\frac{x^4}{2\times 3\times 5}\left(\frac{31}{30}\right)+\cdots\right)\right]$$

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Example 04. Solve

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-4)y=0 \qquad \ldots\ldots(1)$$

Solution.

Let $y=\sum a_k x^{m+k}$

$$\frac{dy}{dx}=\sum a_k(m+k)x^{m+k-1}, \qquad \frac{d^2y}{dx^2}=\sum a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y in (1) we get

$$x^2\sum a_k(m+k)(m+k-1)x^{m+k-2}+x\sum a_k(m+k)x^{m+k-1}+$$
$$(x^2-4)\sum a_k x^{m+k}=0$$

$$\sum a_k[(m+k)(m+k-1)+(m+k)-4]x^{m+k}+\sum a_k x^{m+k+2}=0$$

$$\sum a_k(m+k+2)(m+k-2)x^{m+k}+\sum a_k x^{m+k+2}=0 \qquad \ldots\ldots(2)$$

The coefficient of lowest degree term x^m in (2) is obtained by putting k = 0 in first summation only and equating it to zero. Then the indicial equation is

$$a_0(m+2)(m-2)=0 \Rightarrow \qquad m=2,-2$$

The coefficient of next lowest term x^{m+1} in (2) is obtained by putting k = 1 in first summation only and equating it to zero.

$$a_1(m+3)(m-1)=0 \qquad \Rightarrow \qquad a_1=0$$

Equating the coefficient of x^{m+k+2}

$$a_{k+2}(m+k+4)(m+k)+a_k=0 \qquad \text{or} \qquad a_{k+2}=-\frac{a_k}{(m+k+4)(m+k)}$$

$$a_1=a_3=a_5=\cdots=0$$

$$a_2=-\frac{a_0}{m(m+4)}$$

$$a_4=-\frac{a_2}{(m+2)(m+6)}=\frac{a_0}{m(m+2)(m+4)(m+6)}$$

$$a_6=-\frac{a_4}{(m+4)(m+8)}=-\frac{a_0}{m(m+2)(m+4)^2(m+6)(m+8)}$$

Hence $$y=a_0x^m\left[1-\frac{x^2}{m(m+4)}+\frac{x^4}{m(m+2)(m+4)(m+6)}-\frac{x^6}{m(m+2)(m+4)^2(m+6)(m+8)}+\cdots\right] \quad \ldots\ldots (3)$$

Putting m = 2 in (3), we get

$$y_1=a_0x^2\left[1-\frac{x^2}{2\times6}+\frac{x^4}{2\times4\times6\times8}-\frac{x^6}{2\times4\times6^2\times8\times10}+\cdots\right] \quad \ldots\ldots (4)$$

Coefficient of x^4, x^6 etc. in (3) becomes infinite on putting $m=-2$. To overcome this difficulty, we put

$a_0=b_0(m+2)$ in (3) and we get

$$y=b_0x^m\left[(m+2)-\frac{x^2}{m(m+4)}+\frac{x^4}{m(m+2)(m+4)(m+6)}-\frac{x^6}{m(m+2)(m+4)^2(m+6)(m+8)}+\cdots\right]$$

On differentiating (5) w.r.t. 'm', we get

$$\frac{\partial y}{\partial m}=b_0(x^m\cdot\log x)\left[(m+2)-\frac{(m+2)x^2}{m(m+4)}+\frac{x^4}{m(m+4)(m+6)}-\frac{x^6}{m(m+4)^2(m+6)(m+8)}+\cdots\right]+b_0x^m\left[1-\frac{(m+2)x^2}{m(m+4)}\left(\frac{1}{m+2}-\frac{1}{m}-\frac{1}{m+4}\right)+\frac{x^4}{m(m+4)(m+6)}\left(-\frac{1}{m}-\frac{1}{m+4}-\frac{1}{m+6}\right)+\cdots\right]$$

On replacing m by -2 , we get

$$\left(\frac{\partial y}{\partial m}\right)_{m=-2}=(b_0x^{-2}\cdot\log x)\left[0-0+\frac{x^4}{(-2)(2)(4)}-\frac{x^6}{(-2)(2)^2(4)(6)}+\cdots\right]+ b_0x^{-2}\left[1+\frac{x^2}{2^2}+\frac{x^4}{2^2\times4}\left(\frac{1}{4}\right)+\cdots\right]$$

or $$y_2=b_0x^2\log x\left(-\frac{1}{2^2\times4}+\frac{x^2}{2^3\times4^2\times6\times8}+\cdots\right)+b_0x^{-2}\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\times4^2}+\cdots\right)$$

General solution is $y=c_1y_1+c_2y_2$

$$y=c_1x^2\left(1-\frac{x^2}{2\times6}+\frac{x^4}{2\times4\times6\times8}-\frac{x^6}{2\times4\times6^2\times8\times10}+\cdots\right)+$$

$$c_2\left[x^2\log x\left(-\frac{1}{2^2\times4}+\frac{x^2}{2^3\times4\times6}-\frac{x^4}{2^3\times4^2\times6\times8}+\cdots\right)+x^{-2}\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\times4^2}+\cdots\right)\right]$$

Ans.

Case IV. If the roots differ by an integer such that one or more coefficients are indeterminate.

Related math

Example 01: Find the extended power series solution of the differential equation

$$\boldsymbol{x^2y''+4xy'+(x^2+2)y=0\ldots.(1)}$$

Solution:

Let, $y=\Sigma a_k x^{m+k}$ be the required solution of the given equation

$$\frac{dy}{dx}=\Sigma a_k(m+k)x^{m+k-1}$$
$$\frac{d^2y}{dx^2}=\Sigma a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting the value of $y, \frac{dy}{dx}, and\ \frac{d^2y}{dx^2}$ in the given equation.

$$x^2\Sigma a_k(m+k)(m+k-1)x^{m+k-2}+4x\Sigma a_k(m+k)x^{m+k-1}+(x^2+2)\Sigma a_k x^{m+k}=0$$
$$\Sigma a_k(m+k)(m+k-1)x^{m+k}+4\Sigma a_k(m+k)x^{m+k}+\Sigma a_k x^{m+k+2}+\Sigma 2a_k x^{m+k}=0$$

$$\Sigma a_k[(m+k)(m+k-1)+4(m+k)+2]x^{m+k}+\Sigma a_k x^{m+k+2}=0$$
$$\Sigma a_k[(m+k)^2+3(m+k)+2]x^{m+k}+\Sigma a_k x^{m+k+2}=0\ldots.(2)$$

The coefficient of lowest degree term x^m in (2) in obtained by putting $k=0$ in first summation only and equating it to zero. Then the indicial equation is

$$a_0(m^2+3m+2)=0$$
$$a_0\neq0, m^2+3m+2=0 \text{ or } (m+1)(m+2)=0, m=-1,-2$$

When $m=-2, a_1$ becomes indeterminate $\left(\frac{0}{0}\right)$ But in this case we get the identity $a_1(0)=0$ which is satisfied by every value of a_1. Therefore, in this case we can take a_1 as arbitrary constant Equating the coefficient of x^{m+k+2}

$$a_{k+2}[m^2+(2k+4+3)m+(k+2)^2+3(k+2)+2]+a_k=0$$
$$a_{k+2}[m^2+(2k+7)m+k^2+7k+12]+a_k=0$$

$$a_{k+2} = -\frac{1}{m^2 + (2k+7)m + k^2 + 7k + 12} a_k$$

For

$$a_2 = -\frac{1}{m^2 + 7m + 12} a_0 = -\frac{1}{(m+3)(m+4)} a_0$$

$$k = 1 \quad a_3 = -\frac{1}{m^2 + 9m + 20} a_1 = -\frac{1}{(m+4)(m+5)} a_1$$

$$a_4 \qquad = -\frac{1}{m^2 + 11m + 30} a_2 = \frac{1}{(m+3)(m+4)(m+5)(m+6)} a_0$$

$$a_5 = -\frac{1}{m^2 + 13m + 42} a_3 = \left\{\frac{1}{(m+4)(m+5)(m+6)(m+7)} a_1\right\}$$

For $m = -1$

$$a_2 = -\frac{1}{6} a_0, a_3 = \frac{1}{12} a_1, a_4 = \frac{1}{120} a_0, a_5 = \frac{1}{360} a_1$$

Hence for $m = -1$

$$y_1 = x^{-1}\left[1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \cdots\right] a_0 + \left[1 - \frac{1}{12}x^2 + \frac{x^4}{360} + \cdots\right] a_1$$

For $m = -2$

$$a_2 = -\frac{1}{2} a_0, a_3 = -\frac{1}{6} a_1, a_4 = \frac{1}{24} a_0, a_5 = \frac{1}{120} a_1$$

Hence for $m = -2$, second solution is

$$y_2 = x^{-2}\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right] a_0 + \left[\frac{1}{x} - \frac{x}{6} + \frac{x^3}{120} + \cdots\right] a_1$$

$$y_2 = x^{-2}\left[\left\{1 - \frac{x^2}{2} + \frac{x^4}{4} + \cdots\right\} a_0 + \left\{x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right\} a_1\right]$$

$$y_2 = x^{-2}[a_0 \cos\ x + a_1 \sin\ x]$$

Thus the complete solution is $y = Ay_1 + By_2$ (Ans)

<u>Example 02:</u> Find the extended power series solution of the differential equation

x²y''+ 4xy' + (x² + 2) y = 0

$\ldots (1)$

<u>Solution :</u>

Let $y = \Sigma a_k x^{m+k}$ be the required solution of the given equation

$$\therefore \frac{dy}{dx} = \Sigma a_k(m+k)x^{m+k-1}$$
$$\therefore \frac{d^2y}{dx^2} = \Sigma a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting the value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation,

$$x^2\Sigma a_k(m+k)(m+k-1)x^{m+k-2} + 4x\Sigma a_k(m+k)x^{m+k-1} + (x^2+2)\Sigma a_k x^{m+k} = 0$$
$$\Sigma a_k(m+k)(m+k-1)x^{m+k} + 4\Sigma a_k(m+k)x^{m+k} + \Sigma a_k x^{m+k+2} + \Sigma 2a_k x^{m+k} = 0$$
$$\Sigma a_k[(m+k)(m+k-1) + 4(m+k) + 2]x^{m+k} + \Sigma a_k x^{m+k+2} = 0$$
$$\Sigma a_k[(m+k)^2 + 3(m+k) + 2]x^{m+k} + \Sigma a_k x^{m+k+2} = 0$$

……………………………………(2)

The coefficient of lowest degree term x^m in (2) in obtained by putting k=0 in first summation only and equating it to zero. Then the indicial equation is

$$a_0(m^2+3m+2) = 0$$
$$a_0 \neq 0, m^2+3m+2 = 0 \quad \text{or} \quad (m+1)(m+2) = 0, \quad m = -1, -2$$

The coefficient of next lowest degree term x^{m+1} in (2) is obtained by putting k=1 in first summation only and equating it to zero.

$$a_1[m^2+5m+6] = 0 \quad \text{or} \quad a_1(m+2)(m+3) = 0 \Rightarrow a_1 = \frac{0}{(m+2)(m+3)}$$

when m=−2, a_1 becomes indeterminate $\left(\frac{0}{0}\right)$. But in this case we get the identity $a_1(0)=0$ which is satisfied by every value of a_1. Therefore in this case we can take a_1 as arbitrary constant

Equating the coefficient of x^{m+k+2}

$$a_{k+2}[m^2 + (2k+4+3)m + (k+2)^2 + 3(k+2) + 2] + a_k = 0$$
$$a_{k+2}[m^2 + (2k+7)m + k^2 + 7k + 12] + a_k = 0$$

$$a_{k+2} = -\frac{1}{m^2+(2k+7)m+k^2+7k+12}a_k$$

$$For\ k = 0 \qquad a_2 = -\frac{1}{m^2+7m+12}a_0 = -\frac{1}{(m+3)(m+4)}a_0$$

$$k = 1 \qquad a_3 = -\frac{1}{m^2+9m+20}a_1 = -\frac{1}{(m+4)(m+5)}a_1$$

$$a_4 = -\frac{1}{m^2+11m+30}a_2 = \frac{1}{(m+3)(m+4)(m+5)(m+6)}a_0$$

$$a_5 = -\frac{1}{m^2+13m+42}a_3 = \left\{\frac{1}{(m+4)(m+5)(m+6)(m+7)}a_1\right\}$$

$$For \qquad m = -1$$

$$a_2 = -\frac{1}{6}a_0, \quad a_3 = \frac{1}{12}a_1, \quad a_4 = \frac{1}{120}a_0, \quad a_5 = \frac{1}{360}a_1$$

$Hence\ for \quad m = -1$

$$y_1 = x^{-1}\left[1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \cdots\right]a_0 + \left[1 - \frac{1}{12}x^2 + \frac{x^4}{360} + \cdots\right]a_1$$

$for \quad m = -2$

$$a_2 = -\frac{1}{2}a_0, a_3 = -\frac{1}{6}a_1, a_4 = \frac{1}{24}a_0, a_5 = \frac{1}{120}a_1$$

Hence for m= -2, second solution is

$$y_2 = x^{-2}\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right]a_0 + \left[\frac{1}{x} - \frac{x}{6} + \frac{x^3}{120} + \cdots\right]a_1$$

$$y_2 = x^{-2}\left[\left\{1 - \frac{x^2}{2} + \frac{x^4}{4} + \cdots\right\}a_0 + \left\{x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right\}a_1\right]$$

$$y_2 = x^{-2}[a_0 \cos\ x + a_1 \sin\ x]$$

Thus the complete solution is $\boldsymbol{y = Ay_1 + By_2}$

(Ans)

Thank You

MATH ASSIGNMENT

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Bessel's Equation

Introduction:

The term Bessel's function or equation is named for Friedrich Wilhelm (1784-1846); however, Danial Bernoulli is generally credited with being the first to introduce the concept of Bessel's function in 1732. He used the function of zero order as a solution to the problem of an oscillating chain suspended at one end.

Specifically, a Bessel's function is a *solution of the differential equation*---

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

This is called *Bessel's Equation*. Where n is an arbitrary, constant value. We will limit our focus to values where $n \geq 0$.

Applications of Bessel's Equation:

A. **Heat Diffusion:** One common application that results in Bessel's function solution is steady-state temperature in a cylinder.
B. **Wave Propagation:** In this aspect we discuss a radially symmetrical circular drumhead with fixed edge that is directly in the center with an arbitrary force.

Solution of Bessel's Equation:

It is an ordinary differential equation of second order. It is found in the solution to Laplace's equation in cylindrical coordinates:

$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$;

For an arbitrary real or complex number n (the order of the Bessel's function). The most common and important special case is where n is a positive integer.

Dividing this equation by x^2 gives:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0 \text{ ;}$$

In this case $P_1(x) = \frac{1}{x}$ has a pole of first order at $x = 0$. When $n \neq 0$ $P_o(x) = \left(1 - \frac{n^2}{x^2}\right)$ has a pole of second order at $x=0$. Thus, this equation has a regular singularity at 0.

So, the Bessel's equation of order n is,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \text{-------------------(1)}$$

Since, $x=0$ is regular singular point for (1) so let its series solution be,

$$y = \sum_{r=0}^{\infty} a_r\, x^{m+r} = (a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots) \qquad \text{------(2)}$$

So that, $\frac{dy}{dx} = \sum a_r\,(m+r)x^{m+r-1}$ and

$$\frac{d^2y}{dx^2} = \sum a_r\,(m+r)(m+r-1)x^{m+r-2}$$

Substituting these values in the equation (1), we have,

$$x^2\sum a_r\,(m+r)(m+r-1)x^{m+r-2} + x\sum a_r\,(m+r)x^{m+r-1} + (x^2 - n^2)\sum a_r\,x^{m+r} = 0$$

Or, $\sum a_r\,(m+r)(m+r-1)x^{m+r} + \sum a_r\,(m+r)x^{m+r} + \sum a_r x^{m+r+2} - n^2\sum a_r\,x^{m+r} = 0$

Or, $\sum a_r[(m+r)(m+r-1) + (m+r) - n^2]x^{m+r} + \sum a_r x^{m+r+2} = 0$

Or, $\sum a_r[(m+r)^2 - n^2]x^{m+r} + \sum a_r x^{m+r+2} = 0$.

Equating the coefficient of x^m to zero, we get-

$$a_0[(m+0)^2 - n^2] = 0 \qquad \text{where, } r = 0$$

Or, $m^2 = n^2$ i.e. $m = \pm n$ where, $a_0 \neq 0$

Equating the coefficient of x^{m+1}, if $r=1$, then we get- $a_1[(m+1)^2 - n^2] = 0$ i.e. $a_1 = 0$. Since, $(m+1)^2 - n^2 \neq 0$

Equating the coefficient of x^{m+r+2} to zero, to find relation in successive coefficients, we get,

$$a_{r+2}[(m+r+2)^2 - n^2] + a_r = 0 \text{ or, } a_{r+2} = -\frac{1}{(m+r+2)^2-n^2} \cdot a_r$$

Therefore, $a_3 = a_5 = a_7 = \cdots = 0$ since, $a_1 = 0$

If r=0, $a_2 = -\frac{1}{(m+2)^2-n^2} \cdot a_0$

If r=2, $a_4 = -\frac{1}{(m+4)^2-n^2} \cdot a_2 = \frac{1}{[(m+2)^2-n^2][(m+4)^2-n^2]} \cdot a_0$

and so on.

On substituting the values of the coefficients in (2), we have,

$$y = a_0x^m - \frac{a_0}{(m+2)^2-n^2}\, x^{m+2} + \frac{a_0}{[(m+2)^2-n^2][(m+4)^2-n^2]}\, x^{m+4} + \cdots$$

Or, $y = a_0x^m[1 - \frac{1}{(m+2)^2-n^2}\, x^2 + \frac{1}{[(m+2)^2-n^2][(m+4)^2-n^2]}\, x^4 - \cdots]$

For m= n,

$$y = a_0x^n[1 - \frac{1}{4(n+1)}\, x^2 + \frac{1}{4^2-2!(n+1)(n+2)}\, x^4 - \cdots]$$

Where, $a_0$is an arbitrary constant,

For m=-n,

$$y = a_0x^{-n}[1 - \frac{1}{4(-n+1)}\, x^2 + \frac{a_0}{4^2-2!(-n+1)(-n+2)}\, x^4 - \cdots]$$

Where, $a_0$is an arbitrary constant.

Roll: 180047

Bessel's Function, $J_n(x)$:

Bessel functions, first defined by the mathematician Denial Bernoulli and then generalized by Friendrich Bessel, are canonical solutions $y(x)$ of Bessel's differential equation, for an arbitrary complex number 'n', the order of Bessel function.

The Bessel's function is

$x^2 \frac{d^2y}{dx^2}+x\frac{dy}{dx} + (x^2 - n^2)y = 0$ ……………… (1)

Solution of (1) is,

$$y = a_0x^n[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} - \cdots + (-1)^r \frac{x^{2r}}{(2^r r!)2^r(n+1)(n+2)\ldots(n+r)} + \cdots]$$

$$=a_0x^n \sum_{r=1}^{\infty}(-1)^r \frac{x^{2r}}{2^{2r}r!(n+1)(n+2)\ldots.(n+r)}$$

Where a_0 is an arbitrary constant.

If, $a_0=\frac{1}{2^n \Gamma(n+1)}$

The above solution is called Bessel's function denoted by $J_n(x)$

Thus $J_n(x)=\frac{1}{2^n \Gamma(n+1)}\Sigma(-1)^r \frac{x^{n+2r}}{2^{2r}r!(n+1)(n+2)\ldots.(n+r)}$

(Γn+1=n!)

$$J_n(x) = (\frac{x}{2})^n\{\frac{1}{\Gamma(n+1)} - \frac{1}{\lfloor 1\, \Gamma(n+2)}(\frac{x}{2})^2 + \frac{1}{\lfloor 2\, \Gamma(n+3)}(\frac{x}{2})^4 - \frac{1}{\lfloor 3\, \Gamma(n+4)}(\frac{x}{6})^6 + \cdots\}$$

Or, $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r)!}(\frac{x}{2})^{n+2r} \ldots .(2)$

If, n=0, $J_0(x) = \sum \frac{(-1)^r}{(r!)^2}(\frac{x}{2})^{2r}$

Or, $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots$

If, n=1, $J_1(x) = \frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} - \cdots$

Replacing n by $-n$ in (2), we get,

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\ \Gamma(-n+r+1)}(\frac{x}{2})^{-n+2r}$$

General solution of Bessel's equation is

$$y = AJ_n(x) + BJ_{-n}(x)$$

Example 1: Prove that, $J_{-n}(x) = (-1)^n J_n(x)$, where 'n' is positive integer.

Sol: We know,

$$J(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\ \Gamma(-n+r+1)}(\frac{x}{2})^{-n+2r}$$

$= \sum_{r=0}^{n-1} \frac{(-1)^r(\frac{x}{2})^{-n+2r}}{r!\ \Gamma(-n+r+1)} + \sum_{r=n}^{\infty} \frac{(-1)^r(\frac{x}{2})^{-n+2r}}{r!\ \Gamma(-n+r+1)}$

$=0+\sum_{r=n}^{\infty} \frac{(-1)^r(\frac{x}{2})^{-n+2r}}{r!\ \Gamma(-n+r+1)}$, since [-ve integer=∞]

On putting, $r = n + k$,

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k} (\frac{x}{2})^{n+2k}}{(n+k)!\ \Gamma(k+1)}$$

$=(-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{n+2k}}{(n+k)!\ k!}$

$=(-1)^n J_n(x)$

[proved]

Example 2: Prove that, $\frac{d}{dx}\left[J_0(x)\right] = -J_1$

Sol: We have by definition

$J_0(x) = \sum \frac{(-1)^r}{(r!)^2} (\frac{x}{2})^{2r}$

Differentiating it with respect to x, we get,

$$\frac{d}{dx} J_0(x) = \frac{1}{2} \sum_{1}^{\infty} (-1)^r \frac{1}{(r)!\,(r-1)!} (\frac{x}{2})^{2r-1}$$

$=\sum_{J=0}^{\infty} (-1)^{j+1} \frac{1}{(j+1)!(j)!} (\frac{x}{2})^{2j+1}$

where j=r-1

$=-\sum_{J=0}^{\infty} (-1)^{j} \frac{1}{(j+1)!(j)!} (\frac{x}{2})^{2j+1}$

$= -J_1(x)$

[proved]

Example 3: Prove that,

(a) $J_{1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\, sinx$

(b) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} cosx$

Sol: We know that,

$$J_n(x) = \frac{x^n}{2^n \ \Gamma n+1}\left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} \dots\right] \dots\dots (1)$$

(a) Substituting $n=\frac{1}{2}$ in (1) we obtain

$$J_{1/2}(x)$$

$$= \frac{x^{1/2}}{2^{1/2} \ \Gamma 1+\frac{1}{2}}\left[1 - \frac{x^2}{2.2\left(\frac{1}{2}+1\right)} + \frac{x^4}{2.4.2^2\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)} \dots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\ \Gamma\frac{3}{2}}[1 - \frac{x^2}{2.3!} + \frac{x^4}{2.3.4.5} - \dots]$$

$$= \frac{\sqrt{x}}{\sqrt{2}.\ \Gamma\frac{1}{2}.\frac{1}{2}}.\frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \frac{1}{\sqrt{2x}.\ \sqrt{\pi}.\frac{1}{2}} sinx$$

$= \sqrt{\frac{2}{\pi x}} sinx$ [since $\Gamma\frac{1}{2} = \sqrt{x}$] [proved]

(b) Again substituting $n = -\frac{1}{2}$ in (1), we have $J_{-\frac{1}{2}}(x) = \frac{x^{-\frac{1}{2}}}{2^{-\frac{1}{2}}\ \Gamma 1-\frac{1}{2}}[1 - \frac{x^2}{2.2\left(-\frac{1}{2}+1\right)} + \frac{x^4}{2.4.2^2\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right)} - \cdots]$

$$= \frac{\sqrt{2}}{\sqrt{x}\,.\,\Gamma\frac{1}{2}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right]$$

$= \sqrt{\frac{2}{\pi x}}\ cosx$ [since $\Gamma\frac{1}{2} = \sqrt{x}$] [proved]

Roll: 1800035

RECURRENCE FORMULAE

Formulae-1: $xJ'_n = nJ_n - xJ_{n+1}$

Proof: We know that,

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r+1)!} \left(\frac{x}{2}\right)^{x+2r}$$

Differentiating with respect to 'x' we get,

$$J'n = \sum \frac{(-1)^r (n+2r)}{r!\,(n+r+1)!} \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2}$$

$$xJ'n = n \sum \frac{(-1)^r}{r!\,(n+r+1)!} \left(\frac{x}{2}\right)^{n+2r} + x \sum \frac{(-1)^r 2r}{2r!\,(n+r+1)!} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n + x \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!(n+r+1)!} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n + x \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!(n+s+2)!} \left(\frac{x}{2}\right)^{n+2s-1} \quad [\,puting\ r-1=s]$$

$$= nJ_n + x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+1+s+1)!} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$= nJ_n - xJ_{n+1}$$

Example: Prove that, $J'_2(x) = (1 - \frac{4}{x^2})J_1(x) + \frac{2}{x}J_0(x)$

Where $J_n(x)$ is the Bessel function of first kind.

Solution: By recurrence formula 2,

$$xJ'_n = -J_n + xJ_{n-1} \_\_\_\_\_\_\_\_\_\_(1)$$

On putting n=2 in (1), we have

$$xJ'_2 = -2J'_2 + xJ_1$$

$$\text{or, } J'_2 = -\frac{2}{x}J_2 + J_1 \quad \_\_\_\_\_\_\_\_\_\_(2)$$

By recurrence formula 1,

$$xJ'_n = nJ_n - xJ_{n+1} \_\_\_\_\_\_\_\_\_\_(3)$$

From (1) & (3) we have,

$$-nJ_n + xJ_{n-1} = nJ_n - xJ_{n+1}$$

On putting n=1,

$$-J_1 + xJ_0 = J_1 - xJ_2$$

$$\text{Or, } -\frac{1}{x}J_1 + J_0 = \frac{1}{x}J_1 - J_2$$

$$\text{Or, } J_2 = \frac{2}{x}J_1 - J_0 \_\_\_\_\_\_\_\_\_\_(4)$$

Putting the value of J_2 from (4) in (2) we get,

$$J'_2 = -\frac{2}{x}\left(\frac{2}{x}J_1 - J_0\right) + J_1$$

$$= -\frac{4}{x^2}J_1 + \frac{2}{x}J_0 + J_1$$

$$= (1 - \frac{4}{x^2})J_1 + \frac{2}{x}J_0$$

[Proved]

Roll: 1800036

Recurrence Formulae II: $x\,J_n' = -n\,J_n + x\,J_{n-1}$

Proof: We have, $J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Differentiating with respect to x we get,

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r\,(n+2r)}{r!(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$or,\quad x\,J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r\,(n+2r)}{r!(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$\text{or, } x\,J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r\,(2n+2r)-n}{r!(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$\text{or, } x\,J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r\,(2n+2r)}{r!(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} - n\,\sum_{r=0}^{\infty} \frac{(-1)^r}{r!(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$\text{or, } x\,J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r\,2}{r!(n+r)} \left(\frac{x}{2}\right)^{n+2r} - nJ_n$$

$$\text{or, } x\,J_n' = x\,\sum_{r=0}^{\infty} \frac{(-1)^r}{r![(n-1)+r+1]} \left(\frac{x}{2}\right)^{(n-1+2r)} - n\,J_n$$

So, $x\,J_n' = x\,J_{n-1} - n\,J_n$ (Proved)

Problem: Prove that, $\frac{dJ_n}{dx} = \frac{2}{x}[\frac{1}{2}nJ_n \quad -(n+2)J_{n+2} + (n+4)J_{n+4} \dots\dots\dots\dots\dots.]$

Solution: The recurrence formula is,

$x J_n' = -n J_n + x J_{n-1}$

So that, $J_n' = -\left(\frac{n}{x}\right) J_n + x J_{n-1}$ ……………… (1)

We know, $J_{n-1} = \frac{2}{x}[nJ_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \dots\dots\dots\dots]$

Putting this value of J_{n-1} in (1) we get,

$J_n' = \frac{n}{x}J_n - \frac{2}{x}[n J_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \dots\dots\dots\dots]$

$= \frac{2}{x}[\frac{1}{2}nJ_n \quad -(n+2)J_{n+2} + (n+4)J_{n+4} \dots\dots\dots\dots\dots.]$ (Proved)

Roll:1800037

Formula III. $2J'_n = J_{n-1} - J_{n+1}$

Proof: We know that

$x J'_n = n J_n - x J_{n+1}$ . . . (1) Recurrence formula I

$x J'_n = -n J_n - x J_{n-1}$ . . . (2) Recurrence formula II

Adding (1) and (2), we get,

$2\,x\,J'_n = n\,J_n - x\,J_{n+1} - n\,J_n + x\,J_{n-1}$

Or, $2\,x\,J'_n = -\,x\,J_{n+1} + x\,J_{n-1}$

Or, $2\,J'_n = J_{n-1} - J_{n+1}$ [Proved.]

Example: Prove that, $4\,J_n''(x) = J_{n-2}(x) - 2\,J_n(x) + J_{n+2}(x)$

Solution: We know that the recurrence formula,

$$2\,J'_n = J_{n-1} - J_{n+1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

On differentiating again, we have

$$2\,J''_n = J'_{n-1} - J'_{n+1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Replacing n by n-1 and n by n+1 in (1) we have,

$$2\,J'_{n-1} = J_{n-2} - J_n$$

Or, $J'_{n-1} = \frac{1}{2}\,J_{n-2} - \frac{1}{2}J_n \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$

And, $2J'_{n+1} = J_n - J_{n+2}$

Or, $J'_{n+1} = \frac{1}{2}\,J_n - \frac{1}{2}J_{n+2} \qquad \ldots\ldots\ldots\ldots\ldots.. (4)$

Putting the values of J'_{n-1} and J'_{n+1} from (3) and (4) in (2) we get,

$$2\,J_n'' = \frac{1}{2}[J_{n-2} - J_n] - \frac{1}{2}\,[J_n - J_{n+2}]$$

Or, $4\,J_n'' = J_{n-2} - J_n - J_n + J_{n+2}$

Or, $4\,J_n'' = J_{n-2} - 2\,J_n + J_{n+2}$ [proved]

Formula-IV: $2nJ_n = x\,(J_{n-1} + J_{n+1})$

Proof: We know that,

$x\,J'_n = n\,J_n - x\,J_{n+1}$ . . . (1) Recurrence formula I

$x\,J'_n = -n\,J_n - x\,J_{n-1}$ . . . (2) Recurrence formula II

subtracting (2) from (1), we get,

$0 = 2\,n\,J_n - x\,J_{n+1} - x\,J_{n-1}$

Or, $2nJ_n = x\,(J_{n-1} + J_{n+1})$ [proved]

Example: prove that, $\frac{1}{2}x\,J_n = (n+1)\,J_{n+1} - (n+3)\,J_{n+3} + (n+5)\,J_{n+5} - \ldots\ldots\ldots\ldots$

Solution: We know from recurrence formula-IV that,

$2nJ_n = x\,(J_{n-1} + J_{n+1})$

Putting n+1 for n we get,

$2(n+1)J_{n+1} = x\,(J_n + J_{n+2})$

Or, $\frac{1}{2}x\,J_n = (n+1)\,J_{n+1} - \frac{1}{2}x\,J_{n+2}$ ……………………… (1)

Putting n+2 for n in (1) we get,

$\frac{1}{2}x\,J_{n+2} = (n+3)\,J_{n+3} - \frac{1}{2}x\,J_{n+4}$ ……………………. (2)

Putting this value in (1) we get,

$\frac{1}{2}x\ J_n = (n+1)\ J_{n+1} - (n+3)\ J_{n+3} + \frac{1}{2}x\ J_{n+4}$ ………………. (3)

Putting n+4 for n in (1) , we get

$\frac{1}{2}x\ J_{n+4} = (n+5)\ J_{n+5} - \frac{1}{2}x\ J_{n+6}$

Putting this value in (3) we get,

$\frac{1}{2}x\ J_n = (n+1)\ J_{n+1} - (n+3)\ J_{n+3} + (n+5)\ J_{n+5} - \frac{1}{2}x\ J_{n+6}$

Proceeding so on we get,

$$\frac{1}{2}xJ_n = (n+1)\ J_{n+1} - (n+3)\ J_{n+3} + (n+5)\ J_{n+5} - \dots\dots\dots$$

Which proves the required result.

Roll: 1800039

Recurrence Formulae V.

The Recurrence formulae V. is $\frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1}$

Proof: We know from Recurrence formulae I. is

$xJ_n' = nJ_n - xJ_{n+1}$

Multiplying by x^{-n-1} , we obtain,

$$x^{-n}J_n' = nx^{-n-1}J_n - x^{-n}J_{n+1}$$

- ➢ $x^{-n}J_n' - nx^{-n-1}J_n = -x^{-n}J_{n+1}$
- ➢ $\frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1}$ [Proved]

Example: Prove that, $\int J_3(x)dx + J_2(x) + \frac{2}{x} J_1(x) = 0$

Solution: We know from Recurrence formulae V.

$$\frac{d}{dx}\left(x^{-n}J_n(x)\right) = -x^{-n} J_{n+1}(x)$$

Integrating above relation, we get

$$x^{-n}J_n(x) = -\int x^{-n}J_{n+1}(x)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Taking n = 2 in (1), we get

$$\int x^{-2}J_3(x)dx = -x^{-2}J_2(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Again, $\int J_3(x)dx = \int x^2\left(x^{-2}\right)J_3(x)dx$

$$= x^2 \int\left(x^{-2}\right)J_3(x)dx - \int 2x \int\left(x^{-2}J_3(x)\right)dx \ldots\ldots (3)$$

Putting the value of $\int x^{-2}J_3(x)dx$ from (2) in (3), we get

$$\int J_3(x)dx = x^2(-x^{-2}J_2) - \int 2x(-x^{-2}J_2)dx$$

$$= -J_2 + 2\int x^{-1}J_2\,dx$$

$$= -J_2 + 2(-x^{-1}J_1)$$

On using (1), again when n=1

Hence, $\int J_3(x)dx + J_2 + \frac{2}{x}J_1 = 0$ [proved]

Roll: 1800040

Recurrence formula VI: $\frac{d}{dx}\left(x^n J_n\right) = x^n J_{n-1}$

Proof .1 *from recurrence formula II*

We know that , $xJ'_n = -nJ_n + xJ_{n-1}$

Or, $xJ_n' + nJ_n = xJ_{n-1}$

Multiplying it by x^{n-1} *, we get*

$$x^n J_n' + nx^{n-1} J_n = x^n J_{n-1}$$

This can be written as

$$\frac{d}{dx}(x^n J_n) = x^n J_{n-1}$$

Which is the required result

Proof .2 we have,

$$\frac{d}{dx}(x^n J_n) = [x^n \sum_0^\alpha \frac{(-1)^r}{(r)!\Gamma(n+r+1)} (\frac{x}{2})^{n+2r}]$$

$$= \frac{d}{dx}[\sum_0^\alpha \frac{(-1)^r}{(r)!\,\Gamma(n+r+1)} 2^n (\frac{x}{2})^{2n+2r}]$$

$$= 2^n \sum_0^\alpha \frac{(-1)^r (n+r)}{(r)!\,\Gamma(n+r+1)} (\frac{x}{2})^{2n+2r-1}$$

$$= x^n \sum_0^\alpha \frac{(-1)^r}{(r)!\,\Gamma(n+r)} (\frac{x}{2})^{n+2r-1}$$

$$= x^n J_{n-1}$$

[proved]

Roll: 1800043

Equations reducible to bessel's equation

There are some differential equations which can be reduced to bessel's equation and therefore can be solved.

We shall reduce the following differential eq. to bessel's equation,

$X^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (k^2x^2 - n^2)y = 0$ ……………………………. (1)

Putting t=kx, $\frac{dt}{dx}$=k (1) becomes,

$\frac{t^2}{k^2} * k^2 * \frac{d^2y}{dx^2} + \frac{t}{k} * k * \frac{dy}{dx} + (t^2 - n^2)y = 0$

Its solution is $y = c_1J_n(t) + c_2J_{-n}(t)$

Hence solution of (1) is $y = c_1J_n(kx) + c_2J_{-n}(kx)$

Example: Find the general solution of the differential equation

x2y″+2xy′ +(x2−1)y =0.

Solution. We make the substitution:

y=x1−22z=x−12z,
⇒y′=−12x−32z+x−12z′,
⇒y″=34x−52z−12x−32z′−12x−32z′+x−12z″=34x−52z−x−32z′+x−12z″.

Put these expressions back into the equation:

x2y″+2xy′+(x2−1)y=0,⇒
x2(34x−52z−x−32z′+x−12z″)+2x(−12x−32z+x−12z′)+(x2−1)x−12z=0,

⇒34x−12z−x−12z′+x32z″−x−12z+2x12z′+x32z−x−12z=0,

⇒x32z″+x−12z′+(−54x−12+x32)z=0,

⇒x2z″+xz′+(x2−54)z=0.

Indeed, we see that

n2=v2+14(a−1)2=1+14(2−1)2=1+14=54.

Thus, the general solution for the function z(x) can be written in the form

z(x)=C1J√52(x)+C2Y√52(x).

Then the solution for the original function y(x) is given by

y(x)=x−12z(x)=1√x[C1J√52(x)+C2Y√52(x)],

where C1 and C2 are arbitrary constants.

Roll:1800041

ORTHOGONALITY OF BESSEL FUNCTIONS:

$\int_0^1 \boldsymbol{x J_n(\alpha\, x) \cdot J_n(\beta x) dx = 0}$ where α and β are the roots of $\boldsymbol{J_n(x) = 0}$

Proof: We know that,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(\alpha^2 x^2 - n^2\right) = 0 \qquad \ldots\ldots(1)$$

$$x^2 \frac{d^2z}{dx^2} + x\frac{dz}{dx} + \left(\beta^2 x^2 - n^2\right) = 0 \qquad \ldots\ldots(2)$$

Solutions of (1) and (2) are $y = J_n(\alpha x)$, $z = J_n(\beta x)$ respectively.

Multiplying (1) by $\frac{z}{x}$ *and* $-\frac{y}{x}$ *and add , we get*

$$x\left(z\frac{d^2y}{d^2x} - y\frac{d^2z}{dx^2}\right) + \left(z\frac{dy}{dx} - y\frac{dz}{dx}\right) + (\alpha^2 - \beta^2)xyz = 0$$

$$\frac{d}{dx}[x\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)] + (\alpha^2 - \beta^2)xyz = 0 \qquad \ldots\ldots\ldots\ldots(3)$$

Integrating (3) w.r.t 'x' between the limits o and 1, we get

$$[x\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)]_0^1 + (\alpha^2 - \beta^2)\int_0^1 xyzdx = 0$$

$$(\beta^2 - \alpha^2)\int_0^1 xyzdx = [x\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)$$

$$=(z\frac{dy}{dx} - y\frac{dz}{dx})_{x=1} \qquad \ldots\ldots(4)$$

Putting these values of $y = J_n(\alpha x), \frac{dy}{dx} = \alpha J_n{}'(\alpha x), z = J_n(\beta x), \frac{dz}{dx} = \beta\, ,J_n{}'(\beta x)$ in (4) we get

$$(\beta^2 - \alpha^2) - \int_0^1 x J_n(\alpha x).J_n(\beta x)dx = [\alpha' J_n(\alpha x)J_n(\beta x)\text{-} J_n{}'(\beta x)\, J_n(\alpha x)]_{x=1}$$

$$= \alpha J_n'(\alpha)J_n(\beta)\text{-}\beta\, J_n'(\beta)J_n(\alpha) \qquad \ldots\ldots(5)$$

Since α,β are the roots of $J_n(x) = 0, so\ J_n(\alpha) = J_n(\beta) = 0$.

Putting the values of $J_n(\alpha) = J_n(\beta) = 0\ in\ (5)$, we get,

$$(\beta^2 - \alpha^2) - \int_0^1 x J_n(\alpha x).J_n(\beta x)dx = 0$$

$$\int_0^1 x J_n(\alpha x).J_n(\beta x)dx = 0 \qquad \textit{[proved].}$$

Example: *Prove that*

$$\int_0^1 x\,[J_n(\alpha x)]^2 dx = \frac{1}{2}\,[J_{n+1}(\alpha)\,]^2$$

Solution. From (5) ,we know that

$$(\beta^2-\alpha^2) - \int_0^1 x J_n(\alpha x).J_n(\beta x)dx = \alpha J_n'(\alpha)J_n(\beta) - \beta J_n'(\beta)\,J_n(\alpha)$$

When $\beta=\alpha$

We also know that $J_n(\alpha)=0$.let β be a neighbouring value of α, which tends to α.

Then,

$$\lim_{\beta\to\alpha}\int_0^1 x J_n(\alpha x).J_n(\beta x)dx = \lim_{\beta\to\alpha}\frac{0+\alpha J_n'(\alpha).J_n'(\beta)}{\beta^2-\alpha^2}$$

As the limit is of the form $\frac{0}{0}$, we apply L' hospital' s rule

$$\int_0^1 x J_n^2(\alpha x)dx = \lim_{\beta\to\alpha}\frac{0+\alpha J_n'(\alpha).J_n'(\beta)}{2\beta} = \frac{1}{2}[J_n'(\alpha)]^2 \qquad \text{.........proved}$$

Roll: 1800042

A GENERATING FUNCTION FOR $J_n(x)$

Prove that $J_n(x)$ is the coefficient of z^n in the expansion of $e^{\frac{x}{2}(z-\frac{1}{z})}$

Proof: We know that $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$

$$e^{\frac{xz}{2}} = 1 + \frac{xz}{2} + \frac{1}{2!}\left(\frac{xz}{2}\right)^2 + \frac{1}{3!}\left(\frac{xz}{2}\right)^3 + \cdots \qquad \ldots(1)$$

$$e^{-\frac{x}{2z}} = 1 - \frac{x}{2z} + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 - \frac{1}{3!}\left(\frac{x}{2z}\right)^3 + \cdots \qquad \ldots(2)$$

Multiplying (1) and (2), we get

$$e^{\frac{x}{2}(z-\frac{1}{z})} = \left[1 + \left(\frac{xz}{2}\right) + \frac{1}{2!}\left(\frac{xz}{2}\right)^2 + \frac{1}{3!}\left(\frac{xz}{3}\right)^3 + \cdots\right] \times$$

$$\left[1 - \frac{x}{2z} + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 - \frac{1}{3!}\left(\frac{x}{2z}\right)^3 + \cdots\right] \qquad \ldots(3)$$

The coefficient of z^n in the product of (3)

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} \ldots$$

$$= J_n(x)$$

Similarly coefficient of z^{-n} in the product (3) = $J_{-n}(x)$

$$e^{\frac{x}{2}(z-\frac{1}{z})} = J_0 + zJ_1 + z^2J_2 + z^3J_3 + \ldots + z^{-1}J_{-1} + z^{-2}J_{-2} + z^{-3}J_{-3} + \ldots$$

$$= \sum_{n=-\infty}^{\infty} z^n J_n(x)$$

For this reason $e^{\frac{x}{2}(z-\frac{1}{z})}$ is known as the generating function of Bessel functions.

Roll:1800049

Trigometric expansion involving Bessel function

To show that

Cos (x sinϕ) = [J_0 (x)<2 [cos 2ϕ J_2(x)+cos 4ϕ J_4(x)+…..],

Sin (x cos ϕ) = 2[sinϕ J_1(x) + sin 3ϕ J_3(x)+……], Cos (x cos ϕ)= $J_0(x) - 2$[cos 2 ϕ J_2(x)-cos 4 ϕ J_1(x)+…….],

Sin (x cos ϕ)=2[cos ϕ $J_1(x) - \cos 3\phi$ J_3(x)+……]

Proof: we have,

$e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{-\infty}^{\infty} t^n J_n(x)$

Putting t=e^{iQ} , so that $\frac{1}{2}$ (t-$\frac{1}{t}$) = i sin ϕ, we get,

$e^{xisin\phi} = \sum_{-\infty}^{\infty} t^n J_n(x)$

$= J_0(x) + J_1(x)\, e^{iQ} + J_{-1}(x)\, e^{-iQ} + J_2(x)\, e^{i2Q} + J_{-2}(x)\, e^{-2iQ} + \ldots\ldots + J_n(x)\, e^{niQ} + J_{-n}(x)\, e^{-niQ} + \ldots\ldots\ldots$

$= J_0(x) + J_1(x)\, [e^{iQ} - e^{-iQ}] + J_2(x)\, [e^{i2Q} + e^{-2iQ}] + \ldots\ldots + J_{-n}(x)$

$= J_0(x) + J_1(x)$ 2i sin $\boldsymbol{\phi}$ + J_2(x) 2i cos 2$\boldsymbol{\phi}$+…….(1)$\boldsymbol{\phi}$

Equating real and imaginary parts, we get,

Cos (x sinϕ) = [J_0 (x)+2cos2$\boldsymbol{\phi}$ J_2(x)+……(2)

And sin(x sin$\boldsymbol{\phi}$)=2 sin$\boldsymbol{\phi}$ $J_1(x)$+2 sin3$\boldsymbol{\phi}$ J_3(x)+…….(3)

Then putting $\frac{1}{2}\pi - \boldsymbol{\phi}$ for $\boldsymbol{\phi}$ (1), (2),(3) we get,

cos^{ixcosQ}=$J_0(x)$+2i cos$\boldsymbol{\phi}$$J_1(x)$- 2cos x2$\boldsymbol{\phi}$ J_2(x)-2i cos3$\boldsymbol{\phi}$ J_3(x)+…….(4)

Cos (x cos ϕ)= $J_0(x) - 2cos2\phi\, J_2$(x)-2cos 4 ϕ J_1(x)+…….(5)

Sin (x cos ϕ)=2 cos ϕ $J_1(x) - 2\cos 3\phi$ J_3(x)+……(6)

Example:01: Show that,

Cos $x=J_0(x)-2J_2(x)+2J_4(x)-.....$

Sin $x=2J_1(x)-2J_3(x)+2J_5(x)-....$

Solution: we know,

$\cos(x \sin\phi) = Jo(x) + 2cos2\phi J2(x) + 2cos4\phi J4(x) + \cdots ....$

$\text{Sin}(x \sin\phi) = 2sin\phi J1(x) + 2sin3\phi J3(x) + 2sin5\phi J5x + \cdots$

Putting the value of $\phi = \frac{\pi}{2}, weget,$

Cos $x=J_0(x)-2J_2(x)+2J_4(x)-...............$

And sin $x=2J_1(x)-2J_3(x)+2J_5(x)-....$

Roll:1800044

<u>**Trigometric expansion involving Bessel function**</u>

Prove that $[J_0(x)]^2+2\ [J_1(x)]^2+2\ [[\ J_2(x)]^2+.......=1$

Solution: We know,

$\cos(x\sin\phi) = Jo(x)+2J_2(x)\cos2\phi+2\ J_4(x)\cos 4\phi + \cdots(1)$

$\text{Sin}(x\sin\phi) = 2\ \cos\boldsymbol{\phi} J_1(x)+2sin3\phi J3(x)+......(2)$

It would be noted that,

$$\int_0^{\pi} cos^2 n\boldsymbol{\phi}\ d\boldsymbol{\phi}=\int_0^{\pi} sin^2 n\boldsymbol{\phi}=\int_0^{\pi}(1+\cos 2n\phi)d\boldsymbol{\phi}=\frac{\pi}{2}$$

And

$$\int_0^{\pi} \cos n\phi \cos m\phi = \int_0^{\pi} \sin n\phi \sin m\phi = 0 \ m\neq n$$

Now, sqaring (1) and integrating with respect to $\boldsymbol{\phi}$ between the limits 0 to π, we get by the use of above integrals

$$e^{xi\ sin\phi}=\cos(x \sin\boldsymbol{\phi}) + i \sin(x \sin\boldsymbol{\phi})$$

$$[J_0(x)]^2+2\ [[J_4(x)]^2\pi+.....=\int_0^{\pi} sin^2(xsin\phi)d\boldsymbol{\phi}$$

Again sqaring (2) and integrating with respect to $\boldsymbol{\phi}$ between the limits 0 to π, we similarly get

$$[J_1(x)]^2\pi+[J3(x)]^2\pi+.....=\int_0^{\pi} sin^2(xsin\phi)d\boldsymbol{\phi}$$

Adding these,

$$\pi[\{J_o(x)\}^2+2\ \{J_1(x)\}^2+2\ [\{J(x)\}^2\pi+.........]$$

$$=\int_0^{\pi}[cos^2(x \sin\boldsymbol{\phi})+sin^2(x \sin\boldsymbol{\phi})]=d\boldsymbol{\phi}$$

$$=\int_0^{\pi} d\varphi = \pi$$

$$[J_0(x)]^2+2\ [J_1(x)]^2+2\ [[J_2(x)]^2+.......=1 \text{ (proved)}$$

Example:1:show that, $d/dx(J^2_n+J^2_{n+1})=2(n/x.J^2_n-(n+1)/x.J^2_{n+1})$

Solution: Recurrence formula (1) and (2) is,

$X J'_n=nJ_{n-}xJ_{n+1}$.................(1)

$X J'_n=-nJ_n+xJ_{n-1}$................(2)

Putting (n+1)for n in (2),we get,

$X J'_{n-1}=-(n+1)J_{n+1}+xJ_n$

Now, $d/dx(J^2_n+J^2_{n+1})=2J_nJ'_n+2J_{n+1}J'_{n+1}$

$=2J_n.1/x(n J_n-xJ_{n+1})+2J_{n+1}.1/x[-(n+1)J_{n+1}+xJ_n]$ [from (1) and (3)]

$=2[n/x.J^2_n-(n+1)/x.J^2_{n+1}]$

Roll : 1800045

BESSEL'S INTEGRAL

prove that

(a) $J_0(x)=\frac{1}{\pi}\int_0^\pi \cos(x \sin\theta)\, d\theta$

Proof. We know that,

$\cos(x\sin\theta)=J_0+2J_2\cos 2\theta+2J_4\cos 4\theta+\ldots$ ……..…(1)

$\sin(x\sin\theta)=2J_1\sin\theta+2J_3\sin 3\theta+2J_5\sin 5\theta+\ldots$ …(2)

(a) Integrating (1) between the limits 0 and π, we have

$$\int_0^{\pi} \cos\ (x \sin\theta)\, d\theta = \int_0^{\pi} (J_0 + 2\, J_2 \cos 2\theta + 2\, J_4 \cos 4\theta + \ \dots\ )\, d\theta$$

$$= J_0 \int_0^{\pi} d\theta + 2\, J_2 \int_0^{\pi} \cos 2\theta\, d\theta + 2\, J_4 \int_0^{\pi} \cos 4\theta\, d\theta + \dots$$

$$= J_0\, \pi + 0 + 0$$

$$J_0\,(x) = \frac{1}{\pi} \int_0^{\pi} \cos\ (x\ sin\theta)\, d\theta$$ [Proved.]

Example: Prove that , $J_{1/2}(X) = \sqrt{\frac{2}{\pi X}}\ \sin X$

<u>Solution: </u>We have

$$J_n(X) = \frac{X^n}{2^n \Gamma(n+1)} \left[1 - \frac{X^2}{2.2(n+1)} + \frac{X^4}{2.4.2^2.(n+1)(n+2)} - \dots\right]$$

Putting n=1/2, we get

$$J_{1/2}(X) = \frac{X^{\frac{1}{2}}}{2^{\frac{1}{2}} . \Gamma\left(\frac{3}{2}\right)} \left[1 - \frac{X^2}{2.3} + \frac{X^4}{2.3.4.5} - \dots\dots\right]$$

$$= \frac{X^{\frac{1}{2}}}{2^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)} \cdot \frac{1}{X} \cdot \left[X - \frac{X^3}{(3)!} + \frac{X^5}{(5)!} + \dots\right]$$

$$= \sqrt{\left(\frac{2}{\pi X}\right)} \cdot \sin X \qquad \text{as } \Gamma\,(1/2) = \sqrt{\pi}$$

Roll:1800046

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prove that

(b) $J_n\,(x) = \frac{1}{\pi} \int_0^{\pi} \cos\ (n\theta - x\ sin\theta)\, d\theta$

Multiplying (1) by cos nθ integrating between the limits 0 and π, we have

$\int_0^{\pi} \cos(x \sin\theta) \cos n\theta \, d\theta = \int_0^{\pi} [J_0 \cos n\theta + 2 J_2 \cos 2\theta \cos n\theta + 2 J_4 \cos 4\theta \cos n\theta + \ldots] \, d\theta$

$= 2 J_0 \int_0^{\pi} \cos n\theta \, d\theta + 2 J_2 \int_0^{\pi} \cos 2\theta \cos n\theta \, d\theta + \ldots$

$= 0$ if n is odd ...(3)

$= \pi J_n$ if n is even ...(4)

Again multiplying (2) by sin nθ and integrating between the limits 0 and π, we have

$\int_0^{\pi} \sin(x \sin\theta) \sin n\theta \, d\theta = \int_0^{\pi} (2 J_1 \sin\theta \sin n\theta + 2 J_3 \sin 3\theta \sin n\theta + \ldots) \, d\theta$

$= 2 J_1 \int_0^{\theta} \sin\theta \sin n\theta \, d\theta + 2J_3 \int_0^{\pi} \sin 3\theta \sin n\theta \, d\theta + \ldots = 0$ if n is even ...(5)

$= \pi J_n$ if n is odd ...(6)

Adding (3) and (6) or (4) and (5), we get

$\int_0^{\pi} [\cos(x \sin\theta) \cos n\theta + \sin(x \sin\theta) \sin n\theta] \, d\theta = \pi J_n$

or, $\int_0^{\pi} \cos(n\theta - x \sin\theta) \, d\theta = \pi J_n$

or, $J_n = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin\theta) \, d\theta$ [proved.]

Example: $J_{-1/2}(X) = \sqrt{\frac{2}{\pi X}} \cos X$

Solution: Again, putting n= $-1/2$, we get

$J_{-1/2}(X) = \frac{X^{-1/2}}{2^{-1/2} . \Gamma\left(\frac{1}{2}\right)} [1 - \frac{X^2}{2} + \frac{X^4}{2.3.4} - \ldots\ldots]$

$$= \sqrt{\left(\frac{2}{\pi X}\right)} . [1 - \frac{X^2}{(2)!} + \frac{X^4}{(4)!} + \ldots]$$

$$= \sqrt{\left(\frac{2}{\pi X}\right)} . cosX$$

$\int_0^{\pi} \cos n\Phi \, d\Phi = [\frac{\sin n\Phi}{n}]_0^{\pi} = 0$ when n is an integer.

Roll: 1800034

Practice problem

Problem-1: Solve the differential equation $x^2y'' + xy' + (3x^2 - 2)y = 0$

Solution:

This equation has order √2 and differs from the standard Bessel equation only by factor 3 before x^2. Therefore, the general solution of the equation is expressed by the formula

$y(x) = C_1 J_{\sqrt{2}}(\sqrt{3}x) + C_2 Y_{\sqrt{2}}(\sqrt{3}x)$,

where C_1, C_2 are constants, $J_{\sqrt{2}}(\sqrt{3}x)$ and $Y_{\sqrt{2}}(\sqrt{3}x)$ are Bessel functions of the 1st and 2nd kind, respectively.

Problem-2: Solve the equation $x^2y''+xy' -(4x^2+12)y =0$.

Solution. This equation differs from the modified Bessel equation by factor 4 in front of x^2. The order of the equation is $v=1\sqrt{2}$. Then the general solution is written through the modified Bessel functions in the following way:

$y(x)=C_1I_{1\sqrt{2}}(2x)+C_2K_{1\sqrt{2}}(2x)$,

where C_1 and C_2 are arbitrary constants.

Problem-3: Prove that, $\int J_3(x)dx + J_2(x) + \frac{2}{x} J_1(x) = 0$

Solution: We know from Recurrence formulae V.

$$\frac{d}{dx}\left(x^{-n}J_n(x)\right) = -x^{-n} J_{n+1}(x)$$

Integrating above relation, we get

$$x^{-n}J_n(x) = -\int x^{-n}J_{n+1}(x)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Taking n = 2 in (1), we get

$$\int x^{-2}J_3(x)dx = -x^{-2}J_2(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Again, $\int J_3(x)dx = \int x^2(x^{-2})J_3(x)dx$

$$= x^2\int(x^{-2})J_3(x)dx - \int 2x\int(x^{-2}J_3(x))dx \ldots\ldots (3)$$

Putting the value of $\int x^{-2}J_3(x)dx$ from (2) in (3), we get

$$\int J_3(x)dx = x^2(-x^{-2}J_2) - \int 2x(-x^{-2}J_2)dx$$

$= -J_2 + 2\int x^{-1}J_2\,dx$

$= -J_2 + 2(-x^{-1}J_1)$

On using (1), again when n=1

Hence, $\int J_3(x)dx + J_2 + \frac{2}{x}J_1 = 0$ [proved]

RAJSHAHI UNIVERSITY OF ENGINEERING & TECHNOLOGY

Assignment

Math 2101

Legendres' Polynomials

Submitted To,

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Topic : Legendre's Polynomials Part-1

Legendre's Equation

A solution which is regular at finite points is called a Legendre function of the first kind, while a solution which is singular at is called a Legendre function of the second kind. If. is an integer, the function of the first kind reduces to a polynomial known as the **Legendre polynomial.**

The equation is named for **Adrien-Marie Legendre** who proved in **1785** that it is solvable in integers x, y, z, not all zero, if and only if −bc, −ca and −ab are quadratic residues modulo a, b and c, respectively, where a, b, c are nonzero, square-free, pairwise relatively prime integers, not all positive or all negative.

The differential equation $\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$ ...(1)

is known as Legendre's equation. The above equation can also be written as

$$\frac{d}{dx}\left\{\left(1-x^2\right)\frac{dy}{dx}\right\}+n\,(n+1)\,y=0 \qquad n\,\varepsilon\,1$$

This equation can be integrated in series of ascending or descending powers of $x.\,i.\,e.,$ series in ascending or descending powers of x can be found which satisfy the equation (1).

Let the series in descending powers of x be

$$y=\ x^m(a_0+a_1x^{-1}+a_2x^{-2}+\cdots) \qquad \text{...(2)}$$

or

$$y=\sum_{r=0}^{\infty}a_rx^{m-r}$$

so that

$$\frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(m-r)^{m-r-1}$$

and

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(m-r)(m-r-1)x^{m-r-2}$$

Substituting these in **(1)**, we have

$$(1-x^2)\sum_{r=0}^{\infty} ar(m-r)(m-r-1)x^{m-r-2} - 2x\sum_{r=0}^{\infty} a_r(m-r)x^{m-r-1}$$
$$+ n(n+1)\sum_{0}^{\infty} a_r x^{m-r} = 0$$

or

$$\sum_{r=0}^{\infty} a_r(m-r)(m-r-1)x^{m-r-2} + \{n(n+1) - 2(m-r) - (m-r)(m-r-1)\}x^{m-r}a_r$$
$$= 0$$

or

$$\sum_{r=0}^{\infty}[(m-r)(m-r-1)x^{m-r-2} + \{n(n+1) - (m-r)(m-r+1)\}x^{m-r}]a_r \equiv 0 \qquad ...(3)$$

The equation **(3)** is an identity and therefore coefficients of various powers of x must vanish. Now equating to zero the coefficients of x^m from the above we have ($r = 0$)

$$a_0\{n(n+1) - m(m+1)\} = 0$$

But $a_0 \neq 0.$ as it is the coefficient of the very first term in the series

Hence, $n(n+1) - m(m+1) = 0$ ...(4)

i.e., $n^2 + n - m - m^2 - m = 0$ or $(n^2 - m^2) + (n - m) = 0$

or $(n-m)(n+m+1)=0$

which gives $m=n$ or $m=-n-1$ ...(5)

This is important as it determines the index.

Next, equating to zero the coefficient of x^{m-1} by putting $r=1$

$$a_1[n(n+1)-(m-1)m]=0$$

or $$a_1[(m+n)(m-n-1)=0$$

which gives $$a_1=0 \quad ...(6)$$

Since $(m+n)(m-n-1)\neq 0$ by (5)

Again, to find a relation in successive coefficients a_r, etc., equating the coefficient of x^{m-r-2} to zero, we get

$$(m-r)(m-r-1)a_r+[n(n+1)-(m-r-2)(m-r-1)]a_2+2=0$$

Now $n(n+1)-(m-r-2)(m-r-1)=n^2+n-(m-r-1-1)(m-r-1)$

$$=-[(m-r-1)^2-(m-r-1)-n^2-n]$$

$$=-[(m-r-1+n)(m-r-1-n)-(m-r-1+n)]$$

$$=-[(m-r-1+n)(m-r-1-n-1)]$$

$$=(m-r+n-1)(m-r+n-2)$$

or $(m-r)(m-r-1)a_r-(m-r+n-1)(m-r-2)a_{r+2}=0$

or

$$a_{r+2}=\frac{(m-r)(m-r-1)}{(m-r+n-1)(m-r-n-2)}a_r \quad ...(7)$$

Now since $a_1=a_2=a_5=a_7=\cdots=0$

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Topic : **Legendre's Polynomials Part-2**

Legendre's polynomials

Two cases are given below

Case I: When $\boldsymbol{m = n}$

$$a_{r+2} = \frac{(n-r)(n-r-1)}{(2n-r-1)(r+2)} a_r$$

so that, $a_2 = -\frac{n(n-1)}{(2n-1)\times 2} a_0$

$$a_4 = -\frac{(n-2)(n-3)}{(2n-3)\times 4} a_2 = \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-2)\times 2\times 4} a_0$$

and so on and $a_1 = a_3 = a_5 = \cdots = 0$

Hence the series (2) becomes

$$y = a_0 \left[x^n - \frac{n(n-1)}{(2n-1)\times 2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)\times 2\times 4} x^{n-4} - \cdots \right]$$

which is the solution of (1)

Case II: When $\boldsymbol{m = -(n+1)}$, we have

$$a_{r+2} = \frac{(n+r+1)(n+r+2)}{(r+2)(2n+r+3)} a_r$$

so that, $a_2 = \frac{(n+1)(n+2)}{2(2n+3)} a_0$

$$a_4 = \frac{(n+1)(n+2)(n+3)(n+4)}{2\times 4(2n+3)(2n+5)} a_0$$

and so on.

Hence the series in this case becomes

$$y = a_0 \left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} \right.$$
$$\left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \times 4(2n+3)(2n+5)} x^{-n-5} + \cdots \right]$$

This gives another solution of (1) in a series descending powers of x.

Note: If we want to integrate the Legendre's equation in a series of ascending powers of x, we may proceed by taking

$$y = a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \cdots = \sum_{0}^{\infty} a_r x^{k+r}$$

But integration in descending powers of x is more important than in ascending powers of x.

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Topic **: Legendre's Polynomial $P_n(x)$**

Legendre's Polynomial $P_n(x)$

Definitions:

The Legendre's Equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad \text{...(1)}$$

where n is a non-negative integer.

It is possible to obtain the solution of (1) in terms of descending powers of x. Due to its applications to physical problems, this form of solution of Legendre's differential equation is more important.

For such a solution, let us assume that the Legendre's differential equation (1) has a series solution of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k-m}, \qquad C_0 \neq 0$$

Then, by Frobenius method, we can find two linearly independent solutions of (1) in descending powers of x as:

$$y_1 = a[x^n - \frac{n(n-1)}{(2n-1)2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3).2.4}x^{n-4} \dots] \qquad \text{...(2)}$$

$$y_{2=}b[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5} + \cdots] \qquad \text{...(3)}$$

If we take $a = \frac{1.3.5\dots(2n-1)}{n!}$, the solution (3) is denoted by $P_n(x)$ and is

called Legendre's function of the first kind or Legendre's polynomial of degree n and defined by

$$P_n(x) = \frac{1.3.5\dots(2n-1)}{n!}[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} - + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \cdots]$$

...(4)

Note 1: This is a terminating series.

We can also write $P_n(x)$ in a compact from as:

$$P_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{2^n r!\,(n-2r)!\,(n-r)} x^{n-2r}$$

When n is even ,it contains $\frac{1}{2}n + 1$ terms, the last term being

$$(-1)^{\frac{n}{2}} \frac{n(n-1)(n-2)\dots 1}{(2n-1)(2n-3)\dots(n+1)2.4.6\dots n}$$

When n is odd,it contains $\frac{1}{2}(n+1)$ terms and the last term in this case is

$$(-1)^{\frac{n-1}{2}} \frac{n(n-1)\dots 3.2}{(2n-1)(2n-3)\dots(n+2).2.4\dots(n-1)} x$$

Note. $\boldsymbol{P_n(x)}$ is that solution of Legendre's equation (1) which is equal to unity when $x = 1$.

The first several Legendre's Polynomials are listed below

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

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Topic **: Legendre’s functions of the second kind (Qn)**

Legendre's functions of the second kind (Q_n)

It is another solution of Legendre's equation.

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

When n is a positive integer;

$$y = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \cdots\right]$$

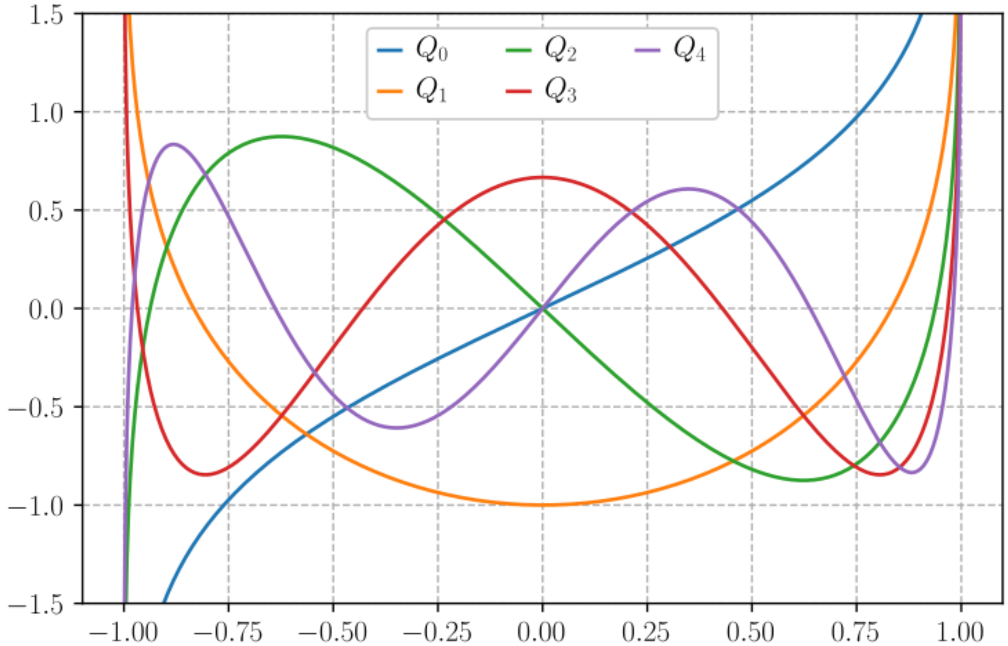
If we take , $a_0 = \frac{n!}{1.3.5\ldots.(2n+1)}$

The above solution is called $Q_n(x)$. *so that*

$$Q_n(x) = \frac{n!}{1.3.5\ldots(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \cdots\right]$$

The series for $Q_n(x)$ is a non-terminating series.

This solution is necessarily singular when $x = \pm 1$



The Legendre functions of the second kind can also be defined recursively via Bonnet's recursion formula

$$Q_n(x) = \begin{cases} \frac{1}{2}\log\frac{1+x}{1-x} & ; n = 0 \\ P_1(x)Q_0(x) - 1 & ; n = 1 \\ \frac{2n-1}{n}xQ_{n-1}(x) - \frac{n-1}{n}Q_{n-2}(x) & ; n \geq 2 \end{cases}$$

Associated Legendre functions of the second kind

The nonpolynomial solution for the special case of integer degree $\lambda = n \in \mathbb{N}_0, and\ \mu = m \in \mathbb{N}_0$ is given by

$$Q_n^m(x) = (-1)^m(1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}Q_n(x).$$

Integral representations

The Legendre functions can be written as contour integrals. For example,

$$P_\lambda(z) = P_\lambda^0(z) = \frac{1}{2\pi i}\int_1^z \frac{(t^2-1)^\lambda}{2^\lambda(t-z)^{\lambda+1}}dt$$

where the contour winds around the points 1 and *z* in the positive direction and does not wind around −1. For real *x*, we have;

$$P_s(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x+\sqrt{x^2-1}\cos\theta)^s d\theta = \frac{1}{\pi}\int_0^1\left(x+\sqrt{x^2-1}(2t-1)\right)^s \frac{dt}{\sqrt{t(1-t)}}, \quad s \in \mathbb{C}$$

Legendre function as characters

The real integral representation of P_s are very useful in the study of harmonic analysis on $L^1(G//\mathrm{K})$ where G//K is the double coset space of SL(2,$\mathbb{R}$). Actually the Fourier transform on $L^1(G//\mathrm{K})$ is given by

$L^1(G//\mathrm{K}) \ni f \to \mathcal{F}$

Where

$$\mathcal{F}(s) = \int_1^\infty f(x)P_s(x)dx, \qquad -1 \le R_s \le 0$$

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Topic **: General Solution of Legendre's Equation**

General Solution of Legendre's Equation

Legendre's equation arises when solving partial differential equations involving the Laplacian in spherical coordinate. The ODE in the *r* direction of spherical coordinate usually takes the form

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \mu y = 0, \qquad -1 < x < 1 \tag{1}$$

Since $p(x) = \frac{-2x}{1-x^2}$, $q(x) = \frac{\mu}{1-x^2}$, and $r(x) = 0$ are analytic at $x = 0$, the equation can be represented by a power series solution of the form

$$y = \sum_{m=0}^{\infty} a_m x^m \tag{2}$$

Differentiating the series solution (2) yields

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

Substituting the funsction and its derivatives into Eq. (1) yields

$$(1 - x^2)\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + \mu\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + \mu\sum_{m=0}^{\infty} a_m x^m = 0 \tag{3}$$

The terms with x^{m-2} can be changed to x^m by replacing *m* with *m* + 2

$$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = \sum_{m+2=2}^{\infty} (m+2)(m+1) a_{m+2} x^m$$

Equation (3) becomes

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{m=2}^{\infty}m(m-1)a_m x^m - 2x\sum_{m=1}^{\infty}ma_m x^{m-1} + \mu\sum_{m=0}^{\infty}a_m x^m = 0$$

$$\sum_{m=2}^{\infty}\{(m+2)(m+1)a_{m+2} - [m(m+1)+2m-\mu]a_m\}x^m + 2a_2 + \mu a_0 + (6a_3 - 2a_1 + \mu a_1)x = 0$$

$$a_2 = -\frac{\mu}{2}a_0, \qquad a_3 = \frac{2-\mu}{6}a_1$$

$$a_{m+2} = \frac{m(m+1)-\mu}{(m+2)(m+1)}a_m$$

For $\mu = n(n+1) \Rightarrow a_{m+2} = \frac{m(m+1)-n^2-n}{(m+2)(m+1)}a_m = -\frac{(n-m)(n+m+1)}{(m+2)(m+1)}a_m$

$$a_2 = -\frac{n(n+1)}{2!}a_0 \qquad a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{3\times 4}a_2 = \frac{(n-2)n(n+1)(n+3)}{4!}a_0$$

$$a_5 = -\frac{(n-3)(n+4)}{4\times 5}a_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$$

The general solution is then

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^2 - \ldots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \ldots$$

In summary: For n = 0, 1, 2, ... the Legrendre equation of order n

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0, \qquad -1 < x < 1$$

has two linearly independent solutions $y_1(x)$ and $y_2(x)$

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^2 - \ldots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \ldots$$

When n is an even inrteger y_1 is a polynomial $P_n(x)$ of degree n and y_2 has the form of an infinite series. Legendre's function of the second kind is defined as

$$Q_n(x) = y_1(x) + y_2(x) \qquad n \text{ even}$$

When n is an odd integer y_2 is a polynocmial $P_n(x)$ of degree n and y_1 has the form of an infinite series. Legendre's function of the second kind for this case is defined as

$$Q_n(x) = - y_1(x) + y_2(x) \qquad n \text{ odd}$$

The general solution of Legrendres' equation is then

$$y(x) = C_1 P_n(x) + C_2 Q_n(x)$$

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Topic **: Rodigue's Formula Part-1**

Rodigue's Formula

$P_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^n - 1)^n$

Proof. Let, v=$(x^2-1)^n$ ..... (1)

Then $\frac{dv}{dx} = n(x^2-1)^{n-1}$(2x)

Multiplying both sides by $(x^2-1), we\ get$

$(x^2-1)\ \frac{dv}{dx} = 2n(x^2-1)^n x$

Or $(x^2-1)\ \frac{dv}{dx} = 2nvx \ldots..(2)$

Now differentiating (2), (n+1) times by Leibnitz's theorem, we have

$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + (n+1)_{C_1}(2x)\frac{d^{n+1}v}{dx^{n+1}} + (n+1)_{C_2}(2)\frac{d^n v}{dx^n} = 2n\left[x\frac{d^{n+1}v}{dx^{n+1}} + (n+1)_{C_1}\frac{d^n v}{dx^n}\right]$

Or $(x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + 2x\left[(n+1)_{C_1} - 1\right]\frac{d^{(n+1)}v}{dx^{n+1}} + 2\left[(n+1)_{C_1}\right]\frac{d^n v}{dx^n} = 0$

Or $(x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + 2x\frac{d^{(n+1)}v}{dx^{n+1}} + n(n+1)\frac{d^n v}{dx^n} = 0$ ...... (3)

If we put $\frac{d^n v}{dx^n} = y, (3) becomes$

$(x^2-1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y = 0$

Or $(x^2-1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$

This shows that y=$\frac{d^n v}{dx^n}$ $is\ a$ solution of Legendre's equation.

∴ $c\frac{d^n v}{dx^n} = p_n(x)$

Where c is a constant.

But $v = (x^2-1)^n(x-1)^n$

So that $\frac{d^n v}{dx^n} = (x+1)^n\frac{d^n}{dx^n}(x-1)^n + n_{C_1}.n(x+1)^{n-1}.\frac{d^{n-1}}{dx^n}(x+1)^n \ldots\ldots. + (x-1)^n\frac{d^n}{dx^n}(x+1)^n = 0$

When $x = 1$, $\frac{d^n v}{dx^n} = 2^n.n!$

All the other terms disappear as $(x-1)$ is a factor in every term except first.

Therefore when $x = 1, (4)$ gives

$$C.2^n.n! = P_n(1) = 1 \qquad\qquad P_n(1) = 1$$

$$C = \frac{1}{2^n.n!}$$

Substituting the value of C from (1) in (5) we have

$$P_n(x) = \frac{1}{2^n.n!}\frac{d^n v}{dx^n}$$

$$P_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

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Topic **: Rodigue’s Formula Part-2**

Rodigue's Formula

Example 1. Let $P_n(x)$ be the Legendre polynomial of degree n. Show that for any function, $f(x)$, for which the nth derivative is continuous,

$$\int_{-1}^{1} f(x)P_n(x)dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} (x^2-1)^n f^n(x)dx$$

Solution:

$$\int_{-1}^{1} f(x)P_n(x)dx = \int_{-1}^{1} f(x).\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^n-1)^n dx$$

$$\left[P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\right]$$

$$= \frac{1}{2^n n!}\int_{-1}^{+1} f(x).\frac{d^n}{dx^n}(x^2-1)^n dx$$

Integrating by parts, we get

$$= \frac{1}{2^n n!}[f(x).\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n - \int f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx]_{-1}^{1}$$

$$= \frac{1}{2^n n!}[0 - \int_{-1}^{+1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx]$$

$$= \frac{(-1)}{2^n n!}\int_{-1}^{+1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$$

Again integrating by parts, we have

$$= \frac{(-1)^2}{2^n n!}[f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n - \int f''(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n dx]_{-1}^{+1}$$

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{+1} f''(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n dx$$

Integrating (n-2) times, by parts, we get

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{+1} f^n(x)(x^2-1)^n dx$$ **Proved.**

Example 2. Show that if m<n,

$$\int_{-1}^{1} x^m P_n(x)dx = 0$$

and

$$\int_{-1}^{1} x^n P_n(x)dx = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

Solution:

Rodrigue's formula is

$$P_n(x) = \frac{1}{2^n(n)!}\frac{d^n}{dx^n}(x^2-1)^n$$

Now, $\int_{-1}^{1} x^m P_n(x)dx = \frac{1}{2^n n!}\int_{-1}^{1} x^m \frac{d^n}{dx^n}(x^2-1)^n dx$

$= \frac{1}{2^n n!}[x^m \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n]_{-1}^{1} - \frac{1}{2^n n!}\int_{-1}^{1} mx^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2 -$
$1)^n dx$

$$= 0 - \frac{m}{2^n n!}\int_{-1}^{1} x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$$

$$= (-1)^m \frac{m!}{2^n n!}\int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n dx$$

$since\ m < n \quad \ldots(1)$

Integrating by parts m times

$$= (-1)^m \frac{m!}{2^n n!}[\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n]_{-1}^{1}$$

= 0

Again in the second part, m=n; therefore proceeding as above integrating by parts n times, we get,

$$\int_{-1}^{1} x^n P_n(x)dx = \frac{(-1)^n n!}{2^n n!}\int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}}(x^2-1)^2 dx$$

$$= \frac{1}{2^n}\int_{-1}^{1} (1-x^2)^n dx$$

$$= \frac{2}{2^n}\int_{0}^{1} (1-x^2)^n dx$$

$$= \frac{2}{2^n}\int_{0}^{\frac{\pi}{2}} cos^{2n}\theta cos\theta \, d\theta \qquad \text{where } x = \sin\theta$$

$$= \frac{2}{2^n}\frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{2\Gamma(\frac{2n+3}{2})}$$

$$= \frac{2^{n+1} .n!.n!}{2^n(2n+1)(2n-1)(2n-3).......3.1.n.(n-1)(n-2).....2.1}$$

Expanding $\Gamma(\frac{2n+3}{2})$ and multiplying numerator and denominator by n!

$$= \frac{2^{n+1}(n!)^2}{(2n+1)(2n-1)(2n-3).........3.1.2n.(2n-2)(2n-4).....4.2}$$

$$= \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

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Section : A

Session : 2018-2019

Subject : Math 2101

Topic **: LEGENDRE POLYNOMIALS**

Legendre Polynomials

$p_n(x) = \frac{1}{2^n.n!} \cdot \frac{d^n}{dx^n}(x^2-1)^{\ n}$ (Rodrigue's formula)

If n = 0, $p_0(x) = \frac{1}{2^0.0!} = 1$

If n=1, $p_1(x) = \frac{1}{2^1.1!}\frac{d}{dx}(x^2-1) = \frac{1}{2}(2x) = x$

If n=2, $p_2(x) = \frac{1}{2^2.2!}\frac{d^2}{dx^2}(x^2-1)^{\ 2} = \frac{1}{8}\frac{d}{dx}[2(x^2-1)(2x)]$

$= \frac{1}{2}[(x^2-1).1+2x.x] = \frac{1}{2}(3x^2-1)$

Similarly, $p_3(x) = \frac{1}{2}(5x^3-3x)$

$p_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

$p_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

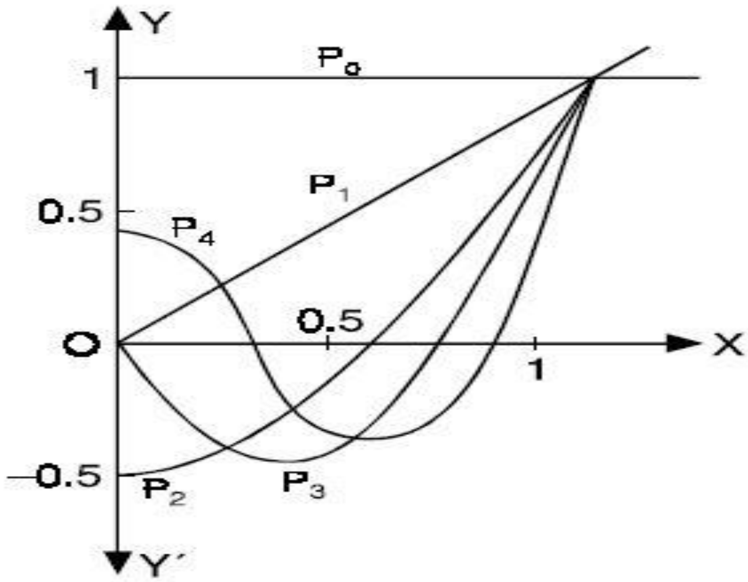
$p_6(x) = \frac{1}{16}(231x^6-315x^4+105x^2-5)$

……………………………………………………….

$p_n(x) = \sum_{r=0}^{N} \frac{(-1)^r(2n-2r)!}{2^n.r!(n-r)!(n-2r)!} x^{n-2r}$

Where N = $\frac{n}{2}$ if n is even.

N = $\frac{1}{2}$(n-1) if is odd.



Note. we can evaluate p_n(x) by expanding $(x^2-1)^n$ by Binomial theorem.

$(x^2-1)^n = \sum_{r=0}^{r=n} \boldsymbol{n_{C_r}} (\boldsymbol{x^2})^{\boldsymbol{n-r}} (\boldsymbol{-1})^{\boldsymbol{r}}$

$= \sum_{r=0}^{r=n} (\boldsymbol{-1})^{\boldsymbol{r}} \frac{\boldsymbol{n}!}{\boldsymbol{r}!(\boldsymbol{n-r})!} \boldsymbol{x^{2n-2r}}$

$p_n(x) = \frac{1}{\boldsymbol{2^n}.n!} \frac{d^n}{dx^n} (x^2-1)^n$

$= \frac{1}{\boldsymbol{2^n}.n!} \sum_{r=0}^{r=n} (\boldsymbol{-1})^{\boldsymbol{r}} \frac{\boldsymbol{n}!}{\boldsymbol{r}!(\boldsymbol{n-r})!} \frac{d^n}{dx^n} \boldsymbol{x}^{(\boldsymbol{2n-2r})}$

$= \sum_{r=0}^{N} \frac{(\boldsymbol{-1})^{r}(2n-2r)!}{2^{\boldsymbol{n}}.r!(n-r)!(n-2r)!} x^{\boldsymbol{n-2r}}$

Either x^0 or x^1 is in the last term.

n-2r = 0 or r = $\frac{n}{2}$ (n is even) or   n-2r = 1 or r = $\frac{1}{2}$(n-1)   (n is odd)

Ex-1. Express f(x) = $4x^3 + 6x^2 + 7x + 2$ in terms of *Legendre Polynomials.*

Solution. Let

$$4x^3 + 6x^2 + 7x + 2 \equiv a\,P_3(x) + bP_2(x) + cP_1(x) + dP_0(x) \quad \ldots(1)$$

$$\equiv a\left(\frac{5x^3}{2} - \frac{3x}{2}\right) + b\left(\frac{3x^2}{2} - \frac{1}{2}\right) + c(x) + d(1)$$

$$\equiv \frac{5ax^3}{2} - \frac{3ax}{2} + \frac{3bx^2}{2} - \frac{b}{2} + cx + d$$

$$\equiv \frac{5ax^3}{2} + \frac{3bx^2}{2} + \left(\frac{-3a}{-2} + c\right)x - \frac{b}{2} + d.$$

Equating the coefficients of like powers of x, we have

$$4 = \frac{5a}{2}, \text{ or } a = \frac{8}{5}$$

$$6 = \frac{3b}{2} \text{ or } b = 4$$

$$7 = \frac{-3a}{2} + c \quad \text{or} \quad 7 = \frac{-3}{2}\left(\frac{8}{5}\right) + c \quad \text{or} \quad c = \frac{47}{5}$$

$$2 = \frac{-b}{2} + d \text{ or } 2 = \frac{-4}{2} + d \text{ or } d = 4$$

Putting the values of a,b,c,d in (1), we get

$$4x^3 + 6x^2 + 7x + 2 = \frac{8}{5}P_3(x) + 4P_2(x) + \frac{47}{5}P_1(x) + 4P_0(x)$$

Ans.

Example-2. Prove that $P_n(1)=1$

Solution. We know that

$$(1-2xz+z^2)^{-1/2}=1+zP_1(x)+z^2P_2(x)+z^3P_3(x)+\ldots z^nP_n(x)+\ldots$$

Substituting1 for x in the above equation,we get

$$(1-2z+z^2)^{-1/2}=1+zP_1(1)+z^2P_2(1)+z^3P_3(1)+\ldots z^nP_n(1)$$

$$[(1-z)^2]^{-1/2}=\sum_{n=0}^{\infty} z^n p_n(1) \text{ or} (1-z)^{-1}=\sum z^n p_n(1)$$

or

$$\sum z^n p_n(1)=(1-z)^{-1}=1+z^2+z^3+\ldots+z^n+\ldots$$

Equating the coefficients z^n on both sides we get

$p_n(1)=1$ Proved

Example-3. Show that

(i) $P_{2n}(0)=(-1)^n\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$ (ii) $P_{2n+1}(0)=0.$

Solution. We know that

$$\sum z^2 p_{2n}(x)=(1-2xz+z^2)^{-1/2}$$

$$\sum z^2 p_{2n}(0)=(1+z^2)^{-1/2}$$

$$=1+(-1/2)z2+\frac{(-1/2)}{2!}(z^2)+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(z^2)^3$$

$$+\ldots+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots(-\frac{1}{2}-n+1)}{n!}(z^2)^n+\ldots$$

Equating the coefficient of z^{2n} both sides we get

$$p_{2n}(0)=\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)-\left(\frac{5}{2}\right)\ldots(-\frac{1}{2}-n+1)}{n!}$$

$$=(-1)^n\frac{1.3.5\ldots(2n-1)}{2^n.n!}$$

$$=(-1)^n\frac{1.3.5\ldots(2n-1)}{2.4.6.8\ldots2n}$$ Proved

coefficient of $P_{2n+1}(0)=0.$ Proved

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Section : A

Session : 2018-2019

Subject : Math 2101

Topic : **A GENERATING FUNCTION OF LEGENDRE'S POLYNOMIAL**

A GENERATING FUNCTION OF LEGENDRE'S POLYNOMIAL

Prove that P_n(x) is the coefficient of z^n in the expansion of

$(1 - 2\text{xz} + z^2)^{-1/2}$ in ascendig powers of z.

Proof. $(1 - 2\text{xz} + z^2)^{-1/2} = [1 - \text{z}(2\text{x} - \text{z})]^{-1/2}$

$$= 1+\frac{1}{2}z(2x-z)+\frac{-\frac{1}{2}(-\frac{3}{2})}{2!}z^2(2x-z)^2+....$$

$$+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)...\left(-\frac{1}{2}-n+1\right)}{n!}(-z^n)(2x-z)^n+... ...(1)$$

Now coefficient of z^n in

$$\frac{(-\frac{1}{2})\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)...(-\frac{1}{2}-n+1)}{n!}(-z^n)(2x-z)^n$$

$$=\frac{(-\frac{1}{2})\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)...(-\frac{1}{2}-n+1)}{n!}(-1^n)(2x)^n$$

$$=\frac{1.3.5...(2n-1)}{2^n.n!}(2)^n.\,x^n$$

$$=\frac{1.3.5...(2n-1)}{n!}\,x^n$$

Coefficient of z^n in

$$\frac{(-\frac{1}{2})\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)...(-\frac{1}{2}-n+2)}{(n-1)!}(-z)^{n-1}(2x-z)^{n-1}$$

$$=\frac{(-\frac{1}{2})\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)...(-\frac{1}{2}-n+2)}{(n-1)!}(-1)^{n-1}[-(n-1)(2x)^{n-2}]$$

$$=\frac{1.3.5...(2n-3)}{2^{n-1}.(n-1)!}(2)^{n-2}(n-1)x^{n-1}$$

$$=\frac{1.3.5...(2n-3)}{2.(n-1)!}(n-1)x^{n-2}$$

$$=\frac{1.3.5...(2n-3)}{2.(n-1)!}\cdot\frac{(2n-1)}{(2n-1)}(n-1)x^{n-2}$$

$$=\frac{1.3.5...(2n-3)(2n-1)}{n!}\cdot\frac{n(n-1)}{2(2n-1)}x^{n-2}$$

Coefficient of z^n in

$$\frac{(-\frac{1}{2})\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots(-\frac{1}{2}-n+3)}{(n-2)!}z^{n-2}(2x-z)^{(n-2)}$$

$$= \frac{(-\frac{1}{2})\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots(-\frac{1}{2}-n+3)}{(n-2)!}.(-1)^{n-2}.\frac{(n-2)(n-3)}{2!}(2x)^{n-4}$$

$$= \frac{1.3.5\dots(2n-5)}{2^{n-2}(n-2)!}.\frac{(n-2)(n-3)}{2!}(2x)^{n-4}$$

$$= \frac{1.3.5\dots(2n-5)(2n-3)(2n-1)}{4(n-2)!}.\frac{(n-2)(n-3)}{2(2n-3)(2n-1)}x^{n-4}$$

$$= \frac{1.3.5\dots(2n-1)}{4n(n-1)(n-2)!}.\frac{n(n-1)(n-2)(n-3)}{2(2n-3)(2n-1)}x^{n-4}$$

$$= \frac{1.3.5\dots(2n-1)}{n!}.\frac{n(n-1)(n-2)(n-3)}{2.4(2n-3)(2n-1)}x^{n-4}$$

And so on.

Thus coefficient of z^n in the expansion of (1)

$$= \frac{1.3.5\dots(2n-1)}{n!}.[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-3)(2n-1)}x^{n-4} - \dots]$$

$= P_n(x)$

Thus coefficients of z, z^2, z^3 … etc. in (1) are $P_1(x), P_2(x), P_3(x)$….

Hence

$$(1-2xz+z^2)^{-1/2} = P_0(x) + zP_1(x) + z^2P_2(x) + z^3P_3(x) + \dots + z^nP_n(x) + \dots$$

i.e., $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{n=\infty} P_n(x).z^n$.

Proved.

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Section : A

Session : 2018-2019

Subject : Math 2101

Topic **: Orthogonality of Legendre Polynomials**

Orthogonality of Legendre Polynomials

The **Legendre** polynomials $P_m(x)$ and $P_n(x)$ are said to be orthogonal in the interval $-1 \le x \le 1$ provided

$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$ m ≠ n

First Method:

The Legendre's equation can be written as ,

$$\frac{d}{dx}\left\{ (1-x^2)\,\frac{dy}{dx}\right\} + n(n+1)y=0$$

Since P_n is a solution of the equation ,we have

$$\frac{d}{dx}\left\{ (1-x^2)\,\frac{dP_n}{dx}\right\} + n(n+1)\,P_n=0 \qquad \text{.. } \mathbf{(1)}$$

Similarly,

$$\frac{d}{dx}\left\{ (1-x^2)\,\frac{dP_m}{dx}\right\} + m(m+1)\,P_m=0 \qquad \text{... } \mathbf{(2)}$$

Multiplying (1) b P_m (2) by P_n and then subtracting ,we get when m ≠ n,

$$P_m\frac{d}{dx}\left\{ (1-x^2)\,\frac{dP_n}{dx}\right\} - P_n\frac{d}{dx}\left\{ (1-x^2)\,\frac{dP_m}{dx}\right\} + P_m P_n[\, n(n+1) - m(m+1)\,]=0 \text{ .... } \mathbf{(3)}$$

Now,

$$=\int_{-1}^{1} P_m\frac{d}{dx}\{ (1-x^2)\,\frac{dP_n}{dx}\}\, dx$$

$$=\left[P_m(1-x^2)\,\frac{dP_n}{dx}\right]_{-1}^{1} - \int_{-1}^{1} \frac{dP_m}{dx}(1-x^2)\frac{dP_n}{dx}\, dx$$

.... *integrating by parts.*

$$=\int_{-1}^{1} \frac{dP_n}{dx}.\frac{dP_m}{dx}(1-x^2)\, dx$$

and

$\int_{-1}^{1} P_n \frac{d}{dx}\{(1-x^2)\frac{dP_m}{dx}\}$ dx = -$\int_{-1}^{1} \frac{dP_m}{dx}.\frac{dP_n}{dx}(1-x^2)\,dx$

Integrating now (3) w.r.t x, we get

$\int P_m \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx$ - $\int P_n \frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx$ + *(n-m)(m+n+1)* $\int P_n P_m dx$ = *0*

Or,

-$\int_{-1}^{1} \frac{dP_m}{dx}.\frac{dP_n}{dx}(1-x^2)\,dx$ + $\int_{-1}^{1} \frac{dP_m}{dx}.\frac{dP_n}{dx}(1-x^2)\,dx$ +*(n-m)(n+m+1)* $\int_{-1}^{1} P_n P_m dx$ = *0*

*A*nd this gives,

(n-m)(n+m+1) $\int_{-1}^{1} P_m P_n dx$ = *0*

Or, $\int_{-1}^{1} P_m P_n dx$ = *0*

Second Method:

From **Rodrigue's** formula , we have

$P_n(x) = \frac{1}{2^n!}\frac{d^n}{d\,x^n}(x^2-1)^n$

And,

$P_m(x) = \frac{1}{2^m!}\frac{d^m}{d\,x^m}(x^2-1)^m$

without any loss of generality, we can suppose that m>n.

Consider now,

$I_{m,n} = \int_{-1}^{1} P_m(x)\, P_n(x)\, dx$

$= \frac{1}{2^{m+n}(m)!(n)!} \int_{-1}^{1} \frac{d^m}{d\,x^m} (x^2-1)^m \quad \frac{d^n}{d\,x^n} (x^2-1)^n\, dx$

$= \frac{1}{2^{m+n}(m)!(n)!} \left[\frac{\mathrm{d}^{m-1}}{\mathrm{d\,x}^{m-1}} (x^2-1)^m \quad \frac{\mathrm{d}^n}{\mathrm{d\,x}^n} (x^2-1)^n \right]_{-1}^{1}$

$- \frac{1}{2^{m+n}(m)!(n)!} \int_{-1}^{1} \frac{d^{m-1}}{d\,x^{m-1}} (x^2-1)^m \quad \frac{d^{n+1}}{d\,x^{n+1}} (x^2-1)^n\, dx$

……………………………………..Integrating by parts.

Now, $\frac{\mathrm{d}^{m-1}}{\mathrm{d\,x}^{m-1}}$ in its every term contains factors (x-1) and (x+1) both.Hence in the limits **-1 to 1** , its every term vanishes.

$I_{m,n} = - \frac{1}{2^{m+n}(m)!(n)!} \int_{-1}^{1} \frac{d^{m-1}}{d\,x^{m-1}} (x^2-1)^m \quad \frac{d^{n+1}}{d\,x^{n+1}} (x^2-1)^n\, dx$

Integrating now (n -1) times times , we get

$I_{m,n} = - \frac{(-1)^n}{2^{m+n}(m)!(n)!} \int_{-1}^{1} \frac{d^{m-n}}{d\,x^{m-n}} (x^2-1)^m \quad \frac{d^{2n}}{d\,x^{2n}} (x^2-1)^n\, dx$

But,

$\frac{d^{2n}}{d\,x^{2n}} (x^2-1)^n\, dx = (2n)!$

So,

$I_{m,n} = \frac{(-1)^n\,(2n)!}{2^{m+n}\,(m)!(n)!} \int_{-1}^{1} (x^2-1)^m\, dx$

$= \frac{(-1)^n\,(2n)!}{2^{m+n}\,(m)!(n)!} \left[\frac{\mathrm{d}^{m-n-1}}{\mathrm{d\,x}^{m-n-1}} (x^2-1)^m \right]_{-1}^{1}$

$=0$

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Section : C

Session : 2016-2017

Subject : Math 2101

Topic : **Orthogonality of Legendre Polynomials**

Orthogonality of Legendre Polynomials

Example 1: Prove that, $\int_{1}^{+1} \mid P_n(x)/^2 dx = \frac{2}{2n+1}$

Solution: We know that,

$$(1-2xz+z^2)^{-1/2} = \Sigma 2^n P_n(x)$$

Squaring both sides we get,

$$(1-2xz+z^2)^1 = \Sigma z^{2n} P_n^2(x) + 2\Sigma z^{m+n} P_m(x)\cdot P_n(x)$$

Integrating both sides between -1 and +1, we have

$$\int_{-1}^{+1} \Sigma z^{2n}\cdot P_n^2(x)dx + \int_{1}^{+1} 2\Sigma z^{m+n}\cdot P_m(x)\cdot P_n(x)dx$$
$$= \int_{-1}^{+1} (1-2xz+z^2)^1 dx \int_{1}^{+1} \Sigma z^{2n} P_n^2(x)dx + 0 = \int_{1}^{+1} \frac{1}{1-2x+z^2} dx \; or,$$
$$\sum z^{2n} \int_{-1}^{+1} P_n^2(x)dx = -\frac{1}{2z}[log\; log\;(1-2xz+z^2)\;]_1^{+1}$$
$$= -\frac{1}{2z} log\; \frac{1-2z+z^2}{1+2z+z^2} = -\frac{1}{2z} log\; \left(\frac{1-z}{1+z}\right)^2 = \frac{1}{z} log\; \frac{1+z}{1-z}$$
$$= \frac{1}{z}[log\;(1+z) - log\;(1-z)]$$

$$= \frac{1}{z}\left[\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots + \frac{z^{2n+1}}{2n+1} + \cdots\right) - \left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \cdots - \frac{z^2 n+1}{2n+1} - \cdots\right)\right]$$
$$= \frac{2}{z}\left[z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots + \frac{z^{2n+1}}{2n+1} + \cdots\right] = 2\left[1 + \frac{z^2}{3} + \frac{z^4}{5} + \cdots + \frac{z^{2n}}{2n+1} + \cdots\right]$$

Equating the coefficient of Z^{2n} on both sides, we have

$$\int_{-1}^{+1} [P_n(x)]^2 dx = \frac{2}{2n+1} \;\; Hence \int_{-1}^{+1} P_3^2(x)dx = \frac{2}{2\times 3+1} = \frac{2}{7}$$

Proved.

Example 2: Assuming that a polynomial f(x) of degree n can be written as Show that

$$f(x) = \sum_{0}^{1} C_m P_m(x) \; C_m = \frac{2m+1}{2}\int_{1}^{1} f(x)P_m(x)dx$$

Solution: $f(x) = \sum_0 \quad C_m P_m(x) = C_0P_0(x) + C_1P_1(x) + C_2P_2(x) + C_3P_3(x) + C_4P_4(x) + \cdots + C_mP_m(x) + \cdots$

Multiplying both sides with $P_m(x)$ we get

$$P_m(x)f(x) = C_0P_0(x)P_m(x) + C_1P_1(x)P_m(x) + C_2P_2(x)P_n(x) + \cdots + C_mP_m^2(x) + \cdots \int_{-1}^{+1} f(x)P_m(x)dx$$

$$= \int_{-1}^{+1} [C_0P_0(x)P_m(x) + C_1P_1(x)P_m(x) + C_2P_2(x)P_m(x) + \cdots + C_mP_m^2(x) + \cdots]dx = \left[0 + 0 + \cdots + C_m\frac{2}{2m+1} + \cdots\right] = \frac{2C_m}{2m+1}$$

$$C_m = \frac{2m+1}{2}\int_{1}^{+1} f(x)P_m(x)dx$$

Proved

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Section : C

Session : 2017-2018

Subject : Math 2101

Topic **: Orthogonality of Legendre Polynomials**

Orthogonality of Legendre Polynomials

Example 3. Using Rodrigues formula for Legendre function, prove that $\int_{-1}^{+1} x^m \ P_n\,(x)\,dx = 0$, where m, n are positive integers and m < n.

Solution:

$$\int_{-1}^{+1} x^m\, P_n\,(x)\,dx = \int_{-1}^{+1} x^m \frac{1}{2^n\, n!} \frac{d^n}{dx^n}\,(x^2-1)^n\,dx$$

$$= \frac{1}{2^n\, n!} \int_{-1}^{+1} x^m \frac{d^n}{dx^n}\,(x^2-1)^n\,dx$$

On integrating by part we get,

$$= \frac{1}{2^n\, n!}\left[\left\{x^m \frac{d^{n-1}}{dx^{n-1}}\,(x^2-1)^n\right\}_{-1}^{+1} - \int_{1}^{+1} m x^{m-1} \frac{d^{n-1}}{dx^{n-1}}\,(x^2-1)^n dx\right]$$

$$= 0 - \frac{m}{2^n\, n!} \int_{-1}^{+1} x^{m-1} \frac{d^{n-1}}{dx^{n-1}}\,(x^2-1)^n\,dx$$

$$\int_{-1}^{+1} x^m\, P_n\,(x)\,dx = -\frac{(-1)^2 m(m-1)}{2^n\, n!} \int_{-1}^{+1} x^{m-2} \frac{d^{n-2}}{dx^{n-2}}\,(x^2-1)^n\,dx$$

Integrating m 2 times we get,

$$= (-1)^m \frac{m(m-1)\ldots 1}{2^n\, n!} \int_{-1}^{+1} \frac{d^{n-m}}{dx^{n-m}}\,(x^2-1)^n\,dx = \frac{(-1)^m\, m!}{2^n\, n!} \int_{-1}^{+1} \frac{d^{n-m}}{dx^{n-m}}\,(x^2-1)^n\,dx$$

$$= \frac{(-1)^m\, m!}{2^n\, n!}\left[\frac{d^{n-m-1}}{dx^{n-m-1}}\,(x^2-1)^n\right]_{-1}^{+1} = 0$$

(Proved)

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Session : 2016-2017

Subject : Math 2101

Topic **: Recurrence Formulae**

LEGENDRE POLYNOMIALS

Recurrence Formulae For $P_n(x)$
Case 1 :
Formula I. $nPn = (2n-1)xP_{n-1} - (n-1)P_{n-2}$
Solution. We know that (1-2xz+z^2)$^{-1/2}$ $= \sum z^n P_n(x)$

Differentiating w.r.t. 'z', we get

$$-\frac{1}{2}(1-2xz+z^2)^{-\frac{3}{2}}(-2x+2z) = \sum nz^{n-1}P_n(x)$$

Multiplying both sides by (1-2xz+z^2)$, we\ get$

$$(1-2\text{xz}+z^2)^{-\frac{1}{2}}(x-z) = (1-2\text{xz}+z^2)\sum nz^{n-1}\,P_n(x)$$

$$(x-z)\sum z^n\,P_n(x) = (1-2\text{xz}+z^2)\sum nz^{n-1}\,P_n(x)$$
...(1)
Equating the coefficients of z^{n-1} from both sides, we get

$$xP_{n-1} - P_{n-2} = nP_n - 2x(n-1)P_{n-1} + (n-2)P_{n-2}$$

or
n$P_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$ Proved

Formula II. x$P\acute{n} - \acute{P}_{n-1} = nP_n$

Solution. We know that (1-2xz+z^2)$^{-\frac{1}{2}} = \sum z^n P_n(x)$
...(1)
Differentiating (1) with respect to z, we get
$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \sum nz^{n-1}P_n(x)$
or (x-z)(1-2xz+z^2)$^{\frac{3}{2}}$= $\sum nz^{n-1}P_n(x)$

...(2)
Differentiating (1) with respect to x, we get

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2z) = \sum z^n \acute{P}_n(x)$$

or z(1-2xz+z^2)$^{-\frac{3}{2}} = \sum z^n \acute{P}_n(x)$

Dividing (2) by (3), we get

$$\frac{x-z}{z} = \frac{\sum nz^{n-1}P_n(x)}{\sum z^n \acute{P}_n(x)}$$

Or

(x-z)$\sum z^n \acute{P}_n(x) = \sum nz^n P_n(x)$ ...(3)

Equating coefficients of z^n from both sides, we get

x$\acute{P}_n(x) - \acute{P}_{n-1}(x) = nP_n(x)$ proved

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Subject : Math 2101

Topic : **Recurrence Formulae**

Recurrence Formulae

Formula III. $P'_n - xP'_{n-1} = nP_{n-1}$

Solution:

$nP_{n-1} = (2n-1)xP_{n-1} - (n-1)P_{n-2}$ **(Recurrence formula I)**

Defferentiating the above formula w.r.t 'x', we get

$nP'_n = (2n-1)P_{n-1} + (2n-1)P'_{n-1} - (n-1)P'_{n-2}$

Or $n|P_n' - xP'_{n-1}| - (n-1)|xP'_{n-1} - P'_{n-2}| = (2n-1)P_{n-1}$

Or $n|P'_n - xP'_{n-1}| - (n-1)|(n-1)P'_{n-1} - P'| = (2n-1)P_{n-1}$

From formula II

$n|P'_n - xP'_{n-1}| = (n-1)^{)2+(}2n-1)P_{n-1} = nP_{n-1}$

$P'_n - x P'_{n-1} = nP_{n-1}$

(Proved)

Formula IV. $P'_{n+1} - P'_{n-1} = (2n+1)P_n$

Solution:

$nP_n = (2n-1)xP_{n-1} - (n-1)P_{n-2}$

Replacing n by (n+1)

$(n+1)P_{n+1} = (2n+2-1)xP_n - nP_{n-1}$

Or $(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$ ....(1)

Differentiating (1) w.r.t 'x',we get

$(n+1)P'_{n+1} = (2n+1)P_n + (2n+1)xP_n' - nP'_{n-1}$ ....(2)

$x P'_n - P'_{n-1} = nP_n$ (Recurrence formula II) ....(3)

Substituting the value of x p' from (3) into (2) we get

$(n+1)P'_{n+1} = (2n+1)P_n - +(2n+1)|nP_n - +P'_{n+1}| - nP'_{n-1}$

Or $(n+1)P'_{n+1} - (n+1)P'_{n-1} = (2n+1)P_n$

Or $P'_{n+1} - P'_{n-1} = (2n+1)P_n$

(Proved)

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Session : 2016-2017

Subject : Math 2101

Topic : **Recurrence Formulae**

Recurrence Formulae

Recurrence Formulae For $P_n(x)$

Case 3 :

Formula V. $(x^2-1)\acute{P}_n = n[xP_n - P_n - 1]$

Solution : $\acute{P}_n - x\acute{P}_{n-1} = nP_n - 1$ ...(1)

$x\acute{P}_n - \acute{P}_{n-1}$ = nP_n ... (2)

Multiplying (2) by x and subtracting from (1), We get

$(1-x^2)\acute{P}_n$ =$n\ (P_{n-1}) - xP_n$ Proved.

Formula VI. $(x^2-1)\acute{P}_n = (n+1)(P_{n-1}) - xP_n$

Solution : n$P_n = (2n-1)xP_{n-1} - (n-1)\ P_{n-2}$

Replacing n by (n+1), we get

$(n+1)P_{n+1} = (2n+2-1)xP_n - nP_{n-1}$

$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$

which can be written as

$(n+1)(P_{n+1} - xP_n) = n(xP_n - P_{n-1})$

But $(x^2-1)\acute{P}_n = n(xP_n - P_{n-1})$ ...(2)

From (1) and (2) we get

or $(x^2-1)\acute{P}_n = (n+1)(P_{n+1} - xP_n)$ Proved.

Example. Prove that

$\int_{-1}^{+1} x^2\ P_{n+1}(x)\ P_{n-1}(x)dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

Solution : The recurrence formula I is

$2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$

Replacing n by (n+1) and (n-1), we have

(2n+3)x $P_{n+1} = (n+2)P_{n+2} + (n+1)P_n$ ...(1)

$(2n-1)xP_{n-1} = nP_n + (n-1)P_{n-2}$ ...(2)

Multiplying (1) and (2) and integration in the limits -1 to +1, we have

$(2n+3)(2n-1)\ \int_{-1}^{+1} x^2\, P_{n+1}(x)\, P_{n-1}(x)dx = n(n+1)\int_{1-}^{1} P_n^2\ dx + n(n+$
$2)\ \int_{-1}^{+1} P_n . P_{n+2}$dx
$+(n^2-1)\int_{-1}^{+1} P_n\, P_{n-2}$dx+(n-1) (n-2) $\int_{-1}^{+1} P_{n+2}. P_{n-2}$dx

$=n(n+1)\int_{-1}^{1} P_n^2 dx + 0 + 0 + 0$

= n(n+1).$\frac{2}{(2n+1)}$

or $\int_{-1}^{+1} x^2\, P_{n+1}\ \ P_{n-1} dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$ Proved.

THE END

Rajshahi University of Engineering & Technology

Assignment on Mathematics

Topic: **Solution of linear equations of second and higher orders with constant co-efficient.**

Submitted by : 1800061-1800083 **Department** : Civil Engineering **Course no** : Math-2121 **Date** : 06/01/2021	**Submitted to** : Mst. Rupale Khatun Assistant Professor Department of Mathematics

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Linear Differential Equation:

A differential equation of the form-

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + p_n y = X$$

Where $p_1, p_2, \ldots\ p_n$ and X are functions of x or constants, is called a linear differential equation of n^{th} order.

Here, $p_1, p_2, \ldots\ p_n$ are all constants (not functions of x), and X is some function of x, then the equation is a linear differential equation with constant coefficients.

The Operator D:

It is usual to write- D for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$, … , D^n for $\frac{d^n}{dx^n}$.

And in terms of terms of the operator D the differential equation (1) can be written as

$$[D^n + p_1 D^{n-1} + p_2 D^{n-2} + \cdots + p_n]y = X$$

It can be proved that D can be treated as an algebraic operator of quantity in several respects.

A Theorem:

If $y = y_1, y = y_2, \ldots, y = y_n$ linear independent solutions of

$$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n]y = 0$$

.. .. (1)

then $y = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$ is the general or complete solution of the differential equation, where $C_1, C_2, \ldots, C_n$ are n arbitrary constants.

Let us denote the given equations (1) by $f(D)\ \ y = 0$,

where, $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n$.

Since $y = y_1, y = y_2, \ldots, y = y_n$ are solutions of the equation, so, $f(D)\,y_1 = 0\,, f(D)\,y_2 = 0, \dots\,, f(D)\,y_n = 0.$ .. .. (2)

Now putting $y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$ in (1), we have

$$D^n(C_1y_1 + \cdots + C_ny_n) + a_1D^{n-1}(C_1y_1 + C_2y_2 + \cdots + C_ny_n) + \cdots$$
$$+a_n(C_1y_1 + C_2y_2 + \cdots + C_ny_n) = 0$$

or, $C_1(D^ny_1 + a_1D^{n-1}y_1 + \cdots + a_n) + C_2(D^ny_2 + a_1D^{n-1}y_2 + \cdots + a_n) + \cdots$
$$+C_n(D^ny_n + a_1D^{n-1}y_n + \cdots + a_n) = 0$$

or, $C_1f(D)y_1 + C_2f(D)y_2 + \cdots + C_nf(D)y_n = 0$

or, $C_1.0 + C_2.0 + \cdots + C_n.0 = 0$ [by (2)]

Since, (1) is satisfied by $y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$, it is a solution of (1). Also, since it contains n arbitrary constants, it is the general or complete solution of the equation.

Roll:1800062

Auxiliary equation:

Consider the differential equation

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)y = 0 \quad \dots\dots\dots(1)$$

Where $a_1, a_2, \dots, a_n$ are all constants .

Let y = e^x be a solution of this equation. Then putting

$y=e^{mx}, Dy = me^{mx}, D^2y = m^2e^{mx}, D^ny = m^ne^{mx}$, the equation becomes

$$(m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_n)e^{mx}=0$$

Hence e^{mx} will be a solution of (1) if m is a root of the algebraic equation

$$m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_n = 0 \quad \dots\dots(2)$$

This Equation in m is called the Auxiliary equation.

NOTE: It is observed that the auxiliary equation f(m)=0 gives the same values of m as the equation f(D)=0 gives of D.

Hence f (D) =0, i.e., $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n = 0$

Can in general be regarded as the auxiliary equation

SOLUTION OF EQUATION (1) OF THE ARTICLE

Case 1: When all the roots of auxiliary equation are real and different.

If $m_1, m_2, \ldots, m_n$ be the n different roots of (2), then $y=e^{m_1 x}$, $y=e^{m_2 x}$ ..., $y = e^{m_n x}$ are all independent solutions of (1). Therefore the general solution of (1) is

$$y=C_1 e^{m_1 x}+C_2 e^{m_2 x}+C_3 e^{m_3 x}+\ldots+C_n e^{m_n x}$$

Example 1: Solve $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+15y=0$

Solution: Given equation can be written as

$$(D^2 - 8D + 15)\, y = 0$$

Here auxiliary equation is

$$m^2 - 8m + 15 = 0$$

$$\Rightarrow (m - 3)(m - 5) = 0$$

$$\therefore m = 3, 5$$

Hence, the required solution is $y = C_1 e^{3x} + C_2 e^{5x}$ **ANSWER**

Example 2: Solve $\frac{d^3y}{dx^3}-13\frac{dy}{dx}-12y=0$

Solution: Given equation can be written as

$$(D^3 - 13D - 12)\, y = 0$$

Here auxiliary equation is

$$m^3 - 13m - 12 = 0$$

$$\Rightarrow (m + 1)(m + 3)(m - 4) = 0$$

$\therefore$ m = -1, -3, 4

Hence, the required solution is $y = C_1e^{-x} + C_2e^{-3x} + C_3e^{4x}$

ANSWER

Example 3: Solve $\frac{d^3y}{dx^3}+6\frac{d^2y}{dx^2}+11\frac{dy}{dx}+6y=0$

Solution: Given equation can be written as

$$(D^3 + 6D^2 + 11D + 6)y = 0$$

Here auxiliary equation is

$$m^3 + 6m^2 + 11m + 6 = 0$$

$$\Rightarrow (m+1)(m+2)(m+3) = 0$$

$$\therefore m = -1, -2, -3$$

Hence, the required solution is $y = C_1e^{-x} + C_2e^{-2x} + C_3e^{-3x}$

ANSWER

Example 4: Solve $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+5y=0$

Solution: Given equation can be written as

$$(D^2 - 6D + 5)y = 0$$

Here auxiliary equation is

$$m^2 - 6m + 5 = 0$$

$$\Rightarrow (m-1)(m-5) = 0$$

$$\therefore m = 1, 5$$

Hence, the required solution is $y = C_1e^{x} + C_2e^{5x}$

ANSWER

Example 5: Solve $\frac{d^2y}{dx^2}-3\frac{dy}{dx}-4y=0$

Solution: Given equation can be written as

$$(D^2 - 3D - 4)y = 0$$

Here auxiliary equation is

$$m^2 - 3m - 4 = 0$$

$\Rightarrow (m + 1)(m - 4) = 0$

$\therefore m = -1, 4$

Hence, the required solution is $y = C_1 e^{-x} + C_2 e^{4x}$ **ANSWER**

Roll:1800063

Case II: Auxiliary equation having equal roots.

We have shown in case [5.5, that when $m_1, m_2, \ldots, m_n$ are all different, then general solution is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 X} + \cdots + c_n e^{m_n x}.$$

But if $m_1 = m_2$ (two roots equal) then this becomes

$$y = (C_1 + C_2) e^{m_1 x} + C_3 e^{m_3 x} + C_n e^{m_n x},$$

Which clearly contains only $n - 1$ arbitrary constants (since $C_1 + C_2$ is equivalent to only one arbitrary constant)

Therefore there is no longer a general solution.

Consider an equation $(D - m_1)^2 y = 0$, ... (1)

A differential equation of second order having both the roots equal

Put $(D - m_1)\, y = v$, then (1) becomes

$(D - m_1)\, v = 0$ or $\frac{dv}{dx} = m_1 v$,

Separating the variables, $\frac{dv}{v} = m_1\, dx$.

Integrating, $\log v = \log C + m_1 x$ or, $v = Ce^{m_1 x}$

or, $(D - m_1)\, y = Ce^{m_1 x}$ as $v = (D - m_1)\, y$

or, $\frac{dy}{dx} - m_1 y = Ce^{m_1 x}$

which is a linear equation of first order, its I.F. $= e^{-mx}$

$\therefore \quad ye^{-m_1 x} = \int Ce^{m_1 x} e^{-m_1 x}\, dx + C_2$

Or, $y = (Cx + C_2) e^{m_1 x}$.

Therefore the most general solution is

$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots a_n)y = 0,$

When two roots of A.E. are equal, is

$y = (C_1 + C_2 x)\, e^{m_1 x} + C_3 e^{m_3 x} + \ldots + C_n e^{m_n x}$.

Cor. In case three roots are equal, i.e. $m_1 = m_2 = m_3$. The general solution is

$y = (C_1 + C_2 x + C_3 x^2) e^{m_1 x} + C_4 e^{m_4 x} + \cdots + C_n e^{m_n x}$.

Ex. 1. Solve $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$.

Solution: A.E. is $D^4 - D^3 - 9D^2 - 11D - 4 = 0$.

i.e. $(D+1)^3 (D-4) = 0,\ D = -1, -1, -1, 4.$

Hence the general solution is

$y = (C_1 + C_2 x + C_3 x^2) e^{-x} + c_4 e^{4x}.$

Ex. 2. Solve $(D^3 - 2D^2 - 4D + 8)\, y = 0$

Solution: Auxiliary equation is

$D^3 - 2D^2 - 4D + 8 = 0$ or, $(D+2)(D+2)^2 = 0,$

$D = -2, 2, 2.$

$\therefore\ y = (C_1 + C_2 x) e^{2x} + C_2 e^{-2x}$

Roll:1800064

Case III. Auxiliary equation having imaginary roots.

let $\alpha \pm i\beta$ be the imaginary roots of an equation of second order (since imaginary roots occur in pairs).

Then its general solution is

$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$

$=e^{\alpha x}\left[C_1 e^{i\beta x} + C_2 e^{-i\beta x}\right]$

$=e^{\alpha x}[C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)]$

$=e^{\alpha x}[(C_1 + C_2)\cos\beta x + (C_1 - C_2)\, i\sin\beta x]$

$=e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

Note: The above result after suitably adjusting constants may also be written as

$y= e^{\alpha x} A\cos(\beta x + B)$ or $y = e^{\alpha x} A\sin(\beta x + B)$

Imaginary roots repeated. If auxiliary equation has two equal pairs of imaginary roots, i.e., if α+iβ and α-iβ occur twice, then general solution is obtained as

$y=e^{\alpha x}[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x]$

Cor. If a pair of roots of the auxiliary equation occur in the form of quadratic surd $\alpha \pm \sqrt{\beta}$, where β is +ive, then the corresponding term in the solution may be written as

$e^{\alpha x}[C_1 \cosh x\sqrt{\beta} + C_2 \sinh x\sqrt{\beta}]$

or $C_1 e^{\alpha x}\cosh(x\sqrt{\beta} + C_2)$ or $C_1 e^{\alpha x}\sinh(x\sqrt{\beta} + C_2)$

Ex. 1. Solve $(D^4 + 5D^2 + 6)y = 0$

Solution. Auxiliary equation is $(D^4 + 5D^2 + 6)=0$

i.e., $(D^2 + 3)(D^2 + 2) = 0 \quad \therefore D = \pm\sqrt{3}i, \pm\sqrt{2}i$

Hence the complete solution is

$y=C_1\cos\sqrt{3}x + C_2\sin\sqrt{3}x + C_3\cos\sqrt{2}x + C_4\sin\sqrt{2}x.$

Ex. 2. Solve $(D^4 - D^3 - D + 1)x = 0$

Solution: Auxiliary equation is $(D^4 - D^3 - D + 1) = 0$

or, $(D^3 - 1)(D - 1) = 0$ or $(D - 1)^2 (D^2 + D + 1) = 0$

or $D=1,1,-\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$

Hence the complete solution is

$y=(C_1 + C_2 x)e^x + e^{-x/2}[C_3\cos\frac{\sqrt{3}}{2}x + C_4\sin\frac{\sqrt{3}}{2}x]$

Roll:1800065

Synopsis of the forms of solutions

To solve an equation of the from

$$(D^n) + a_1 D^{n-2} + a_2 D^{n-2} + \cdots + a_n)\mathrm{y} = 0$$

Find the roots of the auxiliary equation, viz.

$$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n = 4$$

2. Putting the General Solution as follows:

Roots of Auxilary Equation	**Complete Solution**
Case 1: All roots $m_1, m_2, m_3, \ldots$ m_n real and different.	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots \cdot C_n e^{m_n x} x^r$
Case 2: $m_1 = m_2$ but other roots real and different	$y = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_2 x} + \cdots + C_n e^{n_n x}$ Corresponding part of the general solution is
Case 3:(Image Roots) 1. $\alpha \pm i\beta$, a pair of imaginary roots.	or $e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$ or or $C_1 e^{\alpha x} \cos(\beta x - C_2)$ $C_1 e^{\alpha x} \sin(\beta x + C_2)$ Corresponding part of general solution is
2. $(\alpha \pm i\beta), (\alpha \pm i\beta)$ repeated twice.	$y = e^{\alpha} \cdot [(C_1 + C_2 x) \cos \beta x$ $+ (C_3 + C_1 x) \sin \beta x]$

Roll:1800066

Ex-1: Solve $\frac{d^4y}{dx^4} - a^4y=0$

Solution: The auxiliary equation is $(D^4 - a^4) = 0$

Or $(D^2+a^2)(D^2 - a^2) = 0$; $D = \pm a, \pm ai$.

$\therefore$ solution is $y = C_1 e^{ax} + C_2 e^{-ax} + (C_3 \cos ax + C_4 \sin ax)$

Ex-2: Solve $\frac{d^4y}{dx^4} + m^4y=0$

Solution: The auxiliary equation is $(D^4 + m^4) = 0$

Or $(D^2+a^2)^2 - 2m^2D^2 = 0$

Or $(D^2 - \sqrt{2mD} + m^2)(D^2 + \sqrt{2mD} + m^2) = 0$

When $(D^2 - \sqrt{2mD} + m^2) = 0$; $D = \frac{m \pm mi}{\sqrt{2}}$

When $(D^2 + \sqrt{2mD} + m^2) = 0$; $D = \frac{-m \pm mi}{\sqrt{2}}$

i.e. , roots of the auxiliary equation are $\frac{m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}}\, i$, $-\frac{m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}}\, i$.

Hence the solution is $y = e^{(m/\sqrt{2})x}\, C_1 \cos(\frac{m}{\sqrt{2}}x + C_2) + e^{(-m/\sqrt{2})x}\, C_3 \cos(\frac{m}{\sqrt{2}}x + C_4)$.

5.9: General solution of $(D^n + a_1 D^{n-1} + \ldots a_n)\, y = X$ ....(1)

To show that if $y = Y$ is a complete solution of $(D^n + a_1 D^{n-1} + \ldots + a_n)\, y = X$ ....(2)

And $y = u$ is a particular solution of (1), then $y = Y + u$ is a general solution of (1).

Since $y = Y$ is a solution of (2) , we have

$(D^n + a_1 D^{n-1} + \ldots + a_n)(Y) = 0$ ....(3)

Since $y = u$ is a solution of (1) , we have

$(D^n + a_1 D^{n-1} + \ldots + a_n)(u) = X$ ....(4)

Adding (3) and (4) , we have

$(D^n + a_1 D^{n-1} + \dots + a_n)(Y + u) = X$

This shows that y = Y + u is a solution of (1). Now Y being a general solution of (2) contains n arbitrary constants and as such Y + u also contains n arbitrary constants. Therefore y = Y + u is a general solution of (1)

Note 1:

In the general solution y = Y + u of the equation (1), Y is called the Complementary Function (C.F.) and u is called the Particular Integral (P. I.) , thus The General Solution = C.F. + P.I.

2:

The solution Y of (2) can be determined by the methods discussed above . The problem is now the particular integral u of (1). We give below certain methods of finding u.

Ex: Define the complementary function and particular integral for the linear differential equation with constant.

f(D) y = X.

Complementary Function: The solution which contains a number of arbitrary constants equal to the order of the differential equation is called the complementary function (C.F.) of a differential equation.

Particular Intrgular for the linear differential equation with constants f(D)y=X.

Assuming

$f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots\dots + a_n$

in $D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots\dots + a_n y = X$.

Differential equation will be

$f(D)y = X$

⇨ $y = \frac{1}{f(D)} X$

Above value of y will be P.L. of L.D.E.

Roll:1800067

Meaning of the symbol $\frac{1}{f(D)}$

Definition. The expression $\frac{1}{f(D)}X$ is defined to be that function of x, free from arbitrary constants which operated upon by f(D) gives X.

Thus, $f(D).\frac{1}{f(D)}X = X$.

For example, $\frac{1}{D^2+3D}(2 + 6x) = x^2$ $[\because (D^2 + 3D)x^2 = 2 + 6x]$

Therefore f(D) and $\frac{1}{f(D)}$ are inverse operators (i.e., they cancel each other's effect on the function on which they operate)

Thus, the symbol $\frac{1}{D}$ stands for integration.

Determination of the particular integral (P.I) of $f(D)$ y=X

Clearly $\frac{1}{f(D)}X$ will be the solution of (1) if it satisfies (1).

So, putting $\frac{1}{f(D)}$ X for y in (1), we get

$f(D).\frac{1}{f(D)}X = X$ i.e., X = X, which is true.

It means that $\frac{1}{f(D)}y$ is a particular solution of (1).

Therefore, to find the particular solution of f(D) y=X, we should find the value of $\frac{1}{f(D)}$ X.

Note: we know that in solving f(D) y= 0, f(D)= 0 forms the auxiliary equations, which can be resolve into linear factors (real or imaginary). The partial fractions will be of the form $\frac{1}{D-x}$ where x is real or imaginary.

General method of getting particular integral

Theorem. If X is a function of x, then $\frac{1}{D-\alpha}X = e^{\alpha x}\int Xe^{-\alpha x}\,dx.$

Proof. Let $y = \frac{1}{D-\alpha}X$

On operating by (D-α), we get (D-α) y = X

Or $\left(\frac{d}{dx}-\alpha\right)y = X$ or $\frac{dy}{dx}-\alpha y = X.$

which is a linear differential equation whose I.F. = $e^{-\int \alpha dx} = e^{-\alpha x}$ and hence its solution is given by $ye^{-\alpha x} = \int Xe^{-\alpha x}dx$, after omitting constant of integration, since P.I. is required.

$\therefore \quad y = e^{\alpha x}\int Xe^{-\alpha x}dx$

Thus, $\frac{1}{D-\alpha}X = e^{\alpha x}\int Xe^{-\alpha x}\,dx$ .... (1)

Similarly, $\frac{1}{D+\alpha}X = e^{\alpha x}\int Xe^{-\alpha x}\,dx$ .... (2)

Remark 1. Since we require only a particular integral, we shall never add a constant of integration after integration is performed in connection with any method of finding P.I. Hence P.I. will never contain any arbitrary constant.

Remark 2. The above method can be used to evaluate P.I. in any problem, since shorter methods depending upon the special form of function X are available (to be discussed later on, the above general method, however must be used for problems in which X is of the forms, sec αx, cosec αx, $\sec^2\alpha x$, $\text{cosec}^2\alpha x$, tan αx, cot αx or any other form not covered by shorter methods (employed for special forms).

Example. Solve $\frac{d^2y}{dx^2}+9y = \sec 3x$

Solution. Auxiliary equation is $m^2+9=0$ or $m=\pm 3i$,

$$\text{C.F.} = C_1\cos 3x + C_2\sin 3x$$

$$\text{P.I.} = \frac{1}{D^2+9}\sec 3x = \frac{1}{(D+3i)(D-3i)}\sec 3x = \frac{1}{6i}\left[\frac{1}{D-3i}-\frac{1}{D+3i}\right]\sec 3x$$

$$=\frac{1}{6i}\cdot\frac{1}{D-3i}\cdot\sec 3x - \frac{1}{6i}\cdot\frac{1}{D+3i}\cdot\sec 3x \qquad \text{.......... (1)}$$

Now, $\frac{1}{D-3i}\sec 3x = e^{3ix}\int e^{-3ix}\sec 3x\,dx$ $\left[\because \frac{1}{D-\alpha}X = e^{\alpha x}\int Xe^{-\alpha x}\,dx\right]$

$$=e^{3ix}\int\frac{\cos 3x - i\sin 3x}{\cos 3x}dx = e^{3ix}\int(1-i\tan 3x)dx = e^{3ix}\left(x+\frac{i}{3}\log\cos 3x\right)$$

Changing I to -I, we have $\frac{1}{D+3i}\sec 3x = e^{-3ix}\left(x-\frac{i}{3}\log\cos 3x\right)$

Putting these values in (1), we get

$$\text{P.I.} = \frac{1}{6i}[e^{3ix}(x + \frac{i}{3}\log\cos 3x) - e^{-3ix}(x - \frac{i}{3}\log\cos 3x)$$

$$= \frac{x}{6i}e^{3ix} + \frac{e^{3ix}\log\cos 3x}{18} - \frac{xe^{-3ix}}{6i} + \frac{e^{-3ix}}{18}\log\cos 3x$$

$$= \frac{x}{3}\frac{e^{3ix} - e^{-3ix}}{2i} + \frac{1}{9}.\frac{e^{3ix} + e^{-3ix}}{2}\log\cos 3x = \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x\,.\log\cos 3x$$

Hence, complete solution is $y = C_1\cos 3x + C_2\sin 3x + \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x\,.\log\cos 3x$

(Ans.)

Example. Solve $(D^2 + 4)y = \tan 2x$

Solution. Auxiliary equation is $m^2 + 4 = 0$ or $m = \pm 2i$

$$\text{C.F.} = C_1\cos 2x + C_2\sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}\tan 2x = \frac{1}{(D+2i)(D-2i)}\tan 2x = \frac{1}{4i}[\frac{1}{D-2i} - \frac{1}{D+2i}]\tan 2x$$

$$= \frac{1}{4i}[e^{2ix}\int e^{-2ix}\tan 2x\,dx - e^{-2ix}\int e^{2ix}\tan 2x\,dx] \qquad [\because \frac{1}{D-\alpha}X = e^{\alpha x}\int Xe^{-\alpha x}\,dx]$$

$$= \frac{1}{4i}[e^{2ix}\int(\cos 2x - i\sin 2x)\tan 2x\,dx - e^{-2ix}\int(\cos 2x + i\sin 2x)\tan 2x\,dx]$$

$$= \frac{1}{4i}[e^{2ix}\int\{\sin 2x - i(\sec 2x - \cos 2x)\}dx - e^{-2ix}\int\{\sin 2x + i(\sec 2x - \cos 2x)\}dx]$$

$$= \frac{1}{4i}[-\frac{\cos 2x}{2} - i\{\frac{\log(\sec 2x + \text{ta}2x)}{2} - \frac{\sin 2x}{2}\}](\cos 2x + i\sin 2x) - \frac{1}{4i}\left[-\frac{\cos 2x}{2} + i\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right](\cos 2x - i\sin 2x)$$

$$= \frac{1}{4i}\left[-\frac{\cos^2 2x}{2} + \frac{\sin 2x}{2}\log(\sec 2x + \tan 2x) - \frac{\sin^2 2x}{2} + \frac{\cos^2 2x}{2} - \frac{\sin 2x}{2}\log(\sec 2x + \tan 2x) + \frac{\sin^2 2x}{2} + i - \frac{\sin 2x\cos 2x}{2} - \frac{\cos 2x}{2}\log(\sec 2x + \tan 2x) + \frac{\cos 2x\sin 2x}{2} - \frac{\sin 2x\cos 2x}{2} - \frac{\cos 2x}{2}\log(\sec 2x + \tan 2x) + \frac{\sin 2x\cos 2x}{2}\right]$$

$$= -\frac{1}{4i}i\cos 2x\log(\sec 2x + \tan 2x) = -\frac{1}{4}\cos 2x\log(\sec 2x + \tan 2x)$$

Hence, complete solution is $y = C_1\cos 2x + C_2\sin 2x - \frac{1}{4}\cos 2x\log(\sec 2x + \tan 2x)$

(Ans.)

Roll:1800068

Corollary. If n is a positive integer, then $\frac{1}{(D-\alpha)^n}e^{\alpha x}=\frac{x^n}{n!}e^{\alpha x}$

Proof L.H.S.

$$=\frac{1}{(D-\alpha)^n}e^{\alpha x}=\frac{1}{(D-\alpha)^{n-1}}\frac{1}{(D-\alpha)}e^{\alpha x}=\frac{1}{(D-\alpha)^{n-1}}e^{\alpha x}\int e^{\alpha x}e^{-\alpha x}dx$$

[Using the theorem with X=$e^{\alpha x}$]

$$=\frac{1}{(D-\alpha)^{n-1}}e^{\alpha x}x=\frac{1}{(D-\alpha)^{n-2}}\frac{1}{(D-\alpha)}xe^{\alpha x}=\frac{1}{(D-\alpha)^{n-2}}e^{\alpha x}\int xe^{\alpha x}e^{-\alpha x}\,dx$$

[Using the theorem with X=$xe^{\alpha x}$]

$$\frac{1}{(D-\alpha)^{n-2}}e^{\alpha x}\int xdx=\frac{1}{(D-\alpha)^{n-2}}e^{\alpha x}.\frac{x^2}{2!} \qquad \cdots (i)$$

$$=\frac{1}{(D-\alpha)^{n-3}}\frac{1}{(D-\alpha)}e^{\alpha x}\frac{x^2}{2!}=\frac{1}{(D-\alpha)^{n-3}}e^{\alpha x}\int\{e^{\alpha x}.(\frac{x^2}{2!})e^{-\alpha x}\}dx$$

[Using the theorem with X=$e^{\alpha x}(\frac{x^2}{2!})$]

$$=\frac{1}{(D-\alpha)^{n-3}}\frac{e^{\alpha x}}{2!}\int x^2dx=\frac{1}{(D-\alpha)^{n-3}}\frac{x^3}{3!}e^{\alpha x} \qquad \cdots (ii)$$

Continuing as before and noting (i) and (ii), we finally obtain

$$\frac{1}{(D-\alpha)^n}e^{\alpha x}=\frac{1}{(D-\alpha)^{n-n}}\frac{x^n}{n!}e^{\alpha x}=\frac{x^n}{n!}e^{\alpha x} \cdots (iii)$$

Working rule of finding the particular integral (P.I.) i.e. $\frac{1}{f(D)}X$.

There are two following ways to obtain P.I.

Method 1. The operator $\frac{1}{f(D)}$ **maybe factored into linear factors**;

Then, $\quad P.I.=\frac{1}{D-\alpha_1}.\frac{1}{D-\alpha_2}.\frac{1}{D-\alpha_3}\cdots\frac{1}{D-\alpha_n}X$

On operating with the first symbolic factor, beginning αt the right, there is obtained

$$P.I.=\frac{1}{D-\alpha_1}.\frac{1}{D-\alpha_2}\cdots\frac{1}{D-\alpha_{n-1}}e^{\alpha_n x}\int Xe^{-\alpha_n x}dx;$$

Then, on operating with them 2nd and remaining factors in succession, taking them from right to left , required P.I. can be obtained.

Method 2. The operator $\frac{1}{f(D)}$ **may be decomposed into it partial fractions, then**

$$\text{P.I.} = \left[\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \frac{A_3}{D-\alpha_3} + \cdots + \frac{A_n}{D-\alpha_n}\right] X$$

$$= A_1 e^{\alpha_1 x}\int X e^{-\alpha_1 x}dx + \cdots + A_n e^{\alpha_n x}\int X e^{-\alpha_n x}dx.$$

Of this two methods, the latter is generally used in practice.

Solved problems based on method 1 & 2

1. Solve $(D^2 + \alpha^2)y = \cot \alpha x$.

Sol. Here the auxiliary equation is $D^2 + \alpha^2 = 0$ so that $D = 0 \pm i\alpha$.

$\therefore$ C. F. $= e^{\alpha x}(c_1 \cos \alpha x + c_2 \sin \alpha x) = c_1 \cos \alpha x + c_2 \sin \alpha x$, c_1, c_2 being arbitrary constants

Now, $\text{P.I.} = \dfrac{1}{D^2+\alpha^2}\cot \alpha x = \dfrac{1}{(D+\alpha i)(D-\alpha i)}\cot \alpha x$

$[\text{As } D^2 + \alpha^2 = D^2 - (\alpha i)^2 = (D+\alpha i)(D-\alpha i)]$

$\dfrac{1}{2i\alpha}\left[\dfrac{1}{(D-\alpha i)} - \dfrac{1}{(D+\alpha i)}\right]\cot \alpha x$, on resolving into partial fractions

Now, $\dfrac{1}{(D-\alpha i)}\cot \alpha x = e^{i\alpha x}\displaystyle\int e^{-i\alpha x}\cot \alpha x \, dx = e^{i\alpha x}\int (\cos \alpha x - i \sin \alpha x)\frac{\cos \alpha x}{\sin \alpha x}dx$

$[\text{by Euler's theorem, } e^{-i\alpha x} = (\cos \alpha x - i \sin \alpha x)]$

$= e^{i\alpha x}\displaystyle\int \left(\frac{\cos^2 \alpha x}{\sin \alpha x} - i\cos \alpha x\right)dx = e^{i\alpha x}\int \left(\frac{1-\sin^2 \alpha x}{\sin \alpha x} - i\cos \alpha x\right)dx$

$= e^{i\alpha x}\displaystyle\int (\operatorname{cosec} \alpha x - \sin \alpha x - i \sin \alpha x)\, dx = e^{i\alpha x}\left[\left(\frac{1}{\alpha}\right)\log\tan\left(\frac{\alpha x}{2}\right) + \left(\frac{1}{\alpha}\right)\cos(\alpha x) - \left(\frac{i}{\alpha}\right)\sin \alpha x\right]$

$= e^{i\alpha x}\left[\left(\dfrac{1}{\alpha}\right)\log\tan\left(\dfrac{\alpha x}{2}\right) + \left(\dfrac{1}{\alpha}\right)\{\cos(\alpha x) - i\sin \alpha x\}\right]$

$= e^{i\alpha x}\left[\left(\dfrac{1}{\alpha}\right)\log\tan\left(\dfrac{\alpha x}{2}\right) + \left(\dfrac{1}{\alpha}\right)e^{-i\alpha x}\right]$ by Euler's theorem

$\therefore \dfrac{1}{(D-\alpha i)}\cot \alpha x = \dfrac{1}{\alpha}\left[e^{i\alpha x}\log\tan\left(\dfrac{\alpha x}{2}\right) + 1\right]$ $\cdots (1)$

Replacing{ i by –i in (1), $\frac{1}{(D+\alpha i)}\cot \alpha x = \frac{1}{\alpha}\left[e^{-i\alpha x}\log\tan\left(\frac{\alpha x}{2}\right) + 1\right]$ $\cdots (2)$

Using 1and 2,

$\text{P.I.} = \dfrac{1}{2i\alpha}\left[\dfrac{1}{\alpha}\left\{e^{i\alpha x}\log\tan\left(\dfrac{\alpha x}{2}\right) + 1\right\} - \dfrac{1}{\alpha}\left\{e^{-i\alpha x}\log\tan\left(\dfrac{\alpha x}{2}\right) + 1\right\}\right]$

$= \frac{1}{\alpha^2}.\frac{e^{i\alpha x}-e^{-i\alpha x}}{2i}\log\tan\left(\frac{\alpha x}{2}\right) = \frac{1}{\alpha^2}.\sin \alpha x \log\tan\left(\frac{\alpha x}{2}\right)$

Hence the required general solution is y= C.F.+P.I.

$y = (c_1 \cos \alpha x + c_2 \sin \alpha x) + \frac{1}{\alpha^2}.\sin \alpha x \log\tan\left(\frac{\alpha x}{2}\right)$, where c_1 and c_2 are arbitray constant

(Ans)

2. Solve $\frac{d^2y}{dx^2} + y = \sec^2 x$.

Let, $D = \frac{d}{dx}$, Then the given equation is $(D^2 + 1)y = \sec^2 x$.

Its auxiliary equation is $D^2 + 1 = 0$ so that $= \pm i$ and $\text{C.F.} = c_1 \cos x + c_2 \sin x$

and $\text{P.I.} = \frac{1}{D^2 + 1} \sec^2 x = \frac{1}{(D+i)(D-i)} \sec^2 x = \frac{1}{2i}\left[\frac{1}{(D-i)} - \frac{1}{(D+i)}\right] \sec^2 x \quad \cdots (1)$

Now, $\frac{1}{(D-i)} \sec^2 x = e^{ix} \int e^{-ix} \sec^2 x dx = \int \left(\frac{\cos x - i \sin x}{\cos^2 x}\right) dx$

$= e^{ix} \int (\sec x - i\sec x \tan x)\, dx = e^{ix}[\log(\sec x + \tan x) - i\sec x] \quad \cdots (2)$

Replacing i by $-$ i in (2), $\frac{1}{(D+i)} \sec^2 x = e^{-ix}[\log(\sec x + \tan x) + i\sec x] \quad \cdots (3)$

From 2 αnd 3, we hαve

$$\text{P.I.} = \frac{1}{2i}\left[e^{ix} \log(\sec x + \tan x) - e^{ix} i\sec x - e^{-ix} \log(\sec x + \tan x) - e^{-ix} i\sec x\right]$$
$$= \frac{e^{ix} - e^{-ix}}{2i} \log(\sec x + \tan x) - \frac{e^{ix} - e^{-ix}}{2} \sec x$$
$$= \sin x \log(\sec x + \tan x) - \cos x \sec x \ = \sin x \log(\sec x + \tan x) - 1$$

Hence the required solution is $y = c_1 \cos x + c_2 \sin x + \sin x \log(\sec x + \tan x) - 1$

(Ans)

Roll:1800069

Particular Integral

When X is of the form e^{ax}, where a is any constant

By successive differentiation, we find that

$e^{ax} = e^{ax}$ .....(1)

$De^{ax} = ae^{ax}$ ......(2)

$D^2e^{ax} = a^2e^{ax}$ ......(3)

………………………….. ……..

$D^n ae^{ax} = a^n e^{ax}$ .....(n)

If $f(D) = (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n)$, then multiplying (1),(2),(3).....(n) by a_n, a_{n-1}.....,1 respectively and adding,

We obtain

$f(D)\, e^{ax} = f(a)\, e^{ax}$

Now operating on both sides by $\frac{1}{f(D)}$

$\frac{1}{f(D)} f(D)\, e^{ax} = \frac{1}{f(D)} f(a)\, e^{ax}$

$e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$

or $\frac{1}{f(a)} e^{ax} = \frac{1}{f(D)} e^{ax}$ [dividing by $f(a) \neq 0$]

Therefore,

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad \text{,provided that } f(a) \neq 0.$$

Example 1 , Solve $\frac{d^2y}{dx^2} + 2k\frac{dy}{dx} + k^2 y = 0$.

Solution :

Auxiliary equation is $D^2 - 2kD + k^2 = 0$,

$(D-k)^2 = 0$ or $D = k,k$.

$\therefore$ C F . $= (c_1 + c_2 x)\, e^{kx}$

P.I $= \frac{1}{D^2 - 2kD + k^2} e^x = \frac{1}{1 - 2k + k^2} e^x = \frac{1}{(1-k)^2} e^x$

Hence the general solution is

$y = (c_1 + c_2 x)\, e^{kx} + \frac{1}{(1-k)^2} e^x,\ k \neq 1.$

Example 2 , Solve $(D^2 + 3D + 5)\, y = e^{2x}$

Solution:

Given equation is ,$(D^2 + 3D + 5)\, y = e^{2x}$

A.E. is $D^2 + 3D + 5 = 0$

$\therefore \quad D = \frac{-3\pm\sqrt{9-20}}{2} = \frac{-3}{2} \pm \frac{\sqrt{11}}{3} i$

C. F. $= e^{-\frac{3}{2}x}(C_1 \cos\frac{\sqrt{11}}{2}x + C_2 \sin\frac{\sqrt{11}}{2}x)$

P.I. $= \frac{1}{D^2+3D+5} e^{2x}$

Putting D = 2 in f(D)

$= \frac{1}{4+6+5} e^{2x}$

$= \frac{1}{15} e^{2x}$

Hence the general solution is

$y = CF + PI$

$y = e^{-\frac{3}{2}x}(C_1 \cos\frac{\sqrt{11}}{2}x + C_2 \sin\frac{\sqrt{11}}{2}x) + \frac{1}{15} e^{2x}.$

Example 3 Solve $\frac{d^2y}{dx^2}+6\frac{dy}{dx} + 9y = 5e^{3x}$.

Solution:

Given equation is , $(D^2+6D+9)\, y = 5e^{3x}$

A.E. $= (D^2+6D+9) = 0$

$\therefore (D + 3)^2 = 0$

$\therefore D = -3, -3$ are the roots.

$\therefore$ C.F. $= (C_1+C_2x)\, e^{3x}$

And P.I. $= \frac{1}{(D+3)^2} 5e^{-3x}$

$= 5e^{3x}\{\frac{1}{(3+3)^2}\}$ ....(1)

$= \frac{5e^{3x}}{36}$

Hence the general solution is

$$y = CF + PI$$

$$y = (C_1+C_2x)\, e^{-3x} + \frac{5e^{3x}}{36}.$$

Example 4 , Solve $(D^2+4D+3)y = e^{-3x}$.

Solution :

Given equation is

$$(D^2+4D+3)y = e^{-3x}$$

A.E. is $(D^2+4D+3) = 0$

Or, $(D+1)(D+3) = 0 \quad \therefore D = -1, -3$

C.F. = $(C_{1}e^{-x} + C_2 e^{-3x})$

P.I. $= \frac{1}{(D+1)(D+3)} e^{-3x}$

Putting D = -3 in f(D) then

P.I. = $\frac{1}{0} e^{-3x}$ [method fails]

P.I. = $\frac{e^{-3x}}{(-3+1)(D-3+3)}\{1\} = \frac{e^{-3x}}{(-2)}\frac{1}{D}\{1\} = \frac{e^{-3x}}{-2}x$

Hence the general solution is

$$y = CF + PI$$

$$y = (C_{1}e^{-x} + C_2 e^{-3x}) - \frac{e^{-3x}}{2}x.$$

Roll:1800070

To show that $1/f(D^2)\text{Sin } ax = 1/f(-a^2)\text{Sin } ax$ except when $f(-a^2)=0$

By succesive differentiation,we get

$\sin ax = \sin ax$, -----(1)

$D\sin ax = a\cos ax$,

$D^2\sin ax = -a^2\sin ax$, --------(2)

$D^3\sin ax = -a^3\cos ax$,

$D^4\sin ax = a^4\sin ax$

Or$(D^2)\sin ax = (-a^2)^2\sin ax$, ---------(3)

Similarly $(D^2)^n\sin ax = (-a^2)^n\sin ax$.

Thus $f(D^2)\sin ax = f(-a^2)\sin ax$.

Operating by $1/f(D^2)$ on both sides,we get

$1/f(D^2)\ f(D^2)\ \sin ax = 1/f(D^2)\ f(-a^2)\sin ax$

i.e.. $\sin ax = f(-a2).1/f(D^2)\sin ax$.

Dividing by $f(-a^2)$,we get

$1/f(D^2)\sin ax = 1/f(-a^2)\sin ax$, if $f(-a^2)=0$.

Similarly $1/f(D^2)\cos ax = 1/f(-a^2)\cos ax$.

Important. It follows from the result above that we put---a^2 in place of D^2. We cannot put anything in place of D.

Thus for D^2. We cannot put anything in place of D.

Thus for D^2 put $-a^2$.

For $D^3 = D^2.D$ put $-a^2D$.

For $D^4 = D^2.\ D^2$ put $-a^2(-a^2)$,i.e..,a^4 etc.

Thus ultimately f(D) becomes linear in D say of the form (D+x). Then we proced as follows :

$1/(D+a)\ \sin ax = (D-a)/(D+a)(D-a) – \sin ax$

$=(D-a)/D^2-a^2\ \sin ax = D-a/(-a^2-a^2)\sin ax$

Putting $-a^2$ for D^2 in the denominator

$=1/-a^2-a^2(d/dx \sin ax - a\sin ax)$ as $D=d/dx$

$=1/-a^2-a^2(a\cos ax - a\sin ax)$.

And thus the particular integral in case of sin ax and cos ax can be completely evaluated.

Roll:1800071

Ex.1: Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\sin 2x$.

Solution: The auxiliary equation is,

$$D^2+D+1=0$$

Now solving this equation we get,

$$D = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\, i$$

$\therefore$ Complementary Function $= e^{-(\frac{1}{2})} \times C_1 \cos\{\frac{1}{2}\sqrt{3}x + C_2\}$

Particular Integral $= \frac{1}{D^2+D+1} \sin 2x$

$$= \frac{1}{-4+D+1} \sin 2x$$

$$= \frac{1}{D-3} \sin 2x$$

$$= \frac{D+3}{D^2-9} \sin 2x$$

$$= -\frac{1}{13}(2\cos 2x + 3\ \sin 3x)$$

Hence the complete solution is,

$$y = e^{-(\frac{1}{2})} \times C_1 \cos\{\frac{1}{2}\sqrt{3}x + C_2\} - \frac{1}{13}(2\cos 2x + 3\ \sin 3x)\ .(\text{Ans})$$

Ex.2: Solve $(D^2+1)^2y=\cos 3x$.

Solution: The auxiliary equation is,

$$(D^2+1)^2=0,$$

$$D= \pm i, \pm i$$

$\therefore$ Complementary Function = $(C_1+C_2x)\cos x + (C_3+C_4x)\sin x$

Particular Integral $= \frac{\cos 3x}{(D^2+1)^2}$

$= \frac{\cos 3x}{(-9+1)^2} = \frac{1}{64}\cos 3x$

Hence the complete solution is,

$y = (C_1+C_2x)\cos x + (C_3+C_4x)\sin x + \frac{1}{64}\cos 3x$.(Ans)

Ex.3: Solve $(D^3+D^2+D+1)y = \sin 2x$

Solution : The auxiliary equation is,

$(D^2+1)(D+1)=0$,

$D= -1, \pm i$

$\therefore$ Complementary Function $=C_1e^{-x} + C_2\cos(x+C_3)$

Particular Integral $= \frac{1}{(D^2+1)(D+1)}\sin 2x$

$= \frac{1}{(-4+1)(D+1)}\sin 2x$

$= -\frac{1}{3}\frac{D-1}{D^2-1}\sin 2x$

$= -\frac{1}{3}\frac{D-1}{-4-1}\sin 2x$

$= \frac{1}{15}[2\cos 2x - \sin 2x]$

Hence the complete solution is,

$y = C_1e^{-x} + C_2\cos(x+C_3) + \frac{1}{15}[2\cos 2x - \sin 2x]$.(Ans)

Ex.4: Prove that the solution of the differential equation $\frac{d^2y}{dx^2} + 4y = \sin ax$ when $a \neq 2$,under the conditions $y=0$ and $\frac{dy}{dx} = 0$ when $x=0$ is $y= \frac{2\sin ax - a\sin 2x}{2(4-a^2)}$.

Solution : The auxiliary equation is,

$D^2+4 = 0$,

$D = \pm 2i$

$\therefore$ Complementary Function = $C_1 \sin(2x+C_2)$

Particular Integral $= \frac{1}{D^2+4} \sin ax = \frac{\sin ax}{4-a^2}$

$\therefore$The general solution is,

$y = C_1 \sin(2x+C_2) + \frac{\sin ax}{4-a^2}$ ………..(1)

so that $\frac{dy}{dx} = 2C_1 \cos(2x+C_3) + \frac{a\cos ax}{4-a^2}$ ………..(2)

But $y = 0$ when $x = 0$,

$\therefore$ (1) gives, $0 = C_1 \sin C_2$ ………..(3)

Again $\frac{dy}{dx} = 0$ when $x = 0$

$\therefore$ (2) gives, $0 = 2C_1 \cos C_2 + \frac{a}{4-a^2}$ ………..(4)

From (3), $C_1=0$ or $C_2=0$ but if $C_1=0$,(4) does not hold.

Hence $C_2=0$ and then from (4), $C_1 = -\frac{a}{2(4-a^2)}$

Putting these values of C_1 and C_2 in (1),the required solution is

$y = -\frac{a\sin 2x}{2(4-a^2)} + \frac{\sin ax}{4-a^2} = \frac{2\sin ax - a\sin 2x}{2(4-a^2)}$

This proves the result.

Roll:1800072

Exceptional case of $\frac{1}{f(D)} e^{ax}$ when f(a)=0

We have from, $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$ if $f(a) \neq 0$

But if $f(a) = 0$, this becomes infinite and our method fails.

Now, $f(a) = 0$ means that $(D-a)$ is a factor of $f(D)$.

Therefore let $f(D) = (D-a)\,\varphi(D)$

Such that $\varphi(a) \neq 0$ ……….. (1)

So, $\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)\ \varphi(D)} e^{ax}$

$= \frac{1}{D-a} \cdot \frac{1}{\varphi(a)} e^{ax}$ as $\varphi(a) \neq 0$

$= \frac{1}{\varphi(a)} \cdot \frac{1}{D-a} e^{ax}$

$= \frac{1}{\varphi(a)} \cdot e^{ax} \int e^{-ax} e^{ax}\, dx$ $\quad [\frac{1}{D-a} \cdot X = e^{ax} \cdot \frac{1}{D}(e^{-ax}.x)]$

$= \frac{1}{\varphi(a)} e^{ax} \int dx = \frac{xe^{ax}}{\varphi(a)}$ …………..(2)

Now differentiating both sides of (1) with respect to D ,

f'(D)= (D-a) φ' (D) + φ (D)

putting D=a , f'(a) = 0 + φ(a)

It means φ(a)= f'(a)

Hence (2) becomes $\frac{1}{f(D)} e^{ax} = \frac{xe^{ax}}{f'(a)}$ or $x. \frac{1}{f'(D)} e^{ax}$

Again if f'(a) = 0 and f''(a) ≠ 0 then D-a is a factor repeated twice and applying the above Result once again

we get,

$\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(D)} e^{ax}$ and so on.

Roll:1800073

Exceptional case of $\frac{1}{f(D^2)}$ sin(ax) when f(-a²)=0.

From 5.5, $\frac{1}{f(D^2)} \sin(ax) = \frac{1}{f(-a^2)} \sin(ax)\, , f(-a^2) \neq 0$

But if $f(-a^2)=0$ it becomes infinite and our method fails.

Now $f(-a^2)=0$ means that D^2+a^2 is a factor of $f(D^2)$

Let $f(D^2)=(D^2+a^2)\varphi(D^2)$ such that $\varphi(D^2) \neq 0$.

Now $\frac{1}{f(D^2)}(\cos ax + i\sin ax) = \frac{1}{f(D^2)} e^{ax}$

$= x\frac{1}{f\prime(D^2)} e^{ax}$

$= x\,\frac{1}{f'(D^2)}(\cos ax + i\sin ax)$

Equating real and imaginary part we have,

$\frac{1}{f(D^2)}\cos ax = x\,\frac{1}{f\prime(D^2)}\cos ax$

And $\frac{1}{f(D^2)}\sin ax = x\frac{1}{f\prime(D^2)}\sin ax.$

In case f'($-a^2$)=0 and f"($-a^2$)$\neq$ 0, D^2+a^2 is a twice repeated factor of f(D^2). Applying the above result once again, we get

$$\frac{1}{f(D^2)}\sin ax = x^2\,\frac{1}{f''(x^2)}\sin ax$$

$$\text{And}\ \frac{1}{f(D^2)}\cos ax = a^2\,\frac{1}{f''(D^2)}\cos ax.$$

Ex 1: solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$.

Solution: Auxilliary equation is

$D^2-3D+2=0$, i.e $(D-2)(D-1)=0$

So C.F $C_1e^x + C_2e^{2x}$.

P.I=$\frac{e^x}{D^2-3D+2}$ (case of failure)

=x$\frac{e^x}{2D-3}$ multiplying by x and differentiating the deno. w. r. t D.

=x$\frac{e^x}{2.1-3}$ $- xe^x$.

Hence the complete solution is y=$C_1e^x + C_2e^{2x} - xe^x$.

Roll:1800074

Examples of exceptional case of $\frac{1}{f(D^2)}$sin ax when f($-a^2$)=0

$\frac{1}{f(D^2)}$ sin ax= $\frac{1}{f(-a^2)}$sin ax , $f(-a^2) \neq 0$

But if , $f(-a^2)$=0,It becomes infinite and this method fails .

<u>EXAMPLE 2</u>: Solve $(D^2 + 4D + 3)y = e^{-3x}$

<u>Solution:</u> Auxiliary equation is

$(D^2 + 4D + 3)$=0

or,(D+3)(D+1)=0

$C.F. = C_1e^{-x} + C_2e^{-3x}$

$P.I. = \frac{e^{-3x}}{D^2+4D+3}$ [case of failure]

$= x\frac{e^{-3x}}{2D+4}$ [multiplying by x and differentiating the denominator w.r.t D]

$=x\frac{e^{-3x}}{2(-3)+4}$

$=-\frac{1}{2}xe^{-3x}$

Hence the general solution is

$Y=C_2e^{-x} + C_2e^{-3x} - \frac{1}{2}xe^{-3x}$ (Ans)

<u>EXAMPLE 3: </u>Solve $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$

<u>Solution</u>: Auxiliary equation is

$D^3 + 3D^2 + 3D + y = 0$

or,$(D+1)^3 = 0$

0r,D=-1,-1,-1

C.F.=$(C_1 + C_2x + C_3x^2)e^{-x}$

P.I.=$\frac{e^{-x}}{(D+1)^3}$ (case of failure)

=$x\frac{e^{-x}}{3(D+1)^2}$ [multiplying by x and differentiating the denominator w.r.t D

(this is again a case of failure)]

$=x^2\frac{e^{-x}}{6(D+1)}$ [multiplying by x and differentiating the denominator w.r.t D

(this is again a case of failure)]

$=x^3\frac{e^{-x}}{6}$ [multiplying by x and differentiating the denominator w.r.t D]

Hence the complete solution is

$Y=(C_1 + C_2x + C_3x^2)e^{-x} + x^3\frac{e^{-x}}{6}$

(ANS)

<u>EXAMPLE 4:</u> Solve $\mathbf{2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + y = e^x + 1}$

<u>Solution</u>: Auxiliary solution is

$2D^3-3D^2+1=0$

or, $(D-1)(D-1)(2D+1)=0$

or, $D=1,1,-1$

C.F. $=(C_1+C_2x)e^2+C_3e^{-x/2}$

P.I. $=\frac{e^x}{2D^3-3D^2+1} + \frac{1}{2D^3-3D^2+1}$ [first term case of failure]

$=x\frac{e^x}{6D^2-6D} + \frac{e^x}{0-3.0+1}$ [differentiating the denominator of the first and multiplying by x (again case of failure)]

$=x^2\frac{e^x}{12D-6} + 1$ [again differentiating the denominator and multiplying by x]

$=\frac{1}{6}x^2e^x+1$

Hence the complete solution is

$Y=(C_1+C_2x)e^2+C_3e^{-x/2}+\frac{1}{6}x^2e^x+1$ (ANS)

<u>EXAMPLE 5:</u> $\mathbf{(D^3-2D^2-5D+6)y=e^{3x}}$

<u>Solution:</u> Auxiliary equation is

$(D-3)(D^2+D-2)=0$

Or, $(D-3)(D+2)(D-1)=0$

Or, $D=3,-2,1$

C.F.$=C_1e^x+C_2e^{-2x}+C_3e^{3x}$

P.I.$=\frac{1}{(D-3)(D+2)(D-1)}\,e^{3x}$

$=\frac{1}{(D-3)(3+2)(3-1)}e^{3x}$

$=\frac{1}{10(D-3)}\,e^{3x}$

$=-\frac{1}{10}\,e^{3x}x$ as it a case of failure.

The complete solution is

Y=C.F.+P.I.= $C_1e^x+C_2e^{-2x}+C_3e^{3x}-\frac{1}{10}\,e^{3x}x$ (ANS)

<u>EXAMPLE 6:</u> (a) Solve $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=e^{2x}+e^{-2x}$

<u>Solution:</u> Auxiliary equation is

$(D^2+4D+4)=0$

Or, $(D+2)^2=0$

C.F.$=(C_1+C_2x)e^{-2x}$

P.I.$=\frac{e^{2x}}{(D+2)^2}+\frac{e^{-2x}}{(D+2)^2}$ second is a case of failure

$=\frac{e^{2x}}{(2+2)^2}+\frac{e^{-2x}}{2}\,x^2$ [differentiating denominator of the second twice w.r.t D and multiplying by x^2]

$=\frac{e^{2x}}{16}+\frac{1}{2}x^2e^{-2x}$

Hence the complete solution is

Y=C.F.+P.I.= $(C_1+C_2x)e^{-2x}+\frac{e^{2x}}{16}+\frac{1}{2}x^2e^{-2x}$ (ANS)

<u>EXAMPLE 6:</u> (b)Solve $\frac{d^2x}{dt^2}=x+e^t+e^{-t}$

<u>Solution:</u> Auxiliary equation is

$D^2-1=0$

Or, $D=\pm 1$

C.F.=$C_1e^{-t}+C_2e^{t}$

P.I.=$\frac{e^t+e^{-t}}{(D^2-1)}$

=$\frac{e^t}{(D^2-1)}+\frac{e^{-t}}{(D^2-1)}$ Exceptional Case

=$t\frac{e^t}{2D}$ - $t\frac{e^{-t}}{2D}$

=$\frac{1}{2}t(e^t-e^{-t})$

The general solution is

X= $C_1e^{-t}+C_2e^{t}+\frac{1}{2}t(e^t-e^{-t})$ (ANS)

Roll:1800075

7.Solve: $(\mathbf{D^2+a^2})\mathbf{y}=\mathbf{\sin ax}$

Solution: A.E. is $D^2+a^2=0$, $D=\pm ai$

$\therefore$ C.F.=$C_1\cos(ax+C_2)$

P.I. = $\frac{\sin ax}{D^2+a^2}$ case of failure

= $x\frac{\sin ax}{2D}$ multiplying by x and differentiating the denominator w.r.t. D

=$-\frac{x}{2a}\cos ax$

Hence y=$C_1\cos(ax+C_2)-\frac{x}{2a}\cos ax$ is the complete solution.

8.Solve: $\mathbf{\frac{d^6y}{dx^6}+y=\sin\frac{3}{2}x\ \sin\frac{1}{2}x}$

Solution: A.E. is $D^6+1=0$

Or $(D^2+1)\ (D^4-D^2+1)=0$

Or $(D^2+1)\ [(D^2+1)^2-3D^2]=0$

Or $(D^2+1)\left(D^2-\sqrt{3D}+1\right)\left(D^2+\sqrt{3D}+1\right)=0$

When $D^2+1=0 \quad \therefore D=\pm i$

When $D^2-\sqrt{3D}+1=0 \quad \therefore D=\frac{\sqrt{3}\pm i}{2}$

When $D^2+\sqrt{3D}+1=0 \quad \therefore D=\frac{\sqrt{3}\pm i}{2}$

Hence C.F.=$C_1\cos(x+C_2)+C_3e^{\frac{1}{2}\sqrt{3}x}\cos\left(\frac{1}{2}x+C_4\right)+C_5e^{-\frac{1}{2}\sqrt{3}x}\cos(\frac{1}{2}x+C_6)$

Now, $\sin\frac{3}{2}x\ \sin\frac{1}{2}x=\frac{1}{2}(\cos x-\cos 2x)$

P.I. = $\frac{1}{2}.\frac{\cos x}{D^6+1}-\frac{1}{2}.\frac{\cos 2x}{D^6+1}$ (first term case of failure)

$=\frac{1}{2}x\ \frac{\cos x}{6D^5}-\frac{1}{2}.\frac{\cos 2x}{(-4)^3+1}$

$=\frac{1}{2}x.\frac{\cos x}{6(-1)^2D}+\frac{1}{126}\cos 2x$

$=\frac{1}{12}x\sin x+\frac{1}{126}\cos 2x$ as $\frac{1}{D}$ means integration

Hence the complete solution is y=C.F.+ P.I.

9.Solve: $\mathbf{\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=e^x+\cos x}$

Solution: A.E. is $D^3-3D^2+4D-2=0$

Or $(D-1)\ (D^2-2D+2)=0$

Or $(D-1)\ [(D-1)^2+1]=0$

Or $((D=1)1\pm i$

C.F. =$C_1e^x+e^x(C_2\cos x+C_3\sin x)$

P.I. = $\frac{1}{D^3-3D^2+4D-2}\ e^x+\frac{1}{D^3-3D^2+4D-2}\cos x$ first term in case of failure

$=x\frac{1}{3D^2-6D+4}e^x+\ \frac{1}{(-1)D-3(-1)+4D-2}\cos x$

$=x\ e^x+\frac{1}{3D+1}\cos x=xe^x+\frac{3D-1}{9D^2-1}\cos x$

$=xe^x+\frac{1}{10}(3\sin x+\cos x)$

Hence the complete solution is y=C.F.+ P.I.

10.Solve: $\mathbf{(D^3-5D^2+7D-2)y=e^{2x}\cos hx}$

Solution: A.E. is $D^3-5D^2+7D-3=0$

Or $(D-1)(D^2-4D+3)=0$

Or $(D-1)(D-3)(D-1)=0$

$\therefore$ C.F. = $(C_1 + C_2x)e^x + C_3e^{3x}$

P.I. = $\frac{e^{2x}\cos hx}{(D-1)^2(D-3)} = \frac{e^{2x}.\frac{1}{2}(e^x+e^{-x})}{(D-1)^2(D-3)}$

$= \frac{1}{2}.\frac{e^{3x}}{(D^3-5D^2+7D-3)} + \frac{1}{2}.\frac{e^x}{D^3-5D^2+7D-3}$

$= \frac{1}{2}\text{ x }\frac{e^{3x}}{3D^2-10D+7} + \frac{1}{2}\text{ x }\frac{e^x}{3D^2-10D+7}$

$= \frac{1}{2}\text{ x }\frac{e^{3x}}{3.3^x-10.3+7} + \frac{1}{2}\text{ x}^2\ \frac{e^x}{6D-10}$

$= \frac{1}{8}\text{ x } e^x - \frac{1}{8}\text{ x}^2e^x$

Hence the complete solution is y=C.F.+ P.I.

11.Solve: $[D^4 + (m^2 + n^2)D^2 + m^2 n^2]y = \cos\frac{1}{2}(m+n)x\cos\frac{1}{2}(m-n)x$

Solution: A.E. is $(D^2 + m^2)(D^2 + n^2) = 0$

$\therefore D = \pm mi\ , \pm ni$

C.F. = $C_1\cos(mx + C_2) + C_3\cos(nx + C_4)$

P.I. = $\frac{\cos\frac{1}{2}(m+n)x\ \cos\frac{1}{2}(m-n)x}{D^4+(m^2+n^2)D^2+m^2n^2}$

$= \frac{1}{2}\ \frac{\cos mx+\cos nx}{D^4+(m^2+n^2)D^2+m^2n^2}$ cases of failure

$= \frac{1}{2}\text{x}\frac{\cos mx+\cos nx}{4D^3+2(m^2+n^2)D}$

$= \frac{1}{2}\text{ x }\frac{1}{2D}\ [\frac{\cos mx}{-2m^2+(m^2+n^2)} + \frac{\cos nx}{+2n^2+(m^2-n^2)}]$

$= \frac{x}{4(m^2-n^2)}[-\frac{\cos mx}{D} + \frac{\cos nx}{D}]$

$= \frac{x}{4(m^2-n^2)}\ [-\frac{1}{m}\sin mx + \frac{1}{n}\sin nx]$

Hence the complete solution is y=C.F.+ P.I.

Roll:1800076

$1/f(D)\ X^m$, where m is a positive integer.

Consider $1/(D-a)\ x^m$

$= -1/a(1-D/a)\ x^m$

$= -1/a(1-D/a)^{-1}X^m$

$=1/a(1+D/a+D^2/a^2+\text{--------}+D^m/a^m+\text{---})X^m$

$=-1/a\{(x^m+mx^{m-1}/a)+m(m-1)x^{m-2}/a^2+\text{----}+m!/a^m\}$

Therefore to evaluate $1/f(D)x^m$ expand $[f(D)]^{-1}$ in ascending powers of D, retaining terms as far D^m and operate each term on x^m.

We need not retain terms containing D^{m+1},D^{m+2} etc. as

$D^{m+1}x=0$,$D^{m+2}x^m=0$ etc.

Ex. Solve $(D^3+2D^2+D)y=e^{2x}+x^2+x$.

Solution. A.E is $D(D+1)^2=0$,i.e, D=0,-1,-1.

$C.F=C_1+(C_2+C_3x)e^{-x}$

$P.I=e^{2x}/D(D+1)^2 + 1/D(1+D)^2\ (x^2+\times)$

$=e^{2x}/2(2+1)^2 + 1/D(1+D)^{-2}(x^2+x)$

$=e^{2x}/18 + 1/D[1-2D+3D^2—](x^2+x)$

$=e^{2x}/18 +1/D[x^2+x-4x-2-6]$

$=e^{2x}/18 +x^3/3-3x^2/2+4x.$

The complete solution is y=C.F.+P.I.

Roll:1800077

To show that $\frac{1}{f(D)}(e^{ax}V)$ - $e^{ax}\frac{1}{f(D+a)}V$.

Where V is function of x.

We have on successive differentiation (by parts),

$D(e^{ax}V) = e^{ax}DV + ae^{ax} = e^{ax}(D+a)V$,

$D^2(e^{ax}V) = e^{ax}D^2V + ae^{ax}\ V + a^2e^{ax} + ae^{ax}DV$

$= e^{ax}(D^2 + 2aD + a^2)\ V\ = e^{ax}(D+a)^2\ V$

Similarly, $D^3(e^{ax}V) = e^{ax}(D+a)^3\ V$

And $D^n(e^{ax}V) = e^{ax}(D+a)^n\ V$

Therefore $f(D)(e^{ax}V) = e^{ax}\ f(D+a)\ V$.

Taking the inverse operators, we have

$$\frac{1}{f(D)}(e^{ax}\ V) = e^{ax}\frac{1}{f(D+a)}V.$$

Thus we find that operator $\frac{1}{f(D)}$ on $(e^{ax}\ V)$ is equivalent to $\frac{1}{f(D+a)}$ on V taking e^{ax} outside .

Therefore in practice take out e^{ax} and put (D+a) in place of D and then find $\frac{1}{f(D+a)}$ V as usual .

Ex. 1. Solve $\frac{d^2y}{dx^2} - 9y = 6e^{3x} + xe^{3x}$ **.**

Solution : Auxiliary equation is $D^2 - 9 = 0$, $D = \pm 3$.

$$C.F. = C_1e^{3x} + C_2e^{-3x}$$

$$P.I. = \frac{1}{D^2-9}e^{3x}(6+x) = \frac{1}{(D+3)^2-9}(6+x)$$

$$= e^{3x}\frac{1}{D^2+6D}(6+x) = e^{3x}\frac{1}{6D}(1+\frac{1}{6}D)^{-1}(6+x)$$

$$= e^{3x}\frac{1}{6D}(1-\frac{1}{6}D - \ldots\ldots)(6+x)$$

$$= e^{3x}\frac{1}{6D}(6 + x - \frac{1}{6}) = \frac{1}{36}e^{3x}(35x + 3x^2)$$

Hence the complete solution is

$$y = C_1e^{3x} + C_2e^{-3x} + \frac{1}{36} e^{3x}(35x + 3x^2).$$

Roll:1800078

Ex.2: Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = xe^x + e^x$ **.**

<u>Solution:</u> A.E. is $D^3-3D^2+3D-1=0$,

i.e. $(D-1)^3=0$ or $D = 1, 1, 1$.

$\therefore$ C.F. = $(C_1 + C_2x + C_3x^2)e^x$.

P.I. = $\frac{1}{(D-1)^3} e^x (x+1)$

$=e^x \frac{1}{(D+1-1)^3}(x+1)$

$=e^x \frac{1}{D^3}(x+1)$

$=e^x \frac{1}{D^2}\frac{(x+1)^2}{2}$

$=e^x.\frac{1}{D}\frac{(x+1)^3}{6}$

$=e^x.\frac{(x+1)^4}{24}$

Hence the general solution is y=C.F. + P.I.

Ex.3: Solve $(D^3 - 7D - 6)y = e^{2x}.x^2$

<u>Solution:</u> A.E. is $D^3 - 7D - 6 = 0$;

i.e. $(D+1)(D^2 - D - 6) = 0$ or $(D+1)(D-3)(D+2) = 0$

$\therefore$ C.F. $=C_1e^{-x} + C_2e^{3x} + C_3e^{-2x}$.

P.I. = $\frac{e^{2x}.x^2}{D^3-7D-6} = e^{2x} \frac{1}{(D+2)^3-7(D+2)-6} x^2$

$= e^{2x} \frac{1}{D^3+6D^2+5D-12} x^2$

$= -\frac{e^{2x}}{12} \left(1 - \frac{5}{12}D - \frac{1}{2}D^2 - \frac{1}{12}D^3\right)^{-1} x^2$

$= -\frac{e^{2x}}{12} \left(1 + \frac{5}{12}D + \frac{1}{2}D^2 + \frac{25}{12^2}D^2\right) x^2$

$= -\frac{e^{2x}}{12} \left(x^2 + \frac{5}{6}x + \frac{97}{72}\right)$ etc.

Ex. 4 : Solve $\frac{d^3 y}{dx^3}$ -2 $\frac{dy}{dx}$ +4y = e^x cos x.

Solution: Auxiliary equation is $D^3 - 2D + 4 = 0$,

i.e. $(D+2)(D^2 - 2D + 2) = 0$

or $(D+2)[(D-1)^2 + 1] = 0$

$D = -2, 1 \pm i$. C.F.= $C_1 e^{-2X} + C_2 e^x \cos(x + C_3)$.

P.I. = $\frac{1}{D^3-2D+4} e^x \cos x$

$= e^x \frac{1}{(D+1)^3-2(D+1)+4} \cos x$

$= e^x \frac{1}{D^3+3D^2+D+3} \cos x$

$= e^x . x \frac{1}{3D^2+6D+1} \cos x$

$= xe^x \frac{1}{-3+6D+1} \cos x$

$= xe^x \frac{1}{6D-2} \cos x$

$= \frac{1}{2} xe^x \frac{3D+1}{9D^2-1} \cos x$

$= -\frac{1}{20} xe^x (-3 \sin x + \cos x)$

$= \frac{1}{20} xe^x (3 \sin x - \cos x)$

Hence the complete solution is y= C.F. + P.I.

Ex. 5 : Solve $\frac{d^2 y}{dx^2}$ -2 $\frac{dy}{dx}$ +4y = e^x cos x.

Solution: A.E. is $D^2 - 2D + 4 = 0$,

Or, $[(D-1)^2 + 3]=0$ Or, $D= 1\pm\sqrt{3}i$

C.F.= $e^x\ [C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}\,x]$

P.I. = $\frac{1}{D^2-2D+4} e^x \cos x$

$= e^x \frac{1}{(D+1)^2-2(D+1)+4} \cos x$

$= e^x \frac{1}{D^2+3} \cos x$

$= e^x \frac{1}{-1+3} \cos x$

$= \frac{1}{2} e^x \cos x$,etc.

Roll:1800079

Topic: $\frac{1}{f(D)}\{xV\}$, ***where V is any function of x***

We have , $D^n(x)V = xD^nV + nD^{n-1}V$ by Leibnitz's rule

$$= xD^nV + \frac{d}{dD}(D^n)V \quad \text{as } \frac{d}{dD}D^n = nD^{n-1}$$

$$\therefore f(D)\{xV\} = xf(D)V + f'(D)V.$$

Taking the inverse operator , we get

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \left[\frac{d}{dD}\frac{1}{f(D)}\right]V.$$

Or, $\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$

Topic: $\frac{1}{f(D)}(x^mV)$, ***where V is some function of x.***

Now V can have the following different forms:

1)V has the form of x^m, then x^mV becomes x^{m+n} which can be evaluated

2)V has the form e^{ax}, then $x^m V$ becomes $x^m e^{ax}$ which can be evaluated .

3)V has the form cos ax or sin ax ,then $x^m V$ becomes $x^m \cos ax$ or $x^m \sin ax$,i.e. it is real or imaginary part of $x^m e^{ax}$,which can be easily evaluated.

Ex.1 .Solve $\frac{d^4y}{dx^4} - y = x\sin x$

Solution:

Auxiliary equation is

$$D^4 - 1 = 0, (D^2 + 1)(D^2 - 1) = 0, D = \pm i, \pm 1.$$

Hence C.F.=$C_1\cos x + C_2\sin x + C_3e^x + C_4e^{-x}$

P.I.= $\frac{1}{D^4-1}x\sin x$.

=Imaginary part of $\frac{1}{D^4-1}xe^{ix}$

=I.P. of $e^{ix}\frac{1}{(D+i)^4-1}x$

=I.P. of $e^{ix}\frac{1}{D^4+4iD^3-6D^2-4iD}x$

=I.P. of $-e^{ix}\frac{1}{4iD}[1 - \frac{3}{2}iD - D^2 + \frac{1}{4}iD^3]^{-1}x\ (\because \frac{1}{i} = -i)$

=I.P. of $-e^{ix}\frac{1}{4iD}[x + \frac{3}{2}i]$

=I.P. of $\frac{i}{4}(\cos x + i\sin x)[\frac{1}{2}x^2 + \frac{3}{2}ix]$

=$\frac{1}{8}x^2\cos x - \frac{3}{2}x\sin x$

Hence the complete solution is

y=$C_1\cos x + C_2\sin x + C_3e^x + C_4e^{-x} + \frac{1}{8}x^2\cos x - \frac{3}{8}x\sin x$.

Ex.2. Solve $\frac{d^2y}{dx^2} + 4y = x\sin x$

Solution:

A.E. is $D^2 + 4 = 0$, $D = \pm 2i$

C.F. =$C_1\cos 2x + C_2\sin 2x$

$$\text{P.I.} = \frac{1}{D^2+4} x \sin x$$

$$= \text{I.P. of } \frac{1}{D^2+4} x e^{ix}$$

$$= \text{I.P. of } e^{ix} \frac{1}{(D+i)^2+4} x$$

$$= \text{I.P. of } e^{ix} \frac{1}{D^2+2Di+3} x$$

$$= \text{I.P. of } \frac{1}{3} e^{ix}\left(1 + \frac{2}{3}Di + \frac{1}{3}D^2\right)^{-1} x$$

$$= \text{I.P. of } \frac{1}{3} e^{ix}\left(1 - \frac{2}{3}Di\right) x$$

$$= \text{I.P. of } \frac{1}{3}(\cos x + i \sin x)\left(x - \frac{2}{3} i\right)$$

$$= \frac{1}{9}(3x \sin x - 2\cos x).$$

The complete solution is y=C.F.+P.I.

Roll:1800080

Ex:-2 Solve $(d^2y/dx^2)+4y= x\sin x$

$C.F=C_1Cos2x+C_2Sin2x$

$P.I= 1/(D^2+4)x\sin x$

$=I.P \text{ of } 1/(D^2+4)\ xe^{ix}$

$= I.P \text{ of } (1/(D^2+i)^2+4\ )x$

$=I.P.\text{of } e^{ix}\ (1/(D^2+2Di+3))x$

$=I.P.\text{of } 1/3\ e^{ix}\,(1+2/3Di+3)^{-1}x$

$=I.P.\text{of } 1/3e^{ix}\,(1-2/3Di)x$

$=I.P.\text{of } 1/3\ \ (Cosx + i\sin x\ )(x- 2/3i)$

$=1/9\ (3x\sin x-2Cos\ x\ )$

The complete solution is y= C.F+P.I

EX:-3 (a) Solve $(d^2y/dx^2)- 4(dy/dx)+4y= 3x^2e^{2x} \sin 2x$

Solution :- A.E is $D^2-4D+4=0$, $(D-2)^2=0$

$C.F= (C_1+C_2x)\, e^{2x}$

$P.I =(3x^2e^{2x}\sin 2x)/(D-2)^2=3e^{2x}.$

$(1/(D+2-2)^2)\, X^2 \sin 2x$

$=3e^{2x}(1/D^2)\, x^2 \sin 2x$

$=I.P.of\ 3e^{2x}\, (1/D^2)\, x^2e^{2ix}$

$=I.P.of\ 3e^{2x}.e^{2ix}\, (1/D^2)x^2e^{2ix}$

$= I.P.of\ 3e^{2X}\ e^{2ix}(1/(D+2i)^2)\, x^2$

$= I.p.of\ 3e^{2x}e^{2ix}\, (1/D^2+4iD-4)x^2$

$= I.P.of\ -3e^{2x}\, e^{2ix}(1/4)\, (1-iD-1/4D^2)^{-1}\, x^2$

$= I.P\ of\ -3e^{2x}\, e^{2ix}.¼\, (1+iD+1/4D^2-D^2)\, X^2$

$= I.p.of\ -3e^{2x}e^{2ix}\, 1/4\, (X^2+2ix-3/2)$

$=I.P.of – ¾\, e^{2x}\, (\cos 2x+i \sin 2x)$

$(X^2+2ix-3/2)$

$= I.P.of – ¾\, e^{2x}(\cos 2x+\sin 2x)$

$(X^2+2ix-3/2)$

$= -3/8\, e^{2x}\, [(2x^2 -3) \sin 2x+\cos 2x]$

The complete solution is y=C.F+P.I

Ex:-4 Solve $(D^2+2D^2+1)y= x^2 \cos x$

Solution :- The auxiliary equation is

$(D^2+2D^2+1)=0$ i.e.$(D^2+1)^2=0$

D= +/-i ; +/-i.

:. $C.F.= (C_1+C_1x) \cos x+ (C_3+C_4X) \sin x$

$P.I.= 1/(D^4+2D^2+1)\, X^2 \cos x=$ real part

Of $1/(D^2+1)^2\, X^2\, e^{ix}$

But $[\, 1/(D^2+1)^2\,]\, x^2e^{ix}$

$= e^{ix} 1/[(D+1)^2+1]^2 x^2$

$=e^{ix} 1/(D^2+2iD)^2x^2$

$=e^{ix} 1/-4D^2(1- ½ iD)^{-2} x^2$

$= -1/4 \; e^{ix} (1/D^2) \; (1-1/2 iD)^{-2} x^2$

$= -1/4 e^{ix} 1/D^2 (1+iD-3/4 D^2 – ½ iD^3$

$+ 5/10 D^4 +\ldots\ldots)X^2$

$= -1/4 e^{ix} 1/D^2 [x^2+ 2ix -3/2]$

$= -1/4 e^{ix} [x^4/12 + i(x^3/3) -3/4 x^2]$

$= -1/4 (Cos x +i sinx) [x^4/12 -3/4 x^2 +$

$i x^3/3]$

P.I.= real part of the above

$= - 1/4 [(X^4/12 – ¾ \; X^2) Cos x -1/3 \; X^3 Sin x]$

The complete solution is y=C.F+P.I

Roll:1800081

Miscellaneous Examples

1.Solve: $(D^2 - 5D + 6)y = 4e^x + 5$

Solution: A.E. is $(D - 3)(D - 2) = 0$

$C.F.=C_1e^{2x} + C_2e^{3x}$

$P.I.=\frac{4e^x+5e^{0x}}{D^2-5D+6}=\frac{4e^x}{1-5+6}+\frac{5}{6}=2e^x+\frac{5}{6}$

Hence $y = C_1e^{2x} + C_2e^{3x} + 2e^x + \frac{5}{6}$.

2.Solve: $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = e^{2x}$

Solution: A.E. is $D^3 - 6D^2 + 11D - 6 = 0$

or $(D-1)(D^2 - 5D + 6) = 0$

or $(D-1)(D-2)(D-3) = 0$

$\therefore$ C.F.$=C_1e^x + C_2e^{2x} + C_3e^{3x}$

P.I.$= \frac{e^{2x}}{D^3-6D^2+11D-6}$ (case of failure)

$= x\frac{e^{2x}}{3D^2-12D+11}$

$= x\frac{e^{2x}}{12-24+11}$

$= -xe^{2x}$

Hence $y = C_1e^x + C_2e^{2x} + C_3e^{3x} - xe^{2x}$.

3.Solve: $\mathbf{\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = e^{-2x}}$

Solution: A.E. is $(D-1)(D^2 + D - 2) = 0$

or$(D-1)(D+2)(D-1) = 0$

$\therefore$ C.F.$=(C_1 + C_2x)e^x + C_3e^{-2x}$

P.I.$= \frac{e^{-2x}}{D^3-3D+2}$ (case of failure)

$=x\frac{e^{-2x}}{3D^2-3}$

$=\frac{1}{9}xe^{-2x}$

Hence $y = (C_1 + C_2x)e^x + C_3e^{-2x}\frac{1}{9}xe^{-2x}$.

4.Solve: $\mathbf{\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x} + \sin 2x}$

Solution: A.E. is $D^2 - 4D + 4 = 0$

or $(D-2)^2 = 0$

or $D = 2,2$

$\therefore$ C.F. $= (C_1 + C_2x)e^{2x}$

P.I. $= \frac{e^{2x}}{D^2-4D+4} + \frac{\sin 2x}{D^2-4D+4}$ first term, case of failure

$=x\frac{e^{2x}}{2D-4} + \frac{\sin 2x}{-4-4D+4}$ first term, again case of failure

$=x^2 \frac{e^{2x}}{2} - \frac{1}{4} \cdot \frac{1}{D} \sin 2x$

$= \frac{1}{2} x^2 e^{2x} + \frac{1}{8} \cos 2x$

Hence $y = (C_1 + C_2 x)e^{2x} + \frac{1}{2} x^2 e^{2x} + \frac{1}{8} \cos 2x.$

3.Solve: $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = e^{-2x}$

Solution: A.E. is $(D-1)(D^2 + D - 2) = 0$

or $(D-1)(D+2)(D-1) = 0$

$\therefore$ C.F.$=(C_1 + C_2 x)e^{x} + C_3 e^{-2x}$

P.I.$= \frac{e^{-2x}}{D^3 - 3D + 2}$ (case of failure)

$= x \frac{e^{-2x}}{3D^2 - 3}$

$= \frac{1}{9} x e^{-2x}$

Hence $y = (C_1 + C_2 x)e^{x} + C_3 e^{-2x} \frac{1}{9} x e^{-2x}.$

4.Solve: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x} + \sin 2x$

Solution: A.E. is $D^2 - 4D + 4 = 0$

or $(D-2)^2 = 0$

or $D = 2,2$

$\therefore$ C.F. $= (C_1 + C_2 x)e^{2x}$

P.I. $= \frac{e^{2x}}{D^2 - 4D + 4} + \frac{\sin 2x}{D^2 - 4D + 4}$ first term, case of failure

$= x \frac{e^{2x}}{2D - 4} + \frac{\sin 2x}{-4 - 4D + 4}$ first term, again case of failure

$= x^2 \frac{e^{2x}}{2} - \frac{1}{4} \cdot \frac{1}{D} \sin 2x$

$= \frac{1}{2} x^2 e^{2x} + \frac{1}{8} \cos 2x$

Hence $y = (C_1 + C_2 x)e^{2x} + \frac{1}{2} x^2 e^{2x} + \frac{1}{8} \cos 2x.$

Roll:1800083

$$\frac{1}{f(D)}.e^{ax}.\Phi(x) = e^{ax}.\frac{1}{f(D+a)}.\Phi(x)$$

$$D[e^{ax}\Phi(x)] = e^{ax}D\Phi(x) + ae^{ax}\Phi(x) = e^{ax}(D+a)\Phi(x)$$

$$D^2[e^{ax}\Phi(x)] = D[e^{ax}(D+a)\Phi(x)] = e^{ax}(D^2+aD)\Phi(x) + ae^{ax}(D+a)\Phi(x)$$

$$= e^{ax}(D^2+2aD+a^2)\Phi(x) = e^{ax}(D+a)^2\Phi(x)$$

Similarly, $D^n[e^{ax}\Phi(x)] = e^{ax}(D+a)^n\Phi(x)$

$$f(D)[e^{ax}\Phi(x)] = e^{ax}(D+a)\Phi(x)$$

$$e^{ax}\Phi(x) = \frac{1}{f(D)}[e^{ax}f(D+a)\Phi(x)] \qquad ......(1)$$

Put $f(D+a)\Phi(x) = X$, so that $\Phi(x)=\frac{1}{f(D+a)}X$

Substituting these values in (1),we get

$$e^{ax}\frac{1}{f(D+a)}X = \frac{1}{f(D)}[e^{ax}.X]$$

$$\Rightarrow \frac{1}{f(D)}[e^{ax}.\Phi(x)] = e^{ax}\frac{1}{f(D+a)}\Phi(x)$$

MISCELLANEOUS EXAMPLES

EX.5 Solve $(D^2+1)y = e^{2x}\sin x + e^{\frac{x}{2}}\sin(\frac{1}{2}\sqrt{3x})$

Solution. A.E. is $D^3+1=0, (D+1)(D^2-D+1)=0$

$$D = -1, \frac{1\pm\sqrt{3}i}{2}$$

$$\text{C.F.} = C_1e^{-x} + C_2e^{\frac{x}{2}}\cos(\frac{1}{2}\sqrt{3x}+C_3)$$

P.I. corresponding to $e^{2x}\sin x$

$=\frac{e^{2x}\sin x}{D^3+1}=e^{2x}\frac{1}{(D+2)^3+1}\sin x$

$=e^{2x}\frac{1}{D^3+6D^2+12D+9}\sin x=e^{2x}\frac{1}{-D-6+12D+9}\sin x$

$=e^{2x}\frac{1}{11D+3}\sin x=e^{2x}\frac{11D-3}{11^2-3^2}\sin x$

$=\frac{1}{180}e^{2x}(11\cos x-3\sin x)$

P.I. corresponding to $e^{\frac{x}{2}}\sin(\frac{1}{2}\sqrt{3x})$

$=e^{\frac{x}{2}}\frac{1}{\left(D+\frac{1}{2}\right)^3+1}\sin x\frac{\sqrt{3}}{2}$

$=e^{\frac{x}{2}}\frac{1}{D^3+\frac{3}{2}D^2+\frac{3}{4}D+\frac{9}{8}}\sin\frac{\sqrt{3}}{2}x$ (case of failure)

$=e^{\frac{x}{2}}x\frac{1}{3D^2+3D+\frac{3}{4}}\sin\frac{\sqrt{3}}{2}x$ differentiating the dominator w.r.t D and multiplying by x

$=e^{\frac{x}{2}}x\frac{1}{-3\times\frac{3}{4}D+\frac{3}{4}}\sin\frac{\sqrt{3}}{2}x$

$=e^{\frac{x}{2}}\frac{x}{3}\frac{1}{D-\frac{1}{2}}\sin\frac{\sqrt{3}}{2}x$

$=e^{\frac{x}{2}}\frac{x}{3}\frac{1}{-\frac{3}{4}-\frac{1}{4}}(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}x+\frac{1}{2}\sin\frac{\sqrt{3}}{2}x)$

$=-\frac{x}{6}e^{\frac{x}{2}}(\sqrt{3}\cos\frac{\sqrt{3}}{2}x+\sin\frac{\sqrt{3}}{2}x)$

Thus the P.I.$=\frac{1}{180}e^{2x}(11\cos x-3\sin x)-\frac{x}{6}e^{\frac{x}{2}}(\sqrt{3}\cos\frac{\sqrt{3}}{2}x+\sin\frac{\sqrt{3}}{2}x)$

The complete solution is y=C.F.+P.I.

Ex.6 Solve $(D^4+D^2+1)y=e^{\frac{-1}{2}}\cos(x\frac{\sqrt{3}}{2})$

Solution. The auxiliary equation is $(D^4+D^2+1)=0$

Or,$[((D^2+1)^2-D^2]=0$ or, $(D^2-D+1)(D^2+D+1)=0$

When $D^2-D+1=0, D=\frac{1\pm\sqrt{3}i}{2}$

And when $D^2+D+1=0,\ D=\frac{-1+\sqrt{3}i}{2}$

Then C.F. $=C_1e^{\frac{x}{2}}\cos(\frac{1}{2}\sqrt{3x}+C_2)+C_3e^{-\frac{x}{2}}\cos(\frac{1}{2}\sqrt{3x}+C_4)$

P.I. $=\frac{1}{(D^4+D^2+1)}e^{-\frac{x}{2}}\cos(\frac{1}{2}\sqrt{3}x)$

$=e^{-\frac{x}{2}}\frac{1}{\left(D-\frac{1}{2}\right)^4+\left(D-\frac{1}{2}\right)^2+1}\cos(\frac{1}{2}\sqrt{3}x)$

$=e^{-\frac{x}{2}}\frac{1}{D^4-2D^3+\frac{5}{2}D^2-\frac{3}{2}D+\frac{21}{16}}\cos(\frac{1}{2}\sqrt{3}x)$

$=e^{-\frac{x}{2}}x\frac{1}{4D^3-6D^2+5D-\frac{3}{2}}\cos(\frac{1}{2}\sqrt{3}x)$

[multiplying by x and differentiating the denominator w.r.t. D]

$=e^{-\frac{x}{2}}x\frac{1}{4D\left(-\frac{3}{4}\right)-6\left(-\frac{3}{4}\right)+5D-\frac{3}{2}}\cos(\frac{1}{2}\sqrt{3}x)$

$=e^{-\frac{x}{2}}x\frac{1}{2D+3}\cos(\frac{1}{2}\sqrt{3}x)$

$=e^{-\frac{x}{2}}x\frac{2D-3}{4D^2-9}\cos(\frac{1}{2}\sqrt{3}x)$

$=e^{-\frac{x}{2}}x\frac{1}{4\left(-\frac{3}{4}\right)-9}[-2\frac{\sqrt{3}}{2}\sin(\frac{1}{2}\sqrt{3}x)-3\cos(\frac{1}{2}\sqrt{3}x)]$

$=\frac{e^{-\frac{x}{2}}x}{-12}[-\sqrt{3}\sin(\frac{1}{2}\sqrt{3}x)-3\cos(\frac{1}{2}\sqrt{3}x)]$

$=\frac{e^{-\frac{x}{2}}x}{4}[\frac{1}{\sqrt{3}}\sin(\frac{1}{2}\sqrt{3}x)+\cos(\frac{1}{2}\sqrt{3}x)]$

The complete solution is y=C.F. + P.I.

Ex. 7. Solve $(D^4+2D^3-3D^2)y=x^2+3e^{2x}+4\sin x$

Solution.

The auxiliary equation is, $(D^4+2D^3-3D^2)=0$

$D^2(D^2+2D-3)=0$

$D^2(D+3)(D-1)=0$

$\therefore D=0,0,-3,1$

Hence, C.F. = $(C_1+C_2x)+C_3e^x+C_4e^{-3x}$

P.I. = $\frac{1}{D^2(D^2+2D-3)}(x^2+3e^{2x}+4\sin x)$

$=\frac{3e^{2x}}{20}+\frac{4\sin x}{-2(D-2)}-\frac{1}{3D^2}\left(1-\frac{2}{3}D-\frac{1}{3}D^2\right)^{-1}x^2$

$=\frac{3}{20}e^{2x}-\frac{2(D-2)}{D^2-4}\sin x-\frac{1}{3D^2}\left(1+\frac{2}{3}D+\frac{1}{3}D^2+\frac{4}{9}D^2\dots\right)x^2$

$= \frac{3}{20}e^{2x} - \frac{2}{5}(\cos x + 2\sin x) - \frac{1}{3D^2}\left(x^2 + \frac{4}{9}x + \frac{14}{9}\right)$

$= \frac{3}{20}e^{2x} - \frac{2}{5}(\cos x + 2\sin x) - \left(\frac{1}{36}x^4 + \frac{2}{27}x^3 + \frac{7}{27}x^2\right)$

Therefore, the complete solution is y=C.F. + P.I.

Ex. 8. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 8e^{3x}\sin 2x$

Solution. Auxiliary equation is, $D^2 - 6D + 13 = 0$

$$D = \frac{6 \pm \sqrt{36 - 52}}{2} = 3 \pm 2i$$

$$\text{C.F.} = e^{3x}[C_1\cos 2x + C_2\sin 2x]$$

P.I. $= \frac{1}{D^2-6D+13} 8e^{3x}\sin 2x$

$= 8e^{3x}\frac{1}{(D+3)^2-6(D+3)+13}\sin 2x$

$= 8e^{3x}\frac{1}{D^2+4}\sin 2x$ (case of failure)

$= 8e^{3x}\frac{1}{2D}\sin 2x$

$= -2xe^{3x}\cos 2x$

Hence, the complete solution is,

$$y = -2xe^{3x}\cos 2x + e^{3x}[C_1\cos 2x + C_2\sin 2x]$$

Ex 10. Solve $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 19\frac{dy}{dx} + 20y = xe^x + 2e^{-4x}\sin x$

Solution. Auxiliary equation is $D^3 - 2D^2 - 19D + 20 = 0$

$$(D-1)(D^2 - D - 20) = 0$$

$$(D-1)(D-5)(D+4) = 0$$

$$\therefore D = 1, 5, -4$$

C.F. = $C_1e^x + C_2e^{5x} + C_3e^{-4x}$

P.I. corresponding to xe^x

$= \frac{1}{(D-1)(D-5)(D+4)} xe^x$

$= e^x\frac{1}{[(D+1)-1][(D+1)-5][(D+1)+4]}x$

$=e^x \frac{1}{D(D-4)(D+5)} x$

$=e^x \frac{1}{D} \frac{1}{D^2+D-20} x$

$=-e^x \frac{1}{20} \frac{1}{D} \left(1-\frac{1}{20}D-\frac{1}{20}D^2\right)^{-1} x$

$=-e^x \frac{1}{20} \frac{1}{D} \left(1+\frac{1}{20}D\right) x$

$=-e^x \frac{1}{20} \frac{1}{D} \left(x+\frac{1}{20}\right)$

$=-\frac{1}{20} e^x (\frac{x^2}{2}+\frac{1}{20}x)$

P.I. corresponding to $2e^{-4x}$sinx

=Imaginary part of $\frac{2e^{-4x}e^{ix}}{D^3-2D^2-19D+20}$

= Imaginary part of $\frac{2e^{x(i-4)}}{(i-4)^3-2(i-4)^2-19(i-4)+20}$

= Imaginary part of $\frac{e^{x(i-4)}}{1+22i}$

= Imaginary part of $\frac{e^{4x}(\cos x+i \sin x)}{1^2+22^2}(1+22i)$

$=\frac{e^{-4x}}{485}(\sin x-22\cos x)$

Hence P.I.= $=-\frac{1}{20} e^x (\frac{x^2}{2}+\frac{1}{20}x)=\frac{e^{-4x}}{485}(\sin x-22\cos x)$

Therefore, the complete solution is y=C.F. + P.I.

∴The general solution is,

$y = C_1 \sin(2x+C_2) + \frac{\sin ax}{4-a^2}$ ………..(1)

so that $\frac{dy}{dx} = 2C_1 \cos(2x+C_3) + \frac{a\cos ax}{4-a^2}$ ………..(2)

But y = 0 when x = 0,

∴ (1) gives, $0 = C_1 \sin C_2$ ………..(3)

Again $\frac{dy}{dx} = 0$ when x = 0

∴ (2) gives, $0 = 2C_1 \cos C_2 + \frac{a}{4-a^2}$ ………..(4)

From (3), $C_1=0$ or $C_2=0$ but if $C_1=0$,(4) does not hold.

Hence $C_2=0$ and then from (4), $C_1= -\frac{a}{2(4-a^2)}$

Putting these values of C_1 and C_2 in (1),the required solution is

$$y = -\frac{a \sin 2x}{2\,(4-a^2)} + \frac{\sin ax}{4-a^2} = \frac{2 \sin ax - a \sin 2x}{2(4-a^2)}$$

This proves the result.

END

Rajshahi University of Engineering & Technology

Course No. MATH 2101

Course Teacher: **Mst. Rupale Khatun**

Assignment

Solution of linear equations of second & higher orders with variable co-efficient

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Submitted By

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Solution of Linear Equations of Second and Higher Orders with Variable Co-efficient

Method : Cauchy -Euler Equations

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a^n y = f(x) \cdots\cdots\cdots\cdots (1)$$

where, a_1, a_2 are constants; $f(x)$= function of x.

It can be also written as –

$$x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-2} D^{n-2} + \cdots + a_n y = f(x) \cdots\cdots\cdots\cdots (2)$$

where, $D \equiv \frac{d}{dx}$

These equations are known as ***Cauchy -Euler Equations***.

Conversion: Linear differential equation with variable co-efficient to constant co-efficient.

Substitution: In order to solve (2) introduce a new independent variable z such that-

$x = e^z$ or, $\log x = z$ so that $\frac{1}{x} = \frac{dz}{dx} \cdots\cdots\cdots\cdots\cdots (3)$

Now,

$$xD = x\frac{d}{dx} \equiv \frac{d}{dz} = D_1 \;;\quad x^2D^2 = D_1(D_1 - 1)y \text{ and so on.}$$

Proceeding likewise, we can show that

$$x^nD^n = x^n\frac{d^ny}{dx^n} = D_1(D_1 - 1)(D_1 - 2)\cdots\cdots\cdots(D_1 - n + 1)y$$

$$\cdots\cdots\cdots\cdots (4)$$

Substituting the above values of $x, xD, x^2D^2, x^3D^3, \cdots, x^nD^n$ in equation (1) and thus changing the independent variable from x to , we have

$$\left[D_1(D_1 - 1)\cdots(D_1 - n + 1) + \cdots\cdots + a_{n-2}D_1(D_1 - 1) + a_{n-1}D_1 + a_n\right]y = Z$$

Or, $f(D_1)y = Z \quad \cdots\cdots\cdots\cdots (5)$

Where, Z is now a function of z only.

By solving (5) we will be able to get the auxiliary equation. And by solving auxiliary equation, all the roots of auxiliary equation can be found.

Depending on the different cases of the roots of the auxiliary equation (A.E), the complementary function (C.F) and the particular integral (P.I) can be found.

So, the required general solution will be –

$$y = \text{C.F} + \text{P.I}$$

Solution of Linear Equations of Second Order with Variable Co-efficient

The general (standard) form of the linear equations of the second order –

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Q = R$$

where, P, Q and R are functions of x or constants.

Examples:

(1) Solve the differential equation, $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = logx.$

Solution:

Given equation, $[x^2D^2 + 2xD]y = logx \cdots\cdots\cdots\cdots (1)$

where, $D = \frac{d}{dx}$

Let, $x = e^z$ or, $z = logx$ and $D_1 \equiv \frac{d}{dz}$

Then, equation (1) becomes –

$[D_1(D_1 - 1) + 2D_1]y = z$

Or, $(D_1^2 + D_1)y = z$

Its auxiliary equation is, $D_1^2 + D_1 = 0$

So that, $D_1 = 0, -1$

$\therefore C.F = c_1 e^{0\times z} + c_2 e^{(-1)\times z} = c_1 + c_2(e^z)^{-1} = c_1 + c_2 x^{-1}$; where c_1 and c_2 are arbitrary constants.

$$\therefore P.I = \frac{1}{D_1^2 + D_1} = \frac{1}{D_1}(1 - D_1 + \cdots)z = \frac{1}{D_1}(z - 1)$$

$$= \frac{1}{D_1(1 + D_1)} z = \frac{1}{D_1}(1 + D_1)^{-1} z$$

So, $P.I = \frac{1}{2} \times z^2 - z = \frac{1}{2}(logx)^2 - logx$

As, $x = e^z$ or, $z = logx$

$\therefore$ The required solution is,

$$\boldsymbol{y = c_1 + c_2 x^{-1} + \frac{1}{2}(logx)^2 - logx.}$$

(2) Solve the differential equation, $\boldsymbol{(x^2D^2 + 7xD + 13)y = logx}$.

<u>Solution:</u>

Given equation, $(x^2D^2 + 7xD + 13)y = logx \cdots (1)$

where, $D = \frac{d}{dx}$

Let, $x = e^z$ or, $z = logx$ and $D_1 \equiv \frac{d}{dz}$

Then, equation (1) becomes –

$[D_1(D_1 - 1) + 7D_1 + 13]y = z$

Or, $(D_1^2 + 6D_1 + 13)y = z$

Its auxiliary equation is, $D_1^2 + 6D_1 + 13 = 0$

So that, $D_1 = -3 \pm 2i$

$\therefore C.F = e^{-3z}(c_1 cos2z + c_2 sin2z) = x^{-3}[c_1 \cos(2logx) + c_2 \sin(2logx)]$; where c_1 and c_2 are arbitrary constants.

$$\therefore P.I = \frac{1}{D_1^2 + 6D_1 + 13} z$$

$$= \frac{1}{13[1 + (16/13)D_1 + (1/13)D_1^2]} z$$

$$= \frac{1}{13}\left[1 + \left(\frac{6}{13}D_1 + \frac{1}{13}D_1^2\right)\right]^{-1} z$$

$$= \frac{1}{13}\left[1 - \left(\frac{6}{13}D_1 + \frac{1}{13}D_1^2\right) + \cdots\right] z$$

$$= \frac{1}{13}\left(z - \frac{6}{13}\right)$$

$$= \frac{1}{13}\left(logx - \frac{6}{13}\right)$$

$$= \frac{1}{169}(13logx - 6)$$

∴ The required solution is,

$$\boldsymbol{y = x^{-3}[c_1cos(2logx) + c_2sin(2logx)] + \frac{1}{169}(13logx - 6).}$$

(3) Solve the differential equation, $\boldsymbol{(x^2D^2 - xD + 2)y = xlogx}$

Solution:

Given equation, $(x^2D^2 - xD + 2)y = xlogx \cdots\cdots (1)$

where, $D = \frac{d}{dx}$

Let, $x = e^z$ or, $z = logx$ and $D_1 \equiv \frac{d}{dz}$

Then, equation (1) becomes –

$[D_1(D_1 - 1) - D_1 + 2]y = ze^z$

Or, $(D_1^2 - 2D_1 + 2)y = ze^z$

Its auxiliary equation is, $D_1^2 - 2D_1 + 2 = 0$

So that, $D_1 = 1 \pm i$

$\therefore C.F = e^z(c_1cosz + c_2sinz) = x[c_1\cos(logx) + c_2\sin(logx)]$; where c_1 and c_2 are arbitrary constants.

$$\therefore P.I = \frac{1}{D_1^2 - 2D_1 + 2} ze^z$$

$$= e^z \frac{1}{(D_1+1)^2 - 2(D_1+1) + 2} z$$

$$= e^z \frac{1}{{D_1}^2 + 1} z$$

$$= e^z (1 + D_1^2)^{-1} z$$

$$= e^z (1 - \cdots\cdots) z$$

$$= e^z z$$

$$= xlogx$$

∴ The required solution is,

$$\boldsymbol{y = x[c_1 cos(logx) + c_2 sin(logx)] + xlogx.}$$

(4) Solve the differential equation, $(x^2D^2 + xD)y = 12logx$.

Solution:

Given equation, $(x^2D^2 + xD)y = 12logx \cdots\cdots (1)$

where, $D = \frac{d}{dx}$

Let, $x = e^z$ or, $z = logx$ and $D_1 \equiv \frac{d}{dz}$

Then, equation (1) becomes –

$[D_1(D_1 - 1) + D_1]y = 12z$

Or, $D_1^2 y = 12z$

Its auxiliary equation is, $D_1^2 = 0$

So that, $D_1 = 0,0$

$\therefore C.F = c_1 + c_2 z = c_1 + c_2 \log x$; where c_1 and c_2 are arbitrary constants.

$$\therefore P.I = \frac{1}{D_1^2} 12z$$

$$= 12 \frac{1}{D_1{}^2} z$$

$$= 12 \frac{1}{D_1} \frac{z^2}{2}$$

$$= 12 \times \frac{z^3}{6}$$

$$= 2(logx)^3$$

$\therefore$ The required solution is,

$$\boldsymbol{y = c_1 + c_2 logx + 2(logx)^3}.$$

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5 . solve (D^2-2D+5)y=0

6 . solve (D^2+4)y= x^2

7 . solve (D^2- 2kD+k^2)y= e^x

8 . solve (D^2-6D+9)y=1+x+x^2

Solution of problem 5 :

(D^2-2D+5)y=0 … (1)

Let, y= e^{mx} be the trial solution of equation (1)

Now, from (1)

m^2-2m+5=0

Or, m=$\frac{2 \pm\sqrt{4-20}}{2}$

=$\frac{2\pm\sqrt{-16}}{2}$

=$\frac{2\pm 4i}{2}$

=1±2i

Since the general solution : y= e^x(A cos 2x + B sin2x).

Solution of problem 6:

(D^2+4)y= x^2 ... (1)

Let, y= e^{mx} be the trial solution of equation (1)

Now, from (1)

m^2+4=0

0r, m= $\pm$ 2i

y_c= A cos 2x + B sin 2x

Now, y_p= $\frac{1}{D^2+4}x^2$

=$\frac{1}{4}$(1- $\frac{D^2}{4}$) x^2

=$\frac{1}{4}$(x^2- $\frac{2}{4}$)

=$\frac{1}{8}$(2x^2- 1)

Since, the general solution:

y= A cos 2x + B sin 2x+ $\frac{1}{8}$(2x^2- 1).

Solution of problem 7:

$(D^2$- 2kD+$k^2)$y= e^x ... (1)

Let, y= e^{mx} be the trial solution of equation (1)

Now, from (1)

m^2-2km+ k^2=0

Or, $(m-k)^2$=0

Or, m= k,k

y_c= (A + Bx) e^{kx}

Now, $y_p = \frac{1}{D^2 - 2kD + k^2} e^x$

$= \frac{1}{(D-k)^2} e^x$

$= \frac{1}{(1-k)^2} e^x$

Since the general solution :

y= y_c+y_p =(A + Bx) e^{kx} + $\frac{1}{(1-k)^2} e^x$.

Solution of problem 8:

$(D^2$-6D+9)y=1+x+x^2 ... (1)

Let, y= e^{mx} be the trial solution of equation (1)

Now, from (1)

m^2-6m+9=0

Or, $(m-3)^2$=0

Or, m=3,3

y_c= (A + Bx) e^{3x}

Now, y_p=$\frac{1}{(3-D)^2}$(1+x+x^2)

$= \frac{1}{9}\left(1 - \frac{D}{3}\right)^{-2}(1+x+x^2)$

$=\frac{1}{9}(1+ 2\frac{D}{3} + 3\frac{D^2}{9}) (1+x+x^2)$

$= \frac{1}{9} [1+x+x^2+ \frac{2}{3} (1+2X)+ \frac{2}{3}]$

$= \frac{1}{9} (x^2+\frac{7}{3}X+ \frac{7}{3})$

Since the general solution :

$y= y_c + y_p=(A + Bx) e^{3x} + \frac{1}{9} (x^2+\frac{7}{3}X+ \frac{7}{3})$

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Ex. 9: Solve: $\frac{x^2(d^2y)}{dx^2} + \frac{xdy}{dx} + y = \sin(\log x^2)$

Ex. 10: Solve: $\frac{x^2d^2y}{dx^2} - \frac{4xdy}{dx} + 6y = \frac{42}{x^4}$

Ex. 11: Solve: $(x^2D^2 - 3xD + 4)y = 2x^2$

Ex.12: Solve: $(x^2D^2+xD-1)y=x^4$

Solution-9:

We have, $\frac{x^2(d^2y)}{dx^2}+\frac{xdy}{dx}+y=\sin(\log x^2)--(1)$

Let, $x=e^z \;=>\; logx=z$

And D=$\frac{d}{dz}$

From (1) we get,

D(D-1)y+Dy+y=sin(2z)

(D^2+1)y=sin(2z)

A.E. is m^2+1=0

Or, m=$\pm i$

Yc=C_1cosz+C_2sinz

=C_1cos(logx)+C_2sin(logx)

Where C_1,C_2 are the arbitrary constant.

Yp=$\frac{1}{D^2+1}*\sin 2z$

=$\frac{1}{-4+1}sin2z=-\frac{1}{3}\sin(logx^2)$

The complete solution is,

Y=Yc+Yp =C_1cos(logx)+C_2sin(logx)-$\frac{1}{3}\sin\left(logx^2\right)$

Solution-10:

We have, $x^2\frac{dy^2}{dx^2} - 4x\frac{dy}{dx} + 6y = \frac{42}{x^4} - - - (1)$

Let, x=$e^z => logx = z$

And $D = \frac{d}{dz}$

From (1) we get,

D(D-1)y-4Dy+6y=$42e^{-4z}$

Or, (D^2-5D+6)y=$42e^{-4z}$

A.E is,. m^2-5m+6=0

Or, m=2,3

Y_c= $C_1e^{2z} + C_2e^{3z} = C_1x^2 + C_2x^3$

Where C_1,C_2 are the arbitrary constant.

Yp=$\frac{1}{D^2-5D+6}\ 42e^{-4z}$

=42 $\frac{1}{(-4)^2-5(-4)+6}$ =$e^{-4z} = \frac{1}{e^{4z}} = \frac{1}{x^4}$

The complete solution is,**Y=Yc+Yp =C1x²+$C2x^3 + \frac{1}{x^4}$**

Solution-11:

We have, (x^2D^2-3xD+4)y=$2x^2$-----(1)

Let, x=$e^z => logx = z$

From (1) we get,

(D^2-D-3D+4)y=$2e^{2z}$

$(D^2-4D+4)y=2e^{2z}$

Auxiliary equation is,

$(m^2-4m+4)=0$

Or, m= 2,2

$Yc=C_1e^{2z}+ C_2ze^{2z} = C_1x^2 + C_2x^2logx$

Where C_1, C_2 are the arbitrary constant

$Yp=\frac{1}{D^2-4D+4}2e^{2z}$

$=2\frac{1}{(D-2)(D-2)}e^{2z}$

$=2\frac{1}{D-2}\left[\frac{1}{D-2}e^{2z}\right]$

$=2\frac{1}{D-2}*\frac{z^1}{1!}e^{2z}$

$=2\frac{1}{D-2}ze^{2z}$

$=2\frac{e^{2z}}{(D+2)-2}z$

$=2e^{2z}\frac{1}{D}z$

$=2e^{2z}\frac{z^2}{2}$

$=z^2e^{2z} = x^2(logx)^2$

The complete solution is,

Y=Yc+Yp

$=(C_1+C_2logx)x^2+x^2(logx)^2$

Solution-12:

We have, $(x^2D^2+xD-1)y=x^4 \ldots\ldots\ldots\ldots\ldots.(1)$

Let, $x=e^z \Rightarrow logx = z$

From (1) we get,

$(D^2-D+D-1)y= e^{4z}$

Or, $(D^2-1)y= e^{4z}$

Auxiliary equation is,

$m^2-1=0$

Or, $m=\pm 1$

$Yc= C_1e^z + C_2e^{-z} = C_1x + C_2\frac{1}{x}$

Where C_1, C_2 are the arbitrary constant

$Yp=\frac{1}{D^2-1}e^{4z}$

$=\frac{1}{4^2-1}e^{4z}$

$=\frac{1}{7}x^4$

The complete solution is,

$$Y=Yc+Yp = C_1x + C_2\frac{1}{x} + \frac{1}{7}x^4$$

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Example No.13: **FINDTHE COMPLEMENTARY FUNCTION OF** $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0,$

$$y = 0 \text{ and } \frac{dy}{dx} = \frac{d^2y}{dx^2} \text{ when } x = 0$$

Solution:

Here the auxiliary equation is

$$m^2 + 4m + 5 = 0$$

Its root are $(-2 \pm i)$

The complementary function is

$$y = e^{-2x}(Acosx + bsinx) \text{ ....(1)}$$

On putting y = 2 and x = 0 in (1), we get

$$2 = A$$

On putting A = 2 in (1),

we have,

$$y = e^{-2x}\ (2cosx + bsinx) \text{..................(2)}$$

On differentiating (2),

we get

$\frac{dy}{dx}$=$e^{-2x}(2sinx + bcosx)$-2$e^{-2x}(2cosx + bsinx)$=
$e^{-2x}[(-\ 2B-\ 2)\sin x\ +\ (B-\ 4)\cos x]$

=$\frac{d^2y}{dx^2} = e^{-2x}[(-2B-2)\cos x - (B-4\sin x)]$
$-2e^{-2x}[(-\ 2B-\ 2)\sin x\ +\ (B-\ 4)\cos x$

=$e^{-2x}\ [(-\ 4B\ +\ 6)\cos x\ +\ (3B\ +\ 8)\sin x]$

But, $\frac{dy}{dx} = \frac{d^2y}{dx^2}$

$$e^{-2x}[(-2B-2)\sin x\ +\ (B-\ 4)\cos x] = e^{-2x}\ [(-\ 4B\ +\ 6)\cos x\ +\ (3B\ +\ 8)\sin x]$$

On putting x = 0, we get,

$$B-\ 4\ =-\ 4B\ +\ 6 \quad \Rightarrow \quad B = 2$$

(2) becomes,

$$y\ =\ e^{-2x}[2\cos x\ +\ 2\sin x]$$

$$y\ =\ 2e^{-2x}\ [\sin x\ +\ \cos x]$$

<u>Ans.</u>

Example 14: **FINDTHE COMPLEMENTARY FUNCTION OF** $\boldsymbol{\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 5e^{3x}}$

Solution:

$$(D^2 + 6D + 9)y = 5e^x$$

Auxiliary equation is,

m^2 + 6m + 9 = 0 ⇒ (m + 3)2 = 0 ⇒ m = – 3, – 3

$\mathrm{C.F.} = (C_1 + C_2x)\, e^{-3x}$

$\mathrm{P.I.} = \frac{1}{D^2+6D+9} . 5e^{3x} = 5\frac{e^{3x}}{(3^2)+6(3)+9} = \frac{5e^{3x}}{36}$

The complete solution is

$$y = (C_1 + C_2x)e^{-3x} = \frac{5e^{3x}}{36}$$

Ans.

Example 15: **FINDTHE COMPLEMENTARY FUNCTION OF** $\boldsymbol{\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - log2}$

Solution:

$$(D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x} - log2$$

Auxiliary equation is,

$(m_2 - 6m + 9) = 0 \Rightarrow (m-3)2 = 0, \Rightarrow m = 3, 3$

$$C.F. = (C_1 + C_2x)e^{3x}$$

P.I. = $\frac{1}{D^2 - 6D + 9} 6e^{3x} + \frac{1}{D^2 - 6D + 9} 7e^{-2x} + \frac{1}{D^2 - 6D + 9}(-log2)$

$= x\frac{1}{2D-6} 6e^{3x} + \frac{1}{4+12+9} 7e^{-2x} - log2 \frac{1}{D^2 - 6D + 9} e^{0x}$

$= \frac{x^2}{2} 6e^{3x} + \frac{7e^{-2x}}{25} \pm log2(\frac{1}{9}) = 3x^2e^{3x} + \frac{7}{25}e^{-2x} - \frac{1}{9}log2$

Complete solution is $y = (C_1 + C_2x)e^{3x} + 3x^2e^{3x} + \frac{7}{25}e^{-2x} - \frac{1}{9}log2$

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Example 16: Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$.

Solution : putting x=e^z $and\ D \equiv \frac{d}{dz}$, the equation becomes

[D(D-1)-2D-4]y=e^{4z} or $(D^2 - 3D - 4)$y=e^{4z}.

Auxiliary eqn is $D^2 - 3D - 4 = 0$ or (D-4)(D+1)=0.

C.F = $C_1e^{4z} + C_2e^{-z} = C_1x^4 + C_2/x\ \ as\ e^z = x$.

P.I =$\frac{e^{4z}}{D^2-3D-4} = z\ \frac{e^{4z}}{2D-3}$

$[\ differentiating\ denominator\ w.r.t\ D\ and\ multiplying\ by\ z]$

=z$\frac{e^{4z}}{2.4-3} = \frac{ze^{4z}}{5} = \frac{1}{5}(\log x)\,x^4.as\ x = e^z$.

Therefore, the required equation is

$$y= C_1x^4 + C_2/x + \frac{1}{5}x^4\log(x).\quad (ans).$$

Example 17: Solve $x^4\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 log\ x$.

Solution : Putting x=e^z $, D \equiv d/dz$, the equation becomes

[D(D-1)-D-3]y= ze^{2z}.

Auxiliary equation is $(D^2 - 2D - 3) = 0\ , (D-3)(D+1) = 0$.

C.F = $C_1e^{3z} + C_2e^{-z} = C_1x^3 + C_2x^{-1}$

P.I =$\frac{ze^{2z}}{D^2-2D-3} = e^{2z}\frac{1}{(D+1)^2-2(D+2)-3}\ z$

= $e^{2z}\frac{1}{D^2+2D-3}z$

= -$\frac{e^{2z}}{3}[1-\frac{2}{3}D-\frac{1}{3}D^2]^{-1}z$

= -$\frac{1}{3}e^{2z}\left(1+\frac{2}{3}D+\cdots\right)z=-\frac{1}{3}e^{2z}(z+\frac{2}{3})$

= -$\frac{1}{3}x^2(\log x+\frac{2}{3})$ as $e^x=x$.

Hence the complete solution is

y = $C_1x^3+C_2x^{-1}-\frac{1}{3}x^2(\log x+\frac{2}{3})$. (ans)

Example 18: Solve $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x$.

Solution : Putting x=e^x , D≡ $\frac{d}{dz}$ $, the\ equation\ becomes$

[D(D-1)+4D-2]y=e^{e^z}.

A.E is $D^2+3D+2=0$ $,\therefore (D+2)(D+1)=0.$

$\therefore C.F=C_1e^{-2z}+C_2e^{-z}=C_1x^{-2}+C_2x^{-1}.$

P.I=$\frac{1}{(D+2)(D+1)}e^{e^z}$.

=($\frac{1}{D+1}-\frac{1}{D+2}$)$e^{e^z}$.

Now let $\frac{1}{D+1}e^{e^z}=u, i.e. (D+1)u=e^{e^z}$,

Or, $\frac{du}{dz}+u=e^{e^z}$, linear , I.F=e^z.

$\therefore ue^z=\int e^z.e^{e^z}dz=\int e^x dx\ \ as\ e^z=x$

=$e^x\ or, u=\frac{1}{x}e^x\ as\ e^z=x.$

Also let $\frac{1}{D+2}e^{e^z} = y \, , (D+2)y = e^{e^z}$

Or $\frac{dv}{dz} + 2v = e^{e^z}, linear \, , I.F = e^{2z}.$

$\therefore ve^{2z} = \int e^{2z}.e^{e^z}dz = \int e^{e^z}.e^z.e^2 dz$

$= \int xe \, dx = \; e^x(x-1)$

Or y=$\frac{1}{e^{2z}}[e^x(x-1)] = \frac{e^x}{x^2}(x-1) = \frac{e^x}{x} - \frac{e^x}{x^2}.$

(1) gives P.I = u-v =$\frac{1}{x}e^x - \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) = \frac{e^x}{x^2}.$

Hence the complete solution is

y = $C_1x^{-2} + C_2x^{-1} + \frac{e^x}{x^2}$. (ans)

Example 19: Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = log\, x$.

Solution : Let x=e^z $and \; D \equiv \frac{d}{dy}$, So the equation becomes [D(D-1)-D+1]y= $loge^z$

A.E = $D^2 - 2D + 1 = 0$

Or, $(D-1)^2 = 0$

Or, D=1,1 .

$\therefore C.F$=$(C_1 + C_2z)e^z = (C_1 + C_2 \log x)x \, . \quad [e^z = x]$

Now P.I=$\frac{1}{D^2-2D+1}z = (1-D)^{-2}z$

=(1+2D+3D^2+…)z

=z+2

$=\log x + 2$.

Hence , the complete solution is ,

$y = (C_1 + C_2 \log x\ )x + \log x + 2.$ (ans)

Israt Jahan Ema

Roll No: 1800090

(20) Solve $x^2 \frac{d^2y}{dx^2}$ -3x $\frac{dy}{dx}$ + 4y =0

Sol^n: Let $\frac{d}{dx} \equiv D$

Then the given equation reduces to $(x^2D^2 -3xD +4)y =0$ ……..(1)

Let,

$x=e^z$

or, $z=\log x$ and $D_1 = \frac{d}{dz}$ ……..(2)

Then

$xD = D_1$

and $x^2D^2 = D_1(D_1-1)$

Hence (1) reduces to ,

$[D_1(D_1-1) - 3D_1 + 4]\, y = 0$

or, $(D_1-2)^2\, y = 0$

Its auxiliary equation is

$(D_1-2)^2 = 0$

Giving $D_1 = 2,2$

The general solution is

$y = (c_1 + c_2 z)e^{2z}$

$= (c_1 + c_2 z)\,(e^z)^2$

$= (c_1 + c_2 \log x)\, x^2$ [Using (2)]

where c_1 and c_2 are arbitrary constant.

(21) Solve $x^2 y_2 + y = 3x^2$

Sol^n: Given that,

$x^2 y_2 + y = 3x^2$

or, $(x^2D^2 + 1)\, y = 3x^2$ …………(1)

where $D \equiv \frac{d}{dx}$

Let,

$x = e^z$

or, $z = \log x$

and $D_1 \equiv \frac{d}{dz}$

so that,

$x^2D^2 = D_1(D_1-1)$

Therefore from equation (1), we get,

$[D_1(D_1-1) + 1]\, y = 3e^{2z}$

or, $(D_1^2 - D_1 + 1) = 3e^{2z}$

Its auxiliary equation is,

$D_1^2 - D_1 + 1 = 0$

so that,

$D_1 = \frac{(1+i\sqrt{3})}{2}$

Therefore,

$$C.F = e^{\frac{z}{2}}\,[c_1 \cos(\frac{z\sqrt{3}}{2}) + c_2 \sin(\frac{z\sqrt{3}}{2})]$$

$$= (e^z)^{\frac{1}{2}}\,[c_1 \cos(\frac{z\sqrt{3}}{2}) + c_2 \sin(\frac{z\sqrt{3}}{2})]$$

$$= x^{\frac{1}{2}}\,[c_1 \cos(\frac{\sqrt{3}}{2}\log x) + c_2 \sin(\frac{\sqrt{3}}{2}\log x)]$$

[as $x = e^z$

or, $z = \log x$]

where c_1 , c_2 being arbitrary constants.

and P.I = $\frac{1}{D_1^2 - D_1 + 1} 3e^{2z}$

$= 3 \frac{1}{2^2 - 2 + 1} e^{2z}$

$= (e^z)^2$

$= x^2$

Hence the required general solution is

y = C.F + P.I

$= x^{\frac{1}{2}} [c_1 \cos(\frac{\sqrt{3}}{2} \log x) + c_2 \sin(\frac{\sqrt{3}}{2} \log x)] + x^2$

(22) Solve $x^2 D^2 - 2y = x^2 + \frac{1}{x}$

Sol^n: Given that,

$(x^2 D^2 - 2)\, y = x^2 + x^{-1}$ …………(1)

where $D \equiv \frac{d}{dx}$

Let,

$x = e^z$

or, $z = \log x$

and $D_1 \equiv \frac{d}{dz}$

Then equation (1) becomes,

$[D_1(D_1-1)-2]\ y = e^{2z}+e^{-z}$

or, $(D_1^2-D_1-2)\ y = e^{2z}+e^{-z}$

Its auxiliary equation is

$D_1^2-D_1-2=0$

so that, $D_1= 2, -1$

Therefore,

$\text{C.F} = c_1e^{2z}+c_2e^{-z}$

$= c_1(e^z)^2+c_2(e^z)^{-1}$

$= c_1x^2+c_2x^{-1}$ $\quad [\text{as} = e^z]$

$$\text{P.I} = \frac{1}{(D^2-D_1-2)}(e^{2z}+e^{-z})$$

$$= \frac{1}{(D_1-2)}\frac{1}{(D_1+1)}e^{2z} + \frac{1}{(D_1+1)}\frac{1}{(D_1-2)}e^{-z}$$

$$= \frac{1}{(D_1-2)}\frac{1}{(2+1)}e^{2z} + \frac{1}{(D_1+1)}\frac{1}{(-1-2)}e^{-z}$$

$$= \frac{1}{3}\frac{z}{1!}e^{2z} - \frac{1}{3}\frac{z}{1!}e^{-z}$$

$$= \frac{1}{3}\log x\,(x^2+\frac{1}{x})$$

Then the required solution is

$$y = c_1x^2+c_2x^{-1}+\frac{1}{3}(x^2+\frac{1}{x})\log x$$

where c_1, c_2 being arbitrary constants.

(23) Solve $x^2 \frac{d^2y}{dx^2}$ + 2$x \frac{dy}{dx}$ = $\log x$

Sol^n: Putting x = e^z, D≡ $\frac{dy}{dx}$ we get,

[D(D-1)+2D] y = z

The auxiliary equation is

D^2+D = 0

or, D(D+1) = 0

so that, D = 0, -1

Therefore,

C.F = c_1+c_2e^{-z}

= c_1+c_2x^{-1}

P.I = $\frac{1}{(D^2+D)}$ z

= $\frac{1}{D}$ $(1+D)^{-1}$ z

= $\frac{1}{D}$ [1-D-…….] z

= $\frac{1}{D}$ (z-1)

= $\frac{z^2}{2} -$ z

= $\frac{(\log x)^2}{2}$ - $\log x$

Therefore the complete solution is

$$y = c_1+c_2x^{-1}+\frac{(\log x)^2}{2} - \log x$$

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Roll: 1800091

Solution of linear equation of higher orders with variable coefficient

Theory:

$x^n\, d^ny/dx^n + x^{n-1}\ d^{n-1}/dx^{n-1}$ +...+$x\frac{dy}{dx}$+y= x

trial solution,

$x = e^z$

or, log x= log e^z

or, logx = zloge

or, logx = z

or,1/x =dz/dx[diff. w.r to x]

$\therefore \frac{dz}{dx} = 1/x$

Now,

$dy/dx = dy/dz \times dz/dx$

$\quad = dy/dz \times 1/x$

Or, $x \times dy/dx = dy/dz = D_1Y$

$d^2y/dx^2 = d/dx(dy/dx) = d/dx(1/x \times dy/dz$

$\quad = -1/x^2\, dy/dz + 1/x \times d(\frac{dy}{dz})/dz \times dz/dx$

$\quad = -1/x^2\, dy/dz + 1/x \times d^2y/dz^2 \times 1/x$

$\quad = -1/x^2\, dy/dz + 1/x^2 \times d^2y/dz^2$

Or, $x^2\, d^2y/dx^2 = -dy/dz + d^2y/dz^2$

$\quad = -D_1y + D_1^2y$

$\quad = D_1(D_1-1)y$

This the solution .

Example 01.

$(x^3D^3+2x^2D^2+2)y=10(x+x^{-1})$ Where $D=dy/dx$

Solution: $(x^3D^3+2x^2D^2+2)y=10(x+x^{-1})$……………………………………………..(1)

Let, $x=e^z$

Or, $z=\log x$

And,

$D_1=d/dz$……………………………………………………………………………………..(2)

Then (1) becomes

$[D_1(D_1-1)(D_2-2)+2D_1(D_1-1)+2]y=10(x+x^{-1})$……………………………………..(3)

For 3 is $D_1^3-D_1^2+2=0$

Giving $D_1=-1,1$

$\therefore C.F = c_1e^{-Z}+e^z[c_2\cos z+c_3\sin z]=c_1x^{-1}+x[c_2\cos \log x + c_3\sin\log x]$, as $x=e^z$

Where c_1,c_2,c_3 are arbitrary constants,

P.I corresponding to 10x

$=10x/ D_1^3-D_1^2+2=5x$

P.I corresponding to $10x^{-1}$

$=10x^{-1}/ D_1^3-D_1^2+2=10x^{-1}/(D_1+1)\,(D_1^2-2D_1+2)$

$=10x^{-1}/(D_1+1)5$
$=2x^{-1}/ D_1+1$
$=2x^{-1}\log x/1!$
$=2x^{-1}\log x$
Required solution is $y= c_1x^{-1}+x[c_2 \cos \log x+c_3\sin \log x]+5x+2x^{-1}\log x$.

Muyeen Ahmad

Roll:1800092

Problem 2. Solve $(x^3D^3 + 3x^2D^2 - 2xD + 2)y = 0$

Solution: Putting $x=e^z$ *and* $D\equiv \frac{d}{dz}$, *the equation becomes*

$[D(D-1)(D-2)+3D(D-1)-2D+2]y=0$

or, $(D^3 - 3D^2 + 2D + 3D^2 - 3D - 2D + 2)y = 0$

or, $(D^3 - 3D + 2)y = 0$

∴ A.E. $= D^3 - 3D + 2 = 0$

or, $(D - 1)^2(D + 2) = 0$

or, $D = 1, 1, -2$

∴ C.F. $=(C_1+C_2z)\, e^z + C_2e^{-2z}$

$=(C_1+C_2 \log x)\, x + \frac{C_2}{x^2}$

Therefore the general solution is

$y = (C_1 + C_2 \log x)\, x + \frac{C_2}{x^2}$ (Answer)

Problem 3: Solve $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = x^2$

Solution: Putting $x=e^z$ and $D \equiv \frac{d}{dz}$ the equation becomes

$[D(D-1)(D-2) - D(D-1) + 2D - 2]y = e^{2z}$

$\Rightarrow [D^3 - 4D^2 + 5D - 2]y = e^{2z}$

Auxiliary equation is $D^3 - 4D^2 + 5D - 2 = 0$

$\Rightarrow (D-1)^2(D-2) = 0$

$\Rightarrow D = 1, 1, 2$

C. F = $(C_1 + C_2 z)\, e^z + C_3 e^{2z}$

= $(C_1 + C_2 \log x)\, x + C_3 x^2$ as $x=e^z$ i. e. $z=\log x$

Now, P. I. $= \frac{1}{(D-1)^2(D-2)} . e^{2z}$

$= \frac{1}{D^3-4D^2+5D-2} . e^{2z}$ (Case of failure)

$= \frac{z}{3D^2-8D+5} . e^{2z}$ (Multiplying by z and differentiating the denominator with respect to D)

$= \frac{ze^{2z}}{3(2)^2-8(2)+5}$

$= \frac{ze^{2z}}{12-16+5}$

$= \frac{ze^{2z}}{1}$

$= ze^{2z}$

$=x^2 \log x$

Hence the complete solution is;

y= $(C_1 + C_2 \log x)\, x + C_3 x^2 + x^2 \log x$ (Answer)

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Example 4: Solve $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 2y = x^3 + 3x$

Solution: Putting $x = e^z$ *and* $D \equiv \frac{d}{dz}$, *the equation becomes*

$$[D(D-1)(D-2)-D(D-1)+2D-2]y = e^{3z} + 3e^z$$

or, $[D^3-4D^2+5D-2]y = e^{3z} + 3e^z$

The A.E. is $D^3-4D^2+5D-2=0$

or, $(D-1)^2(D-2)=0$

or, D = 1, 1, 2

$\therefore$ C.F. $= (C_1+C_2 z)e^z + C_2 e^{2z}$

$=(C_1+C_2 \log x)x + C_2 x^2$ $\quad as, x = e^z$, i.e. z=log x

Also P.I.= $\frac{e^{3z}}{(D-1)^2(D-2)} + 3\frac{e^{z}}{(D-1)^2(D-2)}$

$=\frac{e^{3z}}{(3-1)^2(3-2)} + 3z^2\frac{e^{z}}{(6D-8)}$ multiplying the second term by z^2 *and differentiating the denominator twice w.r.t. D*

$=\frac{1}{4}e^{3z} - \frac{3}{2}z^2e^z$

$=\frac{1}{4}x^3 - \frac{3}{2}(\log x)^2\, x$

Therefore the general solution is

$$y = (C_1 + C_2 log\ x)x + C_2x^2 + \frac{1}{4}x^3 - \frac{3}{2}(\log x)^2\, x \quad \text{(Answer)}$$

Example 5 : Solve $(x^3D^3 + 2xD - 2)y = x^2\log\ x + 3$

Solution: Putting x=e^z *and* D≡ $\frac{d}{dz}$, *the equation becomes*

$[D(D-1)(D-2)+ 2D-2]y=e^{2z}.z + 3$

or, $(D^3 - 3D^2 + 4D - 2)y=z.\ e^{2z} + 3$

or, $(D-1)(D^2 - 2D + 2)= z.\ e^{2z} + 3$

∴ A.E. $= (D-1)(D^2 - 2D + 2) = 0$

or, D = 1, $1\pm i$

∴ C.F. = $C_1e^z + e^z(C_2\cos z + C_3 sinz)$

$= C_1x + x[C_2\cos(\log x) + C_3\sin(\log x)$

∴ P.I. = $\frac{1}{(D-1)(D^2-2D+2)}(z.\,e^{2z} + 3)$

$$=e^{2z} \frac{1}{(D+2-1)[(D+2)^2-2(D+2)+2]} z + 3\frac{1}{(D-1)(D^2-2D+2)} e^{0z}$$

$$= e^{2z} \frac{1}{(1+D)(D^2+2D+2)} z + 3\frac{1}{(-1)(2)}$$

$$=\frac{e^{2z}}{2}\left[1-\frac{D(D+2)}{2}+\frac{D^2(D+2)^2}{4}-\cdots\right](z-1)-\frac{3}{2}$$

$$=\frac{e^{2z}}{2}[(z-1)-\frac{1}{2}.2] - \frac{3}{2}$$

$$=\frac{e^{2z}}{2}(z-2) - \frac{3}{2}$$

$$=\frac{x^2}{2}(\log x - 2) - \frac{3}{2}$$

Therefore the general solution is

$$y = C_1 x + x[C_2\cos(\log x) + C_3\sin(\log x) + \frac{x^2}{2}(\log x - 2) - \frac{3}{2} \textbf{(\textit{Answer})}$$

Example 7: Solve $(x^3D^3 + 3x^2D^2 + xD + 1)y = x + xlnx$

Solution: Putting $x=e^z$ *and* $D\equiv \frac{d}{dz}$, *the equation becomes*

$$[D(D-1)(D-2)+ 3D(D-1)+D+1]y = e^z + z$$

or, $(D^3 - 3D^2 + 2D + 3D^2 - 3D + D + 1)y = e^z + z$

or, $(D^3 + 1)y = e^z + z$

∴ A.E. = $D^3 + 1 = 0$

or, $(D+1)(D^2 - D + 1) = 0$

or, $D = -1, \frac{1}{2}(1 \pm \sqrt{3i})$

∴ C.F. $= +e^{z/2}[C_2 cos\frac{\sqrt{3}}{2}z + C_3 sin\frac{\sqrt{3}}{2}z]$

$$= \frac{C_1}{x} + x^{1/2}[C_2\cos(\frac{\sqrt{3}}{2}\ln x) + C_3\sin(\frac{\sqrt{3}}{2}\ln x)]$$

∴ P.I. $= \frac{1}{(D^3+1)}(e^z + z)$

$= \frac{e^z}{1+1} + (1 - D^3 + \cdots)z$

$= \frac{1}{2} e^z + z$

$= \frac{1}{2} x + \ln x$

Therefore the general solution is

y = $\frac{C_1}{x}$+$x^{1/2}[C_2\cos(\frac{\sqrt{3}}{2}\ln x) + C_3\sin(\frac{\sqrt{3}}{2}\ln x)] + \frac{1}{2}x + \ln x$ (***Answer***)

1800094

❖ **Example 6**: Solve $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + x y = 1$

Solution: Dividing by x, the equation can be written as

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \frac{1}{x}$$

Putting x=e^z and $D \equiv \frac{d}{dz}$ the equation becomes

[D (D-1) (D-2) + 2 D (D-1) - D +1] y= e^{-z}

The auxiliary equation is $D^3 - D^2 - D + 1$=0

$\Rightarrow (D-1)^2$(D+1) =0

⇒D=1, 1, -1

C. F. = $(C_1 + C_2 z)\, e^z + C_3 e^{-z}$

= $(C_1 + C_2 \log x)\, x + C_3 x^{-1}$ as x=e^z i. e. z=log x

Now, P. I. =$\frac{e^{-z}}{(D-1)^2(\mathrm{D}+1)}$ (Case of failure)

=$\frac{ze^{-z}}{3D^2-2D-1}$ (Multiplying by z and differentiating the denominator with respect to D)

=$\frac{ze^{-z}}{3(-1)^2-2(-1)-1}$

=$\frac{ze^{-z}}{4}$

=$\frac{1}{4}\, ze^{-z}$

=$\frac{1}{4x} \log x$

Hence the complete solution is

Y= $(C_1 + C_2 \log x)\, x + C_3 x^{-1} + \frac{1}{4x} \log x$ (Answer)

❖ **Example 7**: Solve $x^3 \frac{d^3y}{dx^3} + 3\, x^2 \frac{d^2y}{dx^2}$ + x$\frac{dy}{dx}$ + y =x $\log x$

Solution: Putting x=e^z and $D \equiv \frac{d}{dz}$ the equation becomes

[D (D-1) (D-2) +3 D (D-1) + D + 1] y= $e^z.z$

⇒$(D^3 + 1)y = e^z.z$

A. E. is $(D^3 + 1) = 0$

⇒ (D +1)($D^2 - D + 1$)=0

So, D=-1,$\frac{1\pm\sqrt{3}i}{2}$

C.F = $C_1 e^{-z} + C_2 e^{\left(\frac{1}{2}\right)z} \cos\left(\frac{1}{2}\sqrt{3}z + C_3\right)$

=$C_1 x^{-1} + C_2\sqrt{x} \cos\left(\frac{1}{2}\sqrt{3} \log x + C_3\right)$

P. I. $=\frac{ze^z}{(D^3+1)}$

$=e^z\frac{1}{(D+1)^3+1}.z$

$=e^z\frac{2}{D^3+3D^2+3D+2}.z$

$=\frac{e^z}{2}\left(\frac{1}{2}D^3+\frac{3}{2}D^2+\frac{3}{2}D+1\right)^{-1}\cdot z$

$=\frac{e^z}{2}\left(1-\frac{2}{3}D\ \ldots\ldots\right).z$

$=\frac{e^z}{2}\left(z-\frac{3}{2}\right)$

$=\frac{x}{2}(\log x-\frac{3}{2})$

Hence the complete solution is

y=$C_1x^{-1}+C_2\sqrt{x}\cos\left(\frac{1}{2}\sqrt{3}\log x+C_3\right)+\frac{x}{2}(\log x-\frac{3}{2})$ (Answer)

❖ **Example 8**: Solve $x^4\frac{d^4y}{dx^4}+\ 2x^3\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}$ - x$\frac{dy}{dx}$ + 1 =x+$\log x$

Solution: Putting x=e^z and $D\equiv\frac{d}{dz}$ the equation becomes

[D (D-1) (D-2) (D-3) +2 D (D-1)(D-2)+ D(D-1)-D +1]y= e^z+z

Or, $(D-1)^4$y=e^z+z

A. E. is $(D-1)^4=0$

Or, D=1, 1, 1, 1

C.F. = $(C_1+C_2z+C_3z^2+C_4z^3)\ e^z$

P. I. = $\frac{e^z}{(D-1)^4}+\frac{1}{(D-1)^4}.z$ (First term case of failure)

$=z^4\frac{1}{4.3.2.1}e^z+(1-D)^{-4}.z$ [Multiplying by $z^4$and differentiating denominator of first term four times]

$=\frac{z^4}{4!}.e^{z}+(1+4D+....).z$

$$=\frac{z^4}{4!}\cdot e^{z}+z+4$$

$=\frac{(\log x)^4 x}{4!}+\log x+4$

Hence the complete solution is

$[C_1+C_2\log x+C_3(\log x)^2+C_4(\log x)^3]\,x+\frac{(\log x)^4 x}{4!}+\log x+4$

(Answer)

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Ex. 9. Solve $x^3\frac{d^3y}{dx^3}+6x^2\frac{d^2y}{dx^2}+8x\frac{dy}{dx}+2y=x^2+3x-4$

Solution.

Putting $x=e^z$ and $D\equiv d/dz$,

The equation is

$[D(D-1)(D-2)+6D(D-1)+8D+2]\,y=e^{2z}+3e^{z}-4$

The A.E. is $(D^3+3D^2+4D+2)=0$

Or, $(D+1)(D^2+2D+2)=0$

Or, $D=-1,\ \frac{-2\pm\sqrt{(4-8)}}{2},$

i e. D= -1, -1$\pm$ *i*

so, C.F. = $C_1 e^{-z} + C_2 e^{-z} \cos(z+C_3)$

P.I. $= \frac{e^{2z}+3e^{z}-4}{D^3+3D^2+4D+2}$

$= \frac{e^{2z}}{2^3+3.2^2+4.2+2} + \frac{3e^{z}}{1^3+3.1^2+4.1+2} - \frac{4}{0+0+0+2}$

$= \frac{e^{2z}}{30} + \frac{3e^{z}}{10} - 2$

$= \frac{x^2}{30} + \frac{3x}{10} - 2$

So, the complete solution is,

$y=C_1x^{-1}+ C_2x^{-1}\cos(\log x+C_3) +\frac{x^2}{30} + \frac{3x}{10} - 2$

Ex. 10. Solve $x^2\frac{d^3y}{dx^3} +2x^2 \frac{d^2y}{dx^2} +8x\frac{dy}{dx}+2y= 10\left(x + \frac{1}{x}\right)$

Solution.

Putting $x=e^z$ and D≡ d/dz,

The equation is,

[D (D-1) (D-2) + 2D (D-1) +2] y = 10 ($e^z + e^{-z}$)

The A.E. is, $D^3-D^2+2=0$

i.e. $(D^2-2D+2) =0$

Or D=-1, $\frac{-2\pm\sqrt{(4-8)}}{2}$

i.e. D= -1, 1$\pm$ *i*

so, C.F. $= C_1 e^{-z} + C_2 e^{z} \cos(z+C_3)$

$= C_1 x^{-1} + C_2 x \cos(\log x + C_3)$

P.I. $= \frac{10e^{z}}{(D+1)(D^2-2D+2)} + \frac{10e^{-z}}{(D-1)(D^2-2D+2)}$

[second term case of failure]

$= \frac{10e^{z}}{(1+1)+(1^2-2.1+2)} + z\frac{10e^{-z}}{(3D^2-2D)}$

[multiplying second term by

z and differentiating its denominator w. r. t. D]

$= 5e^{z} + z \cdot \frac{10e^{-z}}{3.\ (-1)^2(-1)}$

$= 5e^{z} + z2e^{-z}$

$= (5x + 2 \log x\frac{1}{x})$ as $x=e^{z}$, $z = \log x$

Hence the complete solution is ,

$Y = C_1 x^{-1} + C_2 x \cos(\log x + C_3) + 5x + 2 \log x\frac{1}{x}$

2021

MATH ASSIGNMENT

TOPIC: Solution of differential equation when dependent and independent variables are absent

Submitted to: MRS. RUPALE KHATUN

ROLL NO.
1800096,1800097,1800098,1800099,1800100,1800101,1800102,1800103,1800104,1800105,1800106,1800107,1800108,1800109,1800110

6/1/2021

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EQUATIONS THAT DO NOT CONTAIN "y" DIRECTLY :
The equations that do not contain x directly are of the form ,

f($\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots \ldots \frac{dy}{dx}, x$) = 0 ………………….(1)

On substituting $\frac{dy}{dx} = P\ i.e, \frac{d^2y}{dx^2} =, \frac{dP}{dx}, \frac{d^3y}{dx^3} = \frac{d^2P}{dx^2}\ etc.$ in the equation (1), we get

[$\frac{dP^{n-1}}{dy^{n-1}}, \ldots \ldots P, x$] = 0

Example: 83. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + (\frac{dy}{dx})^3 = 0$

Solution. On putting $\frac{dy}{dx} = P, \frac{d^2y}{dx^2} = \frac{dP}{dx}$ in equation (1) becomes

$$\frac{dP}{dx} + P + P^3 = 0\ or\ \frac{dP}{dx} + P(1+P^2) = 0$$
$$\frac{dP}{dx} = -P(1+P^2)\ or\ \frac{dp}{P(1+P^2)} = -dx \Rightarrow \left(\frac{1}{P} - \frac{P}{1+P^2}\right) dP = -dx$$

On integrating, we have

$$\log P - \frac{1}{2}\log(1+P^2) = -x + c_1\ or\ log\frac{P}{\sqrt{1+P^2}} = -x + c_1$$

$$\frac{P}{\sqrt{1+P^2}} = e^{-x+c_1} \text{or} \frac{P^2}{1+P^2} = a^2e^{-2x} \Rightarrow P^2 = (1+P^2)a^2e^{-2x}$$

$$\Rightarrow P^2(1-a^2e^{-2x}) = a^2e^{-2x} \Rightarrow P = \frac{ae^{-x}}{\sqrt{1-a^2e^{-2x}}} \Rightarrow \frac{dy}{dx} = \frac{ae^{-x}}{\sqrt{1-a^2e^{-2x}}}$$

$$dy = \frac{ae^{-x}}{\sqrt{1-a^2e^{-2x}}}dx$$

On integration, we get $y = -\sin^{-1}(ae^{-x}) + b$ **(Ans.)**

NAME: SUMAIYA HASAN
ROLL: 1800097

Example 84: Solve, $\frac{d^2y}{dx^2} = [1 - (\frac{dy}{dx})^2]^{1/2}$

Solution: We have, $\frac{d^2y}{dx^2} = [1 - (\frac{dy}{dx})^2]^{1/2} \ldots \ldots (1)$

Since, DE (1) does not contain y directly and the lowest differential co-efficient is $\frac{dy}{dx}$

Let, $\frac{dy}{dx} = p$ or, $\frac{d^2y}{dx^2} = \frac{dp}{dx}$

Putting these values in (1) we get,

$\frac{dp}{dx}$ = [1 - $p^2]^{1/2}$

Or, $\frac{dp}{dx} = \sqrt{1 - p^2}$

Or, $\frac{dp}{\sqrt{1-p^2}} = dx$

On integrating we have, $\sin^{-1} p = x + c$

Or, $p = \sin(x + c)$

Or, $\frac{dy}{dx} = \sin(x + c)$

On integrating we have,

y = - $\cos(x + c) + c_1$

(Ans.)

Roll:1800098

NAME: MD. RAKIB HOSSAIN

Example85. *Solve:* $x\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx} = 0$

Solution. On putting $\frac{dy}{dx} = P$ and $\frac{d^2y}{dx^2} = \frac{dP}{dx}$ in the given equation, we get

$$x\frac{dP}{dx} + xP^2 - P = 0 \Rightarrow \frac{1}{P^2}\frac{dP}{dx} - \frac{1}{P}\frac{1}{x} = -1 \qquad ...(1)$$

Again putting $\frac{1}{P} = z$ so that $-\frac{1}{P^2}\frac{dP}{dx} = \frac{dz}{dx}$

Equation (1) becomes $-\frac{dz}{dx} - \frac{z}{x} = -1 \Rightarrow \frac{dz}{dx} + \frac{z}{x} = 1$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Hence, solution is $zx = \int x\,dx + c_1$ or $zx = \frac{x^2}{2} + c_1$ or $\frac{1}{P}x = \frac{x^2}{2} + c_1$

$$\Rightarrow \frac{x}{p} = \frac{x^2 + 2c_1}{2} \Rightarrow P = \frac{2x}{x^2 + 2c_1} \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2 + 2c_1} \Rightarrow dy = \frac{2x}{x^2 + 2c_1}dx$$

On integrating, we have $y = \log(x^2 + 2c_1) + c_2$ **Ans**

NAME: DHRUVA

ROLL: 1800099

(1+x)$\frac{d^2y}{dx^2}$+x$\frac{dy}{dx}$+ax=0

Solution: Putting, $\frac{\mathrm{dy}}{\mathrm{dx}} = p \ and \ \frac{d^2y}{dx^2} = \frac{dp}{dx}$

In the given solution we get,

(1+x)$\frac{dp}{dx}$+xp+ax=0

=>(1+x)$\frac{dp}{dx}$=-x(p+a)

=>$\frac{dp}{dx}$= - $\frac{x(p+a)}{(1+x^2)}$

=>$\frac{dp}{(p+a)} = -\frac{xdx}{1+x^2}$

=>∫$\frac{dp}{(p+a)} = -\int \frac{xdx}{1+x^2}$

Here,

z=1+x^2

i.e. $\frac{dz}{dx} = 2x$

i.e.,$\frac{dz}{2} = xdx$

=>log (p+a) =-∫$\frac{dz}{2z}$

=>log (p+a) = -$\frac{1}{2}$$logz$

=>log $c_{1 = \log}$ (p+a) +log√(1+x^2)

=>C_1= (p+a) √ (1+x^2)

$=> C1 = p\sqrt{(1} + x^2) + a\sqrt{(1} + x^2)$

$=> p\sqrt{(1} + x^2) = C1 - a\sqrt{(1} + x^2)$

=> p= $\frac{c1}{\sqrt{(1}+x^2)} - a$

$=> \frac{dy}{dx} = \frac{c1}{\sqrt{(1}+x^2)} - a$

$=> \int dy = C1\int \frac{dx}{\sqrt{(1}+x^2)} - \int adx$

$=> y = c1 * log\left|\sqrt{(1} + x^2) + x\right| + c2 - ax$

$=> y = C2 - ax + c1 * log\left|\sqrt{(1} + x^2) + x\right|$

This is the required ans.

NAME: NAHIAN BIN OBAIED

ROLL: 1800100

Ex. 3: Solve: $(1 + x^2)\frac{d^2y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0.$

Solution: The equation does not contain y directly.

Putting $\frac{dy}{dx} = p,\ \frac{d^2y}{dx^2} = \frac{dp}{dx^2}$, the equation becomes

$(1 + x^2)\frac{dp}{dx} + 1 + p^2$ (Variables separable)

i.e. $\frac{dp}{1+p^2} + \frac{dx}{1+x^2} = o;\ \therefore \tan^{-1} p + \tan^{-1} x = \tan^{-1} c_1$

Or, $\tan^{-1}\frac{p+x}{1-px}=\tan^{-1}c_1$ or. $\frac{p+x}{1-px}=c_1$

i.e. $(p+x)=c_1 \ (1-px)$ or $p=\frac{dy}{dx}=\frac{c_1-x}{1+c_1x}=\frac{1}{c_2}\left[\frac{1+c_1^{\ 2}}{1+c_1x}-1\right]$

Integrating, $y=\frac{1+c_1^{\ 2}}{c_1^{\ 2}}\log(1+c_1x)-\frac{1}{c_1}x+c_2$ this is the general solution.

Name: Sazzatul Amin Rafi

Roll :**1800101**

Exercise: 03

Solve $\frac{d^4y}{dx^4}$ -cot x $\frac{d^3y}{dx^3}$ =0

<u>Solution:</u> *Equation is free from y and lowest differential coefficient is* $\frac{d^3y}{dx^3}$; *hence putting* $\frac{d^3y}{dx^3}$ =*q* ,$\frac{d^4y}{dx^4} = \frac{dq}{dx}$, *the equation becomes* $\frac{dq}{dx} - \cot x.q = 0$

Or $\frac{dq}{q} - \cot x \, dx = 0$

Integrating $\log q - \log \sin x = \log c_1$

Or $q = c_1 \sin x$

i.e. .$\frac{d^3y}{dx^3} = c_1 \sin x$

Integrating,$\frac{d^2y}{dx^2}$= $-c_1 \cos x + c_2$

Integrating again,$\frac{dy}{dx}$ = - $c_1 \sin x + c_2x + c_3$

And then,

$y = c_1 \cos x + \frac{1}{2}c_2x^2 + c_3 + c_4$.

NAME: SANJIDA AKTER

ROLL: **1800102**

PROBLEM = $2x\frac{d^3y}{dx^3}$ x $\frac{d^2y}{dx^2} = \left(\frac{d^2y}{dx^2}\right)^2 - a^2$

<u>SOLUTION:</u> Given that,

$$2x\frac{d^3y}{dx^3} \text{ x } \frac{d^2y}{dx^2} = \left(\frac{d^2y}{dx^2}\right)^2 - a^2 \quad \text{-----------------} \quad (i)$$

The equation does not contain y directly

Also, lowest differential is $\frac{d^2y}{dx^2}$

So put,

$$\frac{d^2y}{dx^2} = p;$$

$$\frac{d^3y}{dx^3} = \frac{dp}{dx}$$

Then equation (i) becomes;

$$2x\frac{dp}{dx} \times p = p^2 - a^2$$

$$\Rightarrow 2p\frac{dp}{dx} = \frac{p^2}{x} - \frac{a^2}{x} \text{ --------------- (ii)}$$

Now,

$$p^2 = z$$

$$\Rightarrow 2p\frac{dp}{dx} = \frac{dz}{dx}$$

From equation (ii)

$$\frac{dz}{dx} = \frac{z}{x} - \frac{a^2}{x}$$

$$\frac{dz}{dx} - \frac{z}{x} = -\frac{a^2}{x}$$

Linear equation,

IF = $e^{\int -\frac{1}{x}dx}$

$=\frac{1}{x}$

$$\therefore \frac{z}{x} = c_1 - \int \frac{a^2}{x^2}dx$$

$$\Rightarrow \frac{z}{x} = c_1 + \frac{a^2}{x^2}$$

$$\Rightarrow z = c_1x + a^2$$

$$\Rightarrow p^2 = (c_1x + a^2)[\because P^2 = z]$$

$$\Rightarrow p = (c_1x + a^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = (c_1x + a^2)^{\frac{1}{2}} \left[\because p = \frac{d^2y}{dx^2} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(c_1x+a^2)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\times c_1} + c_2 [\, by\ integrating]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(c_1x + a^2)^{\frac{3}{2}}}{\frac{3}{2} \times c_1} + c_2$$

$$\Rightarrow y = \frac{(c_1x + a^2)^{\frac{3}{2}+1}}{\frac{3}{2}\left(\frac{3}{2} + 1\right) \times c_1 \times c_1} + c_2x + c_3 \ \ [\, again\ integrating]$$

$$\Rightarrow y = \frac{(c_1x+a^2)^{\frac{5}{2}}}{\frac{3}{2}\times\frac{5}{2}\times c_1^2} + c_2x + c_3$$

$$\Rightarrow 15c_1^2y = 4(c_1x + a^2)^{\frac{5}{2}} + c_2x + c_3 \ \ [\, as\ c_2, c_3\ \ constant\ ]$$

$\therefore$The required general solution is

$$15c_1^2y = 4(c_1x + a^2)^{\frac{5}{2}} + c_2x + c_3$$

NAME: MAHMUD KAISER TUSHAR
ROLL: 1800103

PROBLEM: 5 Solve. $\boldsymbol{e^{\frac{x^2}{2}}\left[x\frac{d^2y}{dx^2} - \frac{dy}{dx}\right] = x^3}$

SOLUTION: On putting $\frac{dy}{dx} = P$ and $\frac{d^2y}{dx^2} = \frac{dP}{dx}$ in the given equation we get,

$$\boldsymbol{e^{\frac{x^2}{2}}\left[x\frac{dP}{dx} - P\right] = x^3}$$

$$\boldsymbol{\Rightarrow x\frac{dP}{dx} - P = \frac{x^3}{e^{\frac{x^2}{2}}}}$$

$$\boldsymbol{\Rightarrow \frac{dP}{dx} - \frac{P}{x} = \frac{x^2}{e^{\frac{x^2}{2}}}}$$

Now,

I.F. $= e^{\int P(x)dx}$

$= e^{\int -\frac{dx}{x}}$

$= \frac{1}{x}$

Now,

$P\times (I.F.) = \int Q \times (I.F.)dx + c_1$

$$\Rightarrow P \times \frac{1}{X} = \int \frac{\boldsymbol{x^2}}{\boldsymbol{e^{\frac{x^2}{2}}}} \times \frac{1}{X} dx + c_1$$

$\Rightarrow\ P \times \frac{1}{X} = \int \frac{dz}{e^z} + c_1$ **Let,**

$\Rightarrow\ P \times \frac{1}{X} = -\frac{1}{e^z} + c_1$ $\qquad$ $z = \frac{\boldsymbol{x^2}}{\boldsymbol{2}}$

$\Rightarrow P = c_1 x - \frac{x}{e^z}$ $\Rightarrow \frac{dz}{dx} = x$

$\Rightarrow\ \frac{dy}{dx} = c_1 x - \frac{x}{e^z}$ $\Rightarrow$ dz = xdx

$$\Rightarrow \int dy = \int (c_1 x - \frac{x}{e^z})dx$$

$\Rightarrow y = c_1 \frac{x^2}{2} - \frac{x^2}{2e^{\frac{x^2}{2}}} + c_2$ **(Ans.)**

NAME : MOHAMMAD LABIB SARWAR
ROLL : 1800104

EQUATIONS THAT DO NOT CONTAIN "x" DIRECTLY :

The equations that do not contain x directly are of the form ,

$f(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots \ldots \frac{dy}{dx}, y) = 0$ …………………(1)

On substituting $\frac{dy}{dx} = P, \frac{d^2y}{dx^2} = \frac{dP}{dx} = \frac{dP}{dx}.\frac{dy}{dx} = \frac{dP}{dy}.P$ in the equation (1), we get

$[\frac{dP^{n-1}}{dy^{n-1}}, \ldots \ldots P, y] = 0$ ………………………..(2)

Equation (2) is solved for P. Let

$P = f_1(y) \Rightarrow \frac{dy}{dx} = f_1(y)\ or, \frac{dy}{f_1(y)} = dx\ \Rightarrow \int \frac{dy}{f_1(y)} = x + c$

Example: 86. Solve $y\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 = \frac{dy}{dx}$

Solution. Put $\frac{dy}{dx} = P, \frac{d^2y}{dx^2} = \frac{dP}{dx} = \frac{dP}{dy}.\frac{dy}{dx} = P\frac{dP}{dx}$ in equation (1)

$yP\frac{dP}{dx} + P^2 = P \Rightarrow y\frac{dP}{dx} = 1 - P$

$$\Rightarrow \frac{dP}{1-P} = \frac{dy}{y} \Rightarrow -\log(1-P) = log y + log c_1$$

$$\Rightarrow \frac{1}{1-P} = c_1 y \Rightarrow P = 1 - \frac{1}{c_1 y}\ or\ \frac{dy}{dx} = \frac{c_1 y - 1}{c_1 y}$$

$$\Rightarrow \frac{c_1 y}{c_1 y - 1} dy = dx \;\Rightarrow \left(1 + \frac{1}{c_1 y - 1}\right) dy = dx$$

y + $\frac{1}{c_1}$log $(c_1 y - 1) = x + c_1$ **(Ans.)**

NAME: NUR AZAM SWAPAN
Roll: 1800105

Solve: y$\frac{d^2y}{dx^2}$ + $(\frac{dy}{dx})^2$ = y^2… (1)

Put $\frac{dy}{dx}$ = P,$\frac{d^2y}{dx^2}$ = $\frac{dP}{dx}$ = $\frac{dP}{dy}\frac{dy}{dx}$ = P$\frac{dP}{dy}$ in (1)

yP$\frac{dP}{dy}$ + P^2 = y^2 or P$\frac{dP}{dy}$ + $\frac{p^2}{y}$ = y … (2)

Put P^2 = z or 2P$\frac{dP}{dy}$ = $\frac{dz}{dy}$ in (2),$\frac{1}{2}\frac{dz}{dy}$ + $\frac{z}{y}$ = y or $\frac{dz}{dy}$ + $\frac{2z}{y}$ = 2y

I.F. = $e^{\int \frac{2}{y} dy}$ = $e^{2\log y}$ = $e^{\log y^2}$ = y^2

Here, the solution is z y^2 =∫ $2y.(y^2)$ dy + c

»$P^2 y^2$ = $\frac{y^4}{2}$ + c

» 2 $P^2 y^2$ = y^4 + k or $\sqrt{2}$yP = $\sqrt{y^4 + k}$… [Put 2c=k]

»$\sqrt{2}$ y$\frac{\text{dy}}{\text{dx}}$ = $\sqrt{y^4 + k}$ or $\sqrt{2}\frac{ydy}{\sqrt{y^4+k}}$ = dx

»$\frac{1}{\sqrt{2}}\frac{dt}{\sqrt{t^2+k}}$ =dx [Put y^2= t, 2y dy = dt]

»$\frac{1}{\sqrt{2}}$sin$h^{-1}\frac{1}{\sqrt{k}}$ = x + c

» sin $h^{-1}\frac{y^2}{\sqrt{k}}$ = $\sqrt{2}x$ + c or y^2 = $\sqrt{k}$sinh($\sqrt{2}$x + c) **Ans.**

NAME : SHAHADAT ISLAM
ROLL : 1800106

Example: 4 Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + (\frac{dy}{dx})^3 = 0$

Solution : On putting $\frac{dy}{dx} = P, \frac{d^2y}{dx^2} = \frac{dP}{dx}$, equation (1) becomes

$$\frac{dP}{dx} + P + P^3 = 0 \; or \; \frac{dP}{dx} + P(1 + P^2) = 0$$

$$\frac{dP}{dx} = -P(1 + P^2) \; or \; \frac{dp}{P(1 + P^2)} = -dx \;\Rightarrow \left(\frac{1}{P} - \frac{P}{1 + P^2}\right) dP = -dx$$

On integrating, we have

logP$-\frac{1}{2}$log$(1 + P^2) = -x + c_1$

$$or\ log\frac{P}{\sqrt{1+P^2}} = -x + c_1$$

$\frac{P}{\sqrt{1+P^2}} = e^{-x+c_1}$ or $\frac{P^2}{1+P^2} = a^2e^{-2x} \Rightarrow P^2 = (1+P^2)a^2e^{-2x}$

$$\Rightarrow P^2(1-a^2e^{-2x}) = a^2e^{-2x} \Rightarrow P = \frac{ae^{-x}}{\sqrt{1-a^2e^{-2x}}} \Rightarrow \frac{dy}{dx} = \frac{ae^{-x}}{\sqrt{1-a^2e^{-2x}}}$$

dy = $\frac{ae^{-x}}{\sqrt{1-a^2e^{-2x}}}dx$

On integration, we get

y = $-\sin^{-1}(ae^{-x}) + b$**(Ans.)**

NAME: MD. SHAMIUL ISLAM

ROLL: 1800107

PROBLEM: $\mathbf{2x\frac{d^3y}{dx^3} \times \frac{d^2y}{dx^2} = \left(\frac{d^2y}{dx^2}\right)^2 - a^2}$

<u>**SOLUTION:**</u> Given that,

$2x\frac{d^3y}{dx^3} \times \frac{d^2y}{dx^2} = \left(\frac{d^2y}{dx^2}\right)^2 - a^2$----------------- (i)

The equation does not contain y directly

Also, lowest differential is $\frac{d^2y}{dx^2}$

So put,

$$\frac{d^2y}{dx^2} = p;$$

$$\frac{d^3y}{dx^3} = \frac{dp}{dx}$$

Then equation (i) becomes;

$$2x\frac{dp}{dx} \times p = p^2 - a^2$$

$\Rightarrow 2p\frac{dp}{dx} = \frac{p^2}{x} - \frac{a^2}{x}$--------------- (ii)

Now,

$$p^2 = z$$

$$\Rightarrow 2p\frac{dp}{dx} = \frac{dz}{dx}$$

From equation (ii)

$$\frac{dz}{dx} = \frac{z}{x} - \frac{a^2}{x}$$

$$\frac{dz}{dx} - \frac{z}{x} = -\frac{a^2}{x}$$

Linear equation,

I.F. $= e^{\int -\frac{1}{x}dx}$

$=\frac{1}{x}$

$$\therefore \frac{z}{x} = c_1 - \int \frac{a^2}{x^2}dx$$

$$\Rightarrow \frac{z}{x} = c_1 + \frac{a^2}{x^2}$$

$$\Rightarrow z = c_1x + a^2$$

$$\Rightarrow p^2 = (c_1x + a^2)[\because P^2 = z]$$

$$\Rightarrow p = (c_1x + a^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{d^2y}{dx^2} = (c_1x + a^2)^{\frac{1}{2}}\left[\because p = \frac{d^2y}{dx^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(c_1x+a^2)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\times c_1} + c_2[by\ integrating]$$

$$\Rightarrow \frac{dy}{dx} = \frac{(c_1x + a^2)^{\frac{3}{2}}}{\frac{3}{2}\times c_1} + c_2$$

$$\Rightarrow y = \frac{(c_1x + a^2)^{\frac{3}{2}+1}}{\frac{3}{2}\left(\frac{3}{2}+1\right)\times c_1 \times c_1} + c_2x + c_3\ \ [again\ integrating]$$

$$\Rightarrow y = \frac{(c_1x+a^2)^{\frac{5}{2}}}{\frac{3}{2}\times\frac{5}{2}\times c_1^2} + c_2x + c_3$$

$$\Rightarrow 15c_1^2y = 4(c_1x + a^2)^{\frac{5}{2}} + c_2x + c_3\ \ [as\ c_2, c_3\ \ constant\ ]$$

∴The required general solution is

$$15c_1^2y = 4(c_1x + a^2)^{\frac{5}{2}} + c_2x + c_3$$

(Ans.)

Roll: **1800108**

Name: Tanvir Ahmed

Question 3: solve

$2y\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2 - 4y^2 = 0 \ldots\ldots\ldots.(i)$

Solution:

Put $\frac{dy}{dx} = P$

$$\frac{d^2y}{dx^2} = \frac{dP}{dx} = \frac{dP}{dy} \cdot \frac{dy}{dx} = p\frac{dP}{dy}$$

Now from (i) *we get*

$2yP\frac{dP}{dy} - 3P^2 = 4y^2$.........................(ii)

Put $p^2 = z$

$\Rightarrow 2P\frac{dP}{dy} = \frac{dz}{dy}[Diff\ w.r.t.y]$

$$\Rightarrow P\frac{dP}{dy} = \frac{1}{2}\frac{dz}{dy}$$

From (ii) *we get*

$$y\frac{dz}{dy} - 3z = 4y^2$$

$$\Rightarrow \frac{dz}{dy} - 3\frac{z}{y} = 4y$$

I.F. $= e^{\int -\frac{3}{y}dy} = e^{-3logy} = e^{log}y^{-3} = y^{-3} = \frac{1}{y^3}$

Hence the solution is,

$$\Rightarrow \frac{z}{y^3} = \int \frac{4y}{y^3} dy$$

$$\Rightarrow \frac{z}{y^3} = 4\int \frac{dy}{y^2}$$

$$\Rightarrow \frac{p^2}{y^3} = -\frac{4}{y} + C$$

$$\Rightarrow p^2 = -4y^2 + Cy^3$$

$$\Rightarrow P = \sqrt{cy^3 - 4y^2}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{Cy^3 - 4y^2}$$

$$\Rightarrow \int dx = \int \frac{dy}{\sqrt{Cy^3 - 4y^2}}$$

$$\Rightarrow x + b = \int \frac{dy}{y\sqrt{Cy - 4}}$$

Let, $cy - 4 = t^2$

$$\Rightarrow c = 2 + \frac{dt}{dy}$$

$$\Rightarrow dy = \frac{2 + dt}{c}$$

$$\Rightarrow x + b_1 = \int \frac{2 + dt}{\frac{(t^2+4)\,.t.c}{c}}$$

$$\Rightarrow x + b_1 = 2\int \frac{dt}{t^2 + 4}$$

$$\Rightarrow x + b_1 = \frac{1}{2}\,.\,2(tan)^{-1}\left(\frac{t}{2}\right) + b_2$$

$$\Rightarrow x + b = (tan)^{-1}\frac{t}{2}\,[b_1 + b_2 = b]$$

$$\Rightarrow tan(x + b) = \frac{t}{2}$$

$$\Rightarrow 2tan(x + b) = \sqrt{Cy - 4}$$

$$\Rightarrow 4tan^2(x + b) = cy - 4$$

$$\Rightarrow 4\{sec^2(x + b) - 1\} = Cy - 4$$

$\Rightarrow 4sec^2(x+b) - 4 = cy - 4$

$\Rightarrow y = \frac{4}{C} sec^2(x+b)$

$y = asec^2(x+b)$ $\quad\quad [\frac{4}{c} = a]$

(Ans.)

Name: Nazmul Huda

Roll: 1800109

Exercise 6: Solve, $y\frac{d^2y}{dx^2} - (\frac{dy}{dx})^2 = y^2 \log y$

Solution: The equation does not contain x directly;

Hence putting $\frac{dy}{dx} = p,\ \ or, \frac{d^2y}{dx^2} = \frac{dp}{dy}.\frac{dy}{dx} = p\frac{dp}{dy}$

The equation becomes,

$$yp\frac{dp}{dy} - p^2 = y^2 \log y, \quad or, p\frac{dp}{dy} - \frac{1}{y}p^2 = y \log y$$

Putting $p^2 = v,\ 2p\frac{dp}{dy} = \frac{dv}{dy}$;

The above equation becomes; $\frac{dv}{dy} - \frac{2}{y}v = 2y \log y$

Linear equation, I.F. = $e^{-\int\frac{2}{y}dy} = \frac{1}{y^2}$

$$\therefore\ v\frac{1}{y^2} = c_1 + \int 2y \log y . \frac{1}{y^2} dy = c_1 + \int \frac{2}{y} \log y\, dy$$

Or, $p^2.\frac{1}{y^2} = c_1 + (\log y)^2$

Or, p = $\frac{dy}{dx} = \pm y\, [\, c_1 + (\log y)^2]^{\frac{1}{2}}$

Or, $\frac{dy}{y\sqrt{[c_1+(\log y)^2]}} = dx$ $\quad\quad$; $\ [\, put\ \log y$ = u, $\frac{1}{y}dy = du\,]$

$\therefore\ \frac{du}{\sqrt{(c_1+u^2)}} = dx$ $\quad$ or, $\log[\, u + \sqrt{(c_1+u^2)}] = \log c_2 + x$

Or, $u + \sqrt{(c_1+u^2)} = c_2 e^x$ $\quad\quad or,\ \log y + \sqrt{[c_1 + (\log y)^2]} = c_2 e^x$

Or, $c_1 + (\log y)^2 = (c_2 e^x - \log y)^2$

Or, $c_1 = {c_2}^2 e^{2x} - 2c_2 e^x \, log\, y$

Or, log y = $k_1 e^x + k_2 e^{-x}$

(Ans.)

Roll: **1800110**

Name: Md. Arifuzzaman

Problem: solve

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 = 0 \dots\dots\dots\dots\dots..(i)$$

Since (i) $contain\ x\ directly\ and\ then$

put $\frac{dy}{dx} = p$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dP}{dx} = \frac{dP}{dy}.\frac{dy}{dx} = p\frac{dP}{dy}$$

Then (i) $becomes,$

$$P\frac{dP}{dy} + p + P^3 = 0$$

$$\Rightarrow \frac{dP}{dy} + 1 + P^2 = 0$$

$$\Rightarrow \frac{dP}{1 + P^2} + dy = 0$$

$$\Rightarrow tan^{-1}P + y = C_1$$

$$\Rightarrow tan^{-1}P = c_1 - y$$

$$\Rightarrow P = \tan(C_1 - y)$$

$$\Rightarrow \frac{dy}{dx} = \tan(C_1 - y)$$

$$\Rightarrow \frac{dy}{tan(C_1 - y)} = dx$$

$$\Rightarrow \cot(C_1 - y)\, dy = dx$$

$$\Rightarrow -logsin(c_1 - y) = x + logc_2$$

$\Rightarrow logsin(c_1 - y) = -x - logC_2$

$\Rightarrow logsin(c_1 - y) = loge^{-x} - logC_2$

$\Rightarrow C_2 \sin(C_1 - y) = e^{-x}$

$\Rightarrow \sin\,(C_1 - y) = \frac{1}{C_2} e^{-x}$

$\Rightarrow C_1 - y = sin^{-1}(be^{-x}) \left[\frac{1}{C_2} = b\right]$

$y = a - \; sin^{-1}(be^{-x})[c_1 = a]$

(Ans.)

MATHEMATICS 2101

RUET

Assignment on

Solution of DE of first order and higher degrees

Department of CIVIL Engineering

Contains –

Submitted by : 1800111, 1800112, 1800113, 1800114, 1800115, 1800116, 1800117, 1800118, 1800119, 1800120, 1800121, 1800122, 1800123, 1800124, 1800125, 1800126, 1800127, 1800128.

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Solution of Differential Equation of First Order and Higher Degree

Definition

The differential equation of first order do not contain differential co-efficient higher than $\frac{dy}{dx}$. In this chapter we shall consider differential equations which involve powers of $\frac{dy}{dx}$. It is usual to denote $\frac{dy}{dx}$ by p. Thus an equation

$$p^n + p_1 p^{n-1} + p_2 p^{n-2} + \cdots \dots \dots . + p_n = 0$$

Where, p_1, p_2, p_n are functions of x and y, is the equation of first order and n^{th} degree.

Types of Equations

It may be possible to solve such equations by one or more of the four methods given below. In each case the problem is reduced to that of solving one or more equations of first order and first degree.

Equations Solvable for p.

Suppose the equation

$$p^n + p_1 p^{n-1} + p_2 p^{n-2} + \cdots \dots \dots . + p_n = 0$$

can be put in the form

$$[p - F_1(x, y)[p - F_2(x, y\ p - F_n(x, y)] = 0.$$

Then equation to zero each factor of the above form, we got n equations of first order and first degree, namely

$\frac{dy}{dx} = F_1(x, y), \frac{dy}{dx} = F_2(x, y) \dots \dots, \frac{dy}{dx} = F_n(x, y)$

If solutions of the above n component equations are given by

$f_1(x, y, c_1) = 0, f_2(x, y, c_2) = 0, \dots\dots f_n(x, y, c_n) = 0.$

then the relation

$f_1(x, y, c_1)\ f_2(x, y, c_2) \dots . f_n(x, y, c_n) = 0.$

is the most general solution of the equation (1).

There is no loss of generality f we take

$c_1 = c_2 = ... = c_n = c$ (say)

Therefore the general solution of the equation s put as

$f_1(x, y, c)\ f_2(x, y, c)\ ....f_n(x, y, c) = 0.$

Roll: 1800112 .

Ex.1. Solve $p^4-(x+2y+1)p^3+(x+2y+2xy)p^2-2xyp=0$

Solution:

On factorization the given equation becomes,

$P(p-1)(p-x)(p-2y)=0$

The component equations of first order and first degree are,

$p=0,\ p=1,\ p=x,\ p=2y$

Or, $\frac{dy}{dx}=0,\ \frac{dy}{dx}=1,\ \frac{dy}{dx}=x,\ \frac{dy}{dx}=2y$

Solutions of these component equations are respectively,

$y-c=0,\ y-x-c=0,\ 2y-x^2-c=0,\ y-ce^x=0$

and therefore the most general solution of the given equation is,

$(y-c)(y-x-c)(2y-x^2-c)(y-ce^x)=0$

Ex. 2. Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

Solution:

The equation can be written as,

$p(\ p^2 + 2xp - y^2p - 2xyp\ ) = 0$

or $p(p+2x)(p-y^2) = 0$

The component equations are

$$\frac{dy}{dx} = 0, \quad \frac{dy}{dx} + 2x = 0, \quad \frac{dy}{dx} - y^2 = 0$$

Solution of these component equations are

$y - c = 3$, $y + x^2 - c = 0$, $xy + yc + 1 = 0$

Therefore the general solution is

$(y - c)(y + x^2 - c)(xy + yc + 1) = 0$.

Ex .3. Solve $xy(p^2+1) = (x^2+y^2)p$

Solution:

The equation can be written as

$xyp^2 - (x^2+y^2)p + xy = 0$

i.e. $(yp - x)(xp - y) = 0$

Thus the component equations are

$y\frac{dy}{dx} - x = 0$, $x\frac{dy}{dx} - y = 0$,

i.e. $ydy - xdx = 0$, $\frac{dy}{y} - \frac{dx}{x} = 0$,

whose solutions are $y^2 - x^2 = c$, $\frac{y}{x} = c$

Hence the general solution is

$(y^2 - x^2 - c) = 0$, $(y - cx) = 0$.

Roll: 1800113

Ex. 4. Solve $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$

Solution:

Writing p for $\frac{dy}{dx}$, the equation becomes

$x^2p^2 + pxy - 6y^2 = 0$,

i.e. $(px + 3y)(px - 2y) = 0$

The component equations are

$x\frac{dy}{dx} + 3y = 0$ and $x\frac{dy}{dx} - 2y = 0$

or $\frac{dy}{y}+3\frac{dx}{y}=0$ and $\frac{dy}{y}-2\frac{dx}{x}=0$

Integrating these $yx^2=c$ and $y/_{x^2}=c$

Hence the solution is $(yx^2-c)(y/_{x^2}-c)=0$.

Ex. 5. Solve $x^2(\frac{dy}{dx})^2-2xy\frac{dy}{dx}+2y^2-x^2=0$

<u>Solution:</u>

Writing p for $\frac{dy}{dx}$, the equation becomes

$x^2p^2-2xyp+2y^2-x^2=0$

Solving for p,

$p=\frac{2xy\pm\sqrt{[4x^2y^2-4x^2(2y^2-x^2)]}}{2x^2}$

$=\frac{y\pm\sqrt{(x^2-y^2)}}{x}$.

the component equations are $\frac{dy}{dx}=\frac{y\pm\sqrt{(x^2-y^2)}}{x}$.

These are homogeneous;

∴ put p= vx, so that

$v+x\frac{dv}{dx}=\frac{v\pm\sqrt{1-v^2}}{1}$, i.e., $x\frac{dv}{dx}=\pm\sqrt{1-v^2}$

or $\frac{dv}{\sqrt{(1-v^2)}}=\frac{dx}{x}$ and $\frac{dv}{\sqrt{(1-v^2)}}=-\frac{dx}{x}$.

Integrating, $\sin^{-1}y=\log cx$ and $\sin^{-1}y=-\log cx$,

i.e., $\sin^{-1}\left(\frac{y}{x}\right)=\pm\log cx$ as $y=\frac{y}{x}$,

which form the required solution.

Ex. 6. Solve $x^2(\frac{dy}{dx})^2+3xy\frac{dy}{dx}+2y^2=0$

<u>Solution:</u>

The equation can be written as

$x^2p^2 + 3xyp + 2y^2 = 0$,   i.e. $(xp + y)(xp + 2y) = 0$.

The component equations are

$x\frac{dy}{dx} + y = 0$ and $x\frac{dy}{dx} + 2y = 0$,

or $\frac{dy}{y} + \frac{dx}{x} = 0$ and $\frac{dy}{y} + 2\frac{dx}{x} = 0$,

Integrating, $xy=c$ and $yx^2=c$,

Hence the solution is $(xy-c)(yx^2-c) = 0$.

Roll: 1800114

Ex. 7. (a) Solve $yp^2 + (x-y)p - x = 0$.

Solution:

We have $(p-1)(yp+x) = 0$.

i.e. $\frac{dy}{dx} = 1$ and $y\frac{dy}{dx} + x = 0$ or $y\,dy + x\,dx = 0$

Integrating, $y = x + c$ and $x^2 + y^2 = c$.

The solution is $(y - x - c)(x^2 + y^2 - c) = 0$.

Ex. 7. (b) Solve $xp^2 + (y-x)p - y = 0$.

Solution:

We have $(p-1)(xp+y) = 0$

i.e. $\frac{dy}{dx} = 1$ and $x\frac{dy}{dx} + y = 0$ or $x\,dy + y\,dx = 0$

Integrating, $y = x-c$ and $xy = -c$.

The solution is $(y - x + c)(xy + c) = 0$.

Roll: 1800115

Ex. 8. Solve $p^2 - p(x^2+xy+y^2) + xy(x+y) = 0$

Solution:

After factorizing, the equation can be written as,

$(p-x)[p^2+px-y(x+y)] = 0$

or $(p-x)(p-y)[p+(x+y)] = 0$

The component equations are

$\frac{dy}{dx} = x, \quad \frac{dy}{dx} = y, \quad \frac{dy}{dx} + x + y = 0.$

Solution of $\frac{dy}{dx} = x$ is $y = \frac{x^2}{2}$ + const., i.e. $2y-x^2=c$,

Solution of $\frac{dy}{dx} = y$ is $\log y = x$ + const., i.e. $y=ce^x$,

and solution of $\frac{dy}{dx} + y = -x$ (linear equation) is

$ye^x = c + \int -xe^x dx$,

i.e. $ye^x = c-(x-1)e^x$

or $y+x-1-ce^{-x} = 0$.

Therefore the complete solution is

$(2y-x^2-c)(y-ce^x)(y+x-1-ce^{-x}) = 0.$

Ex. 9. (a) Solve $p^3(x+2y) + 3p^2(x+y) + (y+2x)p = 0$.

Solution:

On factorizing, the equation is

$p(p+1)(px + 2py + 2x + y) = 0.$

The component equations are

$\frac{dy}{dx} = 0, \quad \frac{dy}{dx} + 1 = 0, \quad \frac{dy}{dx}(x + 2y) + 2x + y = 0$

Solution of $\frac{dy}{dx} = 0$ is $y=c$

Solution of $\frac{dy}{dx} + 1 = 0$, is $y+x = c$.

Solution of $\frac{dy}{dx}(x+2y) + 2x + y = 0$,

i.e. $(x\,dy + y\,dx) + 2y\,dy + 2x\,dx = 0$ is $xy + y^2 + x^2 = c$

Therefore the complete solution of the given equation is

$(y-c)(y+x-c)(xy+y^2+x^2-c) = 0$.

Ex. 9. (b) Solve $p^2+px+py+xy = 0$

Solution:

The given equation is equivalent to

$(p+x)(p+y) = 0$

That is, either

$p+x = 0$ or, $p+y = 0$

In other words,

$\frac{dy}{dx} + x = 0$, or $\frac{dy}{dx} + y = 0$

The solutions of the above factors are

$2y = -x^2+c$ and

$x = -\ln|y|+c$, for c being an arbitrary constant.

Therefore, the general solution of the given equation is

$(2y+x^2-c) . (x+\ln|y|-c) = 0$

Ex. 10. $P^3-(x^2+xy+y^2)\,p^2+(x^3y+x^2y^2+xy^3)\,p-x^3y^3 = 0$.

Solution:

The equation on factorization is

$(p-x^2)(p-y^2)(y-yx) = 0$.

The component equations are

$\frac{dy}{dx} = x^2$, $\frac{dy}{dx} = y^2$, $\frac{dy}{dx} = xy$ ($\equiv \frac{dy}{y} = x\,dx$).

Solutions of these equations are respectively

$3y-x^3 = c$, $xy+cy+1 = 0$, $y=ce^{\frac{1}{2}x^2}$.

Therefore the complete solution is

$(3y-x^3-c)(xy+cy+1)(y-ce^{\frac{1}{2}x^2}) = 0$.

Roll: 1800116

Equations solvable for y

If the equation is solvable for y, we can express y explicitly in terms of x and p. Thus the equations of this type can be put as

$$y = f(x, p) \qquad ...(1)$$

Now differentiating with respect to x, we get

$$\frac{dy}{dx} = p = F\left(x, p, \frac{dp}{dx}\right) \qquad ...(2)$$

which is now an equation in two variables x and p.

Suppose the solution of (2) is,

$$\emptyset\,(x, p, c) = 0. \qquad ...(3)$$

Then eliminating p from (1) and (3), we get the required solution.

If p cannot be easily eliminated, then express values of x and y in terms of the parameter p in the form

$$x = \emptyset_1\,(p, c),\ \ y = \emptyset_2\,(p, c).$$

These two relations together give the complete solution of the given equation.

Lagrange's Equations

To solve the equation

$$y = x\emptyset\,(p) + f\,(p),$$

Differentiating with regard to x, we get

$$p = \emptyset (p) + \{x\emptyset'(p) + f'(p)\}\frac{dp}{dx}$$

or, $p - \emptyset (p) = [x\emptyset'(p) + f'(p)]\frac{dp}{dx}$

or, $\frac{dx}{dp} - x\frac{\emptyset' (p)}{p-\emptyset (p)} = \frac{f'(p)}{p-\emptyset (p)}$.

This is linear equation in x and p and can be solved in the usual way.

Note. In case $\emptyset (p) = p$, the above method fails since $p - \emptyset (p) = 0$ and we do not get a linear equation in x and p. In this case the equation is of Clairaut's form.

Roll: 1800117 **.**

Ex.1: Solve $y = 2px + p^4x^2$

Solution:

Given there, $y = 2px + p^4x^2$ ...(1)

Differentiating with respect to x,

$p = 2p + 2x\frac{dp}{dx} + 2p^4x + 4p^3x^2\frac{dp}{dx}$

or, $(p + 2x\frac{dp}{dx})(1 + 2p^3x) = 0$

We discard the factor, $1 + 2p^3x = 0$

The factor $p + 2x\frac{dp}{dx} = 0$ gives $2\frac{dp}{p} + \frac{dx}{x} = 0$

Integrating, $p^2x = c$ ... ... (2)

From (2), $p^2 = \frac{c}{x}$ Putting this value in (1),

$y = 2px + c^2$ or $y - c^2 = 2px$

Squaring, $(y - c^2)^2 = 4p^2x^2 = 4\frac{c}{x}x^2$

or $(y - c^2)^2 = 4cx$ is the complete solution.

Note. From (2), $x = \frac{c}{p^2}$

∴ (1) gives $y = 2p\frac{c}{p^2} + p^4\frac{c^2}{p^4} = \frac{2c}{p} + c^2$

Thus $x = \frac{c}{p^2}$, $y = \frac{2c}{p} + c^2$ also together constitute the complete solution of (1).

Ex.2: Solve $y = 2px - p^2$

Solution:

The equation is solved for y. Differentiating with respect to x,

$$p = 2p + 2x\frac{dp}{dx} - 2p\frac{dp}{dx} \quad \text{or} \quad p\frac{dp}{dx} + 2x - 2y = 0$$

$$\text{or} \quad \frac{dp}{dx} + \frac{2}{p}x = 2, \quad \text{linear, I.F} = e^{\int\frac{2}{p}dp} = p^2.$$

$$\therefore\ xp^2 = c + \int 2p^2 dp = c + \frac{2}{3}p^3$$

$$\text{or} \quad x = cp^{-2} + \frac{2}{3}p. \qquad \text{...(1)}$$

Also putting this value of x in given equation,

$$y = 2p\left(cp^{-2} + \frac{2}{3}p\right) - p^2$$

$$= 2cp^{-1} + \frac{1}{3}p^2. \qquad \text{...(2)}$$

(1) & (2) together constitute general solution of the given equation.

Roll: 1800118

Ex. 3: Solve $y = -px + x^4p^2$.

Solution.

Given that,

$$y = -px + x^4p^2$$

...(1)

Differentiating with respect to x,

$$p = -p - x\frac{dp}{dx} + 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

i.e. $2p + x\frac{dp}{dx} - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$

or, $\left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$

Rejecting the factor $1-2px^3$, we get

$$x\frac{dp}{dx} + 2p = 0$$

or $\frac{dp}{p} + \frac{2dx}{x} = 0$

Integrating, $p = \frac{c}{x^2}$

Putting this value of p in (1), we get

$$y = -(c/x^2)\,x + x^4(c^2/x^4)$$

or, $y = -c/x + c^2$,

which is the required solution of the equation.

Ex. 4: Solve $x - yp = ap^2$.

<u>Solution.</u>

Solving for y, $y = x/p - ap$.

Differentiating, $p = \frac{1}{p} - \frac{x}{p^2}\frac{dp}{dx} - a\frac{dp}{dx}$,

i.e. $\frac{dp}{dx}(ap^2 + x) = p(1 - p^2)$.

This can be put as $\frac{dx}{dp} - x.\frac{1}{p(1-p^2)} = \frac{ap}{1+p^2}$ ...(1)

which is a linear equation in x and p.

Integrating factor = $e^{\int \frac{dp}{p(1-p^2)}}$,

Now $\int \frac{dp}{p(1-p^2)}$

$= \int \frac{dp}{p(1-p)(1+p)}$

$= \int \{\frac{1}{p} + \frac{1}{2(1-p)} - \frac{1}{2(1+p)}\}dp$

$= \log p - \frac{1}{2}\log(1-p) - \frac{1}{2}\log(1+p) = \log\frac{p}{\sqrt{1-p^2}}$

∴ Integrating factor = $e^{-\log\frac{p}{\sqrt{(1-p^2)}}} = \frac{\sqrt{(1-p^2)}}{p}$.

Solution of (1) is

$\frac{x\sqrt{(1-p^2)}}{p} = c + \int \frac{ap}{1-p^2} . \frac{\sqrt{1-p^2}}{p} dp$

or, $x = \frac{p}{\sqrt{(1-p^2)}}(c + a \sin^{-1} p)$
...(2)

Putting this value of x in the given equation, we get

$y = \frac{1}{\sqrt{(1-p^2)}}(c + a \sin^{-1} p) - ap$
...(3)

(2) and (3) together constitute solution of the given equation

Roll: 1800119

Ex. 5. Solve $p^2 - py + x = 0.$

<u>Solution:</u>

Solving for $y, y = p + x/p.$

Differentiating, $p = \frac{dp}{dx} + \frac{1}{p} - \frac{x}{p^2}\frac{dp}{dx}$

or $\left(p - \frac{1}{p}\right)\frac{dx}{dp} + \frac{x}{p^2} = 1$ or $\frac{dx}{dp} + \frac{1}{p(p^2-1)}x = \frac{p}{p^2-1}$,

which is a linear equation in x and p.

Now proced as in the above expamle or put $a = -1$ in the above example.

Ex. 6. Solve $y = 3x + \log p$.

Solution:

The equqtion is solved for y. Differentiating w. r. t.

x , we get $p = 3 + \frac{1}{p}\frac{dp}{dx}$ or $p(p-3) = \frac{dp}{dx}$

or $dx = \frac{dp}{p(p-3)} = \frac{1}{3}\left[\frac{1}{(p-3)} - \frac{1}{p}\right] dp.$

Integrating, $x = \frac{1}{3}\log\frac{p-3}{p} + \log C_1$

or $\frac{p-3}{p} = ce^{3x}$ or $p = \frac{3}{(1-ce^{3x})}$

Putting this value of p in the given equation, the solution is

$y = 3x + \log\frac{3}{(1-ce^{3x})}$

Ex. 7. Solve $y - 2px = f(xp^2)$

Solution:

Solving for y, $y = 2px + f(xp^2)$

Differentiating w. r. t. x, we get

$p = 2p + 2x\frac{dp}{dx} + f(xp^2)\left[p^2 + x.2p\frac{dp}{dx}\right]$

or $\left(p + 2x\frac{dp}{dx}\right)[1 + pf'(xp^2)] = 0,$

so that $p + 2x\frac{dp}{dx} = 0.$ i. e. $\frac{2dp}{p} + \frac{dx}{x} = 0.$

Integrating, $2\log p + \log x = \log c$, i. e. $p^2x = c.$

Putting $p = \sqrt{c}/\sqrt{x}$ in the given equation,

$y = 2\sqrt{(cx)} + f(c),$

which is the required solution.

Roll: 1800120

Ex. 8 Solve $(\frac{dy}{dx})^3 + m(\frac{dy}{dx})^2 = a(y + mx)$

Solution:

Solving for y, the equntion is

$ay = -amx + mp^2 + p^3 \quad \ldots\ldots..(1)$

Diferentiating w.r.t. 'x' we get

$ap = -am + 2mp\frac{dp}{dx} + 3p^2\frac{dp}{dx}$

$or, \frac{dp}{dx} = \frac{a(p+m)}{2mp + 3p^2}$

$or, adx = \frac{2mp + 3p^2}{p+m}dp = \left(3p - m + \frac{m^2}{p+m}\right)dp$

$Integrating, ax = c + \frac{3}{2}p^2 - mp + m^2 * \log(p+m) \ldots\ldots\ldots\ldots(2)$

so that from (1),

$ay = -m\left[c + \frac{3}{2}p^2 + mp + m^2 * \log(p+m)\right] + mp^2 + p^3 \ldots\ldots\ldots(3)$

(2) and (3) together constitute solution of the equatios.

Ex. 9. Solve $y = x + a \tan^{-1} p$.

Solution:

The equation is solved for y,

$Differentiating, p = 1 + \frac{a}{1+p^2}\frac{dp}{dx}$

$or, a\frac{dp}{dx} = (p-1)(1+p^2)$

$$\text{or,} \frac{a\,dp}{(p-1)(1+p^2)} = dx$$

$$\text{i.e.} \frac{a}{2}\left[\frac{1}{p-1} - \frac{p+1}{p^2+1}\right] dp = dx$$

$$\text{Integrating,} \frac{a}{2}\left[\log(p-1) - \frac{1}{2}\log\,(p^2+1-\tan^{-1} p)\right] = x + c$$

This relation together with the given equation constitutes the solution of the equation.

Ex. 10. Solve $xp^2 - 2yp + ar = 0$

<u>Solution:</u>

$$\text{Solving for y, } y = \frac{1}{2}\frac{ax}{p} + \frac{1}{2}xp$$

$$\text{Diferentiating, } p = \frac{1}{2}\left(\frac{a}{p} - \frac{ax}{p^2}\frac{dp}{dx}\right) + \frac{1}{2}\left(p + x\frac{dp}{dx}\right)$$

$$\text{or, } x\frac{dp}{dx}\left(1 - \frac{a}{p^2}\right) = \left(p - \frac{a}{p}\right)$$

$$\text{or, } \frac{(p^2-a)}{p^2}\left[p + x\frac{dp}{dx}\right] = 0$$

$$\text{or, } x\frac{dp}{dx} = p$$

$$\text{or, } \frac{dp}{p} = \frac{dx}{x}$$

Integrating, $p = cx$

Putting this value of p in the given equation, we have

$c^2x^3 - 2ycx + ax = 0$

i.e. $2y = cx^2 + {}^{a}/_{c}$

which is the required solution.

Equations solvable for 'x':

If x can be expressed explicitly in terms of for 'x'. Such as equation can be out in the form x=f(y,p)………(1)

Differentiate it with respect to y to obtain

$\frac{dx}{dy} = \frac{1}{p} = F(y, p, \frac{dp}{dy})$…………(2)

Which can be solved as an equation in y and p suppose the solution is β (y,p,c)=0

Then eliminating 'p' form '1' and '2' we get the primitive of the equation.

If elimination is not possible then values of x and y are expressed in terms of parameter p together constitute the solution of equation.

Ex. 1. solve $y=3px+6p^2y^2$

<u>Solution:</u>

solving for x, $3x=\frac{y}{p}-6py^2$

Differentiating with respect to y, $\frac{3}{y}=\frac{1}{p}-\frac{y}{p^2}\frac{dp}{dy}-6y^2\frac{dp}{dy}-12py$

i.e. $(1+6p^2y)(2p+y\frac{dp}{dy})=0$

Neglecting the first factor, we get $2p+y\frac{dp}{dy}=0$

Integrating it, $py^2=c$, i. e ,$p=\frac{c}{p2}$

Putting the value of p in the given equation,

$y=3x\frac{c}{y2}+6y^2\frac{c^2}{y^4}$, that is $y^2= 3cx+6c^2$

which is the required solution.

Ex. 2. solve $y=2px+p^2y$

<u>Solution:</u>

solving for x,we get 2x= -py+$\frac{y}{p}$

Differentiating with respect to y, we get

$\frac{2}{p}$= -p-y$\frac{dp}{dy}$+p^{-1}-yp$^{-2}\frac{dp}{dy}$

That is, $\frac{1}{p}$= - $\frac{y}{2}(\frac{1}{p^2})\frac{dp}{dy}$+$\frac{1}{2p}$+$\frac{p}{2}$+ $\frac{y}{2}$ $\frac{dp}{dy}$

y$\frac{dp}{dy}\frac{1}{p^2}$ +y$\frac{dp}{dy}$+($\frac{1}{p}$+p)=0

(y$\frac{dy}{dx}$+p)(1+$\frac{1}{p^2}$)=0

y$\frac{dy}{dx}$ $=$ -p, or $\frac{dp}{p}$+$\frac{dy}{dx}$=0

or, log p +log y =log c (integrating)

putting the value of in the given equation ,the solution is,

y= 2$\frac{c}{y}\mathrm{x} + \frac{c^2}{y^2}\mathrm{y}$,

or, y^2 =2cx+c^2

which is the required solution.

Ex. 3. solve for x=y+alogp

<u>Solution:</u>

x=y+alogp

Differentiating w.r.to y,

$$\frac{1}{p} = 1 + a\frac{1}{p}\frac{dp}{dy}$$

$$\frac{1}{p} - 1 = \frac{a}{p}\frac{dp}{dy}$$

1-p=a$\frac{dp}{dy}$

dy=a$\frac{dp}{1-p}$

$y=a\frac{\log(1-p)}{-1}+c$

y=-alog(1-p) +c

putting the value in the given equation,

x=-alog(1-p)+c+alog p

which is the required solution.

Ex. 4. solve $xp^3=p^2+1$

Solution:

solve for $x=\frac{1}{p}+\frac{1}{p^3}$................(1)

Differentiating with respect to y,

$$\frac{1}{p}=(-\frac{1}{p^2}-\frac{3}{p^4})\frac{dp}{dy}$$

Or, $p^3=-(p^2+3)\frac{dp}{dy}$

$-dy=\frac{(p^2+3)}{p^3}\frac{dp}{dy}$

$\frac{dp(p^2+3)}{p^3}=-dy$, $\frac{dp}{p}+\frac{3dp}{p^3}=-dy$

By integrating, $\ln p-\frac{3}{2p^2}=-y+c$

$$y=\frac{3}{2p^2}-\ln p+c\ldots\ldots\ldots.(2)$$

Equations 1 and 2 together constitute the solution.

Roll: 1800122

Ex. 5. Solve the differential equation (solvable for X), $p^3-2xyp+4y^2=0$

Solution:

We have,

$$p^3 - 2xyp + 4y^2 = 0$$

Solving for x, we get,

$$x = \frac{p^2}{2y} + \frac{2y}{p}$$

Differentiating w.r.to y, we obtain,

$$\frac{dx}{dy} = -\frac{p^2}{2y^2} + \frac{p}{y}\frac{dp}{dy} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} = -\frac{p^2}{2y^2} + \frac{p}{y}\frac{dp}{dy} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

$$\Rightarrow -\frac{1}{p} + \frac{2y}{p^2}\frac{dp}{dy} = -\frac{p^2}{2y^2} + \frac{p}{y}\frac{dp}{dy}$$

$$\Rightarrow \left(\frac{2y}{p^2}\frac{dp}{dy} - \frac{1}{p}\right) = \frac{p^3}{2y^2}\left(\frac{2y}{p^2}\frac{dp}{dy} - \frac{1}{p}\right)$$

$$\Rightarrow \left(\frac{2y}{p^2}\frac{dp}{dy} - \frac{1}{p}\right) - \frac{p^3}{2y^2}\left(\frac{2y}{p^2}\frac{dp}{dy} - \frac{1}{p}\right) = 0$$

$$\Rightarrow \left(\frac{2y}{p^2}\frac{dp}{dy} - \frac{1}{p}\right)\left(1 - \frac{p^3}{2y^2}\right) = 0$$

Omitting the second factor, we have

$$\frac{2y}{p^2}\frac{dp}{dy} - \frac{1}{p} = 0$$

$$\Rightarrow \frac{2y}{p}\frac{dp}{dy} = 1$$

$$\Rightarrow \frac{2dp}{p} = \frac{dy}{y}$$

Integrating,

$$\Rightarrow 2\log p = \log y + \log c$$

$$\Rightarrow \log p^2 = \log(yc)$$

$$\Rightarrow p^2 = yc$$

Putting this value of p^2 in the given equation, we get

$$p(cy - 2xy) = -4y^2$$

$$\Rightarrow p(c - 2x) = -4y$$

$$\Rightarrow p^2(c - 2x)^2 = 16y^2$$

$\Rightarrow yc(c-2x)^2 = 16y^2$

Hence, $16y = c(c-2x)^2$ is the required solution . (Ans.)

Ex. 6. Solve the equation (solvable for x) $p^3y^2 - 2px + y = 0$

<u>Solution:</u>

We have,

$$p^3y^2 - 2px + y = 0$$

Solving for x, we get

$$2x = \frac{y}{p} + p^2y^2$$

Differentiating w.r.to y, we obtain

$$2\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} + 2p^2y + 2y^2p\frac{dp}{dy}$$

$$\Rightarrow \frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} + 2p^2y + 2y^2p\frac{dp}{dy}$$

$$\Rightarrow \frac{1}{p} - 2p^2y = -\frac{y}{p}\frac{dp}{dy}\left(\frac{1}{p} - 2yp^2\right)$$

$$\Rightarrow 1 + \frac{y}{p}\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{y}{dy}\frac{dp}{p} = -1$$

$$\Rightarrow \frac{dp}{p} = -\frac{dy}{y}$$

$$\Rightarrow \frac{dp}{p} + \frac{dy}{y} = 0$$

Integrating,

$$\log p + \log y = \log c$$

$$\Rightarrow \log(py) = \log c$$

$$\Rightarrow yp = c$$

Substituting the value of $p = \frac{c}{y}$ in the given equation, we get

$$\frac{c^3}{y^3}y^2 - 2\frac{c}{y}x + y = 0$$

$$\Rightarrow \frac{c^3}{y} - \frac{2cx}{y} + y = 0$$

$$\Rightarrow c^3 - 2cx + y^2 = 0$$

Hence, $c^3 - 2cx + y^2 = 0$ is the required solution . (Ans.)

Ex. 7. Solve the equation (solvable for x) $(2x - b)p = y - ayp^2$

<u>Solution:</u>

We have,

$$(2x - b)p = y - ayp^2$$

Solving for x, we get

$$2x = b + \frac{y}{p} - ayp$$

Differentiating w.r.to y,

$$\frac{2dx}{dy} = 0 + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - a(p + y\frac{dp}{dy})$$

$$\Rightarrow \quad \frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - ap - ay\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{p} = -\frac{y}{p^2}\frac{dp}{dy} - ap - ay\frac{dp}{dy}$$

$$\Rightarrow \quad p = -y\frac{dp}{dy} - ap^3 - ap^2y\frac{dp}{dy}$$

$$\Rightarrow p + ap^3 = -y\frac{dp}{dy}(1 + ap^2)$$

$$\Rightarrow p(1 + ap^2) = -y\frac{dp}{dy}(1 + ap^2)$$

$$\Rightarrow p + y\frac{dp}{dy} = 0$$

$$\Rightarrow \frac{dp}{p} = -\frac{dy}{y}$$

By integrating,

$\Rightarrow \log p + \log y = \log c$

$\Rightarrow \log(yp) = \log c$

$\therefore yp = c$

Putting the value of $p = \frac{c}{y}$ in the given equation,

$(2x - b)\frac{c}{y} = y - \frac{ac^2}{y}$

Hence, $(2x - b)c = y^2 - ac^2$ is the required solution . (Ans.)

Ex. 8. Solve the equation (solvable for x) $xp^3 = a + bp$

<u>Solution:</u>

Solving for x, we get

$\Rightarrow x = \frac{a}{p^3} + \frac{b}{p^2}$

Differentiating w.r.to y, we get

$\frac{dx}{dy} = -\frac{3a}{p^4}\frac{dp}{dy} - \frac{2b}{p^3}\frac{dp}{dy}$

$\Rightarrow \frac{1}{p} = -\frac{3a}{p^4}\frac{dp}{dy} - \frac{2b}{p^3}\frac{dp}{dy}$

$\Rightarrow p^3 = -3a\frac{dp}{dy} - 2bp\frac{dp}{dy}$

$\Rightarrow p^3 = -(3a + 2bp)\frac{dp}{dy}$

$\Rightarrow dy = -\frac{(3a + 2bp)}{p^3}dp$

$\Rightarrow dy = -\frac{3a}{p^3}dp - \frac{2b}{p^2}dp$

By integrating,

$y = \frac{3a}{2p^2} + \frac{2b}{p} + c$ --------- (1)

From the given equation,

$x = \frac{a}{p^3} + \frac{b}{p^2}$ ----------- (2)

Hence, equations (1) and (2) constitute the required solution. (Ans.)

Roll: 1800123 .

Ex. 9. Solve the differential equation (solvable for x), $p = \tan\left(x - \frac{p}{1+p^2}\right)$

Solution:

when solved for x, the equation becomes

$x = \tan^{-1} p + \frac{p}{1+p^2}$ ....(1)

Differentiating with respect to y,

$$\frac{1}{p} = \frac{1}{(1+p^2)}\frac{dp}{dy} + \frac{(1+p^2) - 2p^2}{(1+p^2)}\frac{dp}{dy}$$

Or,

$$dy = \frac{2pdp}{(1+p^2)^2}$$

Integrating,

$y = c - \frac{1}{(1+p^2)}$ .... (2)

Equations 1 and 2 together constitute the solution.

Ex. 10. Solve the differential equation(solvable for x), $x = a - p(1+p^2)^{\frac{-1}{2}}$

Solution:

Given, $x = a - p(1+p^2)$..............................(1)

Differentiating (1) with respect to y and writing 1/p for dx/dy, we get

$$\frac{1}{p} = 0 - \left[(1+p^2)^{\frac{-1}{2}} + p\left(\frac{-1}{2}\right)(1+p^2)^{\frac{-3}{2}}.2p\right]\frac{dp}{dy}$$

Or,

$$\frac{1}{p} = \frac{dp}{dy}\left[\frac{p^2}{(1+p^2)^{\frac{3}{2}}} - \frac{1}{(1+p^2)^{\frac{3}{2}}}\right] = \frac{dp}{dy}\frac{p^2-(1+p^2)}{(1+p^2)^{\frac{3}{2}}}$$

Or,

$$dy = -(1+p^2)^{\frac{-3}{2}}pdp$$

Integrating, $y = c - \frac{1}{2}\int(1 = p^2)^{\frac{-3}{2}}(2pdp) = c - \frac{1}{2}\int v^{\frac{-3}{2}}\,dv$

Putting $1 + p^2 = v\,, 2pdp = dv$

Or, $y = c + v^{\frac{-1}{2}}$

Or, $y = c + (1+p^2)^{\frac{-1}{2}}$ ………………………………(2)

We now try to eliminate (1) and (2) as follows,

Here, (1) and (2)

$x - a = p/(1+p^2)^{\frac{1}{2}}$ and $y - c = 1/(1+p^2)^{\frac{1}{2}}$

Squaring and adding these,

$$(x-a)^2 + (y-c)^2 = 1$$

Which is the required solution, c being an arbitrary constant.

Roll: 1800124 .

Ex. 11. Solve the differential equation (solvable for X), $y=px+p^3$.

Solution:

We have,

$y=px+p^3$ --------------(1)

Or, $px=y-p^3$

Or, $x=\frac{y}{p}-p^2$

Differentiating equation (1) with respect to y, we get,

$$\frac{dx}{dy}=\frac{1}{p}-\frac{y}{p^2}\frac{dp}{dy}-2p\frac{dp}{dy}$$

Or, $\frac{1}{p}=\frac{1}{p}-\frac{y}{p^2}\frac{dp}{dy}-2p\frac{dp}{dy}$

Or, $\frac{y}{p^2}\frac{dp}{dy}+2p\frac{dp}{dy}=0$

Or, $\frac{dp}{dy}\left(\frac{y}{p^2}+2p\right)=0$

Omitting the second factor, we have

$$\frac{dp}{dy}=0$$

Or, $dp=0$

By integrating,

$p=c$

Putting $p=c$ in equation (1),

$y=cx+c^3$

Hence, $y=cx+c^3$ is the required solution. (Ans.)

Ex. 12. Solve the differential equation (solvable for X), $P^3-4xyp+8y^2=0$.

<u>Solution</u>:

We have, $P^3-4xyp+8y^2=0$ --------------(1)

$X=\frac{2y}{p}+\frac{p^2}{4y}$

Differentiating with respect to y, we get,

$$\frac{dx}{dy}=\frac{1}{p}=\frac{2}{p}-\frac{2y}{p^2}\frac{dp}{dy}+\frac{p}{2y}\frac{dp}{dy}-\frac{p^2}{4y^2}$$

Or, $(\frac{2y}{p^2}\frac{dp}{dy}-\frac{1}{p})(1-\frac{p^3}{4y^2})=0$

Omitting the second factor, we have

i.e., $\frac{2y}{p^2}\frac{dp}{dy}-\frac{1}{p}=0$

or, $\frac{2y}{p^2}\frac{dp}{dy}=\frac{1}{p}$

or, $\frac{2dp}{p}=\frac{dy}{y}$

By integrating,

or, 2 log p=log c+ log y

or, log p^2=log cy

p^2=cy

Putting these value of p^2 in equation (1),

(cy-4xy) p=-8y^2

Or, -8y=(c-4x) p

Or, 64y^2=$(c-4x)^2p^2$

Or, 64y^2=$(c-4x)^2$cy

Hence, 64y=c$(c-4x)^2$ is the required solution. (Ans.)

Ex. 13. Solve the differential equation (solvable for X), x=y+p^2.

Solution:

We have,

x=y+p^2------------(1)

Differentiating equation (1) with respect to y, we get

$$\frac{1}{p} = 1 + 2p\frac{dp}{dy}$$

Or, $\frac{dp}{dy} = \frac{1-p}{2p^2}$

Or, $\frac{2p^2 dp}{1-p} = dy$

Or, -2(p+1+ $\frac{1}{p-1}$) dp = dy

By integrating,

y=c- [p^2+2p+2log (p-1)] --------------(2)

From the given equation,

X=c- [2p+2log (p-1)] --------------(3)

Hence, equations (2) and (3) constitute the required solution. (Ans.)

Ex. 14. Solve the differential equation (solvable for X), y^2 log y=xpy+p^2.

Solution:

We have, y^2 log y=xpy+p^2 -------------(1)

or, xpy = y^2 log y- p^2

or, $x = \frac{ylog\, y}{p} - \frac{p}{y}$

Differentiating equation (1) with respect to y, we get,

$$\frac{1}{p} = (log\, y + y\frac{1}{y})\frac{1}{p} - ylog\, y\frac{1}{p^3}\frac{dp}{dy} - (-\frac{1}{y^2}p + \frac{1}{y}\frac{dp}{dy})$$

Or, $\frac{1}{p} = \frac{log\, y}{p} + \frac{1}{p} + \frac{p}{y^2} - \frac{dp}{dy}\left(\frac{ylog\, y}{p^2} + \frac{1}{y}\right)$

Or, $\frac{p}{y}\left(\frac{ylog\, y}{p^2} + \frac{1}{y}\right) - \frac{dp}{dy}\left(\frac{ylog\, y}{p^2} + \frac{1}{y}\right) = 0$

Or, $\left(\frac{ylog\, y}{p^2} + \frac{1}{y}\right)\left(\frac{p}{y} - \frac{dp}{dy}\right) = 0$

Omitting the first factor, we have

$$\frac{p}{y} = \frac{dp}{dy}$$

Or, $\frac{dp}{p} = \frac{dy}{y}$

By integrating,

Log p=log c+ log y

Or, p=cy

Putting these values in equation (1), we get,

X=$\frac{log\, y}{c} - c$

Or, log y=cx+c^2

Hence, log y=cx+c^2 is the required solution. (Ans.)

Roll: 1800125 .

Clairaut's Equation

y=px+f(p) ……….(1)

The differential equation of the form (1) is known as Clairaut's equation.

To solve y=px+f(p)
Differentiating it with respect to x, we get

p=p+[x+f'(p)] $\frac{dp}{dx}$, i.e. [x+f'(p)]$\frac{dp}{dx}$=0.

Neglecting x+f'(p)=0, we get $\frac{dp}{dx}$=0.

Integrating it, we get p=c.
Putting p=c in (1), the required solution is y=cx+f(c)

Thus, to find the solutions of Clairaut's equation put c for in the equation.

Note: If we eliminate p between x+f'(p)=0 and the given equation, we get an equation involving no constant; this is called the singular solution of equation.

Equation Reducible to Clairaut's Form

It is sometimes possible to reduce a given equation in Clairaut's form with the help of suitable substitutions. The following two substitutions may be noted in this connection:

1. Equation $y^2=pxy+f(p)$

Putting $y^2=Y$, and $x^2=X$, i.e. $\frac{y}{x}\frac{dy}{dx}=\frac{dY}{dX}$, the equation becomes

$y^2 = p\frac{y}{x}.x^2+f(p\frac{y}{x})$ or $Y=\frac{dY}{dX}Y+f(\frac{dY}{dX})$ which is Clairaut's form.

2. Equation $e^{my}(c-mp)=f(pe^{my-ex})$.

This can be reduced to Clairaut's form by the substitutions
$e^{my}=Y$ and $e^{cx}=X$.

Roll: 1800126

Ex. 1. Solve $px-y+p^3=\frac{m^3}{p^3}$

Solution:

The equation is $y=px+p^3-\frac{m^3}{p^3}$

This is of clairaut's form. Hence putting c for p, the solution is,

$y=cx+c^3-\frac{m^3}{c^3}$

Ex. 2. Solve $y=px+p-p^2$

Solution:

The equation is of clairaut's form. Hence putting c for p, the solution is,

$y=cx+c-c^2$

Ex. 3. Solve $(y-px)(p-1)=p$

Solution:

The equation can be written as,

$y-px=\frac{p}{p-1}$

or y=px+$\frac{p}{p-1}$

which is of clairaut's form. Hence putting c for p, the solution is,

y=cx+$\frac{c}{c-1}$

Ex. 4. Solve sinpxcosy=coxpxsiny+p

Solution:

The equation can be written as,

sin(px-y)=p

or y=px-sin^{-1}p

which is of clairaut's form. Hence putting c for p, the solution is,

y=cx-sin^{-1}c

Ex. 5. Solve $\frac{(y-px)^2}{1+p^2}$=a^2

Solution:

The equation can be written as,

y-px=$\pm$a$(1+p^2)^{\frac{1}{2}}$

or, y=px$\pm$a$(1+p^2)^{\frac{1}{2}}$

which is of clairaut's form. Hence putting c for p, the solution is,

y=cx$\pm$a$(1+c^2)^{\frac{1}{2}}$

or, y-cx=$\pm$a$(1+c^2)^{\frac{1}{2}}$

or, $(y-cx)^2$=$a^2(1+c^2)$

Ex. 6. Solve $p^2(x^2-a^2)-2pxy+y^2-b^2$=0

Solution:

The equation can be written as,

$p^2x^2 - 2pxy + y^2 = p^2a^2 + b^2$

Or, $(y - px)^2 = p^2a^2 + b^2$

Or, $y = px \pm (p^2a^2 + b^2)^{\frac{1}{2}}$

which is of clairaut's form. Hence putting c for p, the solution is,

$y = cx \pm (c^2a^2 + b^2)^{\frac{1}{2}}$

Or, $y - cx = \pm(c^2a^2 + b^2)^{\frac{1}{2}}$

Or, $(y - cx)^2 = (c^2a^2 + b^2)$

Or, $c^2(x^2 - a^2) - 2cxy + y^2 - b^2$=0

Roll: 1800127

Ex. 7. Solve p=tan(px-y)

Solution:

The equation can be written as,

$tan^{-1}p$=px-y

Or, y=px-$tan^{-1}p$

The equation is of clairaut's form. Hence putting c for p, the solution is,

y=cx-$tan^{-1}c$

Ex. 8. Solve $(y - px)^2$=1+p^2

Solution:

Here we have y=px$\pm\sqrt{1 + p^2}$

Both the factors are of the Clairaut's form ; their solution are

y=cx$\pm\sqrt{1 + c^2}$

Therefore, the primitive is,

[y-cx-$\sqrt{1+c^2}$][y-cx+$\sqrt{1+c^2}$]=0

Or, $(y-cx)^2$=1+c^2

Roll: 1800128 .

Ex. 9. Solve p^2x(x-2)+p(2y-2xy-x-2)+ y^2+y=0

Solution:

The equation may be written as $(y-px+2p)(y-px+1)=0$.

Each factor is of Clairaut's form.

Hence, putting c for p in each factor, the solution is

$(y-cx+2c)(y-cx+1\rangle=0$.

$Ex. 10. Solve \left(\frac{dy}{dx}\right)^2 (x^2-a^2)-2(\frac{dy}{dx})xy+y^2-b^2=0.$

Solution:

We have $p^2 \mathrm{x}^2-2\mathrm{pxy}+\mathrm{y}^2=\mathrm{a}^2\mathrm{p}^2+\mathrm{b}^2$

or $(\mathrm{y}-\mathrm{px})^2=\mathrm{a}^2\mathrm{p}^2+\mathrm{b}^2$

i.e. y=px$\pm\sqrt{\mathrm{p}^2\mathrm{a}^2+\mathrm{b}^2}$

Both these are in Clairaut's form. Hence, the solution is

y=cx$\pm\sqrt{\mathrm{c}^2\mathrm{a}^2+\mathrm{b}^2}$

Ex. 11. Solve $\mathrm{y}=\mathrm{px}+\mathrm{p}^2$.

Solution:

The given equation is of Clairaut's form.

Hence putting c for p, the solution is

$\mathrm{y}=\mathrm{cx}+\mathrm{c}^2$

Ex. 12. Solve $xp^2 - yp + 2 = 0$.

<u>Solution:</u>

The equation may be written as

$$yp = xp^2 + 2$$

Or, $y = x + \frac{2}{p}$

which is of Clairaut's form.

Hence, putting c for p, the solution is

$$y = cx - e^c$$

Ex. 13. Solve $p = \log(px - y)$.

<u>Solution:</u>

The equation may be written as

$$e^p = px - y$$

⇒y=px−epWhich is of Clairaut's form.

Hence putting c for p, the solution is

$$y = cx - e^c$$

Ex. 14. Solve $y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) = 4\left(\frac{dx}{dy}\right)^2$.

<u>Solution:</u>

The equation may be written as

$$y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) = \frac{4}{\left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow y^2 + x^2p^2 - 2xyp = \frac{4}{p^2}$$

$$\Rightarrow (y - px)^2 = \frac{4}{p^2}$$

$$\Rightarrow y - px = \pm \frac{2}{p}$$

$$\Rightarrow y = px \pm \frac{2}{p}$$

Both these are in Clairaut's form.

Hence putting c for p, the solution is

$$y = cx \pm \frac{2}{c}$$

SOLUTION OF 1ST ORDER ORDINARY DIFFERENTIAL EQUATION BY VARIOUS METHODS

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Ordinary Differential Equation:

An equation involving ordinary derivatives of single independent variable is called ordinary differential equation.

Example:

- $\frac{dy}{dx} = (4x + y + 1)^2$
- $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x$ etc.

1st Order Ordinary Differential Equation:

When the number of the highest derivative in an ordinary differential equation is one, then it is called 1st order ordinary differential equation.

Example:

- $\frac{dy}{dx} = f(ax + by + c)$
- $\frac{dy}{dx} = f({}^{y}/_{x})$
- $\frac{dy}{dx} + p(x)y = Q(x)$
- $\frac{dy}{dx} + p(x)y = Q(x)y^n$
- $\frac{dM}{dy} = \frac{dN}{dx}$ etc.

Thus, we can identify 1st order ordinary differential equation.

Methods of Solving 1st Order Ordinary Differential Equation:

There are commonly 5 methods of solving 1st order ordinary differential equation,

1. Separation of variables,
2. Homogeneous differential equations,
3. Linear differential equations,
4. Bernoulli's equation,
5. Exact differential equations.

Separation of variables method:

If in an equation, it is possible to get all the functions of x and dx to one side and all the functions of y and dy to the other, the variables are said to be separable.

Working rule to solve an equation in which variables are separable:

Step 1: Let $\frac{dy}{dx} = f_1(x)f_2(y)$ be the given function.

[$f_1(x)$ is a function of x alone and f_2(y) is a function of y alone.]

Step 2: Separate the variables, $\left[\frac{1}{f_2(y)}\right] dy = f_1(x)dx$.

Step 3: Integrating both sides of the equation,

$$\int \left[\frac{1}{f_2(y)}\right] dy = \int f_1(x)dx + \text{c} .$$

[here, c is an arbitrary constant that must be added in any one side of the equation according to its suitable form.]

Solving various problems applying separation of variables method:

1. $\boldsymbol{\frac{dy}{dx} = \frac{1+y^2}{1+x^2}}$

Solution:

The 1st order ordinary differential equation is given that*:*

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

Applying separation of variables method,

Or, $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

Integrating both side of the equation,

or, $tan^{-1} y = tan^{-1} x + tan^{-1} c$ *[c being an arbitrary constant]*

or, $tan^{-1} \frac{y-x}{1+xy} = tan^{-1} c$

or, $\frac{y-x}{1+xy} = c$

or, ***y-x=c(1+xy)***

(answer)

2. $\boldsymbol{\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}}$

Solution:

The 1st order ordinary differential equation is given that:

$$\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$$

$$\text{or, } \frac{dy}{dx} = e^x e^{-y} + x^2e^{-y}$$

Applying separation of variables method,

$$\text{or, } e^y dy = (e^x + x^2)dx$$

Integrating both side of the equation,

or, $\boldsymbol{e^y = e^x + \frac{x^3}{3} + c}$ *[c being an arbitrary constant]*

(answer)

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3. $\boldsymbol{sec^2x\ tany\ dx + sec^2y\ tanx\ dy = 0}$

Solution:

The 1st order Ordinary Differential Equation is given that:

$$sec^2x\ tany\ dx + sec^2y\ tanx\ dy = 0$$

Applying separation of variables method,

$$\frac{sec^2x}{tanx}dx + \frac{sec^2x}{tany}dy = 0$$

Integrating the equation,

log tan x + log tan y = A

or, *tan x tan y =e^A = C*   *[C being an arbitrary constant]*

tan x tan y = C .

(answer)

4. $\boldsymbol{y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)}$

Solution:

The 1st order Ordinary Differential Equation is given that:

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$$

Applying separation of variables method,

$$\frac{dx}{x+a} = \frac{dy}{y(1-ay)}$$

or, $\frac{dx}{x+a} = \left(\frac{1}{y} + \frac{a}{1-ay}\right)dy$

Integrating the equation,

$x + a = C\frac{y}{1-ay}$ [C being an arbitrary constant] or, $(x+a)(1-ay)=Cy$

$$\boldsymbol{(x+a)(1-ay)=Cy}$$

(answer)

5. $\boldsymbol{\frac{dy}{dx}=\frac{\sin x+x\cos x}{y(2\log y+1)}}$

Solution:

The 1st order Ordinary Differential Equation is given that:

$$\frac{dy}{dx}=\frac{\sin x+x\cos x}{y(2\log y+1)}$$

Applying separation of variables method,

$$(\sin x + x\cos x)\,dx = (2y\log y + y)\,dy$$

Integrating the equation,

$$-\cos x+\int x\cos x\,dx=2\int y\log y\,dy+\frac{y^2}{2}+c \text{ ....(1)}$$

Now,

$$\int x\cos x\,dx = x\sin x-\int \sin x\,dx$$ [integrating by parts]

or, $\int x\cos x\,dx = x\sin x+\cos x$..........................**(2)**

also,

$$\int y\log y\,dy=\frac{y^2}{2}\log y-\int\{\frac{1}{y}.\frac{y^2}{2}\}dy$$ [integrating by parts]

or, $\int y \log y \, dy = \frac{y^2}{2} \log y - \frac{y^2}{4}$ ***........................(3)***

Using equation (2) and (3), equation (1) reduces to,

$$-cosx + x\, sin\, x + cosx = 2\left\{\frac{y^2}{2} logy - \frac{y^2}{4}\right\} + \frac{y^2}{2} + c$$

or, $x\, sin\, x = y^2\, log\, y + c$ [c being an arbitrary constant]

$$\boldsymbol{x \sin x = y^2 \log y + c}$$

(answer)

6. ***Find the curves passing through (0,1) and satisfying sin*** $\boldsymbol{(\frac{dy}{dx}) = c}$

Solution:

From the given equation, we get,

$$\frac{dy}{dx} = sin^{-1} c$$

Applying separation of variable method,

$$dy = (sin^{-1} c)dx$$

Integrating the equation,

$y = x\, sin^{-1} c + c'$ ***..................(1)*** *[c' being arbitrary constant]*

Since, equation (1) must go through (0,1); we put x=0 and y=1 in equation (1) and get,

$c'=1$

Hence, equation (1) reduces to,

$$y = x \sin^{-1} c + 1$$

or, $\frac{y-1}{x} = \sin^{-1} c$

or, $\sin\{\frac{y-1}{x}\} = c$, which gives the desired curves.

The equation of desired curves is $\boldsymbol{\sin\left\{\frac{y-1}{x}\right\} = c}$

(answer)

7. *Solve* $p = e^{x+y} + x^2 e^{x^3+y}$

Solution:

From the given equation, we get,

$\frac{dy}{dx} = e^y(e^x + x^2 e^{x^3})$ [we know, $p = \frac{dy}{dx}$]

Applying separation of variables method,

$$e^{-y} dy = (e^x + x^2 e^{x^3})dx$$

Integrating the equation,

$$\int e^{-y} dy = \int e^{x}\, dx + \int x^2\, e^{x^3} dx$$

$$or,\ \ -e^{-y} = e^{x} + \frac{1}{3}\int e^{t}\, dt + c \qquad [putting\ x^3 = t]$$

$$or,\ \ -e^{-y} = e^{x} + \frac{1}{3}e^{t} + c$$

$$or,\ \ -e^{-y} = e^{x} + \frac{1}{3}e^{x^3} + c$$

$$\boldsymbol{-e^{-y} = e^{x} + \frac{1}{3}e^{x^3} + c}$$

(answer)

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Method 2

This method is used when separation of variables is difficult and so we have to transform the equation in a form in which variables are easy to separate.

Example:

$$\frac{dy}{dx} = f(ax + by + c)$$

Here it is difficult to separate variables so this equation can be reduced in an equation in which variables can be separated. For this purpose, we can follow these steps:

Step 1: Use of suitable substitution.

$$Let, ax + by + c = v \;\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Step 2: Differentiate equation (1) with respect to x.

$$\frac{d}{dx}(ax + by + c) = \frac{d}{dx}v$$

$$or, a + b\frac{dy}{dx} = \frac{dv}{dx}$$

$$or, \frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right) \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Step 3: Now using equation(2),the given equation can be reduced in such a form where separation of variables is easy.

$$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = f(v)$$

$$or, \frac{dv}{bf(v)+a} = dx \quad [\rightarrow Separation\ of\ variables] \ldots\ldots\ldots (3)$$

Step 4: Integrating both side of equation (3), we have

$$\int \frac{dv}{bf(v)+a} = \int dx$$

$$or, \int \frac{dv}{bf(v)+a} = x + C$$

This is the required solution where C is an integrating constant.

1. *Solve* $\frac{dy}{dx} = (4x + y + 1)^2$

Solution:

Let, $4x + y + 1 = v$

$$or, 4 + \frac{dy}{dx} = \frac{dv}{dx}$$

So the equation becomes,

$$\frac{dv}{dx} - 4 = v^2$$

$$or, \ \frac{dv}{dx} = v^2 + 4$$

Now the variables are separatable and we can write,

$$\frac{dv}{v^2+4} = dx \dots\dots\dots\dots\dots\dots\dots (\mathbf{1})$$

Integrating equation no. (1) we get,

$$\frac{1}{2}\tan^{-1}\frac{v}{2} = x + C$$

$$\therefore \ \boldsymbol{\frac{1}{2} tan^{-1} \frac{4x+y+1}{2} = x + C} \quad [\rightarrow x + y = v]$$

$(answer)$

2. *Solve* $(x + y)^2 \frac{dy}{dx} = a^2$

Solution:

$Let, x + y = v$

$or, \frac{dy}{dx} = \frac{dv}{dx} - 1$

So the equation becomes,

$v^2\left(\frac{dv}{dx} - 1\right) = a^2 or, \frac{dv}{dx} = \frac{v^2+a^2}{v^2}$

$or, dx = \frac{v^2}{a^2+v^2} dv$

$or, dx = \left(1 - \frac{a^2}{a^2+v^2}\right) dv \dots\dots\dots\dots\dots\dots (\mathbf{1})$

Integrating equation no. (1) we get,

$or, x + C = v - a \tan^{-1}\frac{v}{a}$

$or, x + C = (x + y) - a \tan^{-1}\frac{x+y}{a} [\rightarrow x + y = v]$

$\therefore \boldsymbol{y = a \tan^{-1}\frac{x+y}{a} + C}$

$(answer)$

3. *Solve* $\frac{dy}{dx} + 1 = e^{x+y}$

Solution:

$Let, x + y = v$

$or, \frac{dy}{dx} + 1 = \frac{dv}{dx}$

So that,

$$\frac{dv}{dx} = e^{v}$$

$$or, \frac{dx}{dv} = e^{-v}$$

$$or, dx = e^{-v}dv \dots\dots\dots\dots\dots\dots (\mathbf{1})$$

Integrating equation no. (1) we get,

$$x = -e^{-v} + C$$

$$\therefore \boldsymbol{x = -e^{(x+y)} + C}$$

$$(answer)$$

4.*Solve* $\frac{dy}{dx} = \frac{4x+6y+5}{3y+2x+4}$

Solution:

The above equation can be written as

$$\frac{dy}{dx} = \frac{2(2x+3y)+5}{3y+2x+4} \dots\dots\dots\dots\dots\dots (\mathbf{1})$$

$$Let, 2x + 3y = v$$

Differentiating with respect to x,

$$2+3\frac{dy}{dx}=\frac{dv}{dx}$$

$$or, \frac{dy}{dx}=\frac{1}{3}\left(\frac{dv}{dx}-2\right) \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{2})$$

From (1)&(2) we get,

$$\frac{1}{3}\left(\frac{dv}{dx}-2\right)=\frac{2(2x+3y)+5}{3y+2x+4}$$

$$or, \frac{1}{3}\left(\frac{dv}{dx}-2\right)=\frac{2v+5}{v+4}$$

$$or, \frac{dv}{dx}=\frac{3(2v+5)}{v+4}+2$$

$$or, \frac{dx}{dv}=\frac{v+4}{8v+23}$$

$$or, \frac{dx}{dv}=\frac{\frac{1}{8}(8v+23)+4-\frac{23}{8}}{8v+23}$$

$$or, \frac{dx}{dv}=\frac{1}{8}+\frac{9}{8(8v+23)}$$

Separating variables,

$$dx=\left\{\frac{1}{8}+\frac{9}{8(8v+23)}\right\}dv \ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{3})$$

Integrating equation no. (3) we get,

$$x+C=\frac{v}{8}+\frac{9}{64}\log(8v+23)$$

$$or, 8x+8C=2x+3y+\frac{9}{8}\log(16x+24y+23)$$

$[\rightarrow 2x+4y=v \ and \ multiplying \ by \ 8]$

$$or, 3y - 6x + \frac{9}{8}\log(16x + 24y + 23) = 8C$$

$$\therefore \boldsymbol{y - 2x + \frac{3}{8} log(16x + 24y + 23) = C'}$$

$$\left[\rightarrow where\ C' = \frac{8}{3}C\ \ is\ an\ arbitrary\ constant\right]$$

$$(answer)$$

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5. $\boldsymbol{\frac{xdx+ydy}{xdy-ydx} = \sqrt{\frac{a^2-x^2-y^2}{x^2+y^2}}}$

Solution:

Here, we change this to polar co-ordinate by putting,

$$x = rcos\theta, y = rsin\theta, x^2 + y^2 = r^2,\ xdx + ydy = rdr$$

$$\frac{y}{x} = tan\theta, \therefore \frac{xdy-ydx}{x^2} = sec^2\theta d\theta\ or\ xdy - ydx = r^2 d\theta$$

∴The Equation becomes, $\frac{1}{r}\frac{dr}{d\theta} = \sqrt{\frac{a^2-r^2}{r^2}}$

Separating the variables, $\frac{dr}{\sqrt{a^2-r^2}} = d\theta$

Integrating , $sin^{-1}\left(\frac{r}{a}\right) = \theta + c$ or, $r = a\sin(\theta + c)$

$$i.e.\ \sqrt{x^2+y^2} = a\ sin[tan^{-1}\left(\frac{y}{x}\right)+c]$$

(answer)

6. $$\left(\frac{x+y-a}{x+y-b}\right)\frac{dy}{dx} = \frac{x+y+a}{x+y+b}$$

Solution:

Put $x+y=v$

$$so\ that\ 1+\frac{dy}{dx}=\frac{dv}{dx}$$

$$or, \frac{dv}{dx} = 1+\left(\frac{v-a}{v-b}\right)\left(\frac{v-b}{v-a}\right) = \frac{2(v^2-ab)}{v^2+(b-a)v-ab}$$

$$or, 2dx = \left(1+\frac{b-a}{2}\frac{2v}{v^2-ab}\right)dv$$

$$or, 2x+c = v+\frac{b-a}{2}\log(v^2-ab)$$ *[By Integrating]*

$$or, 2x+c = x+y+\frac{1}{2}(b-a)\log\{(x+y)^2-ab\}$$

(answer)

7. $$(2x+y+1)dx+(4x+2y-1)dy=0$$

Solution:

Rewriting the given equation,

$$\frac{dy}{dx} = -(2x+y+1)/(4x+2y-1)$$

$$or, 2x+y=v$$

$$or, 2+\frac{dy}{dx}=\frac{dv}{dx}$$

$$or, \frac{dy}{dx}=\frac{dv}{dx}-2$$

$$or, \frac{dv}{dx}-2=\frac{-(2v+1)}{2v-1}$$

$$or, \frac{dv}{dx}=2-\frac{2v+1}{2v-1}$$

$$or, \frac{dv}{dx}=\frac{2v-1}{2v-1}$$

$$or, \frac{dv}{dx}=1$$

$$or, dv=dx$$

$$or, v=x+c$$ *[By Integrating]*

$$or, 2x+y=x+c$$

$$or, 2x+y-x=c$$

$$\therefore \boldsymbol{x+y=c}$$

(answer)

8. $\boldsymbol{\frac{dy}{dx}=\frac{x-y+3}{2x-2y+5}}$

Solution:

$$Let, x-y=v$$

$$or, 1-\frac{dy}{dx}=\frac{dv}{dx}$$

$$or, 1-\frac{dv}{dx}=\frac{dy}{dx}$$

we can write,

$$1-\frac{dv}{dx}=\frac{v-1}{2v+5}$$

$$or, \frac{dv}{dx} = 1 - \frac{v-1}{2v+5}$$
$$or, \frac{dv}{dx} = \frac{v+6}{2v+5}$$
$$or, dx = \frac{2v+5}{v+6} dv$$
$$or, dx = \frac{2\times(v+6)-7}{v+6} d$$
$$or, dx = \left(2 - \frac{7}{v+6}\right) dv$$
$$or, x + c = 2v - 7\log(v+6)$$ [By Integrating]
$$\therefore \boldsymbol{x + c = 2(x - y) - 7\,log(x - y + 6)}$$
$$(answer)$$

9. $(x + 2y - 1)dx = (x + 2y + 1)dy$

Solution:

Rewriting the given equation,

$$\frac{dy}{dx} = (x + 2y - 1)/(x + 2y + 1)$$

Let,

$$x + 2y = v \; so \; that, 1 + 2\left(\frac{dy}{dx}\right) = \frac{dv}{dx}\text{............................(1)}$$

Or, $\frac{dy}{dx} = (\frac{dv}{dx} - 1)/2$

$\therefore$ (1)Reduces to-

$$\frac{1}{2}\left(\frac{dv}{dx}-1\right)=\frac{v-1}{v+1}$$

$$or, \frac{dv}{dx}=2\left(\frac{v-1}{v+1}\right)+1$$

$$or, \frac{dv}{dx}=\frac{3v-1}{v+1}$$

$$or, dx=\frac{1}{3}\frac{(3v-1)+4}{3v-1}dv$$

$$or, 3dx=\left(1+\frac{4}{3v-1}\right)dv$$

Integrating

$$3x=v+(4/3)\times\log(3v-1)-\left(\frac{4}{3}\right)\times logc$$

c being an arbitrary constant.

$$or, \frac{4}{3}log\frac{3v-1}{c}=3x-v$$

$$or, \frac{4}{3}log\frac{3(x+2y)-1}{c}=3x-(x+2y)$$

$$or, log\frac{3x+6y-1}{c}=\frac{3}{4}\times(2x-2y)$$

$$or, \mathbf{3x+6y-1=ce^{\frac{3(x-y)}{2}}}$$

$(answer)$

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10. $\boldsymbol{x\frac{dy}{dx}-y=x\sqrt{x^2+y^2}}$

Solution :

The equation can be put as

$$xdy - ydx = x\sqrt{x^2 + y^2}dx$$

$$or\,, \frac{xdy - ydx}{x^2} = sec^2\,\theta$$

Changing to polar as above the equation becomes:

$$x^2\,sec^2\,\theta\,d\theta = xrdx$$

$$or, x\,sec^2\,\theta = rdx$$

$$or, rcos\,\theta\,sec^2\,\theta\,d\theta = rdx$$

$$or, sec\,\theta\;\;d\theta = dx$$

$$or, log(sec\,\theta + tan\,\theta) = x + log\,c$$

$$or, sec\,\theta + tan\,\theta = ce^x$$

$$\boldsymbol{\sqrt{1 + \frac{y^2}{x^2}} + \frac{y}{x} = ce^x}$$

(answer)

11. $\boldsymbol{x^4\frac{dy}{dx} + x^3y = -\sec(xy)}$

Solution :

Given that,

$$x^4 \frac{dy}{dx} + x^3 y = -sec(xy)$$

$$or, x^3 \left(x \frac{dy}{dx} + y\right) = -sec(xy)$$

$put\ v = xy$

Differentiating, $\frac{dv}{dx} = x\frac{dy}{dx} + y$

Now,

$$x^3 \frac{dv}{dx} = -secv$$

$$or, \frac{dv}{sec\, v} = \frac{dx}{x^3}$$

$$or, \int cos\, v\, dv = -\int \frac{dx}{x^3} + C$$

$$or, sin\, v = \frac{1}{2x^2} + C$$

$$\boldsymbol{or, \quad sin\, xy = \frac{1}{2x^2} + C}$$

(answer)

12. $\cos(x + y)\, dy = dx$

Solution :

Given,

$$\cos(x + y)\, dy = dx$$

$$or, \frac{dy}{dx} = \sec(x+y)$$

$$put\ x + y = z$$

$$then, 1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

$$or, \frac{dz}{dx} - 1 = \sec z$$

$$or, \frac{dz}{dx} = \sec z + 1$$

$$or, \frac{dz}{1+\sec z} = dx$$

$$or, \int \frac{\cos z}{1+\cos z} dz = \int dx$$

$$or, \int [1 - \frac{1}{1+\cos z}] dz = x + c$$

$$or, \int [1 - \frac{1}{2\cos^2\frac{z}{2} - 1 + 1}]\ dz = x + c$$

$$or, \int [1 - \frac{1}{2}\sec^2\frac{z}{2}] dz = x + c$$

$$or, z - tan\frac{z}{2} = x + c$$

$$or, x + y - tan\frac{x+y}{2} = x + c$$

$$\boldsymbol{y - tan\frac{x+y}{2} = c}$$

(answer)

13. $(dx - dy) = dx + dy$

Solution:

Given,

$(x + y)(dx - dy) = dx + dy$

Rewriting given equation

$(x + y - 1)dx = (x + y + 1)dy$

$\frac{dy}{dx} = \frac{x+y-1}{x+y+1} \dots \dots \dots (\mathbf{1})$

Let $x + y = v$

$1 + \frac{dy}{dx} = \frac{dV}{dx} \dots \dots \dots \dots . (\mathbf{2})$

Or, $\frac{dy}{dx} = \frac{dV}{dx} - 1$ ***…………….(3)***

From eqn (2) and (3)

$\frac{dV}{dx} - 1 = \frac{V-1}{v+1}$

Or, $\frac{dV}{dx} = \frac{2v}{v+1}$

Integrating, *2x+c=v+ log v*

$$\boldsymbol{x + y + c = log(x + y)}$$

(answer)

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Homogeneous Differential Equations:

An equation in the form $\frac{dy}{dx}=f_1(x,y)/f_2(x,y)$ in which $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions of x and y of same degree can be reduced to an equation in which variables are separable by putting ,

$$y = vx$$

$$Or, \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$Or,\ v + x\frac{dv}{dx} = f(v)$$

$$Or,\ x\frac{dv}{dx} = f(v) - v$$

$$Or, \frac{dv}{f(v)-v} = \frac{dx}{x}$$

$$Or, \int \frac{dv}{f(v)-v} = lnx + c$$

Working rules:

Step-1: put $y = vx$,so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Step-2: separate the variables

Step-3: Integrate both the sides.

Step-4: Put $v = \frac{y}{x}$ and simplify.

The following few examples will illustrate the method,

1.solve $x^2ydx-(x^3+y^3)=0$.

Solution:

we have, $\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$ *(Homogeneous)*

Putting, $y = vx, \frac{dy}{dx} = v + x.\frac{dv}{dx}$ the equation becomes

Or, $v + x \frac{dv}{dx} = \frac{v}{1+v^3}$

Or, $x\frac{dv}{dx} = \frac{v}{1+v^3}$

Or, $x\frac{dv}{dx} = -\frac{v^4}{1+v^3}$

Or, $\frac{dx}{x} = -\left[\frac{1}{v^4} + \frac{1}{v}\right] dv$

Integrating, $\log x = \frac{1}{3v^4} - \log v + c$

$$\boldsymbol{\log y = \frac{x^3}{3y^3} + c, \text{ as } y = vx}$$

(answer)

2.solve $y^2+x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

Solution:

The equation is $\frac{dy}{dx} = \frac{y^2}{xy-x^2}$. $(Homogeneous\ )$

Putting,$y = vx,\frac{dy}{dx} = v + x, we\ get$

Or,$v + x\frac{dv}{dx} = \frac{v^2}{v-1}$

Or, x.$\frac{dv}{dx} = \frac{v^2}{v-1} - v$

Or, x.$\frac{dv}{dx} = \frac{v}{v-1}$

Or,$\frac{dx}{x} = \left(1 - \frac{1}{v}\right) dv$

Integrating, $log\, x = v - logv + logc$

Or, $log\, xv = v + log\, c$

Or, $xv = ce^v$

Or, $y = ce^{\frac{y}{x}}$ as $y = vx$

(answer)

3.Solve,$(x^2 + y^2)dx + 2xydy = 0$

Solution:

Given that, $\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$

$$=-\frac{x^2(1+\frac{y^2}{x^2})}{2xy}$$

So, $\frac{dy}{dx} = -\frac{x}{2y}\left(1+\frac{y^2}{x^2}\right) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. (1)$

Let, $\frac{y}{x} = v$

Or, $y = vx$

Or, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Or, $v + x\frac{dv}{dx} = -\frac{1}{2}\frac{1}{v}(1+v^2).$

Or, $x\frac{dv}{dx} = -\frac{1}{2}\frac{1}{v}(1+v^2) - v$

Or, $x\frac{dv}{dx} = \frac{-1-v^2-2v^2}{2v}$

Or, $x\frac{dv}{dx} = \frac{-1-3v^2}{2v}$

Or, $\frac{dv}{dx} = -\frac{1+3v^2}{2v}$

Or, $\frac{dv}{-(1+3v^2)}2v = \frac{dx}{x}$

Or, $-\frac{1}{3}\int\frac{dv}{(1+3v^2)}2v = \int\frac{dx}{x}$

$Or, -\frac{1}{3} ln(1+3v^2) = ln\,x + ln\,c$

$Or, ln\,x + \frac{1}{3} ln(1+3v^2) = ln\,c$

$Or, ln\left(x.(1+3v^2)^{1/3}\right) = ln\,c$

$Or, x(1+3v^2)^{1/3} = c$

$Or, x(1+3\frac{y^2}{x^2})^{1/3} = c$

So, $\boldsymbol{x^2 + 3y^2 = \frac{c^3}{x}}$

(answer)

4. *solve the following equation,*

$$(2xy + x^2)y = 2xy^2 + 3y^2$$

Solution:

we have $(2xy + x^2)y = 3y^2 + 2xy$

$Or,\ \frac{dy}{dx} = \frac{3y^2+2xy^2}{2xy+x^2}$

Putting, $y = vx, \frac{dy}{dx} = v + x.\frac{dv}{dx}$

On substituting, the equation become

$$v + x\frac{dv}{dx} = \frac{3v^2x^2+2vx^2}{2vx^2+x^2} = \frac{3v^2+2v}{2v+1}$$

$$Or, x\frac{dv}{dx} = \frac{3v^2+2v-2v^2-v}{2v+1}$$

$$Or, x\frac{dv}{dx} = \frac{v^2+v}{2v+1}$$

$$Or, \left(\frac{2v+1}{v^2+v}\right)dv = \frac{dx}{x}$$

$$Or, \int\left(\frac{2v+1}{v^2+v}\right)dv = \int\frac{dx}{x}$$

$$Or, v^2 + v = cx$$

$$Or, \frac{y^2}{x^2} + \frac{y}{x} = cx$$

$$\mathbf{Or, y^2 + yx = cx^3}$$

(answer)

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5. $\boldsymbol{(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)dy = 0}$

Solution:

$$(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^2+2xy-y^2}{y^2+2xy-x^2} \qquad [\,y = vx$$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{1+2v-v^2}{v^2+2v-1} \qquad or, dy = v\,dx + x\,dv]$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{1+2v-v^2}{v^2+2v-1} - v = -\frac{v^3+v^2+v+1}{v^3+2v-1}$$

$$\Rightarrow \frac{dx}{x} = -\frac{v^2+2v-1}{v^3+v^2+v+1}dv = -\frac{v^2+2v-1}{(v+1)(v^2+1)}\ dv$$

$$\Rightarrow \frac{dx}{x} \quad = \left(\frac{1}{v+1} - \frac{2v}{v^2+1}\right)dv$$

Integrating, $log\, x = log(v+1) - log(v^2+1) + log\, C$

$$\Rightarrow \frac{x}{v^2+1} = C(v+1)$$

$$\Rightarrow \frac{x}{{y^2}/{x^2}+1} = C\left(\frac{y}{x}+1\right) \qquad [put\ v = {y}/{x}]$$

(answer)

6. $\frac{1}{2x}\frac{dy}{dx} + \frac{x+y}{x^2+y^2} = 0$

Solution :

$$\frac{1}{2x}\frac{dy}{dx} + \frac{x+y}{x^2+y^2} = 0$$

$$\Rightarrow \frac{1}{2x}\frac{(v\,dx+x\,dv)}{dx} + \frac{x+vx}{x^2+v^2x^2} = 0 \qquad [\,y = vx$$

$$\Rightarrow \quad \frac{v}{2x} + \frac{dv}{2\,dx} + \frac{(1+v)}{x(1+v^2)} = 0 \qquad or, dy = v\,dx + x\,dv]$$

$$\Rightarrow \frac{dv}{2\,dx} = -\left\{\frac{1+v}{x(1+v^2)} + \frac{v}{2x}\right\}$$

$$\Rightarrow \frac{dv}{2\,dx} = -\frac{2+2v+v+v^3}{2x(\pi v)}$$

$$\Rightarrow \frac{(1+v^2)\,dv}{v^3+3v+2} = -\frac{dx}{x}$$

Integrating,

$$\frac{1}{3} log(v^3 + 3v + 2) + log\, x = log\, c$$

$$\Rightarrow (v^3 + 3v + 2)x^3 = c^3$$

$$\Rightarrow \boldsymbol{y^3 + 3x^2y + 2x^3 = c^3} \qquad [put\ v = {}^{y}/_{x}]$$

(answer)

7. $\boldsymbol{x\cos\frac{y}{x}(y\,dx + x\,dy) = y\sin\frac{y}{x}(x\,dy - y\,dx) = 0}$

Solution:

$$x\cos\frac{y}{x}(y\,dx + x\,dy) = y\sin\frac{y}{x}(x\,dy - y\,dx) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\left(x\sin\frac{y}{x} + x\cos\frac{x}{x}\right)}{x\left(y\sin\frac{x}{x} - x\cos\frac{y}{x}\right)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v} \qquad [put,\ y = vx; \frac{dx}{dx} = v + x\frac{dv}{dx}\,]$$

$$\Rightarrow \left(\tan v - \frac{1}{v}\right) dv = 2\frac{dx}{x}$$

Integrating,

$$\log\frac{\sec v}{v} = \log C + 2\log x$$

$$\Rightarrow \boldsymbol{\sec\left(\frac{y}{x}\right) = Cxy}$$ [put $v = {}^{y}/_{x}$]

(answer)

8. $\boldsymbol{2y^3\,dx + (x^2 - 3y^2)x\,dy = 0}$

Solution:

$$2y^3\,dx + (x^2 - 3y^2)x\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2y^3}{(x^2-3y^2)x}$$

$$\Rightarrow \frac{x\,dv+v\,dx}{dx} = -\frac{2v^3x^3}{x(x^2-3v^2x^2)}$$ [put, $y = vx; \frac{dx}{dx} = v + x\frac{dv}{dx}$]

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{2v^3}{1-3v^2}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{v-v^3}{1-3v^2}$$

$$\Rightarrow \frac{1-3v^2}{v-v^3}dv = -\frac{dx}{x}$$

Integrating,

$$\log(v - v^3) = -\log x + \log C$$

$$\Rightarrow (v - v^3)x = C$$

$$\Rightarrow \left\{\frac{y}{x} - \left(\frac{y}{x}\right)^3\right\}x = C$$ [put $v = {}^{y}/_{x}$]

$$\Rightarrow x^2y - y^3 = cx^2$$

(answer)

9. $x\frac{dy}{dx} - y = \sqrt{x^2 - y^2}$

Solution:

$$x\frac{dy}{dx} - y = \sqrt{x^2 - y^2}$$

$$\Rightarrow x\left(v + x\frac{dv}{dx}\right) - vx = \sqrt{x^2 - v^2x^2} \qquad [put, y = vx; \frac{dx}{dx} = v + x\frac{dv}{dx}]$$

$$\Rightarrow xv + x^2\frac{dv}{dx} - vx = x\sqrt{1 - v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1-v^2}} = \frac{dx}{x}$$

Integrating, $sin^{-1} v = log\, x + c$

$$\Rightarrow sin^{-1}\frac{y}{x} = log\, x + c \qquad [put\,, v = {}^{y}/_{x}]$$

(answer)

10. $\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$

Solution:

$$\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y(x+y)}{x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{vx(x+vx)}{x^2}$$ [put, $y = vx; \frac{dx}{dx} = v + x\frac{dv}{dx}$]

$$\Rightarrow x\frac{dv}{dx} = -\left(\frac{v+v^2}{x^2} + v\right)$$

$$\Rightarrow x\frac{dv}{dx} = -(2v + v^2)$$

$$\Rightarrow \frac{dv}{2v+v^2} = -\frac{dx}{x}$$

Integrating, $\frac{1}{2}\log\frac{v}{v+2} = -\log x + \log C$

$$\Rightarrow \left(\frac{v}{v+2}\right)x^2 = C^2$$

$$\Rightarrow \left(\frac{\frac{y}{x}}{\frac{y}{x}+2}\right)x^2 = C^2$$ [put , $v = {}^{y}/_{x}$]

$$\Rightarrow \boldsymbol{x^2y = C(y + 2x)}$$

(answer)

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Equations reducible to homogeneous form :

Equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c}$ where $\frac{a}{a\prime} \neq \frac{b}{b\prime} \ldots (1)$ can be reduced to homogeneous form as explained below. Taking x = X+h **and** y = y + k... (2) where X and Y are new variables and h and k are constants to be so chosen that the resulting equation in terms of X and Y may become homogeneous.

From (2) dx = dXand d y = dy, so thatdy/dx = dY/dx.... (3)

Using (2) and (3). (1)becomes

$$\frac{dy}{dx} = \frac{a[X+h]+b[Y+k]+c}{a'[X+h]+b'[Y+k]+c\prime} = \frac{aX+bY+[ah+bk+c]}{a'X+b'Y+[a'h+b'k+c]}\text{...... (4)}$$

In order to make (4) homogeneous, choose h and k so as to satisfy the following two equations ah + bk +c=0 **and** a'h + b'k + c = 0.

Solving (5), h= $\frac{bc'-b'c}{ab'-a'b}$and k=$\frac{ca'-c'a}{ab'-a'b}$

Given that a/a'≠b/b'. Therefore, (ab' - db) ≠0. Hence, h and k given by (6) are meaningful, i.e., h and k will exist. Now, h and k are known. So from (2), we get X = x-h

and

Y = y - k.......(7)

In view of (5),(4) reduces to $\frac{dy}{dx} = \frac{aX+bY}{a'X+b'Y} = \frac{a+b\left(\frac{Y}{X}\right)}{a'+b'\left(\frac{Y}{X}\right)}$

which is surely homogeneous equation in X and Y and can be solved by putting Y/X = v as usual. After getting solution in terms of X and Y, we remove X and Y by using (7) and obtain solution in terms of the original variables x and y.

special case.1:

When $\frac{a}{a\prime} = \frac{b}{b\prime} = \frac{1}{m}$, *then the differential equation can be written as*

$$\frac{dy}{dx} = \frac{ax+by+c}{m(ax+by)+c}$$

Put ax+by = r, so that a+b $\frac{dy}{dx} = \frac{dy}{dx}$

(1)Then becomes 1/b($\frac{dy}{dx} - a) = \frac{b+c}{mb+c}$

1. $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Solution:

Put x=X+h,y=Y+k , where h,k are some constants: then $\frac{dy}{dx} = \frac{dY}{dX}$ then given equation becomes

$$\frac{dY}{dX} = \frac{X + 2Y + (h + 2k - 3)}{2X + Y + (2h + k - 3)}$$

Now choose h,k such that *h+2k-3=0* and *2h+k-3=0* solving these we get h=1,k=1

$\frac{dY}{dX} = \frac{X+2Y}{2X+Y}$ homogenous in X and Y

Put Y=vX

So that $\frac{dY}{dX} = v + X\frac{dv}{dX}$

v+X$\frac{dv}{dX} = \frac{X+2vX}{2X+vX} = \frac{1+2v}{2+v}$ i.e, X$\frac{dv}{dX} = \frac{1+2v}{2+v} - v$

or $\frac{dX}{X} = \frac{2+v}{1-v^2}dv$=($\frac{1}{1-v^2} + \frac{v}{1-v^2}$)$dx1$

integrating ,

logX=2 .$\frac{1}{2}log\frac{1+v}{1-v} - \frac{1}{2}log(1 - v^2) + log\, c$

or *X=C*$\frac{1+c}{1-c}\frac{1}{\sqrt{1-v^2}} = \frac{c\sqrt{1+v}}{(1-v)^{\frac{3}{2}}}$

or $X^2(1 - v)^3 = C^2(1 + v)$

or $X^2(1 - \frac{Y}{X})$^3=$C^2(1 + \frac{Y}{X})$ *as v=Y/X*

or *(X-Y)^3=*$C^2(X + Y)$ $but\ x = X + 1\,, y = Y + 1$

so (x-y)^3=$C^2(x + y - 2)$ is the required solution

(answer)

2. (3x-7y-3)dy/dx =(3y-7x+7)

Solution:

$\frac{dy}{dx} = \frac{3y-7x+7}{3x-7y-3}$

Put x=X+h, y=Y+k where h,k are some constants then

$\frac{dy}{dx} = \frac{dY}{dX}$

And then given equation becomes

$$\frac{dY}{dX} = \frac{3Y - 7X + (3k - 7h + 7)}{3X - 7Y + (2h - 7k - 3)}$$

Choose h,k such that 3h-7k-3=0 and 3k-7h+7=0,which give h=1 ,k=0

$\frac{dY}{dX} = \frac{3Y-7X}{3X-7Y}$ [homogenous]

Put *Y= vX ,dy/dx=v+Xdv/dx*

V+Xdv/dx=$\frac{3vX-7X}{3X-7vX} = \frac{3v-7}{3-7v}$

Or Xdv/dx=$\frac{3v-7}{3-7v} - v = \frac{7(v^1-1)}{3-7v}$

Or 7 $\frac{dX}{X} = \frac{3-7v}{(v^2-1)} dv = -\left(\frac{2}{v-1} + \frac{5}{v+1}\right) dv$

Integrating , *7logX=-2log(v-1)-5log(v+1)+logC*

Or $x^7 (v - 1)^{2(v+1)^5} = C$

Or $X^7 \left(\frac{Y}{X} - 1\right)^2 \left(\frac{Y}{X} - 1\right)^5 = C \; as \; Y = vX$

Or $(Y - X)^2 (Y + X)^6 = C$

$$Or\ (y-x+1)^2(y+x-1)^5 = C\ as\ x - X + 1, y = Y + 0$$

(answer)

3. Solve dyldx = (x + 2y - 3)/(2x + y - 3).

Solution :

Take x = X+h, y = y + k. so that dy/dx = dY/dx.... (1)

. Given equation becomes $\frac{dy}{dx} = \frac{X+2Y+(h+2K-3)}{2X+Y+(2h+k-3)}$......(2)

.. Choose h, k so that h+2k - 3 = 0 and 2h+k - 3 = 0.... (3)

Solving (3), we get h=1, k = 1 so that from (1), we have

X = x-1, and Y = y-1...............(4)

Using (3) in (2), we get $\frac{dy}{dx} = \frac{X+2Y}{2X+Y} = \frac{1+(\frac{2Y}{X})}{2+(\frac{Y}{X})}$

Take Y/X = v. i.e. . Y = vX. Therefore. dY/dX = v + X (dv/dx). ... (6) From (5) and (6), we have

$$V+X\frac{dy}{dx} = \frac{1+2v}{2+v}$$

or $$X\frac{dv}{dx} = \frac{1+2v}{2+v} - v = \frac{1-v^2}{2+v}$$

$$\frac{dX}{X} = \frac{(2+v)dv}{(1-v)(1+v)} = [\frac{1}{2}\left(\frac{1}{1+v} + \frac{3}{2}\left(\frac{1}{(1-v)}\right)\right] dv,$$

resolving into partial fractions

Integrating,

log x + log c = (1/2) [log (1 + v) - 3 log (1 - v)]

2 log (cX) = log 1+v/$(1-\mathrm{v})^3$ or $X^2c^2 = \frac{1+v}{(1-v)^2}$

Or $X^2c^2(1-\frac{Y}{X})$^3=1+Y/X,as v=Y/X

Or $c^2(X-Y)^3 = X+Y$ or $c^2\{x-1-(y-1)\}^2 = x-1+y-1, by\ (4)$

Or c′$c'^{(x-y)^2} = x+y-2,$

$\boldsymbol{taking\ c' = c^2, c'being\ an\ arbitrary\ constant}$

EX.4. SOLVE dy/dx = -(x-y-2)/(x - 2y-3)

SOL. Given equation. Take x = X+h, y = Y + k so that dy/dx = dy/dx.... (1)

The given equation becomes $\frac{dY}{dX} = \frac{X-Y+h-k-2}{X-2Y+h-2k-3}$.. ….(2)

Choose h-k-2=0 and h-2k-3=0 …………(3)

Solving (3), we get h=k, k = -1 so that from (1), we have

X=x-1 and Y=y-1 ……………………………………………(4)

And (2) becomes $\frac{dy}{dx} = -\frac{X-Y}{X-2Y} = -\frac{1-(\frac{Y}{X})}{1-2(\frac{Y}{X})}$ ………………………………(5)

take Y/X=v,i.e Y=vX so that Dy/DX=v+X=dv/dx ………(6)

From (5) and (6) v+X dv/dx =- (1-v)/(1-2v) or X$\frac{dX}{X} = -\frac{1-2v^2)}{2v-1}$dv

Or $\frac{dX}{X} = \frac{2v-1}{1-2v^2}dv$ or $\frac{dX}{X} = [-\frac{1(-4v)}{2(1-2v^2)} - \frac{1}{1-(v\sqrt{2})^2}dv$

Integrating log X= -$\frac{1}{2}log(1-2v^2)$-$\frac{1}{2\sqrt{2}}log\frac{1+v\sqrt{2}}{1-v\sqrt{2}} - \frac{1}{2}\log c$

Or 2logX+log(1-2v^2) + $logc$ =-$\frac{1}{2}$log((1+v√2)/(1-v√2))

$0r$ log{$cX^2(1-2v^2)$=loglog $\left(\frac{1+v\sqrt{2}}{1-v\sqrt{2}}\right)^{1/\sqrt{2}}$

Or,c($x^2(1-2\frac{Y^2}{X^2}$)={$\frac{1-(\frac{Y}{X})\sqrt{2}}{1+(\frac{Y}{X})\sqrt{2}}$})^1/√2 or c($X^2-Y^2) = c(\frac{X-Y\sqrt{2}}{X+Y\sqrt{2}})^{\frac{1}{\sqrt{2}}}$

Or c{(x-1)^2-2(y+1)^2}={$\frac{x-1-(y+1)\sqrt{2}}{x-1+(y+1)\sqrt{2}}$}$^{\frac{1}{\sqrt{2}}}$

Or c$(x^2 - 2y^2 - 2x - 4y - 1) = (\frac{x-y\sqrt{2}-\sqrt{2}-1}{x+y\sqrt{2}-1+\sqrt{2}})^{\frac{1}{\sqrt{2}}}$ c being an arbitrary constant

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5.Solve $(2x - 2y + 5)\frac{dy}{dx} = x - y + 3$

Sol:Theeqn is $\frac{dy}{dx} = \frac{x-y+3}{2(x-y)+5}$................(1)

Let $x - y = v, so\ that\ 1 - \frac{dy}{dx} = \frac{dv}{dx}\ or\ \frac{dy}{dx} = 1 - \frac{dv}{dx}$.

Putting the value in eqn 1,we get $\frac{v+3}{2v+5} = 1 - \frac{dv}{dx}$.

Or,$\frac{dv}{dx} = 1 - \frac{v+3}{2v+5}$

Or,$\frac{dv}{dx}$ =$\frac{v+2}{2v+5}$

Or, $dx = \frac{2v+5}{v+2}dv = \left(2 + \frac{1}{v+2}\right)dv$

Or,$x = 2v + \log(v + 2) + C$,

Or,$x = 2(x - y) + \log(x - y + 2) + C$as $v = x - y$

Or , $2y - x = log(x - y + 2) + C$

Which is the solution.

6.Solve $(2x + y + 3)\frac{dy}{dx} = x + 2y + 3$

Sol: $\frac{dy}{dx} = \frac{x+2y+3}{2x+y+3}$

Let $x = X + h, y = Y + k$,where h ,k are constants.

$$\therefore \frac{dy}{dx} = \frac{dY}{dX}$$

$$\therefore \frac{dY}{dX} = \frac{X + 2Y + (h + 2k + 3)}{2X + Y + (2h + k + 3)}$$

Choosing h,k such that $(h + 2k + 3) = 0, (2h + k + 3) = 0$

Solving these,we get $h = -1, k = -1$

$$\therefore \frac{dY}{dX} = \frac{X + 2Y}{2X + Y}$$

Let $Y = vx$

$$\frac{dY}{dX} = v + X\frac{dv}{dx}$$

$\therefore v + X\frac{dv}{dx} = \frac{X+2vX}{2X+vX}$

Or, $X\frac{dv}{dx} = \frac{1+2v}{2+v} - v$

Or, $\frac{dX}{X} = \frac{2+v}{1-v^2} dv = (\frac{\frac{3}{2}}{1-v} + \frac{\frac{1}{2}}{1+v}) dv$

Integrating, $2\log X = -3\log(1 - v) + \log(1 + v) + logC$

Or, $(X - Y)^3 = C(X + Y); where\ x = X - 1, y = Y - 1$

Or, $(X - Y)^3 = C(x + y - 2)$ is the solution.

7. Solve $\frac{dy}{dx} = \frac{6x-4y+3}{3x-2y+1}$

Sol:

The equation is $\frac{dy}{dx} = \frac{6x-4y+3}{3x-2y+1}$..............(1)

Let 3x-2y=v

$\therefore 3 - 2\frac{dy}{dx} = \frac{dv}{dx}$

Or,$2\frac{dy}{dx} = (3 - \frac{dv}{dx})$

Or,$\frac{dy}{dx} = (\frac{3}{2} - \frac{1}{2}\frac{dv}{dx})$

From (1) we get,

$\left(\frac{3}{2} - \frac{1}{2}\frac{dv}{dx}\right) = \frac{2v+3}{v+1}$

Or,$\frac{3}{2} - \frac{2v+3}{v+1} = \frac{1}{2}\frac{dv}{dx}$

Or,$\frac{1}{2}\frac{dv}{dx} = \frac{3v+3-4v-6}{2v+2}$

Or,$\frac{1}{2}\frac{dv}{dx} = \frac{-v-3}{2v+2}$

Or,$\frac{dv}{dx} = \frac{-(v+3)}{v+1}$

Or,$\frac{dv}{dx} = \frac{-(v+1+2)}{v+1}$

Or,$\frac{dv}{dx} = -1 - \frac{2}{v+1}$

Or,$\frac{dv}{dx} = -1 - \frac{2}{v+1}$

Or,$dx = \frac{2v+5}{v+2}dv = (2 + \frac{1}{v+2})$

Or,$\int dx = \int \left(2 + \frac{1}{v+2}\right) dv$

Or,$x = 2v + \log(v + 2) + C$

Or,$x = 2(x - y) + \log(x - y + 2) + C$

Or, $2y - x = \log(x - y + 2) + C$

Is the required solution.

8. Solve $\frac{dy}{dx} = \frac{3x-2y+1}{6x-4y+1}$

Sol:

The equation is $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$..............(1)

Let x-y=v

$$\therefore 1 - \frac{dy}{dx} = \frac{dv}{dx}$$

Or,$\frac{dy}{dx} = 1 - \frac{dv}{dx}$

From (1) we get,

$1 - \frac{dv}{dx}$=$\frac{v+3}{2v+5}$

Or,$-\frac{v+3}{2v+5} + 1$=$\frac{dv}{dx}$

Or,$\frac{dv}{dx} = 1 - \frac{v+3}{2v+5} = \frac{v+2}{2v+5}$

Or,$dx = \frac{2v+5}{v+2} dv = (2 + \frac{1}{v+2})$

Or,$\int dx = \int \left(2 + \frac{1}{v+2}\right) dv$

Or,$x = 2v + \log(v+2) + C$

Or,$x = 2(x-y) + \log(x-y+2) + C$

Or, $2y - x = \log(x-y+2) + C$

Is the required solution.

9. Solve $\frac{dy}{dx} = \frac{6x-2y-7}{3x-y+4}$

Sol:

The equation is $\frac{dy}{dx} = \frac{6x-2y-7}{3x-y+1}$..............(1)

Let 3x-y=v

$$\therefore 3 - \frac{dy}{dx} = \frac{dv}{dx}$$

Or,$\frac{dy}{dx} = (3 - \frac{dv}{dx})$

From (1) we get,

$$\left(3 - \frac{dv}{dx}\right) = \frac{2v-7}{v+1}$$

Or,$3 - \frac{2v-7}{v+1} = \frac{dv}{dx}$

Or,$\frac{dv}{dx} = \frac{3v+3-2v+7}{v+1}$

Or,$\frac{dv}{dx} = \frac{v+10}{v+1}$

Or,$\frac{dx}{dv} = \frac{(v+1)}{v+10}$

Or,$\frac{dx}{dv} = \frac{(v+10-9)}{v+10}$

Or,$\frac{dx}{dv} = 1 - \frac{9}{v+10}$

Or,$\int dx = \int (1 - \frac{9}{v+10}) dv$

Or,$x = v - 9\log(v + 10) + C$

Or,$x = 2(3\mathrm{x} - \mathrm{y}) + \log(3\mathrm{x} - \mathrm{y} + 1) + C$

Or,$2(3\mathrm{x} - \mathrm{y}) + \log(3\mathrm{x} - \mathrm{y} + 1) + C - x = 0$

Or,$5x - 2y + \log(3\mathrm{x} - \mathrm{y} + 1) + C = 0$

Is the required solution. (ans)

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10. Solve $\frac{dy}{dx} = \frac{3x-4y-2}{3x-y+4}$

Sol:

The equation is $\frac{dy}{dx} = \frac{3x-4y-2}{3x-4y+4}$..............(1)

Let 3x-4y=v

$$\therefore 3 - 4\frac{dy}{dx} = \frac{dv}{dx}$$

Or,$\frac{dy}{dx} = \frac{1}{4}(3 - \frac{dv}{dx})$

From (1) we get,

$$\frac{1}{4}\left(3 - \frac{dv}{dx}\right) = \frac{v-2}{v+4}$$

Or,$\frac{3}{4} - \frac{v-2}{v+4} = \frac{1}{4}\frac{dv}{dx}$

Or,$\frac{1}{4}\frac{dv}{dx} = \frac{3v+12-4v+8}{4(v+4)}$

Or,$\frac{dv}{dx} = \frac{-v+20}{v+4}$

Or,$\frac{dx}{dv} = \frac{(v+4)}{-(v-20)}$

Or,$\frac{dx}{dv} = \frac{-(v-20+24)}{v-20}$

Or,$\frac{dx}{dv} = -1 - \frac{24}{v-20}$

Or,$\int dx = \int(-1 - \frac{24}{v-20})dv$

Or,$x = -v - 24\log(v - 20) + C$

Or,$x = -(3\text{x} - 4\text{y}) + \log(3\text{x} - 4\text{y} + 1) + C$

Or,$4x - 4y - \log(3\text{x} - \text{y} + 1) + C = 0$

Is the required solution. (ans)

11.Solve $(2x + y + 1)dx = (4x + 2y - 1)dy$

Sol:Theeqn is $\frac{dy}{dx} = -\frac{2x+y+1}{4x+2y-1}$................(1)

Let $2x + y = v, so\ that\ 2 + \frac{dy}{dx} = \frac{dv}{dx}\ or\ \frac{dy}{dx} = \frac{dv}{dx} - 2.$

Putting the value in eqn 1,we get $-\frac{v+1}{2v-1} = \frac{dv}{dx} - 2.$

Or,$\frac{dv}{dx} = 2 - \frac{v+1}{2v-1}$

Or,$\frac{dv}{dx} = \frac{4v-2-v-1}{2v-1}$

Or,$\frac{dv}{dx} = \frac{3v-3}{2v-1}$

Or, $dx = \frac{2v-1}{3v-3}dv$

Or, $dx = \frac{2v-1}{3(v-1)}dv$

Or, $\int dx = \int \frac{2v-1}{3(v-1)}dv$

Or, $\int dx = \frac{1}{3}\int \frac{2v-1}{(v-1)}dv$

Or,$3x = 2v + \log(v - 1) + C,$

Or,$x = 2(2x + y) + \log(2x + y - 1) + C$as $v = 2x + y$

Or,$3x + 2y + \log(2x + y - 1) + C = 0$

Which is the solution.

12.Solve $\frac{dy}{dx} = \frac{y-x+1}{y-x-5}$

Sol:

The equation is $\frac{dy}{dx} = \frac{y-x+1}{y-x-5}$..............(1)

Let y-x=v

$$\therefore \frac{dy}{dx} - 1 = \frac{dv}{dx}$$

Or, $\frac{dy}{dx} = \frac{dv}{dx} + 1$

From (1) we get,

$$\frac{dv}{dx} + 1 = \frac{v+1}{v-5}$$

Or, $\frac{dv}{dx} = \frac{v+1-v+5}{(v-5)}$

Or, $\frac{dv}{dx} = \frac{6}{(v-5)}$

Or, $\frac{dx}{dv} = \frac{(v-5)}{6}$

Or, $\int dx = \frac{1}{6}\int (v-5)dv$

Or, $x = \frac{1}{12}v^2 - \frac{5}{6}\mathrm{v} + C$

Or, $x = \frac{1}{12}(y-x)^2 - \frac{5}{6}(\mathrm{y}-\mathrm{x}) + C$

Is the required solution. (ans)

13. Solve $\frac{dy}{dx} = \frac{2x+y+3}{4x+2y+1}$

Sol:

The equation is $\frac{dy}{dx} = \frac{2x+y+3}{4x+2y+1}$..............(1)

Let 2x+y=v

$$\therefore 2+\frac{dy}{dx}=\frac{dv}{dx}$$

Or, $\frac{dy}{dx}=\frac{dv}{dx}-2$

From (1) we get,

$$\left(\frac{dv}{dx}\right)-2=\frac{v+3}{2v+1}$$

Or, $\frac{v+3}{2v+1}+2=\frac{dv}{dx}$

Or, $\frac{dv}{dx}=\frac{v+3+4v+2}{2v+1}$

Or, $\frac{dv}{dx}=\frac{5(v+1)}{v+4}$

Or, $\frac{dx}{dv}=\frac{(v+4)}{5(v+1)}$

Or, $\frac{dx}{dv}=\frac{1}{5}\frac{(v+1+3)}{v+1}$

Or, $5\frac{dx}{dv}=1+\frac{3}{v+1}$

Or, $5\int dx=\int(1+\frac{3}{v+1})dv$

Or, $5x=v+3\log(v+1)+C$

Or, $5x=(2x+y)+3\log(2x+y+1)+C$

Or, $3x-y-3\log(2x+y+1)+C=0$

Is the required solution. (ans)

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A particular case:

A differential equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{-bx + hy + k'}$$

In which co efficient of y in the numerator is equal to the coefficient of x in the denominator with sign changed, can be integrated as follows:

The equation(1) can be written as

- b(xdy+ydx)+(hy+k)dy-(ax+c)dx=0.

Integrating we get -bxy+(1/2 $hy^2 + ky) - \left(\frac{1}{2}ax^2 + cx\right) = A.$

1.solve $\frac{dy}{dx} + \frac{ax+hy+g}{hx+by+f} = 0$

Sol. The equation can be written as

(hx+by+f)dy+(ax+hy+g)dx=0

Or, h(xdy+ydx)+(by+f)dy+(ax+g)dx=0

Integrating, hxy+$\frac{1}{2}by^2 + fy + \frac{1}{2}a\,x^2 + gx = A$

Or, a$x^2 + 2hxy$ +b$y^2 + 2fy + 2gx + c = 0$,

$writing \;\; c = -2A$

2. solve $\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}$

Here coefficient of y in numerator is equal to co efficient of x in the denominator with sign changed, hence written it as

(x+2y-3)dy-(2x-y+1)dx=0

integrating xy+$y^2 - 3y - x^2 - x = c$

3. Solve (2x-y+1)dx+(2y-x-1)dx=0

Sol. The equation is of above type. Hence after regrouping we have

(2x+1)dx+(2y-1)dy-(ydx+xdy)=0

Integrating, $(x^2 + x) + (y^2 - y) - xy = c$

Which is the solution

4. *Solve* $\frac{dy}{dx} + \frac{2x+3y+1}{3x+4y-1} = 0$

Sol. The equation is of above type and can be written as

(3x+4y-1)dy+(2x+3y+1)dx=0

i.e., 3(xdy+ydx) + (4y-1)dy+(2x+1)dx=0

Integrating, 3xy+2$y^2 - y$+$x^2 + x = c$ is the solution

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LINEAR DIFFERENTIAL EQUATION :

A differential equation of the form

$\frac{dy}{dx} + Py = Q$ ....(i)

Is called a linear differential equation, where P & Q are functions of x (but not of y) or constants.

In such case, we multiply both sides of (i) by $e^{\int P\,dx}$ as an integrating factor (I.F) to make the equation readily integrable. So (i) becomes,

$e^{\int P\,dx}\left(\frac{dy}{dx} + Py\right) = Q\,e^{\int P\,dx}$ .... (ii)

The left hand side of (ii) is $\frac{d}{dx}[\,y\,e^{\int P\,dx}]$

Or, $\frac{d}{dx}\left[\,y\,e^{\int P\,dx}\right] = Q\,e^{\int P\,dx}$

Now, integrating both sides we get,

$y\,e^{\int P\,dx} = \int\,Q\,e^{\int P\,dx}dx + C$ which is the solution.

Hence, the required solution is, $y(I.F) = \int\,Q(I.F)dx + C$

1. $\mathbf{(1-x^2)\frac{dy}{dx} - xy = 1}$

Solution:

The equation can be written as, $\frac{dy}{dx} - (\frac{x}{1-x^2})y = \frac{1}{1-x^2}$

Here, $P = -(\frac{x}{1-x^2})$, $Q = \frac{1}{1-x^2}$

Integrating factor = $e^{\int -(\frac{x}{1-x^2})\,dx}$

$= e^{\frac{1}{2}log(1-x^2)}$

$= \sqrt{1-x^2}$

Hence, the solution is ,

$y\sqrt{1-x^2} = \int \frac{1}{1-x^2}\sqrt{1-x^2}\,dx + C$

(answer)

2. $\mathbf{x\frac{dy}{dx} + 2y = x^2 log\, x}$

Solution:

The equation becomes $\frac{dy}{dx} + \frac{2}{x}y = x \log x$

Here, $P = \frac{2}{x}$, $Q = x \log x$

Integrating factor = $e^{\int P\, dx}$

$= e^{\int \frac{2}{x} dx}$

$= e^{2 \log x}$

$= x^2$

Hence , the solution is

$$y.x^2 = \int x \log x \,.x^2\, dx + C$$

Or, $y = \frac{1}{x^2} \int x^3 \log x\, dx + C$

Or, $y = \frac{1}{x^2}\ [\{\log x \,.\int x^3\, dx - \int \frac{1}{x} (\int x^3\, dx) dx\ \} + C]$

Or, $y = \frac{1}{x^2}\ (\frac{x^4}{4} \log x - \frac{x^4}{16}) + C$

Or, $y = \frac{x^2}{4} \left(\log x - \frac{1}{4}\right) + C\ x^{-2}$

(answer)

3. $\boldsymbol{(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x}$

Solution:

The equation is $\frac{dy}{dx} - \left(\frac{3x^2-1}{x^3-x}\right) y = \frac{x^5-2x^3+x}{x^3-x}$

Or, $\frac{dy}{dx} - \left(\frac{3x^2-1}{x^3-x}\right) y = \frac{x(x^4-2x^2+1)}{x(x^2-1)}$

Or, $\frac{dy}{dx} - \left(\frac{3x^2-1}{x^3-x}\right) y = (x^2 - 1)$

Here, P = $-\left(\frac{3x^2-1}{x^3-x}\right)$, Q = $x^2 - 1$

Integrating factor = $e^{\int P\, dx}$

$= e^{\int -\left(\frac{3x^2-1}{x^3-x}\right) dx}$

$= e^{-\log(x^3-x)}$

$= \frac{1}{x^3-x}$

The solution is , $y \,.\, \frac{1}{x^3-x} = \int (x^2 - 1) \frac{1}{x^3-x}\, dx + C$

Or, $\frac{y}{x^3-x} = \int \frac{1}{x}\, dx + C$

Or, $\frac{y}{x^3-x} = \log x + C$

(answer)

4. $\boldsymbol{x\frac{dy}{dx}+y=ax^2+bx+c}$

Solution:

The equation can be written as $\frac{dy}{dx}+\frac{y}{x}=ax+b+\frac{c}{x}$

Here, $P=\frac{1}{x}$, $Q=ax+b+\frac{c}{x}$

Integrating factor = $e^{\int P\,dx}$

$= e^{\int \frac{1}{x}dx}$

$= e^{\log x}$

$= x$

Hence, The solution is ,y.x = $\int\left(\mathrm{ax}+\mathrm{b}+\frac{\mathrm{c}}{\mathrm{x}}\right).x\,dx+C$

Or, xy = $\int(ax^2+bx+c)\,dx$ + C

Or, xy = $(\frac{1}{3}ax^3+\frac{1}{2}bx^2+c)$ + C

(answer)

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5. $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$

Solution:

The equation is $\frac{dy}{dx} - \frac{1}{x(x-1)}y = x(x-1)$

Integral Factor $= e^{-\int \frac{1}{x(x-1)}dx}$

$= e^{\int (\frac{1}{x} - \frac{1}{x-1})dx}$

$= \frac{x}{x-1}$

Hence, $y\frac{x}{x-1} = C + \int x(x-1).\frac{x}{x-1}\, dx$

or, $y\frac{x}{x-1} = C + \int x^2\, dx$

or, $y\frac{x}{x-1} = C + \frac{1}{3}x^3$ *where C is an arbitrary constant*

(answer)

6. $(1 + x)\frac{dy}{dx} + 3y = \frac{1 + x + x^2}{(1 + x)^4}$

Solution :

The equation is $\frac{dy}{dx} + \frac{3}{1+x}y = \frac{1 + x + x^2}{(1 + x)^4}$

Integral Factor $= e^{\int \frac{3}{1+x}dx}$

$= e^{3\log(1+x)}$

$= (1 + x)^3$

$y(1 + x)^3 = C + \int \frac{(1+x+x^2)}{(1 + x)^4}(1 + x)^3\, dx$

$= C + \int \frac{(1+x+x^2)}{(1+x)}\, dx$

$= C + \int \left(\frac{1}{1+x} + x\right) dx$

$= C + \log(1 + x) + \frac{1}{2}x^2$ *where C is an arbitrary constant*

(answer)

7. $x\frac{dy}{dx} + 2y = \frac{dy}{dx} + 4$

Solution:

The equation can be written as $(x-1)\frac{dy}{dx} + 2y = 4$

or, $\frac{dy}{dx} + \frac{2}{x-1}\,y = \frac{4}{x-1}$

Linear Integral Factor $= e^{\int \frac{2}{x-1}dx}$

$= e^{\int 2\,log(x-1)}$

$= (x-1)^2$

Hence, $y\,(x-1)^2 = C + \int \frac{4}{x-1}(x-1)^3\,dx$

or, $\boldsymbol{y\,(x-1)^2 = C + 2(x-1)^2}$ *where C is an arbitrary constant*

(answer)

8. $x\frac{dy}{dx} - 2y = x^2 + sin\,\frac{1}{x^2}$

Solution:

The equation is $\frac{dy}{dx} - \frac{2}{x}y = x + \frac{1}{x}\,sin\,\frac{1}{x^2}$

Integral Factor = $e^{-\int \frac{2}{x} dx}$

= $e^{-2\log x}$

= $\frac{1}{x^2}$

Hence, $y.\frac{1}{x^3} = C + \int x.\frac{1}{x^2}\,dx + \int \frac{1}{x^3}\sin\frac{1}{x^2}\,dx$

or, $y.\frac{1}{x^3} = C + \log x - \frac{1}{2}\int \sin t\,dt$ *[where,* $\frac{1}{x^2} = t$, $\frac{-2}{x^3}\,dx = dt$*]*

or, $\boldsymbol{y.\frac{1}{x^3} = C + \log x + \frac{1}{2}\cos\frac{1}{x^2}}$ *where C is an arbitrary constant*

(answer)

9. $\frac{dy}{dx} - 2y\cos x = -2\sin 2x$

Solution:

Integral Factor = $e^{-\int \cos x\,dx}$

= $e^{-3\sin x}$

Hence, $y.e^{-3\sin x} = C - 2\int \sin 2x.\,e^{-2\sin x}\,dx$

or, $y.\,e^{-3\sin x} = C - 4\int \sin x\cos x.\,e^{-2\sin x}dx$ *[put, -2 sin x = t]*

or, $y.\,e^{-3\sin x} = C - \int t\,e^t dt$

or, $y.\,e^{-3\sin x} = C - e^t\,(t-1)$

So, $\boldsymbol{y = C.\,e^{3\sin x} + (2\sin x + 1)}$ *where C is an arbitrary constant*

(answer)

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10. $\boldsymbol{(x+1)\frac{dy}{dx} - y = e^x(x+1)^2)}$ ______ ***(1)***

Solution :

Step 1:Identifying equation

This is a linear partial differential equation and can be expressed as $\frac{dy}{dx} + Py = Q$,

where P and Q are the functions of **x**

We get from (1),

$\frac{dy}{dx} - \frac{y}{x+1} = e^x(x+1)$ __________ ***(2)***

Step 2:Evaluating integrating factor

Integrating factor = $e^{-\int\frac{dx}{x+1}}$

$= e^{-log(x+1)}$

$= e^{log(x+1)^{-1}} = \frac{1}{x+1}$

Step 3:Solution

$y\frac{1}{x+1} = \int e^x(x+1)\frac{1}{x+1}dx$

$= \int e^x dx$

$\therefore \boldsymbol{\frac{y}{x+1} = e^x + C}$

(answer)

11. $\boldsymbol{sinx\frac{dy}{dx} + 2y = tan^3(\frac{x}{2})}$ _________ *(1)*

Solution:

From (1) we get,

$\frac{dy}{dx} + \frac{2y}{sinx} = \frac{tan^3(\frac{x}{2})}{sinx}$ __________ ***(2)***

Here, $P=\frac{2}{sinx}$ and $Q=\frac{tan^3(\frac{x}{2})}{sinx}$

Integrating factor $= e^{\int\frac{2}{sinx}dx}$

$= e^{2\int cosecx\, dx}$

$= e^{2logtan\frac{x}{2}}$

$= tan^2\frac{x}{2}$

Now we get, $ytan^2\frac{x}{2} = \int tan^2\frac{x}{2}\frac{tan^3\frac{x}{2}}{2sin\frac{x}{2}cos\frac{x}{2}}dx + C$

$$= \frac{1}{2}\int \frac{tan^4\frac{x}{2}}{cos^2\frac{x}{2}}dx + C$$

$$= \frac{1}{2}\int tan^4\frac{x}{2}sec^2\frac{x}{2}dx + C$$

Putting tan $\frac{x}{2} = t\ so\ that\ \frac{1}{2}sec^2\frac{x}{2}dx = dt\ on\ RHS\ we\ get,$

$$ytan^2\frac{x}{2} = \frac{1}{2}\int t^4\,(2dt) + C$$

$$ytan^2\frac{x}{2} = \frac{t^5}{5} + C = \frac{tan^5\frac{x}{2}}{5} + C$$

(answer)

12. $\boldsymbol{cos^2x\frac{dy}{dx}}$*+y=tanx__________(1)*

Solution:

Dividing (1) by cos^2x we get,

$\frac{dy}{dx} + ysec^2x = tanx\ sec^2x$**_________*(2)***

Integral Factor = $e^{\int sec^2xdx}$

$= e^{tanx}$

Now multiplying (2) by the IF and integrating we get,

ye^{tanx} =$\int tanx\ sec^2\ xe^{tanx}dx$__________***(3)***

Let, I =$\int tanx\ sec^2\ xe^{tanx}dx$ and tanx = z

$\therefore sec^2xdx = dz$

$\therefore I = \int ze^z dz$

= $e^z(z-1)$

= $e^{tanx}(tanx-1)$

$(\mathbf{3}) \Rightarrow ye^{tanx} = e^{tanx}(tanx-1)+C$

$\therefore \boldsymbol{y = tanx - 1 + Ce^{-tanx}}$

(answer)

13. $\boldsymbol{\frac{dy}{dx} - \frac{3x^2-1}{x^3-x}y = x^2 - 1}$________***(1)***

Solution:

Integral Factor = $e^{-\int\frac{3x^2-1}{x^3-x}dx}$

= $e^{-\log(x^3-x)}$ = $\frac{1}{x^3-x}$

From (1) $y\frac{1}{x^3-x} = \int\frac{x^3-1}{x^3-x}dx + C$ =$\int\frac{1}{x}dx + C$

$\boldsymbol{y\frac{1}{x^3-x}}$ **= lnx+C**

(answer)

14. $\boldsymbol{\frac{dy}{dx} + 2y\, tanx = sinx}$____________$\mathbf{(1)}$

Solution:

Integral Factor = $e^{2\int tanxdx}$

= $e^{-2logcosx}$

= $e^{cos^{-2}x}$

= $cos^{-2}x$

= sec^2x

Hence the general solution is,

$$ysec^2x = \int sinxsec^2xdx + C$$

$$\boldsymbol{ysec^2x = secx + C}$$

(answer)

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15. $\frac{dy}{dx} + \frac{6y}{x} = -3$

Solution:

We take the equation in form $\frac{dy}{dx} + Py = Q$

Here, P and Q are the function of x

So, the above equation becomes,

$\rightarrow \frac{dy}{dx} + \frac{6}{x}y = -3$ ……….*(1)*

This is now expressed in linear form

Integrating factor = $e^{\int \frac{6}{x} dx}$

= $e^{6 \ln x}$

= x^6

Multiplying equation (1) by I.F we get,

$$x^6 \frac{dy}{dx} + x^5 \cdot 6y = -3$$

$$\Rightarrow \frac{d}{dx}\left(yx^6\right) = -3x^6$$

$$\Rightarrow yx^6 = -\int 3x^6 \, dx$$

$$\Rightarrow y = -\frac{3x}{7} - \frac{c}{x^6}$$ *[where c is an arbitrary constant]*

(answer)

16. $$xcosx\frac{dy}{dx} + y(xsinx + cosx) = 1$$

Solution:

We take the equation in form $\frac{dy}{dx} + Py = Q$

Here, P and Q are the function of ***x***

So, the above equation becomes,

$$\rightarrow \frac{dy}{dx} + \left(tanx + \frac{1}{x}\right)y = \frac{secx}{x} \quad \text{..........(1)}$$

This is now expressed in linear form

Integrating factor = $e^{\int(tanx+\frac{1}{x})dx}$

= $e^{\log secx + \log x}$

= $e^{\log xsecx}$

= $xsecx$

Multiplying equation (1) by I.F we get,

$$xsecx\frac{dy}{dx} + xsecx(tanx + \frac{1}{x})y = sec^2x$$

$$\Rightarrow \frac{d}{dx}(yxsecx) = sec^2x$$

$$\Rightarrow yxsecx = \int sec^2x\,dx$$

$$\Rightarrow yxsecx = tanx + c$$

$$\Rightarrow y = \frac{sinx}{x} - \frac{c\,cosx}{x}$$ *[where c is an arbitrary constant]*

(answer)

17. $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$

Solution:

We take the equation in form $\frac{dy}{dx} + Py = Q$

Here, P and Q are the function of ***x***

So, the above equation becomes,

$$\rightarrow \frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x}{\sqrt{1-x^2}} \quad \text{..........(1)}$$

This is now expressed in linear form

Integrating factor = $e^{\int \frac{2x}{1-x^2}dx}$

$= e^{-\log(1-x^2)}$

$= \frac{1}{1-x^2}$

Multiplying equation (1) by I.F we get,

$$\frac{dy}{dx} + \frac{2x}{1-x^2} y = \frac{x}{(1-x^2)^{3/2}}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{1-x^2}\right) = \frac{x}{(1-x^2)^{3/2}}$$

$$\Rightarrow \frac{y}{1-x^2} = \int \frac{x}{(1-x^2)^{3/2}} dx$$

$\Rightarrow y = \sqrt{1-x^2} + (1-x^2)$ *[where c is an arbitrary constant]*

(answer)

18. $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$

Solution:

We take the equation in form $\frac{dy}{dx} + Py = Q$

Here, P and Q are the function of ***x***

So, the above equation becomes,

$\rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{4x^2}{1+x^2}$ ***..........(1)***

This is now expressed in linear form

Integrating factor = $e^{\int \frac{2x}{1+x^2} dx}$

= $e^{\log(1+x^2)}$

= $1 + x^2$

Multiplying equation (1) by I.F we get,

$$(1+ x^2)\frac{dy}{dx} + 2xy = 4x^2$$

$$\Rightarrow \frac{d}{dx}\{y(1+x^2)\} = 4x^2$$

$$\Rightarrow y(1+x^2) = \int 4x^2\, dx$$

$$\Rightarrow y = \frac{4x^3}{3(1+x^2)} - \frac{c}{1+x^2}$$ [*where c is an arbitrary constant*]

(answer)

19. $sinx\frac{dy}{dx} + 3y = cosx$

Solution:

We take the equation in form $\frac{dy}{dx} + Py = Q$

Here, P and Q are the function of x

So, the above equation becomes,

$$\rightarrow \frac{dy}{dx} + (3cosecx)y = cotx \quad \text{..........(1)}$$

This is now expressed in linear form

Integrating factor = $e^{\int 3cosecx dx}$

= $e^{3\log\tan(x/2)}$

$= tan^3\frac{x}{2}$

Multiplying equation (1) by I.F we get,

$$tan^3\frac{x}{2}\cdot\frac{dy}{dx}+(3cosecx)\cdot tan^3\frac{x}{2}\cdot y = cotx\cdot tan^3\frac{x}{2}$$

$$\Rightarrow \frac{d}{dx}\left(ytan^3\frac{x}{2}\right)=\frac{1-tan^2\frac{x}{2}}{2tan\frac{x}{2}}tan^3\frac{x}{2}$$

$$\Rightarrow ytan^3\frac{x}{2}=\frac{1}{2}\int\{(1-tan^2\frac{x}{2})tan^2\frac{x}{2}\}\,dx$$

$$\Rightarrow ytan^3\frac{x}{2}=\int(1-t^2)t^2\cdot\frac{2dt}{1+t^2}$$

[*Let,* $tan\frac{x}{2}=t$ *so that* $\frac{1}{2}sec^2\frac{x}{2}dx=dt\Rightarrow dx=\frac{2dt}{sec^2\frac{x}{2}}$

$=\frac{2dt}{1+tan^2\frac{x}{2}}=\frac{2dt}{1+t^2}$]

$$\Rightarrow ytan^3\frac{x}{2}=\int\frac{t^2-t^4}{1+t^2}dt$$

$$\Rightarrow ytan^3\frac{x}{2}=\int(-t^2+2-\frac{2}{1+t^2})\,dt$$

$$\Rightarrow ytan^3\frac{x}{2}=-\left(\frac{1}{3}\right)t^3+2t-2\,tan^{-1}t+c$$

$$\Rightarrow y=-\frac{1}{3}+2cot^2\frac{x}{2}-x\,cot^3\frac{x}{2}+c\,cot^3\frac{x}{2}$$

[*where c is an arbitrary constant*]

(answer)

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20. $(x + 2y^3)\, {}^{dy}/_{dx} = y$

Solution :

We take the equation in form $\frac{dx}{dy}$+Px =Q

Here , P and Q are the function of ***y***

So the above equation becomes ,

$\rightarrow \frac{dx}{dy}$=$\frac{x+2y^3}{y}$

$\rightarrow \frac{dx}{dy} - \frac{x}{y}$=$2y^2$ ………………… ***(1)***

This is now expressed in linear form ,

For 1 ,$\int Pdy = -\int \frac{1}{y} dy = -log\ y$

So I.F for (1)

$\rightarrow e^{-\, log\ y}$=$\frac{1}{y}$

Multiply equ^n 1 by I.F we get ,

$$\rightarrow \frac{1}{y}\frac{dx}{dy} - \frac{x}{y^2} = 2y^2\frac{1}{y}$$

$$\rightarrow \frac{d}{dy}\left(\frac{x}{y}\right) = 2y$$

$$\rightarrow \frac{x}{y} = \int 2y dy$$ *[By integration]*

$$\rightarrow \frac{x}{y} = y^2 + C,$$ *where C is an arbitrary constant.*

(answer)

21. $\boldsymbol{y\, log\, y dx + (x - log\, y)dy = 0}$

Solution:

We take the equation in form $\frac{dx}{dy}+Px = Q$

Here , P and Q are the function of ***y***

So the above equation becomes ,

$$\rightarrow \frac{dx}{dy} = -\frac{(x - log\, y)}{y\, log\, y}$$

$$\rightarrow \frac{dx}{dy} + \frac{1}{y\, log\, y}x = \frac{1}{y}$$ **..................*(1)***

This is now expressed in linear form ,

For (1) $\int \frac{1}{y \log y} dy = \log(\log y)$

So ,I.F equal

$$\rightarrow e^{\log(\log y)} = \log y$$

Multiply equ^n (1) by I.F we get ,

$$\rightarrow \log y \frac{dx}{dy} + \frac{1}{y} x = \frac{1}{y} \log y$$

$$\rightarrow \frac{d}{dy}(x \log y) = \frac{1}{y} \log y$$

$$\rightarrow (x \log y) = \int \frac{1}{y} \log y \, dy$$ [By integration]

$\therefore \boldsymbol{x \log x = \frac{1}{2}(\log y)^2 + C}$, where c is an arbitrary constant .

(answer)

22. $\boldsymbol{dx + xdy = e^{-y} \log y dy}$

Solution :

We take the equation in form $\frac{dx}{dy} + Px = Q$

Here , P and Q are the function of ***y***

So the above equation becomes ,

$\rightarrow \frac{dx}{dy} + x = e^{-y} \log y$***..................(1)***

This is now expressed in linear form ,

For 1 $\int dy = y$

∴***I.F=e^y***

Multiply equ^n 1 by I.F we get ,

$$\rightarrow e^y \frac{dx}{dy} + e^y x = e^{-y} \log y \, e^y$$

$$\rightarrow \frac{d}{dy}(xe^y) = e^{-y} \log y \, e^y$$

$$\rightarrow xe^y = \int e^{-y} \log y \, e^y \, dy$$

$$= \log y \, y - \int y \frac{1}{y} dy$$ *[By integration]*

∴ $\boldsymbol{xe^y = y \log y - y + C}$, *where c is an arbitrary constant .*

(answer)

23. *Solve* $\boldsymbol{(1+y^2)dx + (x - \tan^{-1} y)dy = 0}$

Solution :

We take the equation in form $\frac{dx}{dy} + Px = Q$

Here , P and Q are the function of ***y***

So the above equation becomes ,

$$\rightarrow \frac{dx}{dy} = -\frac{x}{1+y^2} + \frac{tan^{-1} y}{1+y^2}$$

$$\therefore \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{tan^{-1} y}{1+y^2} \text{..................}\textbf{\textit{(1)}}$$

This is now expressed in linear form ,

For 1 $\int \frac{1}{1+y^2} dy = \tan^{-1} y$

$\therefore$*I.F=*$e^{tan^{-1} y}$

Multiply equ^n 1 by I.F we get ,

$$\rightarrow \; e^{tan^{-1} y} \frac{dx}{dy} + e^{tan^{-1} y} \frac{x}{1+y^2} = e^{tan^{-1} y} \frac{tan^{-1} y}{1+y^2}$$

$$\rightarrow \frac{d}{dy}(xe^{tan^{-1} y}) = \; e^{tan^{-1} y} \frac{tan^{-1} y}{1+y^2}$$

$$\rightarrow xe^{tan^{-1} y} = \int e^{tan^{-1} y} \frac{tan^{-1} y}{1+y^2} dy \qquad \text{..................}\textbf{\textit{(2)}}$$

Let, t= $tan^{-1} y$

$\therefore dt = \frac{1}{1+y^2} dy$

Now put the value in equ^n 2 we get

$xe^t = \int te^t dt$ *[By integration]*

$= e^t(t-1) + C$

Substitution the value of t we get

$$\rightarrow xe^{tan^{-1}y} = e^{tan^{-1}y}(tan^{-1}y - 1) + C$$

$$\therefore \boldsymbol{x = (tan^{-1}y - 1) + Ce^{tan^{-1}y}}$$ *Here C is an arbitrary constant.*

(answer)

24. $\boldsymbol{x\,log\,x(\frac{dy}{dx}) + y = 2\,log\,x}$

Solution :

We take the equation in form $\frac{dy}{dx}+Py = Q$

Here , P and Q are the function of *x*

So the above equation becomes ,

$$\rightarrow \frac{dy}{dx} + \frac{y}{xlogx} = \frac{2}{x}$$ ……………….***(1)***

This is now expressed in linear form ,

For (1) $\rightarrow \int \frac{1}{xlogx}dx = \int \frac{dt}{t}$ $[By\ integration]$

$$\begin{bmatrix} Let \quad logx = t \\ \therefore \frac{1}{x}dx = dt \end{bmatrix}, by\ differciation = logt$$

$$= log(logx)$$

$\therefore$***I.F*** = $e^{log(logx)} = logx$

Multiply equ^{n} 1 by I.F we get ,

$$\rightarrow logx\frac{dy}{dx}+\frac{y}{x}=\frac{2}{x}logx$$

$$\rightarrow \frac{d}{dx}(ylogx)=\frac{2}{x}logx$$

$$\rightarrow ylogx=\int\frac{2}{x}logxdx\begin{bmatrix}Again\ put, logx=z\\ \therefore \frac{1}{x}dx=dz\end{bmatrix}$$

So, $yz=2\int zdz$ *[By integration]*

$$\rightarrow yz=2\frac{z^2}{2}+C$$

$$\therefore yz=z^2+C$$

Substitution the value of z in equ^n 2, we get;

$\therefore \boldsymbol{ylogx=(logx)^2+C}$ *Here C is an arbitrary constant.*

(answer)

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25. $\boldsymbol{\frac{dy}{dx}+\frac{y}{x}=x^3-3}$

Solution:

we take the equation in form $\frac{dy}{dx}+Px=Q$

Here, P and Q is the function of **x.**

And given is,

$\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$ ....***(1)***

So we get,

P = $\frac{1}{x}$, then integrating factor (I.F.) = $e^{\int \frac{1}{x} dx}$

= e^{lnx}

= *x*

So, **I.F.= x**

Multiplying the equation(1) by I.F., we get,

$$x\frac{dy}{dx} + x\frac{y}{x} = x(x^3 - 3)$$

$$or,\ x\frac{dy}{dx} + y = x^4 - 3x$$

$$or,\ \frac{d}{dx}(xy) = x^4 - 3x$$

$$or,\ d(xy) = (x^4 - 3x)dx$$

integrating both side, we get,

$$or,\ xy = \int (x^4 - 3x)dx$$

$$or,\ y = \frac{x^4}{5} - \frac{3x}{2} + \frac{c}{x}$$

$\boldsymbol{y = \frac{x^4}{5} - \frac{3x}{2} + \frac{c}{x}}$ *where c is an arbitrary constant*

(answer)

26. $\frac{dy}{dx}$ + *ycotx = cosx*

Solution:

we take the equation in form $\frac{dy}{dx} + Px = Q$

Here, P and Q is the function of **x**.

And given is, $\frac{dy}{dx}$ *+ ycotx = cosx ....(1)*

So we get, *P =* $cotx$*,* then integrating factor (I.F.) *=* $e^{\int cotxdx}$

= $e^{ln|sinx|}$

= $sinx$

So, **I.F.=** $\boldsymbol{sinx}$

Multiplying the equation(1) by I.F., we get,

sinx $\frac{dy}{dx}$ *+ y.cotx.sinx = cosx sinx*

or, sinx $\frac{dy}{dx}$ *+ y.*$\frac{cosx}{sinx}$*.sinx = cosx sinx*

or,sinx $\frac{dy}{dx}$ *+ ycosx = cosx sinx*

or,$\frac{d}{dx}(ysinx) = cosx\ sinx$

Integrating both side, we get,

$ysinx = \int cosx\ sinx\ dx$

or, $ysinx = \frac{1}{2}\int 2cosx\ sinx\ dx$

or, $ysinx = \frac{1}{2}\int\ sin2x\ dx$

or, $ysinx = -\frac{1}{2}\left(\frac{1}{2}cos2x\right) + c$

or, $y = -\ \frac{1}{4}\frac{cos2x}{sinx} + c$

$\boldsymbol{y = -\ \frac{1}{4}\frac{cos2x}{sinx} + c}$ *where c is an arbitrary constant*

(answer)

27. $x\frac{dy}{dx} + 2y = x^2logx$

Solution:

we take the equation in form $\frac{dy}{dx} + Px = Q$

Here, P and Q is the function of **x.**

And given is, $x\frac{dy}{dx} + 2y = x^2logx$

or, $\frac{dy}{dx} + \frac{2}{x}y = xlogx$ **...........................*(1)***

$P = \frac{2}{x}$, then integrating factor (I.F.) = $e^{\int\frac{2}{x}dx}$

$= e^{2lnx} = x^2$

So, I.F.=x^2

Multiplying the equation (1) by I.F., we get,

$$x^2\frac{dy}{dx} + x^2\frac{2}{x}y = x^3logx$$

$$or, x^2\frac{dy}{dx} + 2xy = x^3logx$$

$$or, \frac{d}{dx}(x^2y) = x^3logx$$

Integrating both side, we get,

$$x^2y = \int x^3logx\,dx$$

$$Or,\ x^2y = logx\int x^3dx - \int(\frac{d}{dx}logx\int x^3\,dx)dx$$

$$Or,\ x^2y = logx\frac{x^4}{4} - \int\frac{1}{x}\frac{x^4}{4}dx$$

$$Or,\ x^2y = \frac{x^4}{4}logx - \frac{1}{4}\frac{x^4}{4} + c$$

$$Or,\ x^2y = \frac{x^4}{4}logx - \frac{x^4}{16} + c$$

$$\boldsymbol{x^2y = \frac{x^4}{4}logx - \frac{x^4}{16} + c}$$ *where c is an arbitrary constant*

(answer)

28. $\boldsymbol{\frac{dy}{dx} + ysecx = tanx}$

Solution:

we take the equation in form $\frac{dy}{dx} + Px = Q$

Here, P and Q is the function of **x.**

And given is, $\frac{dy}{dx} + ysecx = tanx$**……………………….(1)**

So we get, *P =* $secx$*,* then integrating factor (I.F.) = $e^{\int secxdx}$

$= e^{ln|secx+tanx|}$

$= secx + tanx$

So, ***I.F.=*** $\boldsymbol{secx + tanx}$

Multiplying the equation(1) by I.F., we get,

$(secx + tanx)\frac{dy}{dx} + ysecx(secx + tanx) = tanx(secx + tanx)$

$Or, \frac{d}{dx}(secx + tanx)y = tanx(secx + tanx)$

Integrating both side, we get,

$(secx + tanx)y = \int tanx(secx + tanx)\, dx$

$Or, (secx + tanx)y = \int tanxsecx\, dx + \int tan^2x\, dx$

$Or, (secx + tanx)y = \int tanxsecx\, dx + \int (sec^2x - 1)\, dx$

Or, $(secx + tanx)y$ = *secx + tanx – x + c*

$(\boldsymbol{secx} + \boldsymbol{tanx})\boldsymbol{y}$ = ***secx + tanx – x + c*** *where c is an arbitrary constant*

(answer)

29. $(\boldsymbol{x} + \boldsymbol{a})\frac{\boldsymbol{dy}}{\boldsymbol{dx}} - \boldsymbol{3y} = (\boldsymbol{x} + \boldsymbol{a})^{\boldsymbol{5}}$

Solution:

we take the equation in form $\frac{dy}{dx} + Px = Q$

Here, P and Q is the function of **x.**

And given is,

$(x + a)\frac{dy}{dx} - 3y = (x + a)^5$

Or, $\frac{dy}{dx} - \frac{3}{(x+a)}y = (x + a)^4$***……………………..(1)***

So we get,

P $= -\frac{3}{(x+a)}$, then integrating factor (I.F.) = $e^{-\int\frac{3}{(x+a)}dx}$

$= e^{-3ln(x+a)}$ $= (x + a)^{-3}$

So, ***I.F.=*** $(\boldsymbol{x} + \boldsymbol{a})^{-3}$

Multiplying the equation (1) by I.F., we get,

$(x + a)^{-3}\frac{dy}{dx} - \frac{3}{(x+a)^{-4}}y = x + a$

Or, $\frac{d}{dx}\left\{\frac{y}{(x+a)^{-3}}\right\} = x + a$

Integrating both side, we get,

$\frac{y}{(x+a)^{-3}} = \int x + a \ dx$

Or, $\frac{y}{(x+a)^{-3}} = \frac{x^2}{2} + ax + c$

Or, $y = \frac{x^2+2ax}{2(x+a)^3} + c$

$\boldsymbol{y = \frac{x^2+2ax}{2(x+a)^3} + c}$ *where c is an arbitrary constant*

(answer)

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First order first degree non – linear differential equation

Or, equation reducible to the linear form [Bernoulli's equation]

An equation in the form

$\frac{dy}{dx} + P_{(x)}y = Q_{(x)}y^n \quad ; where\ n \neq 0, 1$

Or, $y^{-n}\frac{dy}{dx} + P_{(x)}y^{1-n} = Q_{(x)}$………………………………………………………… (1)

Now, let,

$y^{1-n} = V$

Or, $(1-n)y^{-n}\frac{dy}{dx} = \frac{dV}{dx}$ [differentiating with respect to x]

Or, $y^{-n}\frac{dy}{dx} = \left(\frac{1}{1-n}\right)\frac{dV}{dx}$

From equation (1), we get,

$\left(\frac{1}{n-1}\right)\frac{dV}{dx} + P_{(x)}V = Q_{(x)}$

Or, $\frac{dV}{dx} + (1-n)\,P_{(x)}V = (1-n)Q_{(x)}$

Which is linear in V and x

It's if $= e^{\int P_{(x)}(1-n)dx}$

$= e^{(1-n)\int P_{(x)}\,dx}$

Hence, required solution is,

$V \cdot e^{(1-n)\int P_{(x)}\,dx} = \int Q_{(x)} \cdot e^{(1-n)\int P_{(x)}\,dx}\,dx + c$ [where c is an arbitrary constant]

Or, $y^{1-n} \cdot e^{(1-n)\int P_{(x)}\,dx} = \int Q_{(x)} \cdot e^{(1-n)\int P_{(x)}\,dx}\,dx + c$

N:B: $\frac{dX}{dy} + P_{(y)}x = Q_{(y)}x^n$ is also in the Bernoulli's form. It can be solved by above method.

1. Solve $x\frac{dy}{dx} - 2y = xy^4$

Solution:

Dividing by xy^4, we get

$$y^{-4}\frac{dy}{dx} - \frac{2}{x}y^{-3} = 1$$

Put $-y^{-3} = v,$

$$3y^{-4}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \quad \frac{dv}{dx} + \frac{6}{x}v = 3$$

Linear, I.F. $= e^{\int \frac{6}{x}dx} = x^6$

Hence, $vx^6 = c + \int 3x^6 dx = c + \frac{3x^7}{7}$

Or, $-y^{-3}x^6 = c + \frac{3}{7}x^7$ is the solution.

2. Solve $\frac{dy}{dx} = x^3y^2 + xy$

Solution:

$$\frac{dy}{dx} = x^3y^2 + xy$$

Or, $y^{-2}\frac{dy}{dx} - xy^{-1} = x^3$ .... (1)

Putting $y^{-1} = v$

So that, $-y^{-2}\left(\frac{dy}{dx}\right) = \frac{dv}{dx}$

Hence equation (1) reduces to $-\left(\frac{dv}{dx}\right) - xv = x^3$

Or, $\left(\frac{dv}{dx}\right) + xv = -x^3$; which is linear whose I.F. $= e^{\int x\, dx} = e^{\frac{x^2}{2}}$

And hence, its solution is

$ve^{\frac{x^2}{2}} = -\int x^3 e^{\frac{x^2}{2}} dx + c = -\int x^2 \cdot e^{\frac{x^2}{2}} x\, dx + c$…………………. (2)

Where c is an arbitrary constant.

Putting $\frac{x^2}{2} = t$, so that $x\, dx = dt$, equation (2) gives that

$$ve^{\frac{x^2}{2}} = -2\int te^t\, dx = -2[te^t - \int e^t dt] + c$$

$$= -2(te^t - e^t) + c = 2e^t(t-1) + c$$

Or, $y^{-1}e^{\frac{x^2}{2}} = 2e^{\frac{x^2}{2}}\left(\frac{x^2}{2} - 1\right) + c$

Or, $y^{-1} = (2 - x^2) + ce^{\frac{x^2}{2}}$ is the solution.

3. Solve $x\frac{dy}{dx} + y = y^2 \log x$

Solution:

Rewritting the given equation $y^2 \frac{dy}{dx} + \frac{1}{x} y^{-1} = \frac{1}{x} \log x$ .... (1)

Putting $y^{-1} = v$ so that $-y^{-2} \frac{dy}{dx} = \frac{dv}{dx}$, then equation (1) gives,

$-\frac{dv}{dx} + \frac{1}{x} y = \frac{1}{x} \log x$

Or, $\frac{dv}{dx} - \frac{1}{x} v = -\frac{1}{x} \log x$ .... (2)

I.F. of equation (2) $= e^{-\int \frac{1}{x} dx} = e^{-\log x} = x^{-1} = \frac{1}{x}$

And hence solution of equation (2) is

$vx^{-1} = -\int x^{-2} \log x \, dx + c$, Where c is an arbitrary constant.

Or, $y^{-1} x^{-1} = -\left[\log x \times \frac{x^{-1}}{-1} - \int \frac{1}{x} \times \frac{x^{-1}}{-1} \, dx\right] + c$

Or, $\frac{1}{y} = \log x + 1 + cx$ is the solution.

4. Solve $\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$

Solution:

Dividing by y^2, the equation becomes,

$y^{-2} \frac{dy}{dx} - 2y^{-1} \tan x = \tan^2 x$

Put $-y^{-1} = v$, $y^{-2} \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore$ equation is $\frac{dv}{dx} + 2\tan x \cdot v = \tan^2 x$, I.F. $= \sec^2 x$

$\therefore$ $v\sec^2 x = c + \int \tan^2 x \cdot \sec^2 x \; dx = c + \frac{1}{3}\tan^3 x$

Or, $-\frac{1}{y}\sec^2 x = c + \frac{1}{3}\tan^3 x$ is the solution.

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5. Solve $\frac{dy}{dx} + y = 3e^x y^3$

Solution:

The given equation is, $\frac{dy}{dx} + y = 3e^x y^3$

Or, $y^{-3}\frac{dy}{dx} + y^{-2} = 3e^x$---------------------- (1)

Let, $y^{-2} = v$

Or, $-2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$

Or, $y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$

Now, putting the value in equation (1), we get,

$-\frac{1}{2}\frac{dv}{dx} + v = 3e^x$

Or, $\frac{dv}{dx} - 2v = -6e^x$ --- (2)

Now, Integrating Factor (I.F) $= e^{\int -2dx}$

$$=e^{-2\int dx}$$

$$=e^{-2x}$$

Now, from equation (2),

$$e^{-2x}\frac{dv}{dx} - e^{-2x}2v = e^{-2x}(-6e^{x})$$

Or, $\frac{d}{dx}(ve^{-2x}) = -6e^{-x}$

Or, $ve^{-2x} = -6\int e^{-x}\,dx$

Or, $y^{-2}e^{-2x} = 6e^{-x} + c$

Or, $\frac{1}{y^2} = 6e^{x} + ce^{2x}$ [Where **c** is an arbitrary constant]

The required solution is $\frac{1}{y^2} = 6e^{x} + ce^{2x}$.

6. Solve $2x^2\frac{dy}{dx} = xy + y^2$

Solution:

The given equation is, $2x^2\frac{dy}{dx} = xy + y^2$

Or, $2x^2\frac{dy}{dx} - xy = y^2$

Or, $\frac{dy}{dx} - \frac{y}{2x} = \frac{y^2}{2x^2}$

Or, $y^{-2}\frac{dy}{dx} - \frac{y^{-1}}{2x} = \frac{1}{2x^2}$------------------------- (1)

Let, $y^{-1} = v$

Or, $-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

Or, $y^{-2}\frac{dy}{dx} = -\frac{dv}{dx}$

Now, putting the value in equation (1), we get,

$-\frac{dv}{dx} - \frac{v}{2x} = \frac{1}{2x^2}$

Or, $\frac{dv}{dx} + \frac{v}{2x} = -\frac{1}{2x^2}$ --- (2)

Now, Integrating Factor (I.F)$= e^{\int \frac{1}{2x}dx}$

$= e^{\frac{1}{2}\int \frac{dx}{x}}$

$= e^{(logx)^{\frac{1}{2}}}$

$= x^{\frac{1}{2}}$

$= \sqrt{x}$

Now, from equation (2),

$\sqrt{x}\,\frac{dv}{dx} + \sqrt{x}\,\frac{v}{2x} = -\frac{\sqrt{x}}{2x^2}$

Or, $\frac{d}{dx}\left(v\sqrt{x}\right) = -\frac{1}{2}x^{-\frac{3}{2}}$

Or, $v\sqrt{x} = -\frac{1}{2}\int x^{-\frac{3}{2}}\,dx$

Or, $y^{-1}\sqrt{x} = -\frac{1}{2}\left(\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\right) + c$

Or, $\frac{\sqrt{x}}{y} = \frac{1}{\sqrt{x}} + c$

Or, $\frac{x}{y} = 1 + c\sqrt{x}$ [Where **c** is an arbitrary constant]

The required solution is $\frac{x}{y} = 1 + c\sqrt{x}$.

7. Solve $(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0$

Solution:

The given equation is, $(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0$

Or, $(x^2 + y^2 + 2x)\, dx = -2y\, dy$

Or, $\frac{dy}{dx} = -\left(\frac{x^2+2x}{2y}\right) - \frac{y}{2}$

Or, $\frac{dy}{dx} + \frac{y}{2} = -\left(\frac{x^2+2x}{2y}\right)$

Or, $y\frac{dy}{dx} + \frac{y^2}{2} = -\left(\frac{x^2+2x}{2}\right)$------------------------ (1)

Let, $y^2 = v$

Or, $2y\,\frac{dy}{dx} = \frac{dv}{dx}$

Or, $y\frac{dy}{dx} = \frac{1}{2}\frac{dv}{dx}$

Now, putting the value in equation (1), we get,

$\frac{1}{2}\frac{dv}{dx} + \frac{v}{2} = -\left(\frac{x^2+2x}{2}\right)$

Or, $\frac{dv}{dx} + v = -(x^2 + 2x)$-- (2)

Now, Integrating Factor (I.F) $= e^{\int 1\, dx}$

$$= e^x$$

Now, from equation (2),

$$e^x \frac{dv}{dx} + e^x v = -e^x(x^2 + 2x)$$

Or, $\frac{d}{dx}(ve^x) = -e^x(x^2 + 2x)$

Or, $ve^x = -\int e^x(x^2 + 2x)dx$

Or, $ve^x = -(x^2 + 2x)\int e^x dx + \int(2x + 2)e^x dx$

Or, $ve^x = -(x^2 + 2x)e^x + \int 2xe^x dx + \int 2\, e^x dx$

Or, $ve^x = -(x^2 + 2x)e^x + 2xe^x - \int 2\, e^x dx + \int 2\, e^x dx$

Or, $y^2 e^x = -x^2 e^x + c$

Or, $(x^2 + y^2)e^x = c$ [Where **c** is an arbitrary constant]

The required solution is. $(x^2 + y^2)e^x = c$.

8.Solve $\boldsymbol{\frac{dy}{dx} - \left(\frac{1}{1+x}\right)\tan y = (1+x)e^x \sec y}$

Solution:

The given equation is, $\frac{dy}{dx} - \left(\frac{1}{1+x}\right)\tan y = (1 + x)e^x \sec y$

Or, $\cos y \frac{dy}{dx} - \left(\frac{1}{1+x}\right)\sin y = (1 + x)e^x$---------- (1)

Let, $\sin y = v$

Or, $\cos y \frac{dy}{dx} = \frac{dv}{dx}$

Now, putting the value in equation (1), we get,

$$\frac{dv}{dx} - \frac{v}{1+x} = (1+x)e^x \text{ -- (2)}$$

Now, Integrating Factor (I.F)$= e^{\int -\frac{1}{1+x}dx}$

$= e^{-\int \frac{1}{1+x}dx}$

$= e^{-\log(1+x)}$

$= (1+x)^{-1}$

Now, from equation (2),

$$(1+x)^{-1}\frac{dv}{dx} - (1+x)^{-1}\frac{v}{(1+x)} = (1+x)^{-1}\{(1+x)e^x\}$$

Or, $\frac{d}{dx}\{v(1+x)^{-1}\} = e^x$

Or, $v(1+x)^{-1} = \int e^x \, dx$

Or, $\sin y \, (1+x)^{-1} = e^x + c$

Or, $\frac{\sin y}{1+x} = e^x + c$ [Where **c** is an arbitrary constant]

The required solution is $\frac{\sin y}{1+x} = e^x + c$.

9. Solve $\frac{dy}{dx} + \frac{xy}{1-x^2} = x\sqrt{y}$

Solution:

The given equation is, $\frac{dy}{dx} + \frac{xy}{1-x^2} = x\sqrt{y}$

Or, $\frac{1}{\sqrt{y}}\frac{dy}{dx} + \frac{x\sqrt{y}}{1-x^2} = x$-------------------------------- (1)

Let, $\sqrt{y} = v$

Or, $\frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{dv}{dx}$

Or, $\frac{1}{\sqrt{y}}\frac{dy}{dx} = 2\frac{dv}{dx}$

Now, putting the value in equation (1), we get,

$$2\frac{dv}{dx} + \frac{xv}{1-x^2} = x$$

Or, $\frac{dv}{dx} + \frac{xv}{2(1-x^2)} = \frac{x}{2}$-- (2)

Now, Integrating Factor (I.F)= $e^{\int \frac{x}{2(1-x^2)}dx}$

$$= e^{\frac{1}{2}\int -\frac{dz}{2z}} \qquad \begin{bmatrix} Let,\ 1-x^2 = z \\ Or, -2xdx = dz \\ Or, xdx = -\frac{dz}{2} \end{bmatrix}$$

$$= e^{-\frac{1}{4}logz}$$

$$= z^{-\frac{1}{4}}$$

$$= (1-x^2)^{-\frac{1}{4}}$$

Now, from equation (2),

$$\left[(1-x^2)^{-\frac{1}{4}}\right]\frac{dv}{dx} + \left[(1-x^2)^{-\frac{1}{4}}\right]\frac{xv}{2(1-x^2)} = \left[(1-x^2)^{-\frac{1}{4}}\right]\frac{x}{2}$$

Or, $\frac{d}{dx}\left\{v\,(1-x^2)^{-\frac{1}{4}}\right\} = \frac{x}{2\sqrt[4]{1-x^2}}$

Or, $\left\{v\,(1-x^2)^{-\frac{1}{4}}\right\} = \frac{1}{2}\int \frac{x}{\sqrt[4]{1-x^2}}\,dx$

Or, $\sqrt{y}\,(1-x^2)^{-\frac{1}{4}} = \frac{1}{2}\int -\frac{z^{-\frac{1}{4}}}{2}\,dz$

$\left[\begin{array}{l} Let,\ 1-x^2 = z \\ Or, -2xdx = dz \\ Or, xdx = -\frac{dz}{2} \end{array}\right]$

Or, $\frac{\sqrt{y}}{(1-x^2)^{\frac{1}{4}}} = -\frac{1}{4}\int z^{-\frac{1}{4}}\,dz$

Or, $\frac{\sqrt{y}}{(1-x^2)^{\frac{1}{4}}} = -\frac{1}{4}\left[\frac{z^{\frac{3}{4}}}{\frac{3}{4}}\right] + c$

Or, $\frac{\sqrt{y}}{(1-x^2)^{\frac{1}{4}}} = -\frac{1}{3}(1-x^2)^{\frac{3}{4}} + c$

Or, $\sqrt{y} = -\frac{1}{3}(1-x^2) + c(1-x^2)^{\frac{1}{4}}$ [Where **c** is an arbitrary constant]

The required solution is, $\sqrt{y} = -\frac{1}{3}(1-x^2) + c(1-x^2)^{\frac{1}{4}}$

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10. Solve $\frac{dy}{dx} + xy = x^3y^3$

Solution:

$\frac{dy}{dx} + xy = x^3y^3$ Let, $y^{-2} = z$

Or, $y^{-3}\frac{dy}{dx}+xy^{-2}=x^3$ $-2\,y^{-3}\frac{dy}{dx}=\frac{dz}{dx}$

Or, $-\frac{1}{2}\frac{dz}{dx}+xz=x^3$

Or, $\frac{dz}{dx}+(-2x)z=-2x^3$

$$\therefore\ I.F.=\ e^{-\int 2xdx}=e^{-x^2}$$

Hence the solution is:

$$ze^{-x^2}=\int(-2x^3)\,e^{-x^2}dx+c$$

$=\int x^2(-2x)\,e^{-x^2}dx+c$ Let, $-x^2=t$

$$=\int -t.e^t dt+c-2xdx=dt$$

$$=-te^t+e^t+c$$

$$=x^2e^{-x^2}+e^{-x^2}+c$$

$\therefore y^{-2}.e^{-x^2}=(1+x^2)e^{-x^2}+c$

$(Ans.)$

11.Solve $\quad x\frac{dy}{dx}+y\log y=xye^x$

Solution:

$$x\frac{dy}{dx}+y\log y=xye^x$$

Or, $\frac{1}{y}\frac{dy}{dx}+\frac{1}{x}\log y=e^x$

Putting $\log y=z,$ $\therefore\frac{1}{y}\frac{dy}{dx}=\frac{dz}{dx}$

$$\therefore \frac{dz}{dx} + \frac{z}{x} = e^x$$

$\therefore\ I.F. = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Hence the solution is:

$$zx = \int x\, e^x dx + c$$

$= xe^x - e^x + c$

$$\therefore x \log y = xe^x - e^x + c$$

(Ans.)

12. Solve $\frac{dy}{dx} + \frac{\sin 2y}{x} = x^2 cos^2 y$

Solution:

$$\frac{dy}{dx} + \frac{\sin 2y}{x} = x^2 cos^2 y$$

Or, $sec^2 y \frac{dy}{dx} + sec^2 y \frac{2 \sin y \cos y}{x} = x^2$

Or, $sec^2 y \frac{dy}{dx} + \frac{2}{x} \tan y = x^2$ Let, $\tan y = z$

Or, $\frac{dz}{dx} + \frac{2}{x} z = x^2$ $sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore\ I.F. = e^{\int \frac{2}{x} dx} = e^{2 \log x} = x^2$

Hence the solution is:

$$z.x^2 = \int x^2\, x^2 dx + c$$

$$= \frac{x^5}{5} + c$$

$$\therefore \quad x^2 \tan y = \frac{x^5}{5} + c$$

(Ans.)

13. Solve $\frac{dy}{dx} + \frac{\sin 2y}{x} = x^2 cos^2 y$

Solution:

$$e^x \frac{dy}{dx} = 2xy^2 + ye^x$$

Or, $e^x \frac{dy}{dx} - ye^x = 2xy^2$

Or, $\frac{1}{y^2}\frac{dy}{dx} + (-\frac{1}{y}) = \frac{2x}{e^x}$ Let, $-\frac{1}{y} = z$

Or, $\frac{dz}{dx} + z = \frac{2x}{e^x} \frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$$\therefore\ I.F. = e^{\int dx} = e^x$$

Hence the solution is:

$$ze^x = \int \frac{2x}{e^x} e^x dx + c$$

$$= x^2 + c$$

$$\therefore\quad -\frac{1}{y} e^x = x^2 + c$$

(Ans.)

14.Solve $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = 2xe^{-x}$

Solution:

$$\frac{1}{y^2}\frac{dy}{dx}+(-\frac{1}{y}) = 2xe^{-x}$$ Let, $-\frac{1}{y} = z$

Or, $\frac{dz}{dx}+z = 2xe^{-x}$ $\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$$\therefore\ I.F. = e^{\int dx} = e^{x}$$

Hence the solution is:

$$z.e^{x} = \int 2xe^{-x}\,e^{x}dx + c$$

$$= x^2 + c$$

Or, $-\frac{1}{y}e^{x} = x^2 + c$

$$\therefore \quad e^{x} + cy + x^2y = 0$$

(Ans.)

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Example 01: $x\frac{dy}{dx} + y^2x = y$

Solution : $x\frac{dy}{dx} + y^2x = y$

⇨ $\frac{dy}{dx} - \frac{y}{x} = -y^2$

⇨ $y^{-2}\frac{dy}{dx} - \frac{y^{-1}}{x} = -1$

Let, $y^{-1} = v$

⇨ $-y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$

⇨ $y^{-2}\frac{dy}{dx} = -\frac{dv}{dx}$

⇨ $-\frac{dv}{dx} - \frac{v}{x} = -1$

⇨ $\frac{dv}{dx} + \frac{v}{x} = 1$

I.F=$e^{\ln x} = x$

The solution becomes,

$x\frac{dv}{dx} + x\frac{v}{x} = x$

⇨ $\frac{d}{dx}(vx) = x$

⇨ vx=$\int xdx$

⇨ $\frac{x}{y} = \frac{x^2}{2} + c$, is the solution.

Example 02:$\frac{dy}{dx} + 2xy = -xy^4$

Solution:$\frac{dy}{dx} + 2xy = -xy^4$

⇨ $y^{-4}\frac{dy}{dx} + 2xy^{-3} = -x$

Let,$y^{-3} = v$

⇨ $-3y^{-4}\frac{dy}{dx} = \frac{dv}{dx}$

⇨ $y^{-4}\frac{dy}{dx} = -\frac{1}{3}\frac{dv}{dx}$

⇨ $-\frac{1}{3}\frac{dv}{dx} + 2xv = -x$

⇨ $\frac{dv}{dx} - 6xv = 3x$

I.F=$e^{\int -6x} = e^{-3x^2}$

The solution becomes,

$e^{-3x^2}\frac{dv}{dx} - e^{-3x^2}6xv = e^{-3x^2}\,3x$

⇨ $\frac{d}{dx}\left(v.e^{-3x^2}\right) = \int 3x\,e^{-3x^2}\,dx$

here,

$3x^2 = z$

⇨ 6xdx =dz

⇨ $3xdx = \frac{dz}{3}$

Then,

⇨ $v\, e^{-3x^2} = \frac{\int e^{-z}}{2} dz$

⇨ $y^3 e^{-3x^2} = -\frac{e^{3x^2}}{2} + C$

⇨ $\frac{1}{y^2} = -\frac{1}{2} + ce^{-3x^2}$, is the solution

3. Solve $\frac{dy}{dx}$=(x^2y^3+xy) = 1

Solution: The equation can be written as,

$\frac{dx}{dy}$ - xy = x^2y^3

Dividing by x^2, $x^{-2}\frac{dx}{dy} - \frac{y}{x} = y^3$

Put, $-\frac{1}{x}$=v, so, $x^{-2}\frac{dx}{dy} = \frac{dc}{dy}$

So, the equation becomes, $\frac{dx}{dy}$+vy=y^3

Linear in v and y, I.F.=$e^{\int ydy}$=$e^{\frac{1}{2}y^2}$

$ve^{\frac{1}{2}y^2}$ = $\int y^3 e^{\frac{1}{2}y^2}$dy+C, $\frac{1}{2}y^2$=t, ydy=dt

= $\int te^t dt$ +C = $2e^t(t-1)$+C

Or, $-\frac{1}{x}e^{\frac{1}{2}y^2} = 2e^{\frac{1}{2}y^2}(\frac{1}{2}y^2-1)$+C

Or, $\frac{1}{x}$ = $(2-y^2)-Ce^{\frac{1}{2}y^2}$ is the solution.

4. Solve tany$\frac{dy}{dx}$ + tanx = cosycos^{2}x

Solution. tany$\frac{dy}{dx}$ + tanx = cosycos^{2}x

secytany$\frac{dy}{dx}$ +secy tanx = cos^{2}xWriting, z=secy, so that $\frac{dz}{dx}$ secy tany $\frac{dy}{dx}$

The equations becomes $\frac{dz}{dx}$ + ztanx = cos^{2}x

I.F. = $e^{\int tanxdx}$ = $e^{logsecx}$=secx

The solution of the equation is

zsecx = ∫ cos^{2}xsecxdx – C

secysecx = ∫ cosxdx – C = sinx-C

secy = (sinx-C)cosx,is the solution

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Example 1: Solve $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

Solation : We have, $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

Or $y^3 \frac{dy}{dx} - xy^2 = -e^{x^2}$ ....(i)

$-y^2 = v$ $2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$

Put Or

$$\therefore \frac{1}{2}\frac{dv}{dx} + xv = -e^{-x^2}$$

Or $$\frac{dv}{dx} + 2xv = -2e^{-x^2} \text{.....................}(ii)$$

Now, I.F $$= e^{\int 2x dx}$$
$$= e^{x^2}$$

Now, From equation (ii) , We get,

Or $$\frac{dv}{dx} \times e^{x^2} = 2xv \times e^{x^2} = -2e^{-x^2} \times e^{x^2}$$

Or $$\frac{d}{dx}\left(ve^{x^2}\right) = -2$$

Or $$ve^{x^2} = -2x + c \quad \left[By - Integrating\right]$$

Or $$-\frac{e^{x^2}}{y^2} = -2x + c \quad Ans:$$

Example 2: Solve The Partial differntial

equation $y \log dx + (x - \log y)dy = 0$

Solation: We have,

$$y \log ydx + (x - \log y)dy = 0$$

Or $$\frac{dx}{dy} = \frac{-(x - \log y)}{y \log y}$$

Or $$\frac{dx}{dx} = \frac{-x + \log y}{y \log y}$$

Or $$\frac{dx}{dy} + \frac{x}{ylogy} = \frac{1}{y}$$

Now, If $= e^{\int \frac{1}{ylog} .dy}$

$$= e^{log(logy)}$$

$$= \log y$$

$\therefore$ Required Solation is $xlogy = \int \frac{1}{y}(log)by$

Or $xlogy = \frac{(logy)^2}{2} = +c$

Example 3: Solve 2xydy- (x^2+ y^2+ 1) dx =0

Solution: Given that,

$$2xydy- (x^2+ y^2+ 1)\, dx=0$$

$$\text{or, } 2xy \frac{dy}{dx} = x^2+ y^2+ 1$$

$$\text{or, } 2xy \frac{dy}{dx} -y^2= x^{2+1}$$

$$\text{or, } 2y \frac{dy}{dx} - \frac{y^2}{x} - \frac{1}{x} + x \text{ ....(i)}$$

Now, Puting y^2=V

$$\text{or, } 2y\frac{dy}{dx} = \frac{dv}{dx} \text{ ....(ii)}$$

From (i) and (ii) we get,

$$\frac{dv}{dx} - \frac{v}{x} = \frac{1}{x} + x \text{ ....(iii)}$$

Now, I.F =e $\int -\frac{1}{x} dx$

$= e^{-logx}$

$= e^{-logx^{-1}}$

$= x^{-1}$

From (iii) We get,

$$\frac{dv}{dx} \times x^{-1} - \frac{v}{x} \times x^{-1} = \frac{1}{x} \times x^{-1} + x \times x^{-1}$$

$$\text{or,}\frac{d}{dx}(v\frac{1}{x})=\frac{1}{x^2}+1$$

By integration,we get,

$$v\frac{1}{x}=-\frac{1}{x}+x+c$$

$$\text{or, }\frac{y^2}{x}=x-\frac{1}{x}+c$$

$$y^2x^{-1}=x-x^{-1}+c$$

Example4: solve $$\frac{dy}{dx}+\tan y\tan x=\cos x\sec y$$

solution: Given that,

$$\frac{dy}{dx}+\tan y\tan x=\cos x\sec y$$

or, $$\cos y\frac{dy}{dx}+\tan x.\sin y=\cos x.....................(i)$$

let, $\sin y=v$

or, $$\cos y\frac{dy}{dx}=\frac{dv}{dx}$$

from(1) we get, $$\frac{dv}{dx}+\tan xv=\cos x...............................(ii)$$

now,I.F = $e^{\int \tan x dx}$

$$= e^{\int \frac{\sin x}{\cos x} dx}$$

Let, cosx = z

Or, -sinxdx = dz

Now, I.F $= e^{\int \frac{-dz}{z}}$

$= e^{-\log z}$

$= e^{\log z^{-1}}$

$= z^{-1}$

$= \frac{1}{\cos x}$

= secx

From (ii) we get ,

$$\frac{dv}{dx} \sec x + \tan x \sec x \text{-} v = \cos x . \sec x$$

Or, $\frac{d}{dx}(V \sec x) = 1$

By integration,

vsecx= x+c

Or, siny secx= x+c

Example 5: solve siny $\frac{dy}{dx} = \cos y(1 - x\cos y)$

Solution: Given, Siny $\frac{dy}{dx} = \cos y(1 - x\cos y)$

or, secy tany $\frac{dy}{dx} = \sec y - x$

or, secy tany $\frac{dy}{dx}$ - secy = - x…………(i)

let, secy =v

or, $\sec y \tan y \frac{dy}{dx} = \frac{dv}{dx}$

from (i)

$$\frac{dv}{dx} - v = -x \quad \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

Now, I.F $= e^{\int - dx}$

$= e^{-x}$

from (ii),

$$\frac{dv}{dx} e^{-x} - ve^{-x} = -xe^{-x}$$

Or, $\frac{d}{dx}(ve^{-x}) = -xe^{-x}$

By integratiag,

$$ve^{-x} = xe^{-x} - e^{-x} + c$$

or, $\sec y\, e^{-x} = xe^{-x} - e^{-x} + c$

or, $\sec y = x - 1 + ce^{x}$

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Solve the following Bernoulli differential equations:

Exercise 1. $2\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$

$2\frac{dy}{dx} - \frac{y}{x} = \frac{y^2}{x^2}$

or, $2y^{-2}\frac{dy}{dx} - y^{-2}\frac{y}{x} = \frac{y^2}{x^2}y^{-2}$

or, $2y^{-2}\frac{dy}{dx} - y^{-1}\frac{1}{x} = \frac{1}{x^2}$........ **(i)**

let,

$y^{-1} = V$

or, $-y^{-2}\frac{dy}{dx} = \frac{dV}{dx}$

or, $y^{-2}\frac{dy}{dx} = -\frac{dV}{dx}$

Substituting the value of $y^{-2}\frac{dy}{dx}$ *and* y^{-1} in equation (i),

$2y^{-2}\frac{dy}{dx} - y^{-1}\frac{1}{x} = \frac{1}{x^2}$

or,-2 $\frac{dV}{dx}$- $\frac{V}{x}$ = $\frac{1}{x^2}$

or, $\frac{dV}{dx}$+ $\frac{V}{2x}$ = - $\frac{2}{x^2}$ …………**(ii)**

This is of the form $\frac{dy}{dx}$+ P(x)y= Q(x)where,

P(x) = $\frac{V}{2x}$

Q(x) = $-\frac{2}{x^2}$

Integrating Factor, I.F = $e^{\int p\, dx}$

= $e^{\int \frac{V}{2x} dx}$

= $e^{\frac{1}{2}\int \frac{1}{x} dx}$

= $e^{ln x^{\frac{1}{2}}}$

= $\sqrt{x}$

multiplying equation (ii) with this integrating factor we get,

$\sqrt{x}\frac{dV}{dx} + \frac{V}{2x}\sqrt{x} = -\frac{2}{x^2}\sqrt{x}$

or, $\sqrt{x}\frac{dV}{dx} + \frac{V}{2\sqrt{x}} = -2\,x^{-\frac{3}{2}}$

or, $\frac{d}{dx}$ (V . $\sqrt{x}$) = -2 $x^{-\frac{3}{2}}$

or, V $\sqrt{x}$= -2 $\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}$ + C (By integrating)

or, $\frac{1}{y}$ = C$x^{-\frac{1}{2}} + x^{-1}$, Which is the complete solution of this Bernoulli differential equation.

Exercise 2. $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

$\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

or, $e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x^2}$........ **(i)**

let,

e^{-y} = V

or, $e^{-y}\frac{dy}{dx} = \frac{dV}{dx}$

or, $e^{-y}\frac{dy}{dx} = -\frac{dV}{dx}$

Substituting the value of $e^{-y}\frac{dy}{dx}$ $and e^{-y}$in equation (i),

$e^{-y}\frac{dy}{dx}$+ $\frac{e^{-y}}{x}$ = $\frac{1}{x^2}$

or,- $\frac{dV}{dx}$+ $\frac{V}{x}$ = $\frac{1}{x^2}$

or, $\frac{dV}{dx}$- $\frac{V}{x}$ = $-\frac{1}{x^2}$...........**(ii)**

This is of the form $\frac{dy}{dx}$+ P(x)y= Q(x)where,

P(x) = $-\frac{1}{x}$

Q(x) = $-\frac{1}{x^2}$

<u>Integrating Factor</u>, I.F. = $e^{\int p(x)\, dx}$

$$= e^{-\int \frac{1}{x}\, dx}$$

$= e^{lnx^{-1}}$

$= \frac{1}{x}$

multiplying equation (ii) with this integrating factor we get,

$\frac{1}{x}\frac{dV}{dx} - \frac{V}{x}.\frac{1}{x} = -\frac{1}{x^2}.\frac{1}{x}$

or, $\frac{1}{x}\frac{dV}{dx} + \frac{V}{x^2} = -x^{-3}$

or, $\frac{d}{dx}(V.\frac{1}{x}) = -x^{-3}$

or, $\frac{V}{x} = -\frac{x^{-2}}{-2} + C$(By integrating)

or, $e^{-y} = \frac{1}{2x} + Cx$,Which is the complete solution of this Bernoulli differential equation.

Exercise 3. $2\frac{dy}{dx} + y = y^3(x-1)$

$2\frac{dy}{dx} + y = y^3(x-1)$

or, $2y^{-3}\frac{dy}{dx} + yy^{-3} = (x-1)$

or, $2y^{-3}\frac{dy}{dx} + y^{-2} = (x-1)$........ **(i)**

let,

$y^{-2} = V$

or, $2y^{-3}\frac{dy}{dx} = \frac{dV}{dx}$

Substituting the value of $2y^{-3}\frac{dy}{dx}$ *and* y^{-2} in equation (i),

or, $2y^{-3}\frac{dy}{dx} + y^{-2} = (x-1)$

or, $\frac{dV}{dx} + V = (x-1)$...........**(ii)**

This is of the form $\frac{dy}{dx}$+ P(x)y= Q(x)where,

P(x) = 1

Q(x) = $(x - 1)$

<u>Integrating Factor</u>, I.F.= $e^{\int p\,dx}$

= $e^{\int dx}$

= $e^{\frac{1}{2}\int \frac{1}{x}\,dx}$ = e^{x}

multiplying equation (ii) with this integrating factor we get,

$e^{x}, \frac{dV}{dx}$ +V .e^{x}= $(x - 1)e^{x}$

or, $\frac{d}{dx}$(V. e^{x}) = $xe^{x} - e^{x}$

or, V. $e^{x} = \int xe^{x}\,dx - \int e^{x}\,dx$(By integrating)

Now, $e^{x} = \int xe^{x}\,dx$

$= x \int e^x \, dx$ - $\int(\frac{dx}{dx}\int e^x \, dx) \, dx$

$$= x\, e^x - \int e^x \, dx$$

V. $e^x = \int x e^x \, dx - \int e^x \, dx$

or, V. e^x = $x\, e^x - \int e^x \, dx - \int e^x \, dx$

or, V. e^x = $x\, e^x - e^x - e^x + C$

or, y^{-2} = $x - 2 + Ce^{-x}$,Which is the complete solution of this Bernoulli differential equation.

Exercise 4. $\cos x \frac{dy}{dx} + y \sin x + 2y^3 = 0$

$\cos x \frac{dy}{dx} + y \sin x + 2y^3 = 0$

or, $\frac{dy}{dx} + y \frac{\sin x}{\cos x} = -\, 2y^3 \frac{1}{\cos x}$

or, $y^{-3} \frac{dy}{dx} + y . y^{-3} \tan x = -\, 2 \sec x$

or, $y^{-3} \frac{dy}{dx} + y^{-2} \tan x = -\, 2 \sec x$........ **(i)**

let,

y^{-2} = V

or, 2 y$^{-3}\frac{dy}{dx} = \frac{dV}{dx}$

or, y$^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dV}{dx}$

Substituting the value of y$^{-3}\frac{dy}{dx}$ $and\ y^{-2}$in equation (i),

$$y^{-3}\frac{dy}{dx} + y^{-2}\tan x = -2\sec x$$

or, $-\frac{1}{2}\frac{dV}{dx}$ + V. $\tan x = -2\sec x$

or,$\frac{dV}{dx}$ − 2V. $\tan x = 4\sec x$…………**(ii)**

This is of the form $\frac{dy}{dx}$+ P(x)y= Q(x)where,

P(x) = $-\tan x$

Q(x) = $-4\sec x$

Integrating Factor, I.F.=$e^{\int p\,dx}$

$$= e^{-\int \tan x \, dx}$$

$$= e^{-2\ln(\sec x)}$$

$$= (\sec x)^{-2}$$

$$= \frac{1}{sec^2 x}$$

$$= cos^2 x$$

multiplying equation (ii) with this integrating factor we get,

$$\frac{dV}{dx} - 2\text{V}. \tan x = 4 \sec x$$

or, $\frac{dV}{dx} cos^2 x - 2\text{V}. \tan x \ \ cos^2 x = 4 \sec x \, cos^2 x$

or, $cos^2 x \ \frac{dV}{dx} - \text{V}. cos^2 x \ \ 2 \ \frac{\sin x}{\cos x} = 4cos^2 x \ . \frac{1}{\cos x}$

or, $cos^2 x \ \frac{dV}{dx}$ – $\text{V}. 2 \ \cos x \, (-\sin x) = 4 \cos x$

or, $\frac{d}{dx}(\text{V} \ . \ cos^2 x \ ) = 4 \cos x$

or, V . $cos^2 x$ = 4 $\sin x$ + C (By integrating) or, $y^{-2}cos^2x$ = 4 $\sin x$ + C ,Which is the complete solution of this Bernoulli differential equation.

Exercise 5. $\frac{dy}{dx} + \frac{y}{3} = e^x y^4$

$\frac{dy}{dx} + \frac{y}{3} = e^x y^4$

or, $y^{-4}\frac{dy}{dx} + \frac{y.y^{-4}}{3} = e^x$

or, $y^{-4}\frac{dy}{dx} + \frac{y^{-3}}{3} = e^x$........ **(i)**

let,

y^{-3} = V

or, -3 $y^{-4}\frac{dy}{dx} = \frac{dV}{dx}$

or, $y^{-4}\frac{dy}{dx} = -\frac{1}{3}\frac{dV}{dx}$

Substituting the value of $y^{-4}\frac{dy}{dx}$ $and\ y^{-3}$in equation (i), or, $y^{-4}\frac{dy}{dx}+\frac{y^{-3}}{3}=e^{x}$

or, $-\frac{1}{3}\frac{dV}{dx}+\frac{V}{3}=e^{x}$

or, $\frac{dV}{dx}-V=-3e^{x}$............**(ii)**

This is of the form $\frac{dy}{dx}$+ P(x)y= Q(x)where,

P(x) $=-1$

Q(x) $=-e^{x}$

<u>Integrating Factor</u>, I.F.$=e^{\int p\,(x)\,dx}$

$=e^{\int -dx}$

$=e^{-x}$

multiplying equation (ii) with this integrating factor we get,

$\frac{dV}{dx} - \text{V} = -3e^x$

or,$e^{-x}\frac{dV}{dx} - \text{V}.e^{-x} = -3e^x.e^{-x}$

or, $e^{-x}\frac{dV}{dx} + \text{V}.(-e^{-x}) = -3$

or, $\frac{d}{dx}(\text{V} \,.\, e^{-x}) = -3$

or, V $e^{-x} = -3x + C$ (By integrating)

or, $y^{-3}e^{-x} = -3x + C$

or, $\frac{1}{y^3}e^{-x} = e^x(C - 3x)$,Which is the complete solution of this Bernoulli differential equation.

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Exact Differential Equation:

A Differential equation of the form $M(x, y)dx + N(x, y)dy = 0$

or, $M + N\frac{dy}{dx} = 0$ is called an exact D.E if it satisfies the following condition

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Working Rules:

i) Integreting M, with respect to x treating y as a constant.

ii) Find out those terms in 'N' which is free from x, then integrate them with respect to y.

iii) Adding i and ii i,e (i) + (ii)) = C

Example:

$xdy + ydx = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial y}{\partial y} = 1$$
$$\frac{\partial N}{\partial X} = \frac{\partial x}{\partial x} = 1$$
as
$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$; So the equation is an exact equation $\int M dx = \int y dx = yx$
N terms free from x is o whose integretingC_1
Therefore, the general solution of the given equation is
$xy + C_1 = C_2$
$xy = C$

1. $(y^4 + 4x^3 + 3x)dx + (x^1 + 4xy^3 + y + I)dy = 0$

Solution: Here $M = y^4 + 4x^3 + 3x \ and \ N = x^1 + 4xy^3 + y + I$

$$\frac{\partial M}{\partial y} = 4y^3 + 4x^3 \ and \ \frac{\partial N}{\partial x} = 4x^3 + 4y^3$$

Since there are equal, the equation is exact.

To find solution if the differential equation, integrating $M \ i, e. y^3 4x^3 y + 3x \ w.r.t. x$, keeping y as constant, we get

$$y^3 x \ \ x^3 y + \frac{1}{2} x^2$$

In $x^4 + \ 4xy^3 + y + 1$, terms free from x are $y + 1$ whose integral with respect to y is $\frac{1}{2} y^2 + y$.

Therefore the general solution is

$$y^4 x + \ x^4 y + \frac{2}{3} x^2 + \frac{1}{2} y^2 + y = C.$$

2. Solve $x(x^2 + y^2 - a^2)\ dx + y\ (x^2 - y^2 - b^2) dy = 0$

Solution. Here $M = x^3 + xy^2 - a^2 x. N = yx^2 - y^3 - b^2 y$

$\frac{\partial M}{\partial y}$ =2xy and $\frac{\partial N}{\partial x}$ =2xy.

Since these are equal, the equation is exact,

Integrating Mw.r.t. x keeping y as constant, we get

$$\frac{1}{4} x^4 + \frac{1}{2} x^2 y^2 - \frac{1}{2} a^2 x^2$$

In N, terms free from x are $-y^3 - b^2 y$ whose integral is

$$-\frac{1}{2}y^4 - \frac{1}{2}b^2y^2$$

Hence the general solution is

$\frac{1}{2}x^4 + \frac{1}{2}x^2y^2 - \frac{1}{2}a^2x^2 - \frac{1}{2}y^4 - \frac{1}{2}b^2y^2$ =const.

or,$x^4 - y^4 + 2x^2y^2 - 2b^2y^2 = C$

3. Solve (x^2–2xy + 3y^2) dx + (4y^3–6xy–x^2) dy=0.

Solution. Here $\frac{\partial M}{\partial y} = -2$+6y, $\frac{\partial N}{\partial x} = 6y$ −2x.

Since these are equal the equation is exact.

Integrating $M, i.e. x2\ -\ 2xy\ + 3y2\ w.r.t. x$keeping yas constant,

we get

$$\frac{1}{3}x^3 - x^2y + 3y^2x$$

In N, term free from x is +4y^3 whose integral is y^4.

Hence the solution is $\frac{1}{3}x^3 - x^2y + 3y^2x + y^4 = C$

Ex. 4. Solve (y—2e^v) dy +(y+xsin x) dx=0).

Solution. Here $\mathrm{M = y + x\sin x.\ \ N = x - 2e^v}$

$\frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x}$ =1; therefore equation is exact.

Integrating $y + x \sin x$with respect to x keeping as constant,

we get $xy\ + \int xsin\ x\ dx = xy - x\ cos\ x + -sin\ x.$

In N, term free from $x\ is - 2e^v$whose integral with respect to $y\ is - 2e^v$

Hence the complete solution is

$xy - x\ cos\ x + sin\ x - 2e^v = C.$

5. Solve $x\ dx + y\ dy = \frac{a^2\ (x\ dy - y\ dx).}{x^2+y^2}$

Solution. The equation can be put as

$$\left(x + \frac{a^2y}{x^2+y^2}\right)dx + \left(y - \frac{a^2x}{x^2+y^2}\right)dy = 0$$

Here $M = x + \frac{a^2y}{x^2+y^2}\ and\ N = x + \frac{a^2y}{x^2+y^2}$

$\therefore\ \ \frac{\partial M}{\partial y} = \frac{(x^2+y^2)a^2 - a^2y.2y}{(x^2+y^2)^2} = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$

and$\frac{\partial N}{\partial x} = \frac{-a^2(x^2+y^2)+2a^2x^2}{(x^2+y^2)^2} = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, the equation is exact.

Integrating $M\ w.r.t.x$ regarding y as constant, we get

$$\frac{1}{2}x^2+a^2y\frac{1}{2}tan^{-1}\frac{x}{y}\quad or\ \frac{1}{2}x^2+a^2tan^{-1}\frac{x}{y}$$

N, term free from x is y whose integral is $\frac{1}{2}y^2$

Hence the solution is $\frac{1}{2}x^2+a^2tan^{-1}\frac{x}{y}+\frac{1}{2}y^2$=const.

or$x^2+\ y^2+2a^2tan^{-1}\frac{x}{y}=C$.

Ex. 6. Solve $\left(1\ +e^{x/y}\right)dx+e^{x/y}\ (1-x/y)\ dy=0$.

Solution. Here $M=1+-e^{x/y}\ and\ N=e^{x/y}\ (1-x/y)$.

$$\therefore\quad \frac{\partial M}{\partial y}=e^{x/y}\left(-\frac{x}{y^2}\right)$$

and$\frac{\partial N}{\partial x}=e^{x/y}\frac{1}{y}\left(1-\frac{x}{y}\right)+e^{x/y}\left(-\frac{1}{y}\right)=e^{x/y}\left(-\frac{x}{y^2}\right)$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, the equation is exact.

Now integrating $1\ +e^{x/y}$with respect to x keeping y as constant,

we get $x+\frac{e^{x/y}}{1/y}i.e.,x+ye^{x/y}$

In $Nie.$, in $e^{x/y}(I\ -x/y)$ there is no term free from x.

Hence the required solution is $x+\ ye^{x/y}=C$.

Ex.7.$[\cos x\tan y+\cos\,(x\ +\ y)]\ dx$

$+[\sin x\ secy+\cos\,(x+y)]\ dy=0$.

Solution. Here $M=\cos x\tan y+\cos\,(x\ +\ y)$,

and$N=\sin x\ sec^2\ y+\cos\,(x+y)$.

Now$\frac{\partial M}{\partial y}=\cos r\ sec^2\ y-\sin\,(x\ +\ y)$,

$\frac{\partial N}{\partial x}$= $\cos x\ sec^2\ y-\sin\,(x-y)$

Since these are equal, the equation is exact.

Now integrating$M, ie.\cos x\tan y-\cos(x+y)$ with respect to x keeping $y$as constant, we get

$$\sin x\tan r+\sin\,(x+1)$$

In N, there is no term free from x

Hence the general solution is

$sin\ x\ tan\ y + sin\ (x+1) = C.$

Ex. 8. $(\cos x \tan y - \sin x \sec y)dx$

$$+(sin\ x\ sec^2 y + cos\ x\ tan^2\ y\ cosec\ y)dy = 0$$

Sol^n^. We have $M = \cos x \tan y - \sin x \sec y$;

$and\ N = \sin x\ sec^2\ y + cos\ x\ tan^2\ y\ cosec\ y.$

$\frac{\partial M}{\partial y} = cos\ x\ sec^2\ y - sin\ x\ sec\ y\ tan\ y,$

$\frac{\partial N}{\partial x} = cos\ x\ sec^2\ y - sin\ x\ tan\ y\ sec\ y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Integrating $M\ ie.\ cos\ x\ tan\ y - sin\ x\ sec\ y$ with regard to keeping y as constant we get $sin\ x\ tan\ y + cos\ x\ sec\ y.$

In N there is no term free from x.

Hence the general solution is

$sin\ x\ tan\ y + cos\ x\ sec\ y = C.$

Ex. 9. Solve $(sin\ x\ cos\ y + e^{2x})\ dx$

$$+(cos\ x\ sin\ y + tan\ y)dy = 0$$

Sol^n^. Here $\frac{\partial M}{\partial y} = -sin\ x\ sin\ y, \frac{\partial N}{\partial x} = -sin\ x\ sin\ y$

Since these are equal, the equation is exact.

Integrating $M i.e., sin\ x\ cos\ y + e^{2x}\ w.r.l.x$, keeping $y$as constant, we get $-\ cos\ x\ cos\ y + \frac{1}{2}e^{2x}$

Also in N the term free from x is $tan\ y$ whose integral $w.r.t\ y$ is $\log sec\ y$.

Hence the solution is

$$-cos\ x\ cos\ y + \frac{1}{2}e^{2x} + log\ sec\ y = C$$

Ex. 10.Solve $(r + sin\theta - cos\theta)dr + r(sin\theta + cos\theta)d\theta = 0$

Sol^n^. Here we have r and θ *in place of usual variables x and y*.

Comparing the given equation with $M\ dr + N\ d\theta = 0, M = r + sin\theta - cos\theta, N = r(sin\theta + cos\theta)$.

So, $\partial M/\partial\theta = \cos\theta + sin\theta = \partial N/\partial r$ .so equation is exact with solution

$$\int M\ dx + \int (\ terms\ in\ N\ not\ containing\ r) = c$$

Or, $\int (r + sin\theta - cos\theta)dr = c$

Or, $r^2/2 + r(\sin\theta - \theta) = c$

Ex. 11. Show $(4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$ is a family of hyperbolas with a common axis and tangent at the vertex.

Sol^n. Given $(4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$ ....*(1)*

Comparing (1) with *M dx +N dy =0* here, *M=4x +3y +1,* *N= 3x +2y +1.*

Here $\partial M/\partial y$ =3=$\partial N/\partial x$ and so (1) is exact .Its solution is

$$\int (4x + 3y + 1)dx + \int (3x + 2y + 1)dy = 0$$

Or, $2x^2 + 3xy + x + y^2 + y + k = 0$, where k is an arbitrary constant. ...(2)

Comparing (2) with standard form of conic section $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, here, $a = 2$, $b = 1$, $h = 3/2$, $g = 1/2$, $f = 1/2$, $c = k$ ....(3)

Then $h^2 - ab = \left(\frac{9}{4}\right) - 2 =$ positive quantity,

Showing that (2) represents a family of hyperbolas ,k being the parameter ,with common axis and tangent at vertex.

Ex. 13. Find the values of constant λ such that $(2xe^y + 3y^2)\left(\frac{dy}{dx}\right) + (3x^2 + \lambda e^y) = 0$ is exact. Further,for this value of λ,solve the equation

Sol^n. Re-writing the given equation ,$(3x^2 + \lambda e^y)dx + (2xe^y + 3y^2)dy = 0$ ...(1)

Comparing (1) with $M\,dx + Ndy = 0$,here$M = 3x^2 + \lambda e^y$ and N=2xe^y +3y^2

Now,for (1) to be exact we must have

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$so that, $\lambda e^y = 2e^y$ giving $\lambda = 2.$

So, (1) becomes $(3x^2 + 2e^y)dx + (2xe^y + 3y^2)dy = 0$ ....(3)

Equation (3) in exact and hence its solution is its solution is

$$\int M\,dx + \int (terms\ in\ N\ not\ containing\ x)dy = c$$

Or, $\int (3x^2 + 2e^x)dx + (3y^2)dy = c$

Or, $x^3 + 2e^x + y^3 = c$.

Reduce to Non exact D.E into exact D.E:

$Mdx + Ndy = 0$

When $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then two case occur to convert the Non exact D.E into exact D.E.

(i) $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$a function of 'x' only, then $e^{\int f(x)dx}$ is an integrating factor.

(ii) $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(x)$,a function of y alone, then $e^{\int -g(x)dx}$ is an integrating factor.

Ex. 14.Solve $(x^2 + y^2 + x)$ dx+xydy=0

Sol. Here $M = x^2 + y^2 + x, N = xy$

$\frac{\partial M}{\partial y}$ =2y,$\frac{\partial N}{\partial x}$ =y. equation is not exact,

However, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x-y}{xy} = \frac{1}{x}$, a function of x alone.

Hence $IF = e^{-} \int \frac{1}{x} dx = e \quad ^{x} = x$

Multiplying by I F., the equation becomes

$(x^2 + y^2 + x^2)dx + x^2y\, dy = 0$, exact now (check up)

Integrating $x^3 + xy^2 + x^2$ with regard to $x$keeping y as constant, we get$\frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{3}x^3$

and in x^2y^2 there is no term free from x. Therefore the solution is $\frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{3}x^3 = C$ or $3x^4 + 4x^3 + 6x^2y^2 = C$

Ex.15. $(\mathbf{2xlogx - xy})\mathbf{dy} + \mathbf{2ydx} = \mathbf{0}$

Sol^n. $\frac{\partial M}{\partial y} = \frac{\partial 2y}{\partial y} = 2 \quad and \quad \frac{\partial N}{\partial x} = \frac{\partial(2xlogx - xy)}{\partial x} = 2 + 2logx - y$

As,$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Hence,we need$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial X}}{N} = \frac{-(2logx - y)}{x(2logx - y)} = -\frac{1}{x}$ which is the function of x alone

So,the I.F is$e^{\int -(\frac{1}{x})dx} = \frac{1}{x}$

Now by multiplying the equation we can get $\frac{2y}{x}dx + (2logx - y)dy = 0$

So now ,by integrating M'with respect to x keeping y constant and adding the integral of the part free from x in N' we can get the solution which is

$2ylogx - \frac{1}{2}y^2 = c$(ans.)

Ex. 16.Solve $(x^2 + y^2 + I)\, dx - 2xy\, dy = 0$

Sol. $\frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = -2y$.not exact,

However, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2x-y}{-2xy} = -\frac{2}{x}$, function of x alone.

$$I.F = e^{-}\int \frac{2}{x}dx = e^{-2\log x} = \frac{2}{x^2}$$

Multiplying by $\frac{1}{x^2}$ the equation becomes

$(1 + \frac{y^2}{x^2} + \frac{1}{x^2})dx - \frac{2y}{x}\, dy = 0$, exact now

Integrating $1 + \frac{y^2}{x^2} + \frac{1}{x^2}$ with regard to x keeping y as constant,

we get$x - \frac{y^2}{x} + \frac{1}{x}$

and in $\frac{2y}{x}$ there is no term free from x.

Hence the solution is$x - \frac{y^2}{x} - \frac{1}{x} = C$ or $x^2 - y^2 = Cx + 1$

Ex. 17. Solve $(x^2 + y^2)\, dx - 2xy\, dy = 0$.

Sol. Just as in the above example, I.F=1/x^2

Hence Multiplying by $\frac{1}{x^2}$the equation becomes

$(1 + y^2/x^2\,)dx - 2y/x\, dy = 0$not exact

∴ Solution is $x - \frac{y^2}{x} = C$ or $x^2 - y^2 = Cx$

Ex. 18. Solve $(x^2 + y^2 + 2x)\, dx + 2y\; dy = 0$.

Sol. $\frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 0$not exact

However, $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2y}{2y}=1$

$\therefore$ I.F=e^x

Multiplying by e^x the equation becomes

$e^x(x^2+y^2+2x)dx\ 2y\ e^x\ dy=0$not exact

This can be written as

$(x^2+2x)\ e^x dx+(y^2e^x dx+e^x.2y\ dy)=0$

Or, $d(x^2e^x)+d(y^2e^x)=0$

$\therefore$Integrating, $x^2e^x+y^2e^x=C$ or $(x^2+y^2)e^x=C$

After, The equation can also be written as

$2y\frac{dy}{dx}+x^2=-x^2+2x$

Putting $y^2-r,\frac{dy}{dx}+r=-x^2+2x$ Linear, I.F=e^x etc

Ex. 19. Solve $\left(\frac{1}{3}y+y^3\frac{1}{2}x^2\right)dx+\frac{1}{4}(x+xy^2)dy=0$

Sol. $\frac{\partial M}{\partial y}=1+y^2,\ \frac{\partial N}{\partial x}=\frac{1}{4}(1+y^2)$ not exact

However , $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{(1+y^2)-\frac{1}{4}(1+y^2)}{\frac{1}{4}x\ (1+y^2)}=\frac{3}{x^2}$ is a function of $x$along.

$\therefore$I.F=$e^{\int\frac{3}{x}dx}=e^{3\log x}=x^3$

Multiplying by x^3 the equation becomes

$(x^3y+\frac{1}{3}x^3y^3+\frac{1}{2}x^5)dx+\frac{1}{4}(x^4+x^4y^2)\ dy=0$exact now

Integrating, $x^3y+\frac{1}{3}x^3y^3+\frac{1}{2}x^5$with respect to x keeping y as constant, we get$\frac{1}{4}x^4y+\frac{1}{12}x^4y^3+\frac{1}{12}x^6$

In $\frac{1}{4}(x^4+x^4y^2)$ there is no term free from x

$\therefore$the solution is $\frac{1}{4}x^4y+\frac{1}{12}x^4y^3+\frac{1}{12}x^6=$ constant

or, $3x^4y+y^3x^4+x^6=C$

Ex. 20. Solve $(y^4+2y)\ dx+(xy^3+2y^4-4x)dy=0$

Sol. $\frac{\partial M}{\partial y}=4y^3+2,\ \frac{\partial N}{\partial x}=y^3-4$, not exact

However, $\frac{\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}}{M}=\frac{4y^3+2-(y^3-4)}{y^4+2y}=\frac{3}{y}\,a$ function of y alone.

$\therefore$ I.F=$e^{-\int \frac{3}{y} dy} = e^{-3\log y} = \frac{1}{y^3}$

Multiplying by $1/y^3$, the equation becomes

$\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right)\ dy =\ 0$,exact now.

Integrating$y + \frac{2}{y^2}\ w.r.t\ x$ keeping$y$ as constant, we have

$$yx + \frac{2}{y^2}x$$

In $x + 2y - \frac{4x}{y^3}$,the term free from $x$is $2y$; So inteerating $2yw.r.t.y$; we get y^2

Therefore the solution is $yx + \frac{2}{y^2}x + y^2 = C$.

Ex. 21. Solve $(3x^2y^4 + 2xy)\ dx + (2x^3y^3 - x^2)dy = 0$

Solution.Here $\frac{\partial M}{\partial y} = 12x^2y^3 + 2x,\ \frac{\partial N}{\partial x} = 6x^2y^3 - 2x,$

Now, $\frac{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}{M} = \frac{6x^2y^3+4x}{y(3x^2y^3+2x)} = \frac{2}{y}$ function of y alone.

$\therefore$ I.F=$e^{-\int 2/\ \ dy} = e^{-2\log y} = \frac{1}{y^2}$

Multiplying by $\frac{1}{y^2}$ the equation becomes

$\left(3x^2y^2 + \frac{2x}{y}\right) dx + \left(2x^3y - \frac{x^2}{y^2}\right)\ dy = 0$,exact now.

Integrating $3x^2y^2 + \frac{2x}{y}\ w.r.t\ x$ keeping y as constant, we get

$$x^3y^3 + \frac{x^2}{y^2}$$

In $2x^3y - \frac{x^2}{y^2}$,there is no term free from x

Hence the solution is $x^3y^2 + \frac{x^2}{y} = C$.

or$x^3y^2 + x^2 = Cy$.

Ex. 22. Solve $(2xy^4e^4 + 2xy^3 + y)\ dx + (x^2y^4e^4 - x^2y^2 - 3x)dy = 0$

Sol. We have$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{4}{y}$ $\qquad\qquad \therefore I.F = \frac{1}{y^4}$

Solution is $x^2e^4 = \frac{x^2}{y} + \frac{x}{y^3} = C$

Ex 23.$y(x^2 + e^x)dx - e^x dy = 0$

Sol.

$(x^2y^2 + e^x y)dx + (-e^x)dy = 0$

Or, $\frac{\partial M}{\partial y} = \frac{\partial(x^2y^2+e^x y)}{\partial y} = 2yx^2 + e^x$ $\quad$ $but, \frac{\partial N}{\partial x} = \frac{\partial(-e^x)}{\partial x} = -e^x$

As, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Now we need to find, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{2(x^2y+e^x)}{y(x^2+e^x)} = \frac{2}{y}$ which is the function of y alone

so,the I.F $= e^{-\int \frac{2}{y} dy} = e^{-2\ln y} = \frac{1}{y^2}$

Multiplying $\frac{1}{y^2}$ the equation becomes,

$\frac{1}{y}(x^2y + e^x)dx - \frac{e^x}{y^2} dy = 0$, which is exact.

Now,as there is no term free from x in N' so, only by integrating $\frac{1}{y}(x^2y + e^x)$ with respect to x keeping y as constant we can have the solution which is

$\frac{x^3}{3} + \frac{e^x}{y} = c$ (ans.)

23.$(2xy^2 - 2y)dx + (3x^2y - 4x)dy = 0$

Now,

$\frac{\partial M}{\partial y} = \frac{\partial(2xy^2-2y)}{\partial y} = 4xy - 2$ and $\frac{\partial N}{\partial x} = \frac{\partial(3x^2y-4x)}{\partial x} = 6xy - 4$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So we need to find, $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-(2xy-2)}{y(2xy-2)} = -\frac{1}{y}$ which is the function of y alone

So the I.F is $e^{-\int(-\frac{1}{y})dy} = y$

Now ,multiplying y the equation becomes,

$$(2xy^3 - 2y^2)dx + (3x^2y^2 - 4xy)dy = 0$$

Now, as there is no term free from x in N', we can find the solution only by integrating M' with respect to x keeping y fixed.So the solution will be ,

$x^2y^3 - 2xy^2 = c$(ans.)

24. $\boldsymbol{(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0}$

Now,$\frac{\partial M}{\partial y}=\frac{\partial(xy^3+y)}{\partial y}=3xy^2+1$ and $\frac{\partial N}{\partial x}=\frac{\partial\{2(x^2y^2+x+y^4)\}}{\partial x}=4xy^2+2$

As,$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ we need to find,

$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=\frac{-(xy^2+1)}{y(xy^2+1)}=-\frac{1}{y}$which is the function of y alone

so, the integrating factor is I.F=$e^{-\int(-\frac{1}{y})dy}=e^{lny}=y$

Now,by multiplying the equation with y we can get,

$$(xy^4+y^2)dx+2(x^2y^3+xy+y^5)dy=0$$

Now, integrating M' with respect to x we can get $\frac{x^2y^4}{2}+y^2x$

And,inN',term free from x is $2y^5$,whose integral is$\frac{y^6}{3}$

So now the solution is $\frac{x^2y^4}{2}+y^2x+\frac{y^6}{3}=c$ (ans.)

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Elementary Discussion on Differential Equations

And Formation of Differential Equations

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Elementary Discussion of Differential Equation & Formation of Differential Equation

Differential Equations

In Mathematics, a differential equation is an equation that contains one or more functions with its derivatives. The derivatives of the function define the rate of change of a function at a point. It is mainly used in fields such as physics, engineering, biology and so on. The primary purpose of the differential equation is the study of solutions that satisfy the equations and the properties of the solutions.

Differential Equation Definition

A **differential equation** is an equation which contains one or more terms and the derivatives of one variable (i.e., dependent variable) with respect to the other variable (i.e., independent variable)

dy/dx = f(x)

Here “x” is an independent variable and “y” is a dependent variable

For example, 1. $\frac{dy}{dx} = \frac{1+x^2}{1-y^2}$, 2. $\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 8y = 0$, 3. $[1 + (\frac{dy}{dx})^2]^{\frac{3}{2}} = k\frac{d^2y}{dx^2}$

A differential equation contains derivatives which are either partial derivatives or ordinary derivatives. The derivative represents a rate of change, and the differential equation describes a relationship between the quantity that is continuously varying with respect to the change in another quantity.

Types of Differential Equations

Differential equations can be divided into several types namely

- Ordinary Differential Equations
- Partial Differential Equations
- Linear Differential Equations
- Non-linear differential equations
- Homogeneous Differential Equations
- Non-homogenous Differential Equations

Differential Equations Solutions

There exist two methods to find the solution of the differential equation.

1. Separation of variables
2. Integrating factor

Separation of the variable is done when the differential equation can be written in the form of dy/dx = f(y)g(x) where f is the function of y only and g is the function of x only. Taking an initial condition, rewrite this problem as 1/f(y)dy= g(x)dx and then integrate on both sides.

Integrating factor technique is used when the differential equation is of the form dy/dx + p(x)y = q(x) where p and q are both the functions of x only.

Order of Differential Equation

The order of the differential equation is the order of the highest order derivative present in the equation. Here some examples for different orders of the differential equation are given.

- dy/dx = 3x + 2, The order of the equation is 1
- $(d^2y/dx^2) + 2\ (dy/dx) + y = 0$. The order is 2
- (dy/dt) +y = kt. The order is 1

First Order Differential Equation

A first order differential equation is an equation of the form $F(t,y,\dot{y})=0$. A solution of a first order differential equation is a function $f(t)$ that makes $F(t,f(t),f'(t))=0$ for every value of t.

Here, F is a function of three variables which we label t, y, and $\dot{y}$. It is understood that $\dot{y}$ will explicitly appear in the equation although t and y need not. The term "first order" means that the first derivative of y appears, but no higher order derivatives do.

Example:

$\dot{y}=t2+1$ is a first order differential equation; $F(t,y,\dot{y})=\dot{y}-t2-1$. All solutions to this equation are of the form $t3/3+t+C$.

The equation from Newton's law of cooling, $\dot{y}=k(M-y)$ is a first order differential equation; $F(t,y,\dot{y})=k(M-y)-\dot{y}$.Higher Order Differential Equation.

Second-Order Differential Equation

The equation which includes second-order derivative is the second-order differential equation. It is represented as;

$d/dx(dy/dx) = d^2y/dx^2 = f''(x) = y''$

Higher order differential equations:

There are two types of higher order differential equation

1.Higher order linear homogenous differential equation.

2.Higher order linear non homogenous differential equation.

Higher order linear homogenous differential equation: Higher order homogenous

Differential equations with constant coefficients is called the characteristic equation

Of the differential equation.

Example:

1.y''+2y''-y'-2y=0

2.y'''-7y''+11y'-5y=0

3.y''''-y'''+2y'=0

4.y'''''+18y'''+81y'=0

5.y''''-4y'''+5y''-4y'+4y=0

Higher order linear non homogenous differential equation: The general solution y(x)

Of the non-homogeneous equation is the sum of the general solution y0(x) of the

Corresponding homogeneous equation and a particular solution y1(x) of the non

Homogeneous equation.

Y(x)=y0(x)+y1(x)

Example:

1.y'''+3y''-10y'=x-3

2.y'''-y'=sin3x

3.y''''-y=2cosx

Degree of Differential Equation

The degree of the differential equation is the power of the highest order derivative, where the original equation is represented in the form of a polynomial equation in derivatives such as y',y", y"', and so on.

Suppose $(d^2y/dx^2) + 2 (dy/dx) + y = 0$ is a differential equation, so the degree of this equation here is 1. Some more examples here:

- dy/dx + 1 = 0, degree is 1
- $(y''')^3 + 3y'' + 6y' – 12 = 0$, degree is 3

Ordinary Differential Equation

An ordinary differential equation involves function and its derivatives. It contains only one independent variable and one or more of its derivatives with respect to the variable.

The order of ordinary differential equations is defined as the order of the highest derivative that occurs in the equation. The general form of n-th order ODE is given as

$F(x, y, y', \ldots., y^n) = 0$

1. (1-x) y"-4xy'+5y = cos x
2. $x\frac{d^3y}{dx^3} - (\frac{dy}{dx})^4 + y = 0$
3. $t^5y^4 – t^3y'' + 6y=0$
4. $\frac{d^2u}{dr^2} + \frac{du}{dr} + u = \cos(r + u)$
5. $\frac{d^2y}{dx^2} = \sqrt{1 + (\frac{dy}{dx})^2}$
6. $\frac{d^2R}{dt^2} = -\frac{k}{R^2}$

Partial Differential Equations

A differential equation involving derivatives of one or more dependent variables with respect to more than one independent variable is called partial differential equation. A **Partial Differential Equation** commonly denoted as PDE is a differential equation containing partial derivatives of the dependent variable (one or more) with more than one independent variable. A PDE for a function u $(x_1, \ldots\ldots x_n)$ is an equation of the form

$$f(x_1, \ldots\ldots, x_n; u, \frac{\partial u}{\partial x1}, \ldots\ldots, \frac{\partial u}{\partial xn}; \frac{\partial^2 u}{\partial x1 \partial x1}, \ldots\ldots, \frac{\partial^2 u}{\partial x1 \partial xn}; \ldots\ldots) = 0$$

Example:

1. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
2. $u_{xx} + u_{yy} = 0$
3. $\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$

Linear Differential Equations

A linear differential equation is a differential equation that is defined by a linear polynomial in the unknown function and its derivatives, that is an equation of the form. This is an ordinary differential equation (ODE). A linear differential equation may also be a linear partial differential equation (PDE), if the unknown function depends on several variables, and the derivatives that appear in the equation are partial derivatives.

Example:

dy/dx + 2y = sin x

dy/dx + y = ex

Non-linear differential equations

A non-linear differential equation is a differential equation that is not a linear equation in the unknown function and its derivatives (the linearity or non-linearity in the arguments of the function are not considered here.

Example: $3+yy'=x-y$

An equation will be non- linear if –
1) Derivatives will be in the form of product.
2) Dependent variables and derivatives will be in the form of product.
3)dependent variables will be in the form of product.
4) Transcendental function of the dependent variables

Homogeneous Differential Equations

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\emptyset(x,y)}$ is called a homogeneous equation if each term of f (x, y) and ϕ (x, y) is of the same degree i.e.,

$$\frac{dy}{dx} = \frac{3xy + y^2}{3x^2 + xy}$$

In such case we put y = vx and $\frac{dy}{dx} = v + x\frac{dy}{dx}$

The reduced equation involves v and x only. This new differential equation can be solved by variables separable method.

Working Rule

Step 1: Put y = vx and $\frac{dy}{dx} = v + x\frac{dy}{dx}$

Step 2 : Separate the variables.

Step 3: Integrate both the sides.

Step 4: Put v = $\frac{y}{x}$ and simplify.

Example:

1. sin(x) $\frac{d^2y}{dx^2}$ + 4 $\frac{dy}{dx}$ + y = 0

2. dy/dx = y(x−y)/x^2

3.f(zx,zy) = $(zx)^2 + (zy)^2 = z^2(x^2 + y^2)$

Non-homogenous Differential Equations

A second order, linear nonhomogeneous differential equation is

y″+p(t)y′+q(t)y=g(t)(1)(1)y″+p(t)y′+q(t)y=g(t)

where g(t)g(t) is a non-zero function. Note that we didn't go with constant coefficients here because everything that we're going to do in this section doesn't require it. Also, we're using a coefficient of 1 on the second derivative just to make some of the work a little easier to write down. It is not required to be a 1.

Before talking about how to solve one of these we need to get some basics out of the way, which is the point of this section.

First, we will call

y″+p(t)y′+q(t)y=0(2)(2)y″+p(t)y′+q(t)y=0

the associated homogeneous differential equation to (1).

Now, let’s take a look at the following theorem.

Theorem:

Suppose that $Y_1(t)Y_1(t)$ and $Y_2(t)Y_2(t)$ are two solutions to (1) and that $y_1(t)y_1(t)$ and $y_2(t)y_2(t)$ are a fundamental set of solutions to the associated homogeneous differential equation (2) then,

$Y_1(t)-Y_2(t)Y_1(t)-Y_2(t)$

is a solution to (2) and it can be written as:

$Y_1(t)-Y_2(t)=c_1y_1(t)+c_2y_2(t)$

Note the notation used here. Capital letters referred to solutions to (1) while lower case letters referred to solutions to (2). This is a fairly common convention when dealing with nonhomogeneous differential equations.

This theorem is easy enough to prove so let’s do that. To prove that $Y_1(t)-Y_2(t)Y_1(t)-Y_2(t)$ is a solution to (2) all we need to do is plug this into the differential equation and check it.

$(Y_1-Y_2)''+p(t)(Y_1-Y_2)'+q(t)(Y_1-Y_2)=0Y_1''+p(t)Y_1'+q(t)Y_1-(Y_2''+p(t)Y_2'+q(t)Y_2)=0g(t)-g(t)=00=0(Y_1-Y_2)''+p(t)(Y_1-Y_2)'+q(t)(Y_1-Y_2)=0Y_1''+p(t)Y_1'+q(t)Y_1-(Y_2''+p(t)Y_2'+q(t)Y_2)=0g(t)-g(t)=00=0$

We used the fact that $Y_1(t)Y_1(t)$ and $Y_2(t)Y_2(t)$ are two solutions to (1) in the third step. Because they are solutions to (1) we know that

$Y_1''+p(t)Y_1'+q(t)Y_1=g(t)Y_2''+p(t)Y_2'+q(t)Y_2=g(t)Y_1''+p(t)Y_1'+q(t)Y_1=g(t)Y_2''+p(t)Y_2'+q(t)Y_2=g(t)$

So, we were able to prove that the difference of the two solutions is a solution to (2).

Proving that

$Y_1(t)-Y_2(t)=c_1y_1(t)+c_2y_2(t)Y_1(t)-Y_2(t)=c_1y_1(t)+c_2y_2(t)$

is even easier. Since $y_1(t)y_1(t)$ and $y_2(t)y_2(t)$ are a fundamental set of solutions to (2) we know that they form a general solution and so any solution to (2) can be written in the form

$y(t)=c_1y_1(t)+c_2y_2(t)y(t)=c_1y_1(t)+c_2y_2(t)$

Well, Y1(t)−Y2(t)Y1(t)−Y2(t) is a solution to (2), as we've shown above, therefore it can be written as

Y1(t)−Y2(t)=c1y1(t)+c2y2(t)Y1(t)−Y2(t)=c1y1(t)+c2y2(t)

So, what does this theorem do for us? We can use this theorem to write down the form of the general solution to (1). Let's suppose that y(t)y(t) is the general solution to (1) and that YP(t)YP(t) is any solution to (1) that we can get our hands on. Then using the second part of our theorem we know that

y(t)−YP(t)=c1y1(t)+c2y2(t)y(t)−YP(t)=c1y1(t)+c2y2(t)

where y1(t)y1(t) and y2(t)y2(t) are a fundamental set of solutions for (2). Solving for y(t)y(t) gives,

y(t)=c1y1(t)+c2y2(t)+YP(t)y(t)=c1y1(t)+c2y2(t)+YP(t)

We will call

yc(t)=c1y1(t)+c2y2(t)yc(t)=c1y1(t)+c2y2(t)

the complementary solution and YP(t)YP(t) a particular solution. The general solution to a differential equation can then be written as.

y(t)=yc(t)+YP(t)y(t)=yc(t)+YP(t)

So, to solve a nonhomogeneous differential equation, we will need to solve the homogeneous differential equation, (2), which for constant coefficient differential equations is pretty easy to do, and we'll need a solution to (1).

This seems to be a circular argument. In order to write down a solution to (1) we need a solution. However, this isn't the problem that it seems to be. There are ways to find a solution to (1). They just won't, in general, be the general solution. In fact, the next two sections are devoted to exactly that, finding a particular solution to a nonhomogeneous differential equation.

Example: $y'' - 2y' - 3y = e^{2t}$

The corresponding homogeneous equation $y'' - 2y' - 3y = 0$ has characteristic equation $r^2 - 2r - 3 = (r + 1)(r - 3) = 0$. So the complementary solution is $y_c = C_1 e^{-t} + C_2 e^{3t}$.

The nonhomogeneous equation has $g(t) = e^{2t}$. It is an exponential function, which does not change form after differentiation: an exponential function's derivative will remain an exponential function with the same exponent (although its coefficient might change due to the effect of the Chain Rule). Therefore, we can very reasonably expect that $Y(t)$ is in the form $A\,e^{2t}$ for some unknown coefficient A. Our job is to find this as yet undetermined coefficient.

Let $Y = Ae^{2t}$, then $Y' = 2A\,e^{2t}$, and $Y'' = 4A\,e^{2t}$. Substitute them back into the original differential equation:

$$(4A\,e^{2t}) - 2(2A\,e^{2t}) - 3(A\,e^{2t}) = e^{2t}$$

$$-3A\,e^{2t} = e^{2t}$$

$$A = -1\,/\,3$$

Hence $Y(t) = \frac{-1}{3} e^{2t}$

Therefore, $y = y_c + Y = C_1e^{-t} + C_2e^{3t} - \frac{1}{3} e^{2t}$

Example: $y'' - 2y' - 3y = 3t^2 + 4t - 5$

The corresponding homogeneous equation is still $y'' - 2y' - 3y = 0$.

Therefore, the complementary solution remains $y_c = C_1\,e^{-t} + C_2\,e^{3t}$.

Now $g(t) = 3t^2 + 4t - 5$. It is a degree 2 (i.e., quadratic) polynomial. Since polynomials, like exponential functions, do not change form after differentiation: the derivative of a polynomial is just another polynomial of one degree less (until it eventually reaches zero). We expect that Y(t) will, therefore, be a polynomial of the same degree as that of g(t). (Why will their degrees be the same?)

So, we will let Y be a generic quadratic polynomial: $Y = At^2 + Bt + C$. It follows $Y' = 2A\,t + B$, and $Y'' = 2A$.

Substitute them into the equation:

$$(2A) - 2(2A\,t + B) - 3(A\,t^2 + B\,t + C) = 3t^2 + 4t - 5$$

$$-3A\,t^2 + (-4A - 3B)\,t + (2A - 2B - 3C) = 3t^2 + 4t - 5$$

The corresponding terms on both sides should have the same coefficients, therefore, equating the coefficients of like terms.

t^2 :	$3 = -3A$		$A = -1$
t :	$4 = -4A - 3B$	$\rightarrow$	$B = 0$
1 :	$-5 = 2A - 2B - 3C$		$C = 1$

Therefore, $Y = -t^2 + 1$, and $y = y_c + Y = C_1 e^{-t} + C_2 e^{3t} - t^2 + 1$.

Normal Equation to Differential Equation

Ex. 1. Find the differential equation of all circles which pass through the origin and whose centers are on the x-axis.

Sol.

We know that the equation of any circle passing through the origin and whose center is on the x-axis is given by,

$$x^2 + y^2 + 2gx = 0, \text{ g being an arbitrary constant. ... (1)}$$

Differentiating (1) w.r.t 'x', we get,

$$2x + 2y(dy/dx) + 2g = 0\text{... (2)}$$

From (1),

$$2gx = -(x^2 + y^2)$$

so that,

$$2g = -(x^2 + y^2)/x\text{... (3)}$$

Substituting for 2g from (3) in (2), we have,

$$2xy\frac{dy}{dx} + x^2 - y^2 = 0$$

Ex. 2. Find the differential equation which has $y = a \cos (mx + b)$ for its integral, a and b being arbitrary constants and m being a fixed constant.

Sol.

Given that,

$y = a \cos(mx + b)$. ... (1)

Differentiating (1) w.r.t 'x', we get,

$dy/dx = -am \sin(mx + b)$. ... (2)

Differentiating (2) w.r.t 'x', we get,

$d^2y/dx^2 = -am^2 \cos(mx + b)$ ... (3)

Or $d^2y/dx^2 = -m^2y$, using (1)

Thus, the required differential equation is,

$$d^2y/dx^2 + m^2y = 0$$

Ex. 3. Find the differential equation from the relation $y = a \sin x + b \cos x + x \sin x$, where a and b are arbitrary constants.

Sol.

Given,

$y = a \sin x + b \cos x + x \sin x$. ... (1)

Differentiating (1) w.r.t 'x',

$dy/dx = a \cos x - b \sin x + \sin x + x \cos x$. ... (2)

Differentiating (2) w.r.t. 'x',

$d^2y/dx^2 = -a\sin x - b\cos x + 2\cos x - x\sin x$

Or $d^2y/dx^2 = 2\cos x - (a\sin x + b\cos x + x\sin x) = 2\cos x - y$, by (1).

$(d^2y/dx^2) + y = 2\cos x$, which is the required differential equation.

Ex. 4. Find the differential equation of the family of curves y ! ex (A cos x " B sin x), where A and B are arbitrary constants.

Sol. Given that,

$y = e^x(A\cos x + B\sin x)$ ... (1)

Differentiating (1),

$$y' = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$$

or $$y' = e^x(-A\sin x + B\cos x) + y \text{ using (1).} \qquad \text{. ... (2)}$$

Differentiating (2) again with respect to x, we get,

$$y'' = -e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) + y' \text{... (3)}$$

Now from (2),

we get,

$$e^x(-A\sin x + B\sin x) = y' - y \text{... (4)}$$

Hence, eliminating A and B from (1), (3) and (4), we get,

$$y'' = -y + y' - y + y' \quad \text{or} \quad y'' - 2y' + 2y = 0$$

Ex. 5. By eliminating the constants, a and b obtain the differential equation for which $xy = ae^x + be^{-x} + x^2$ is a solution.

Sol.

Given that,

$$xy = ae^x + be^{-x} + x^2 \text{... (1)}$$

Diff. (1) w.r.t 'x', we get,

$$xy' + y = ae^x - be^{-x} + 2x \text{... (2)}$$

Diff. (2) w.r.t 'x', we get,

$$xy'' + y' + y' = ae^x + be^{-x} + 2$$

Or $$xy'' + 2y' = (xy - x^2) + 2, \text{ using (1)}$$

Or $$xy'' + 2y' - xy + x^2 - 2 = 0$$

Ex. 6. Find the differential equation corresponding to the family of curves $y = c(x - c)^2$ where c is an arbitrary constant.
Sol. Given that,

$$y = c(x - c)^2 \ldots (1)$$

Diff. (1) w.r.t. 'x', we get,

$$y' = 2c(x - c) \ldots (2)$$

From (1) and (2),

$$y'/y = 2/(x - c) \text{ so that } c = x - (2y/y') \ldots (3)$$

Putting this value of c in (2),

The required equation is $y' = 2\{x - (2y/y')\} \times (2y/y')$ or $(y')^3 = 4y(xy' - 2y)$

Ex. 7. Find the differential equation of all circles of radius a.

Sol. The equation of all circles of radius a is given by,

$$(x - h)^2 + (y - k)^2 = a^2 \ldots (1)$$

where h and k, are to be taken as arbitrary constants.

Diff. (1) w.r.t. 'x', we get,

$$(x - h) + (y - k)y' = 0 \ldots (2)$$

Diff. (2),

$$1 + (y')^2 + (y - k)y'' = 0 \text{ or } y - k = -\{1 + (y')^2\}/y'' \ldots (3)$$

Putting this value of y – k in (2), we get,

$$x - h = -(y - k)y' = \{1 + (y')^2\} \times (y'/y'') \ldots (4)$$

Using (3) and (4), (1) gives the required equation as

$$\frac{\{1+(y')^2\}^2 (y')^2}{(y'')^2} + \frac{\{1+(y')^2\}^2}{(y'')^2} = a^2$$

$$\text{or } \{1 + (y')^2\}^3 = a^2 (y'')^2$$

Exact Differential Equation

A differential equation of type $P(x, y)dx + Q(x, y)dy = 0$ is called an exact differential equation if there exists a function of two variables $u(x, y)$ with continuous partial derivatives such that $du(x, y) = P(x, y)dx + Q(x, y)dy$. The general solution of an exact equation is given by $u(x, y) = C$, (where C is an arbitrary constant.)

This is a type of differential equation that can be solved directly without the use of any of the special techniques in the subject. A first-order differential equation is called exact, or an exact differential, if it is the result of a simple differentiation.

Examples:

$(6x^2 - y + 3)dx + (3y^2 - x - 2)dy = 0.$

We check this equation for exactness:

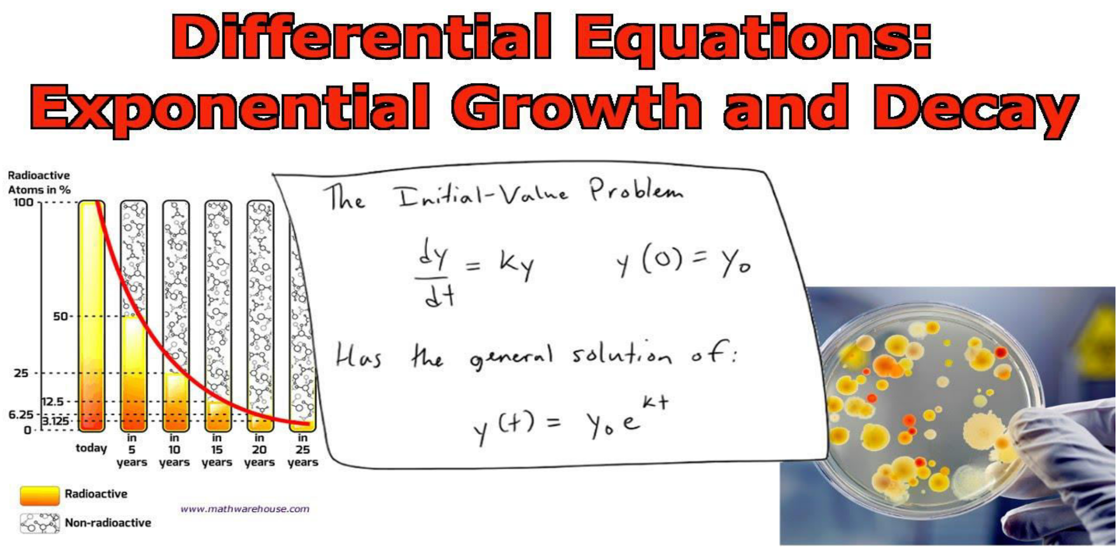
$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(3y^2 - x - 2) = -1, \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(6x^2 - y + 3) = -1.$$

Hence, the given differential equation is exact.

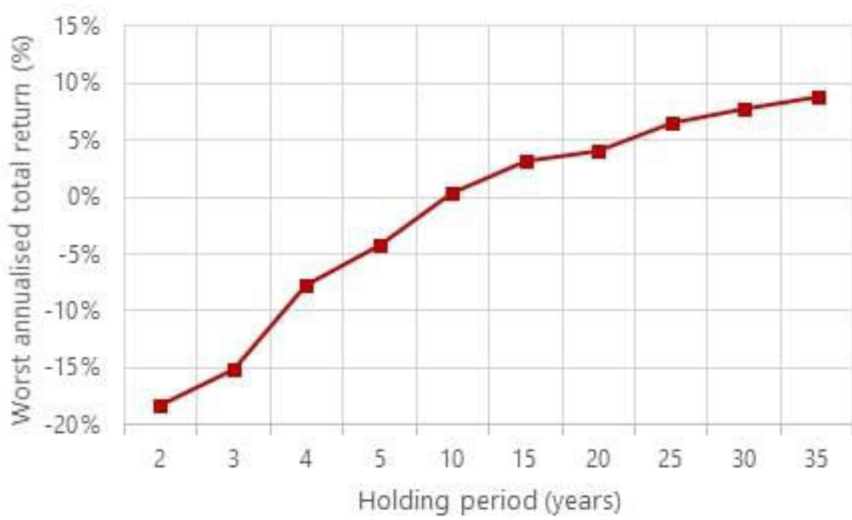
Applications

Let us see some differential equation applications in real-time.

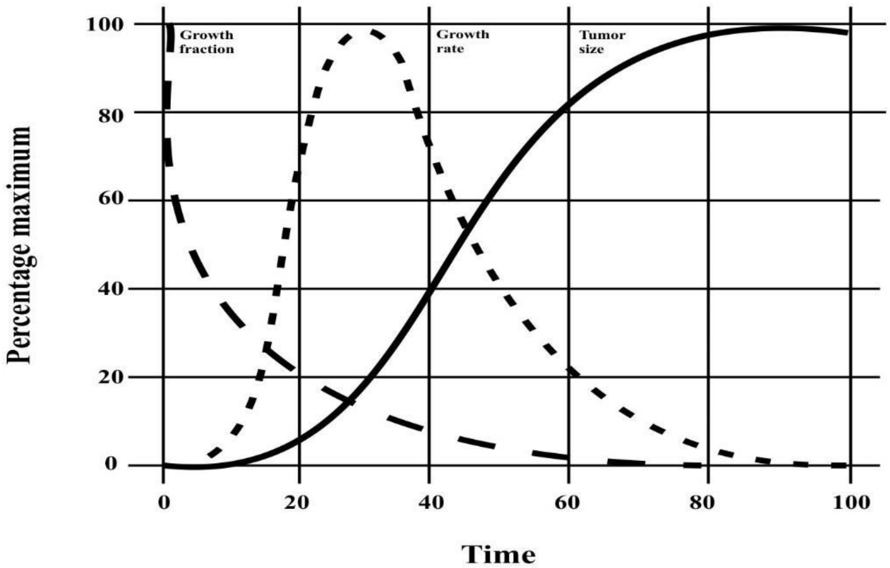
1) Differential equations describe various exponential growths and decays.



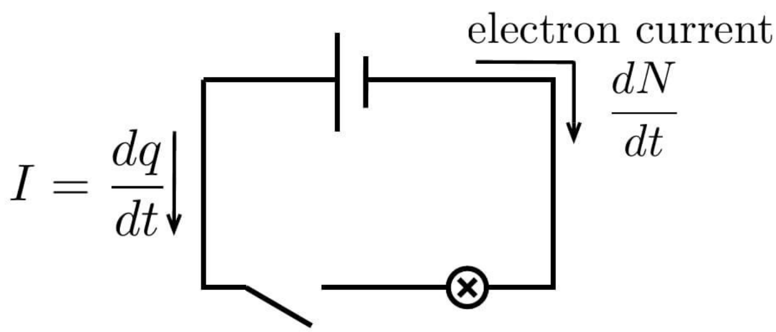
2) They are also used to describe the change in return on investment over time.



3) They are used in the field of medical science for modelling cancer growth or the spread of disease in the body.



4) Movement of electricity can also be described with the help of it.



Drift Velocity

$E = 0$ Random Motion $\qquad$ $E > 0$ Steady Drift

5) They help economists in finding optimum investment strategies.

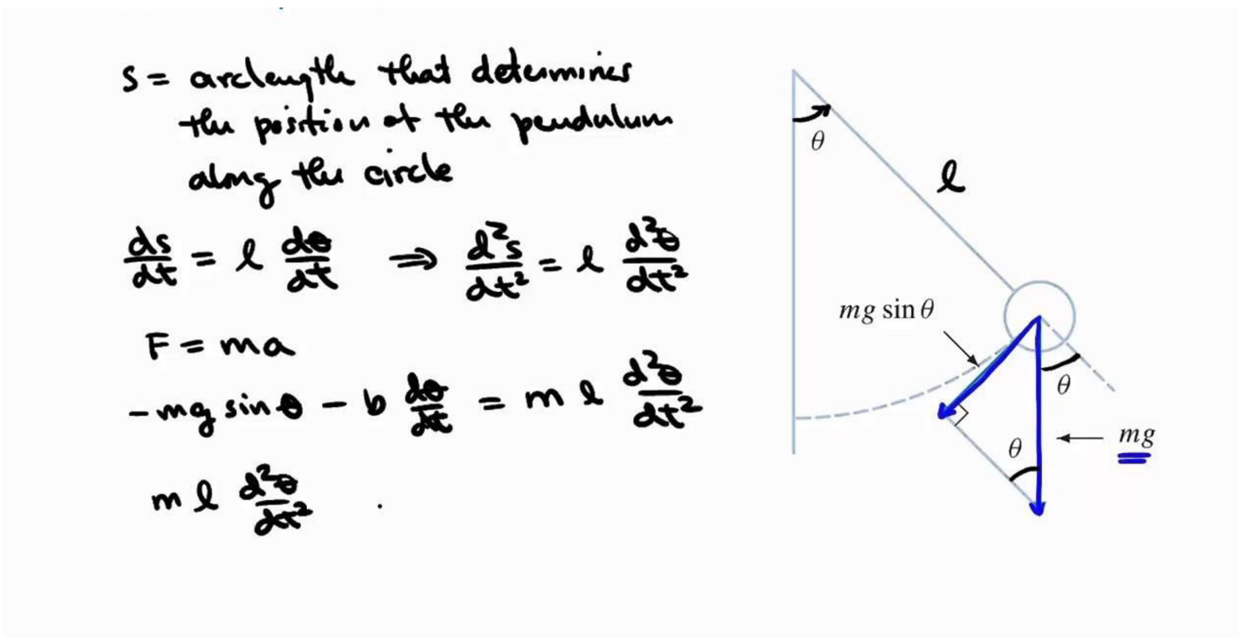
The financial market we consider consists of two tradeable instruments. The price of a risk-free asset $S_0 := (S_0(t), 0 \leq t \leq T)$ is described by the ordinary differential equation

$$\frac{dS_0(t)}{S_0(t)} = rdt, \quad S_0(0) = 1, \tag{2.1}$$

and the dynamics of the risky asset's price $S := (S(t), 0 \leq t \leq T)$ is given by the stochastic differential equation

$$\frac{dS(t)}{S(t-)} = \mu dt + dL(t), \quad S(0) = s_0 > 0, \tag{2.2}$$

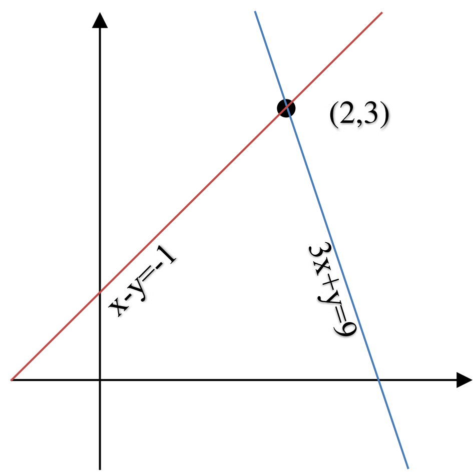
6) The motion of waves or a pendulum can also be described using these equations.



The various other applications in engineering are: heat conduction analysis, in physics it can be used to understand the motion of waves. The ordinary differential equation can be utilized as an application in the engineering field for finding the relationship between various parts of the bridge.

Formation of Differential Equations

For any given differential equation, the solution is of the form $f(x,y,c_1,c_2, \ldots\ldots,c_n) = 0$ where x and y are the variables and c_1 , c_2 ……. c_n are the arbitrary constants. To learn the formation of differential equations in a detailed way, suitable differential equations examples below with few important steps. Differential equation formulas are important and help in solving the problems easily.



To obtain the differential equation from this equation we follow the following steps: -

Step 1: Differentiate the given function w.r.t to the independent variable present in the equation.

Step 2: Keep differentiating times in such a way that (n+1) equations are obtained.

Step 3: Using the (n+1) equations obtained, eliminate the constants (c_1 , c_2 … …. c_n).

Formation of Differential Equation Example

Formation of differential equation (Straight line)

The general equation of a straight line is y = mx + c, where m is the gradient, and y = c is the value where the line cuts the y-axis.

- The differential equation of the family of straight lines
  $y = mx + c$; when
  - (i) m is the arbitrary constant
  - (ii) c is the arbitrary constant
  - (iii) m and c both are arbitrary constants.

(i) When m is an arbitrary constant

$$y = mx + c \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating w.r.t. to x, we get $\frac{dy}{dx} = m$ ……..(2)

Substituting (2) in (1) we get,

$$y = x\frac{dy}{dx} + c$$

So, $x\frac{dy}{dx} - y + c = 0$ is the required differential equation of first order.

(ii) When c is an arbitrary constant

Differentiating equation (1) w.r. to x, we get,

$$\frac{dy}{dx} = m$$

Here c is eliminated from the given equation.

So, $\frac{dy}{dx} = m$ is the required differential equation.

(iii) When both m and c are arbitrary constants

Since m and c are two arbitrary constants differentiating equation (1) twice, we get,

$$\frac{dy}{dx} = m$$

$$\frac{d^2y}{d^2x} = 0$$

Hence m and c are eliminated from the equation

So, $\frac{d^2y}{d^2x} = 0$ is the required differential equation.

Formation of differential equation (Curve)

■The order of the differential equation to be formed is equal to the number of arbitrary constants present in the equation of the family of curves.

■Example: Find the differential equation of the family of curves $y = e^x(a\cos x + b\sin x)$ where a and b are arbitrary constants.

Solution:

Here,

$$y = e^x(a\cos x + b\sin x) \qquad (1)$$

Differentiating (1) w.r.t x, we get,

$$\frac{dy}{dx} = e^x(a\cos x + b\sin x) + e^x(-a\sin x + b\cos x)$$

$$= y + e^x(-a\sin x + b\cos x) \qquad \text{(from (1))}$$

Or, $\frac{dy}{dx} - y = e^x(-a\sin x + b\cos x)$ (2)

Again differentiating, we get,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-a\sin x + b\cos x) + e^x(-a\cos x - b\sin x)$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x(-a\sin x + b\cos x) - e^x(a\cos x + b\sin x)$$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = \left(\frac{dy}{dx} - y\right) - y \qquad \text{(from (1) and (2))}$$

Or, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$, which is the required differential equation.

End of Assignment

Heaven's Light is Our Guide

Rajshahi University of Engineering & Technology

DEPARTMENT OF CIVIL ENGINEERING

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COURSE NO.: MATH 2101

SUBMITTED TO

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ASSIGNMENT

ON

"SOLUTION OF SOME PRACTICAL PROBLEM OF FIRST ORDER FIRST DEGREE DIFFERENTIAL EQUATION"

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Population Growth and Rate of Growth Problems

SAYEM KHAN

1800171

Population Growth and Rate of Growth Problems:

In this age, we are aware of how infection by microorganism such as Escherichia coli (E. coli) causes diseases. Many organisms (such as E. coli) produce a toxin that can cause sickness or even death. Some bacteria can reproduce in our bodies at a surprising fast rate, overwhelming our bodies' natural defenses with the sheer volume of toxin they are producing.

The rate at which the bacterial cultures grow is directly proportional to the current population (until such time as resources become scarce or overcrowding becomes a limiting factor). If we let $P(t)$ to be the population at time t of any compartment say counting bacteria, etc. Where $P(t)$ is not continuous we need to determine the growth (input) rate and death (output) rate for the population.

Let us consider a population of bacteria that produce by simple cell division we assume that the growth rate is proportional to the population present this assumption is consistent with observation of bacteria growth provided there is food and space. We assume death rate is zero.

Therefore.

$$\frac{dp}{dt} = kp$$

$$P(0) = p_0$$

$$k > 0$$

(Growth rate p_0 is at time $t = 0$)

The solution of the equation $\frac{dp}{dt} = kp$ using variable separable we have

$$\frac{dp}{dt} = kp$$

$$dp = kpdt$$

$$\frac{dp}{p} = kdt$$

Integrate both sides of the equation

$$\int \frac{dp}{dt} = \int k\, dt$$

$$In\, P = kt + c$$

$$P(t) = e^{k+c}$$

$$P(t) = e^{kt}e^{c}$$

$$P(t) = p_0 e^{kt}$$

Where,

$$p_0 = e^{c}$$

$$P(t) = p_0 e^{kt} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

Where $P(t)$ is the population at any given time t, p_0 is the initial population, t is the time and k is a constant. Equation (1) is the Malthusian or exponential law of population growth.

Thomas Malthus (1766-1834) published a book on the principle of population as it affects the future improvements of the society in his book; Malthus put forth an exponential growth model for human population and concluded that eventually the population would exceed the capacity to grow an adequate food supply. We refer equation (1) as the general solution of the differential equation

For $k > 0$, equation (1) is called an exponential growth law, and for $k < 0$, it is an exponential decay law.

A problem involving either decay or growth of a particular population requires the use of Malthusian population model (exponential growth or exponential decay rate).

Problem 1:

A freshly incubated bacterial culture of streptococcus (a bacterium found in the throat and on the skin) contains 100 cells. When the culture is checked 60 minutes later, it is determined that there are 450 cells present. Assuming exponential growth:

(a)Determine the number of cells presents at any time t (measured in minutes)

(b) Find the doubling time.

Solution:

This type of problem requires the use of Malthusian population model (exponential growth rate) as stated from the problem.

(a)

Using equation (1) for exponential growth rate we have,

$$P(t) = p_0 e^{kt}$$

Where $P(t)$ is the population at any given time t, p_0 is the initial population, t is the time and k is a constant.

From the given data we have, $P(0) \ = \ 100,$ (initial condition)

Setting $P(t) \ = \ 100,$ and $t \ = \ 0,$ we have

$$100 = P_0$$

$$\Rightarrow P_0 = 100$$

Now the above equation becomes

$$P(t) = 100\ell^{kt}; \text{where (k = constant)}$$

We use the second observation In order to determine the value of the growth constant k;

$$P(60) \ = \ 450$$

$$\Rightarrow t \ = \ 60 \text{ minutes}$$

Therefore, we have,

$$P(t) = P_0 \ell^{kt}$$

$$\Rightarrow 450 = 100\ell^{k(60)}$$

$$\Rightarrow \frac{450}{100} = 1^{60k}$$

$$\Rightarrow \frac{9}{2} = 1^{60k}$$

Take ln of both sides,

$$\ln\left(\frac{9}{2}\right) = 60k$$

$$\Rightarrow k = \frac{1}{60}\ln\left(\frac{9}{2}\right)$$

$$\therefore k = 0.0251$$

We now have a formula representing the number of cells present at any time t as

$$P(t) = 100\ell^{(0.0251)t}$$

(b)

The doubling time is

$$2P(t) = 2(100)$$

$$2P(t) = 20$$

So we have

$$P(t) = P_o \ell^{kt}$$

But,

$$P_0 = 100,$$

And

$$k = 0.0251$$

From the (a) above

$$P(t) = 100\ell^{(0.0251)t}$$

Put

$$P(t) = 200$$

$$\Rightarrow 200 = 100\ell^{(0.0251)t}$$

$$\Rightarrow 2 = \ell^{(0.0251)t}$$

Take ln of both sides:

$$t = \frac{\ln(2)}{0.0251}$$

$$\Rightarrow \text{t} = 27.6$$

$$\Rightarrow \text{t} = 28 \text{ minutes}$$

The population will be double when the time reaches 28 minutes.

From

$$P(t) = 100\ell^{(0.0251)t}$$

$$P(28) = 100\ell^{(0.0251)(28)}$$

$$P(28) = 201$$

The bacterial culture will be double when the time is 28 minutes.

(Ans.)

Problem 2:

A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of the bacteria are observed in the culture and after four hours 3000 strands.

(a) Find an expression for the approximate number of strands of the bacteria presents in the culture at any time t. and

(b) Find the approximate number of strands of the bacteria originally in the culture.

Solution:

(a)

Let $P(t)$ denote the number of bacteria strands in the culture at time t.

Then using exponential growth rate that is

$$\frac{dp}{dt} = kp$$

Which is both linear and separable, then we can use the method of separation of variables to find the solution

$$P(t) = P_0 \ell^{kt}$$

At

$$t = 1 \text{ hour}$$

$$P = 1000$$

$$1000 = P_0 \ell^{k(1)}$$

$$1000 = P_0 \ell^{k} \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

At

$$t = 4 \text{ hours},$$

$$P = 1000$$

$$1000 = P_0 \ell^{k(4)}$$

$$1000 = P_0 \ell^{4k} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)$$

We solve equation (1) and (2) simultaneously to get

$$K = \frac{1}{3} \ln(3)$$

$$K = 0.3662$$

Then we use equation (1) to find P_0

$$1000 = P_0 \ell^{kt}$$

But $K = 0.3662$

$$1000 = P_0 \ell^{0.3662}$$

$$(1000)\ell^{-0.3662} = P_0$$

$$P_0 = 693$$

Therefore the equation $P(t) = P_0 \ell^{kt}$ now becomes:

$$P(t) = 693 \ell^{(0.3662)t} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (3)$$

The above equation is the expression for the approximate number of strands of the bacteria presents at any time t.

(Ans.)

(b)

To find the approximate number of strands of the bacteria originally in the culture.

We require $P(t)$ at $t = 0$.

We substitute $P(0)$ into the equation (3) we have:

$$P(t) = P_0 \ell^{kt}$$

$$P(t) = 693 \ell^{(0.3662)(0)}$$

$$P(0) = 693P(t) = 693\ell^{0}$$

$$P(0) = 693$$

Therefore, the approximate number of strands of the bacteria originally in the culture is 693.

(Ans.)

Problem 3:

If it is assumed that the earth cannot support a population greater than twenty billion persons and that the rate of population growth is proportional to the difference between how closed the world population is to the limiting value, what is the mathematical expression describing the world population as a function of time?

Solution:

If $P(t)$ is the world population according to the described model

$$\frac{dp(t)}{dt} = -k(p(t) - 20)$$

where k is negative to make $\frac{dp(t)}{dt}$ positive since P we must be less than 20 billion. Then solving the equation using variable separable we have

$$\frac{dp(t)}{p(t)-20} = kdt$$

Integrate both sides of the equation

$$\int \frac{dp(t)}{p(t)-20} = \int kdt$$

$$\ln(p(t) - 20) = kt + C$$

$$|p(t) - 20| = P_0 e^{kt}$$

$$|P(t) - 20| = P_0 \ell^{kt}$$

Where $P_0 = \ell^{c}$

Since P(t) is assumed less than 20 billion, thus

$$-(P(t) - 20) = P_0\ell^{kt}$$

By rearranging in order to make P(t) the subject we have:

$$P(t) = 20 - P_0\ell^{kt}$$

The above expression describes a world population as a function of time.

(Ans.)

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Problem 4:

The current population of California is 36 million with a 33% Hispanic population. The rate of people entering the state, by being born or immigrating, is approximately 1 million people per year with 70% of the newcomers Hispanic. The rate of departure of the population, by dying or emigrating, is approximately 0.2 million people per year. Assume that the departing population has the same ethnic mix as California's population at the time of departure. In 20 years, what percent of California will be Hispanic?

Solution:

We use a useful fact applicable to any question where there is input and output through a system:

$$Total\ Rate\ =\ Rate\ In\ -\ Rate\ Out$$

In this case, if we let x represent the number of Hispanics in the state and t represent the time in years from now, then we have

$$\text{Total Rate}\ = \frac{\text{dx}}{\text{dt}}$$

The rate in will be 70% of 1 million or

$$Rate\ In\ =\ 0.7(1) = 0.7$$

The rate out is more difficult. At time t, there are x Hispanics in California. To find the population of California, we take the initial population plus the added population per year times the number of years. Since there are 1 million people entering and 0.2 million leaving the population at time t is

$$P(t)\ =\ 36\ +\ (1\ -\ 0.2)t\ =\ 36\ +\ 0.8t$$

Hence the rate of Hispanics leaving can be found by multiplying the proportion of Hispanics in the state and the rate of people leaving the state.

$$\text{Rate Out} \;= \frac{x}{36+0.8t} \times .02$$

We use the Rate In and Rate Out information to get,

$$\frac{dx}{dt} = 0.7 - \frac{0.2x}{36+.8t}$$

We can rewrite this as,

$$\frac{dx}{dt} + \frac{0.2}{36+.8t}x = 0.7$$

This is a first order linear differential equation with

$$\mu = e^{\int \frac{.2}{36+.8t}dt}$$

$$= e^{\frac{.2}{.8}\ln(36+.8t)}$$

$$= e^{\ln(36+.8t)^{.25}}$$

$$= (36+.8t)^{.25}$$

The solution is

$$x = (36+.8t)^{-.25}\int (36+.8t)^{.25}(.7)dt$$

$$\Rightarrow x = .7(36+.8t)^{-.25}\frac{1}{.8}\frac{1}{1.25}[(36+.8t)^{1.25} + C]$$

$$\Rightarrow x = .7[(36+.8t)^{.25} + C(36+.8t)^{-.25}]$$

Now we have that at time zero, there are 33% of 36 million Hispanics or 11.88 million Hispanics, hence

$$11.88 = .7[(36 + C(36)^{-.25}]$$

Solving for C gives

$$C = -46.61$$

Now plug in 20 for t to get

$$x = .7\left[(36 + .8(20)) - 46.6(36 + .8(20))^{-.25}\right] = 24.25$$

This is the number of Hispanics that will be living in California. To arrive at the percent we divide it by California's population in twenty years.

$$\mathrm{P}(20) = 36 + (0.8)(20) = 52$$

Finally, we have

$$\frac{24.25}{52} = .47 = 47\%$$

Our model predicts that California will be 47% Hispanic in 20 years.

$$\text{Rate Out} = \frac{\text{Number of Hispanics}}{\text{Number of Californians}} \times \text{Rate of all leaving California}$$

(Ans.)

Problem 5:

The population of a community is known to increase at a rate proportional to the number of people present at a time t. If the population has doubled in 6 years, how long it will take to triple?

Solution:

Let $N(t)$ denote the population at time t. Let $N(0)$ denote the initial population (population at $t = 0$).

$$\frac{dN}{dt} = kN\,(t)$$

Solution is

$$N(t) = Ae^{kt}$$

Where,

$$A = N(0)$$

$$Ae^{k} = N(6) = 2N(0) = 2A$$

$$\Rightarrow e^{6k} = 2$$

$$\Rightarrow k = \frac{1}{6} \ln 2$$

Find t when

$$N(t) = 3A = 3N(0)$$

$$\Rightarrow N(0)\, e^{kt} = 3N(0)$$

$$\Rightarrow N(0)e^{kt} = 3e^{\frac{t}{6} ln\, 2}$$

$$\Rightarrow \ln 3 = \frac{(\ln 2)t}{6}$$

∴ t= (9.6 years (approximately 9 years 6 months)

(Ans.)

Problem 6:

Let population of country be decreasing at the rate proportional to its population. If the population has decreased to 25% in 10 years, how long will it take to be half?

Solution:

This phenomenon can be modeled by

$$\frac{dN}{dt} = kN\,(t)$$

Its solution is

$$N(t) = N(0)e^{kt}$$

Where,

$N(0)$ in the initial population

For,

$$t = 10, \quad N(10) = \frac{1}{4}N(0)$$

$$N(0) \; = \; N(0)e^{10k}$$

$$\Rightarrow e^{10k} = \frac{1}{4}$$

$$\Rightarrow k = \frac{1}{10}\ln\frac{1}{4}$$

Set,

$$N(t) = \frac{1}{4}N(0)$$

$$\Rightarrow N(0)e^{\frac{1}{10}\ln\frac{1}{4}} = \frac{1}{2}N(0)$$

$$\Rightarrow t = \frac{ln\frac{1}{2}}{\frac{1}{10}ln\frac{1}{4}} = 8.3, \text{ years approximately.}$$

(Ans.)

Problem 7:

A certain culture of bacteria grows at rate proportional to its size. If the size doubles in 4 days, find the time required for the culture to increase to 10 times to its original size.

Solution:

Let $p(t)$ be the size of the culture after t days.

$$\frac{dp}{dt} = kp$$

we will use the initial condition $p(0) = p_0$

to find the arbitrary constant c, and we will find the additional constant k by using additional condition

$$p(4) = 2p_0$$

we have,

$$p \ = Ce^{kt}$$

from the initial condition $p(0) = p_0 \, , t = 4$

$$2p_0 = \ p_0 e^{4k}$$

$$e^{4k} = 2$$

$$In \ e^{4k} = ln2$$

$$k = 0.173$$

Thus the time is required for the culture to increase 10 times to its original size can be found from

$$10p_0 = \ p_0 e^{0.173t}$$

$$e^{0.173t} = 10$$

$$t = 13.31 \ days.$$

(Ans.)

Problem 8:

Suppose a population of *e-coli* bacteria grows at a rate proportional to the current population. If an initial population of 200 bacteria has grown to 1600 three hours later, find a function for the size of the population at time *t*, and use it to predict when the population size will reach 10,000.

Solution:

We already know that the population at time *t* is given by $p = Ce^{kt}$ for some *C* and *k*. The information about the initial size of the population means that $p(0) = 200$. Thus $C = 200$. Our knowledge of the population size after three hours allows us to solve for*k* via the equation

$$1600 = 200e^{3k}$$

Solving this exponential equation yields

$$k = \frac{\ln 8}{3} \approx .6931$$

The population at time *t* is given by

$$p = 200e^{\frac{\ln 8}{3}t}$$

Solving

$$1000 = 200e^{\frac{\ln 8}{3}t}$$

Yields

$$t = \frac{3ln\ 50}{ln\ 8} \approx 5.644$$

The population is predicted to reach 10,000 bacteria in slightly more than five and a half hours.

(Ans.)

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Problem 9:

The population x of a certain city satisfies the logistic law,

$$\frac{dx}{dt} = \frac{x}{100} - \frac{x^2}{10^8} \dots\dots\dots\dots\dots\dots\dots (1)$$

when time t is measured in years. Given, that the population of this city is 100000 in 1980.determine the population as a function of time for t>1980. In particular, answer the following questions:

(a)What will be the population in 2000?

(b)In what year does the 1980 population double?

(c)Assuming the differential equation applies for all $t > 1980$, how large will the population ultimately be?

Solution:

We must solve the separable differential equation (1) subject to the initial solution,

$$x\,(980) = \ 100000 \dots\dots\dots\dots\dots\dots (2)$$

Separating variable in (1) we obtain

$$\frac{dx}{(10)^{-2}x - (10^{-8})x^2} = dt$$

And hence

$$\frac{dy}{(10)^{-2}x[1-(10)^{-6}x]} = dt$$

Using partial fraction, this becomes

$$100\left[\frac{1}{x} + \frac{10^{-6}}{1-(10)^{-6}x}\right]dx = dt$$

Integrating, assuming $0 < x < 10^6$,we obtain

$$100\,\{\,In\,x - In\,[1 - \ 10^{-6}x]\} = t + \ C_1$$

Hence

$$In\left[\frac{x}{1-10^{-6}x}\right]=\frac{1}{100}+C_2$$

Thus, we find

$$\frac{x}{1-10^{-6}x}=ce^{\frac{t}{100}}$$

Solving this for x, we finally obtain,

$$x=\frac{ce^{t/100}}{1+10^{-6}ce^{t/100}}\ldots\ldots\ldots\ldots\ldots(3)$$

now applying the initial condition to equation (ii) we get

$$10^5=\frac{Ce^{19.8}}{1+(10)^{-6}Ce^{19.8}}$$

From which we obtain

$$C=\frac{10^5}{e^{19.8}[1-10^5(10)^{-6}]}=\frac{10^6}{9e^{19.8}}$$

Substituting this value for c back into (3) and simplifying we get

$$x=\frac{10^6}{1+9e^{\left[19.8-\frac{t}{100}\right]}}\ldots\ldots\ldots\ldots\ldots(4)$$

This gives the population x as a function of time for $t>1980$.

Now,

(a)

Asks for population in the year 2000.Thus we let $t=2000$ in (4) and get

$$x = \frac{10^6}{1+9e^{-.02}} \approx 119495$$

(Ans.)

(b)

Asks for the year in which the population doubles. Thus we let, $x = 200000 = 2.10^5$ in equation (iv) and solving t we get,

$$2.10^5 = \frac{10^6}{1+9e^{19.8-t/100}}$$

From which

$$e^{\frac{19.8-t}{100}} = \frac{4}{9}$$

and hence

$$t \approx 2061$$

(Ans.)

(c)

Asks how large the population will ultimately be assuming the differential equation (i) applies for all t>1980.To answer this, we evaluate $lim\ x\ t \to \infty$ using the solution (iv) of (i) we find

$$\lim_{t\to\infty} x = \lim_{t\to\infty} \frac{10^6}{1+9e^{19.8-t/100}} = (10)^6 = 1{,}000{,}000$$

(Ans.)

Problem 10:

A population grows at the rate of 5% per year. How long does it take for the population to double? Use differential equation.

Solution:

Here, $\frac{dx}{dt} \propto x$ [since increase in population speeds up with increase in population] and let x be the population at any time t

Then

$$\frac{dx}{dt} = rx$$

where r is proportionality constant.

Rewriting,

$$\frac{\mathrm{dx}}{x} = r\,dt$$

Integrating,

$$lnx = rt + C$$

where C is the integration constant

Exponentiating on both sides with e,

$$\Rightarrow x = e^{\mathrm{rt+c}}$$

$$\Rightarrow x = ke^{\mathrm{rt}} \text{ where } k = e^{\mathrm{c}}$$

Here r is the rate of increase, and k is the initial population let x_0

Given to find the time t taken to attain double population,

So,

$$x = 2x_0$$

So,

$$x = ke^{\mathrm{rt}}$$

$$\Rightarrow 2x_0 = x_0 e^{0.05t}$$

Then divided by x_0,

we have

$$2 = e^{0.05t}$$

Taking log on both sides,

$$ln2 = ln(e^{0.05t})$$

$$\Rightarrow 0.69314 = 0.05t$$

$$\Rightarrow t = \frac{0.69317}{0.05}$$

$$\Rightarrow t = 13.86294\ years$$

So, it takes 13.86294 years to attain double population.

(Ans.)

Problem 11:

Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches 10,000. When will the owner's friends be allowed to fish?

Solution:

We know it takes the population of fish 6 months to double in size. So, if t represents time in months, by the doubling-time formula, we have 6=(ln2)/k. Then, $k = \frac{In\ 2}{6}$. Thus, the population is given by $y = 500e^{\frac{In\ 2}{6}t}$ .To figure out when the population reaches 10,000 fish, we must solve the following equation:

$$10{,}000 = 500e^{\frac{In\ 2}{6}t}$$

$$20 = e^{\frac{In\ 2}{6}t}$$

$$ln(20) = \frac{In\ 2}{6}t$$

$$t = \frac{6(ln20)}{ln2} \approx 25.93$$

The owner's friends have to wait 25.93 months (a little more than 2 years) to fish in the pond.

(Ans.)

Problem 12:

A herd of llamas has 1000 llamas in it, and the population is growing exponentially. At time t=4 it has 2000 llamas. Write a formula for the number of llamas at arbitrary time t.

Solution:

Here there is no direct mention of differential equations, but use of the buzz-phrase ‘growing exponentially’ must be taken as indicator that we are talking about the situation

$$f(t) = Ce^{kt}$$

where here f(t) is the number of llamas at time t and c, k are constants to be determined from the information given in the problem. And the use of language should probably be taken to mean that at time t=0 there are 1000 llamas, and at time t=4 there are 2000. Then, either repeating the method above or plugging into the formula derived by the method, we find

C= value of f at $t = 0 \; = 1000$

$$k = \frac{lnf(t1)-lnf(t2)}{\mathrm{t1-t2}}$$

$$= \frac{ln1000-ln2000}{0-4}$$

$$= \ln\frac{1000}{2000} - 4$$

$$= \frac{ln\frac{1}{2}}{-4}$$

$$= \frac{ln2}{4}$$

$$\therefore f(t) = 1000e^{\frac{ln2}{4}t}$$

$$\therefore f(t) = 1000 \times 2^{\frac{t}{4}}.$$

This is the desired formula for the number of llamas at arbitrary time t.

(Ans.)

Problem 13:

Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the

growth rate is proportional to the population size.) What is the relative growth rate k? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

Solution:

We measure the population p(t) in millions of people. We have

$$\frac{dp}{dt} = kp$$

$$p = Ce^{kt}$$

we have the initial condition

$$p(t_0) = p_0$$

$$p(0) = ce^0$$

$$C = 2560$$

Now,

$$p(10) = 3040$$

$$p = ce^{kt}$$

$$\Rightarrow 3040 = 2560e^{10k}$$

$$\mathrm{k} = \frac{\ln 1.1875}{10} = 0.01785$$

The relative growth rate is about 1.7% per year and the model is

$$p(t) = 2560e^{0.01785t}$$

$$p(43) = 2560e^{0.01785(43)} = 5360\ million.$$

$$p(70) = 2560e^{0.01785(70)} = 8524\ million.$$

(Ans.)

Mixing Problems

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Mixing Problems:

In these problems we will start with a substance that is dissolved in a liquid. Liquid will be entering and leaving a holding tank. The liquid entering the tank may or may not contain more of the substance dissolved in it. Liquid leaving the tank will of course contain the substance dissolved in it. If Q(t) gives the amount of the substance dissolved in the liquid in the tank at any time t we want to develop a differential equation that, when solved, will give us an expression for Q(t). Note as well that in many situations we can think of air as a liquid for the purposes of these kinds of discussions and so we don't actually need to have an actual liquid but could instead use air as the "Liquid".

The main assumption that we'll be using here is that the concentration of the substance in the liquid is uniform throughout the tank. Clearly this will not be the case, but if we allow the concentration to vary depending on the location in the tank the problem becomes very difficult and will involve partial differential equations, which is not the focus of this course.

The main equation that we'll be using to model this situation is,

$$Rate\ of\ change\ of\ Q(t) = rate\ at\ which\ Q(t)\ enters\ the\ tank - Rate\ which\ Q(t)\ exits\ the\ tank$$

Rate of change of $Q(t)$:

$$Q(t) = \frac{dQ}{dt} = Q'(t)$$

Rate at which Q(t) enters the tank:

$$(flow\ rate\ of\ liquid\ entering)x(concentration\ of\ substance\ in\ liquid\ entering)$$

Rate at which Q(t) exits the tank:

$$(flow\ rate\ of\ liquid\ exiting)x(concentration\ of\ substance\ in\ liquid\ exiting)$$

Problem 1:

A 1500-gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 9 gal/hr and the water entering the tank has a salt concentration of $\frac{1}{5}(1 + cos(t))$ lbs/gal. If a well-mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?

Solution:

Now, to set up the IVP that we'll need to solve to get Q(t) we'll need the flow rate of the water entering (we've got that), the concentration of the salt in the water entering (we've got that), the flow rate of the water leaving (we've got that) and the concentration of the salt in the water exiting (we don't have this yet).

So, we first need to determine the concentration of the salt in the water exiting the tank. Since we are assuming a uniform concentration of salt in the tank the concentration at any point in the tank and hence in the water exiting is given by,

$$Concentration = \frac{Amount\ of\ salt\ in\ the\ tank\ at\ any\ time,t}{Volume\ of\ water\ in\ the\ tank\ at\ any\ time,t}$$

The amount at any time t is easy it's just Q(t). The volume is also pretty easy. We start with 600 gallons and every hour 9 gallons enters and 6 gallons leave. So, if we use t in hours, every hour3gallons enters the tank, or at any time t there is 600 + 3t gallons of water in the tank.

So, the IVP for this situation is,

$$Q'(t) = \ 9\ (\tfrac{1}{5}\{1 + cos(t))\} - \ 6\ (\tfrac{Q(t)}{600+3t}) \qquad [Q(0) = 5]$$

$$Q'(t) = 9\left(\frac{1}{5}\{1 + cos(t)\}\right) - \frac{2Q(t)}{200+t} \qquad [Q(0) = 5]$$

This is a linear differential equation. Solving it we get,

$$Q'(t) + \frac{2Q(t)}{200+t} = \frac{9}{5}(1 + cos(t))$$

$$\mu(t) = e^{\int \frac{2}{200+t} dt} = e^{2ln(200+t)} = (200 + t)2$$

$$\int \left((200 + t)^2 Q(t)\right)' dt = \int \frac{9}{5}(200 + t)^2(1 + cos(t))dt$$

$$(200 + t)^2 Q(t) = \frac{9}{5}\left(\frac{1}{3}(200 + t)^3 + (200 + t)^2 sin(t) + 2(200 + t)cos(t) - 2sin(t)\right) + C$$

$$Q(t) = \frac{9}{5}\left(\frac{1}{3}(200 + t) + sin(t) + \frac{2cos\,(t)}{200+t} - \frac{2sin\,(t)}{(200+t)^2}\right) + \frac{C}{(200+t)^2}$$

So, here's the general solution. Now, apply the initial condition to get the value of the constant, c.

$$5 = Q(0) = \frac{9}{5}\left(\frac{1}{3}(200) + 2/200\right) + \frac{C}{(200+t)^2}$$

$$C = -4600720$$

So, the amount of salt in the tank at any time t is.

$$Q(t) = \frac{9}{5}\left(\frac{1}{3}(200 + t) + sin(t) + \frac{2cos(t)}{(200+t)} - 2sin(t)\frac{1}{(200+t)^2}\right) - \frac{4600720}{(200+t)^2}$$

Now, the tank will overflow at t = 300 hrs. The amount of salt in the tank at that time is.

$$Q(300) = 279.797\ lbs.$$

(Ans.)

Problem 2:

A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well-mixed solution leaves the tank at 3 gal/hr as well. When the amount of pollution in the holding tank reaches 500 ounces the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 gal/hr while the outflow is increased to 4 gal/hr. Determine the amount of pollution in the tank at any time t.

Solution:

Here are the two IVP's for this problem.

$$Q1'(t) = (3)(5) - (3)(Q1(t)/800) \quad Q1(0) = 2 \quad 0 \le t \le tm$$

$$Q2'(t) = (2)(0) - (4)\left(\frac{Q2(t)}{[800-2(t-tm)]}\right) \quad Q2(tm) = 500 \quad tm \le t \le te$$

The first one is fairly straight forward and will be valid until the maximum amount of pollution is reached. We'll call that time tm. Also, the volume in the tank remains constant the simplified version of the IVP's that we'll be solving.

$$Q1'(t) = 15 - \frac{3Q1(t)}{800} \quad Q1(0) = 2 \quad 0 \le t \le tm$$

$$Q2'(t) = -\frac{2Q2(t)}{400-(t-tm)} \quad Q2(tm) = 500 \quad tm \le t \le te$$

The first IVP is a fairly simple linear differential equation so we'll leave the details of the solution to you to check. Upon solving you get.

$$Q1(t) = 4000 - 3998e^{\frac{-3t}{800}}$$

Now, we need to find tm. This isn't too bad all we need to do is determine when the amount of pollution reaches 500. So, we need to solve.

$$Q1(t) = 4000 - 3998e^{\frac{-3t}{800}} = 500 \Rightarrow tm = 35.4750$$

So, the second process will pick up at 35.475 hours. For completeness sake here is the IVP with this information inserted.

$Q2'(t) = -\frac{2Q2(t)}{435.475-t} \qquad Q2(35.475) = 500 \qquad 35.475 \le t \le 435.475$

This differential equation is both linear and separable and again isn't terribly difficult to solve so I'll leave the details to you again to check that we should get.

$Q2(t) = \frac{(435.476-t)^2}{320}$

So, a solution that encompasses the complete running time of the process is

$$Q(t) = \begin{cases} 4000 - 3998e^{\frac{-3t}{800}}, & 0 \le t \le 35.475 \\ \frac{(435.476-t)^2}{320}, & 35.475 \le t \le 435.4758 \end{cases}$$

(Ans.)

Problem 3:

Uranium disintegrates at a rate proportional to the amount present at any instant. If m_1 and m_2 grams of uranium are present at time t_1 and t_2 respectively, show that half-life of uranium is

$$\frac{(t_1 - t_2)\log 2}{\log\frac{m_1}{m_2}}$$

Solution:

Let m be the amount of uranium at any time t.

$$\frac{dm}{dt} = -km$$

$$\therefore \int_{m_1}^{m_2} \frac{dm}{m} = -k \int_{t_1}^{t_2} dt = \log \frac{m_1}{m_2} = k\,(t_2 - t_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let the mass m reduce to $\frac{m}{2}$ in time t.

Also,

$$\int_{m}^{\frac{m}{2}} \frac{dm}{m} = -k \int_{0}^{t_2} dt$$

$$\therefore \log \frac{m}{2} - \log m = -kt$$

$$\Rightarrow kt = \log 2$$

$$\Rightarrow k = \frac{\log 2}{t}$$

Substituting the value of k in (1), we get,

$$\log \frac{m_1}{m_2}\ \frac{\log 2}{t} (t_2 - t_1)$$

$$\Rightarrow t = \frac{(t_2 - t_1) \log 2}{\log \frac{m_1}{m_2}}.$$

(Proved)

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Problem 4:

A 1000 gallon holding tank that catches runoff from some chemical process initially has 800 gallons of water with 2 ounces of pollution dissolved in it. Polluted water flows into the tank at a rate of 3 gal/hr and contains 5 ounces/gal of pollution in it. A well-mixed solution leaves the tank at 3 gal/hr as well. When the amount of pollution in the holding tank reaches 500 ounces the inflow of polluted water is cut off and fresh water will enter the tank at a decreased rate of 2 gal/hr while the outflow is increased to 4 gal/hr. Determine the amount of pollution in the tank at any time t.

Solution:

Here are the two IVP's for this problem.

$$Q1'(t) = (3)(5) - (3)(Q1(t)/800) \quad Q1(0) = 2 \quad 0 \leq t \leq tm$$

$$Q2'(t) = (2)(0) - (4)\left(\frac{Q2(t)}{[800 - 2(t - tm)]}\right) \quad Q2(tm) = 500 \quad tm \leq t \leq te$$

The first one is fairly straight forward and will be valid until the maximum amount of pollution is reached. We'll call that time tm. Also, the volume in the tank remains constant the simplified version of the IVP's that we'll be solving.

$$Q1'(t) = 15 - \frac{3Q1(t)}{800} \quad Q1(0) = 2 \quad 0 \leq t \leq tm$$

$$Q2'(t) = -\frac{2Q2(t)}{400 - (t - tm)} \quad Q2(tm) = 500 \quad tm \leq t \leq te$$

The first IVP is a fairly simple linear differential equation so we'll leave the details of the solution to you to check. Upon solving you get.

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Now, we need to find tm. This isn't too bad all we need to do is determine when the amount of pollution reaches 500. So, we need to solve.

$$Q1(t) = 4000 - 3998e^{\frac{-3t}{800}} = 500 \Rightarrow tm = 35.4750$$

So, the second process will pick up at 35.475 hours. For completeness sake here is the IVP with this information inserted.

$$Q2'(t) = -\frac{2Q2(t)}{435.475 - t} \quad Q2(35.475) = 500 \quad 35.475 \leq t \leq 435.475$$

This differential equation is both linear and separable and again isn't terribly difficult to solve so I'll leave the details to you again to check that we should get.

$$Q2(t) = \frac{(435.476-t)^2}{320}$$

So, a solution that encompasses the complete running time of the process is

$$Q(t)=\begin{cases} 4000 - 3998e^{\frac{-3t}{800}}, & 0 \le t \le 35.475 \\ \frac{(435.476-t)^2}{320}, & 35.475 \le t \le 435.4758 \end{cases}$$

(Ans.)

Problem 5:

A tank has pure water flowing into it at 10 litre/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Initially, the tank contains 10 kg of salt in 100 litre of water. How much salt will there be in the tank after 30 minutes?

Solution:

we consider the rate of change of the amount of salt in the tank. Let S be the amount of salt in the tank at any time t. If we can create an equation relating dS dt to S and t, then we will have a differential equation which we can, ideally, solve to determine the relationship between S and t. To describe $\frac{dS}{dt}$, we use the concept of concentration, the amount of salt per unit of volume of liquid in the tank. In this example, the inflow and outflow rates are the same, so the volume of liquid in the tank stays constant at 100 litre. Hence, we can describe the concentration of salt in the tank by

$$Concentration\ of\ salt\ = \frac{S}{100}\ kg/litre$$

Then, since mixture leaves the tank at the rate of 10 $litre/min$, salt is leaving the tank at the rate of

$$\frac{S}{100}\ (10\ litre/min)\ = \frac{S}{10}$$

This is the rate at which salt leaves the tank, so

$$\frac{dS}{dt} = -\ \frac{S}{10}$$

This is the differential equation we can solve for S as a function of t. Notice that since the derivative is expressed in terms of a single variable, it is the simplest form of separable differential equations, and can be solved as follows,

$$\int \frac{ds}{S}\ = -\int \frac{1}{10}\ dt$$

$$ln\,|S| = -\frac{1}{10}t + C$$

$$S = Ce^{-\frac{1}{10}t}$$

where C is a positive constant. Note that we have used the fact that S ≥ 0 to eliminate the absolute value symbol.

Since $S = 10$ when $t = 0$, we find $C = 10$ and finally we have

$$S = 10e^{-\frac{1}{10}t}$$

We can see from this that as t goes to infinity, the amount of salt in the tank goes to zero.

Also, after 30 minutes, there will be

$$S = 10e^{-3}$$

$= 0.49787068\ kg$ of salt in the tank.

(Ans.)

Problem 6:

A tank has pure water flowing into it at 10 litre /min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 litre/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 litre of water. How much salt is in the tank after 30 minutes?

Solution:

Here the setup is very similar the previous example. The only difference from the previous example is the addition of 0.1 kg/min of salt to the tank.

As a result, we can modify our differential equation to take this into account,

$$\frac{dS}{dt} = -\frac{S}{10} + 0.1 = -0.1S + 0.1$$

Here we see the effect of the outflow as a negative term and the addition of salt as a positive term which we sum to get the net rate of change of salt.

Yet again, this equation is clearly separable, since there is no t variable on the right hand side. We thus solve in the standard way:

$$\int \frac{dS}{-0.1S + 0.1} = \int dt$$

$$-10\, ln\, | - 0.1S + 0.1| = t + C$$

$$-0.1S + 0.1 = Ce^{-0.1t}$$

$$S = 1 + Ce^{-0.1t}$$

Here C may be positive or negative, depending on the initial conditions. We see that as t approaches infinity, S approaches 1 kg regardless of the initial conditions. For this example, there were initially 10 kg of salt in the tank, so we can solve for C and find C=9. Thus,

$$S = 1 + 9e^{-0.1t}$$

After 30 minutes, there will be 1.448 kg of salt in the tank.

Unlike the previous example, the amount of salt in the tank does not go to zero as t goes to infinity: S goes to 1. Notice that if there was 1 kg of salt in the tank, then the outflow rate of salt will be $(0.01\, kg/litre)(10\, litre/min) = 0.1\, kg/min$, which will exactly balance the inflow rate of salt.

(Ans.)

Problem 7:

A tank has pure water flowing into it at 12 litre/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 litre/min. Initially, the tank contains 10 kg of salt in 100 litre of water.

Solution:

In this case, the inflow rate is greater than the outflow rate. As a result, the volume is not constant. Using the initial conditions and the flow rates, we can say that the volume V of liquid in the tank is

$$V = 100 + 2t$$

after t minutes. The concentration of salt after t minutes is then

$$\frac{S}{V} = \frac{S}{100 + 2t}$$

and the rate of change of S is

$$\frac{dS}{dt} = -\frac{S}{100+2t}\left(10\frac{litre}{min}\right)$$

$$= \frac{10S}{100 + 2t}$$

Once again, this is a separable differential equation, and we can solve it:

$$\int \frac{dS}{S} = -\int \frac{10}{100+2t} dt$$

$$ln\, S = -5\, ln(100 + 2t) + C$$

$$S = C(100 + 2t)^{-5}$$

Note we have used the fact that S ≥ 0 and V = 100+2t ≥ 0 to eliminate absolute value symbols from the equation. With the initial conditions, we can solve for C,

$$10 = C(100 + 0)^{-5}$$

$$C = 10^{11}$$

and thus

$$S = \frac{10^{11}}{(100 + 2t)^5}$$

After 30 minutes, there will be 0.953674 kg of salt in the tank. Notice this is more than in example 1 due to the fact that the increased inflow rate dilutes the salt, and reduces the outflow rate of salt, so the amount of salt in the tank will be greater than in Problem 7.

(Ans.)

Problem 8:

Under certain conditions, cane sugar is converted into dextrose at a rate, which is proportional to the amount unconverted at any time. If out of 75 grams of sugar at t = 0, 8 grams are converted during the first 3 minutes, find the amount converted in $1\frac{1}{2}$ hours.

Solution:

Let M be the amount of cane sugar initially, m be the amount of cane sugar converted into dextrose. Then according to problem,

$$\frac{dm}{dt} = K(M - m)$$

$$\Rightarrow \frac{dm}{dt} + Km = KM$$

which is the linear differential equation.

$$\textbf{I.F.} = e^{\int K dt}$$

$$= e^{kt}$$

Solution is

$$m.e^{kt} = \int K\, M e^{kt} dt = M e^{kt} + C$$

$$m = M + C e^{-kt} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

(a) At $t = 0, m = 0, M = 75$

(1) becomes,

$$m = 75 - 75e^{-kt} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$
$$\Rightarrow 0 = 75 + C$$
$$\therefore C = -75$$

(b) At $t = 30, m = 8$

$$8 = 75 - 75e^{-30k}$$

$$\Rightarrow 67 = 75e^{-30k}$$

$$\Rightarrow e^{-30k} = \frac{67}{75} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

(c) At $t = 90$, (2) becomes

$$m = 75 - 75e^{-90k}$$

$$\Rightarrow m = 75 - 75\left(\frac{67}{75}\right)^3 \text{ from (3)}$$

$$\Rightarrow m = 75 - \frac{67^3}{75^2}$$

$$\Rightarrow m = 75 - \frac{300763}{5625}$$

$$\Rightarrow m = 75 - 53.45$$

$$\therefore m = 21.55$$

(Ans.)

Radioactive-Decay Problems

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Radioactive-Decay Problems:

In many circumstances, for a limited period of time, the rate of reaction of a chemical process can be considered to be proportional only to the amount Q of the chemical that is present at a given time t. The differential equation governing such a process then has the form

$$\frac{dQ}{dt} = kQ \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

where, $k \geq 0$ is a constant of proportionality. This is a homogeneous linear first order differential equation. An analogous situation applies to the radioactive decay of an isotope for which the decay takes place at a rate proportional to the amount of radioactive isotope that is present at any given instant of time. The equation governing the amount Q of the isotope as a function of time t is also of the form shown in (1), but instead of the amount growing as in the previous case, it is decreasing, so as in this case the constant of proportionality is usually denoted by a positive number λ, the equation for radioactive decay takes the form

$$\frac{dQ}{dt} = -\lambda Q \dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

It is not difficult to see by inspection that the general solution of (2) is

$$Q = Q_0 e^{-\lambda t}$$

half-life where Q_0 the amount of the isotope present at the start when $t = 0$. The so-called half-life T_h of an isotope is the time taken for half of it to decay away, so setting $Q = \frac{1}{2} Q_0$ in the above result shows the half-life to be given by $T_h = \frac{1}{\lambda} ln2$.

Problem 1:

Radium is known to decay at a rate proportional to the amount present. If the half-life of radium is 1600 years. What percentage of radium will remain in a given sample after 800 years?

Solution:

We have $x = x_0 e^{kt}$,

where x_0 is the amount of radium at $t = 0$.

Since the half-life of radium is 1600 years.

Therefore, half of the original amount of radium exists after 1600 years,

$$\frac{x_0}{2} = x_0 \, e^{1600t}$$

$$k = -\frac{\log 2}{1600}$$

the amount of radium present in a given sample after 800 years is given by

$$x = x_0 e^{800k}$$

$$= x_0 \ e^{\frac{-log2}{2}}$$

$$= \ x_0 \ 2^{-0.5}$$

$$= \frac{x_0}{\sqrt{2}}$$

So,

$$x = 0.707 \ x_0 \ (approx.)$$

Hence $\frac{x}{x_0}.100 = 70.7\%$ of the original amount of radium would remain after 800 years.

(Ans.)

Problem 2:

A certain radioactive material is known to decay at a rate proportional to the amount present. If initially 0.5 gm of the material is present and 0.1 percent of the original mass has decayed after one week, find the half-life of the material.

Solution:

Let x(t) denote the amount of material present at time t.

Then

$$\frac{dx}{dx} = kt, \text{ where } k < 0$$

$$\Rightarrow x = x_0 e^{kt},$$

since $x = 0.5 \ at \ t = 0$,

So,

$$x_0 = 0.5$$

Thus,

$x = 0.5 \ e^{kt}$, at any time t.

If we take the time unit to be 1 week, then 0.1% of x_0 has decayed.

If $t = 1$, then 99.9% of x_0 remains.

So,

$$x = .999x_0$$

$$\Rightarrow x = \frac{.999}{2}$$

$$\therefore x = .4995\ gms$$

Now,

$$0.4995 = 0.5$$

$$\Rightarrow e^{k} = 0.99$$

$$\Rightarrow k = \log(0.999)$$

$$\therefore k = -0.001$$

Thus,

$$x = 0.5e^{-0.001t}, \text{ in t weeks.}$$

Now half-life is the time t associated with the decay of one half the original mass. Here the original mass is 0.5 gm.

So when $x = 0.5x_0 = \frac{1}{4}$

the time t is given by $\frac{1}{4} = 0.5e^{-0.001\,t}$

$$-0.001t = \log\frac{1}{2}$$

$$\Rightarrow\ t = \frac{log2}{0.001}$$

$$\Rightarrow\ t = 10^{3}log2$$

$$\Rightarrow\ t = 301\ weeks$$

hence, $t = 301\ weeks = 5.78\ years$

(Ans.)

Problem 3:

Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 200 years.

(a) What percentage of the original radioactive nuclei will remain after 1000 years?

(b) In how many years will only one fourth of the original number remain?

Solution:

Here,

$$x = x_0e^{kt}\ \text{, where } x(0) = x_0$$

We are given that

$$x = x_0 - \frac{x_0}{10}$$

$$\Rightarrow x = \frac{9x_0}{10}, \textit{when } t = 200$$

So,

$$\frac{9}{10}\, x_0 = x_0\, e^{200k}$$

$$\Rightarrow e^k = (\frac{9}{10})^{\frac{1}{200}}$$

Thus,

$$x = x_0\, (\frac{9}{10})^{\frac{t}{200}} \text{, at any time, t.}$$

(a)

When t=1000,

$$x = x_0\, (\frac{9}{10})^5$$

So,

$$\frac{x}{x_0}.\,100 = \frac{9^5.100}{10^5}.\ 100$$

$$= 59.05\%\ (approx.)$$

Hence, 59.05% of the original radioactive nuclei will remain after 1000 years.

(Ans.)

(b)

Now,

$$0.25\, x_0 = x_0\, (\frac{9}{10})^{\frac{t}{200}}$$

$$\Rightarrow \log(0.25) = \frac{t}{200} \log(\frac{9}{10})$$

Hence, $t = \frac{200\, log4}{\log(\frac{10}{9})}$ = 2631 years. (approx.).

(Ans.)

Problem 4:

Uradium is known to decay at a rate proportional to the amount present. If the half-life of radium is 1800 years. What percentage of radium will remain in a given sample after 800 years?

Solution:

We have,

$$x = x_0 e^{kt}$$

where x_0 is the amount of radium at t=0. since the half-life of radium is 1600 years, therefore , half of the original amount of radium exists after 1600 years,

$$\frac{x_0}{2} = x_0\, e^{1800t}$$

$$\Rightarrow k = -\frac{\log 2}{1800}$$

the amount of radium present in a given sample after 800 years is given by

$$x = x_0 e^{800k}$$

$$= x_0\ e^{\frac{-\,log2}{2}}$$

$$= x_0\ 2^{-0.5}$$

$$= \frac{x_0}{\sqrt{2}}$$

So,

x=0.707 x_0 (approx.)

Hence $\frac{x}{x_0}.100 = 70.7\%$ of the original amount of radium would remain after 800 years.

(Ans.)

Problem 5:

A fragment of bone is discovered to contain 20% of the usual the 14_C concentration. The relative amount of 14_C in the bone has decreased to 20% of its original value.

Solution:

$$x(t) = 0.20x_0$$

$$k = \frac{ln2}{5730}$$

$$0.20x_0 = x_0 e^{-\left[\frac{ln2}{5730}\right]t}$$

$$t = 5730\ \frac{-ln\ (0.20)}{ln2}$$

$$t = 13300\ year.$$

(Ans.)

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Problem 6:

The rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. Half of the original number of radio-active nuclei have undergone disintegration in a period of 1500 years.

1.what percentage of the original radioactive nuclei will remain after 4500 years?

2.In how many years will only one-tenth of the original number remain?

Solution:

Let, x be the amount of radioactive nuclei present after t years .

Then $\frac{dx}{dt}$ represents the rate at which the nuclei decay.

So,

$$\frac{dx}{dt} = -kx$$

By integrating,

$$x = ce^{-kt}$$

applying the initial condition

$$x = x_0 e^{-kt}$$

when t=1500

$$\frac{1}{2}x_0 = x_0 e^{-1500k}$$

$$\Rightarrow e^{-k} = (\tfrac{1}{2})^{1/1500}$$

$$\Rightarrow k = \frac{ln2}{1500}$$

$$\therefore k = 0.00046.$$

now,

$$x = x_0 e^{-0.00046t}$$

$$\Rightarrow x = x_0 (\frac{1}{2})^{\frac{t}{1500}}$$

Each of the two equivalent expressions gives the number x of radioactive nuclei that are present at time t.

Here $t = 4500$

$$x = x_0 (\frac{1}{2})^3 = x_0 (\frac{1}{8})$$

Thus one-eighth or 12.5% of the original number remain after 4500 years.

$$\frac{1}{10} = (\frac{1}{2})^{\frac{t}{1500}}$$

$$\Rightarrow \ln\frac{1}{10} = \ln (\frac{1}{2})^{\frac{t}{1500}}$$

$$\Rightarrow \ln\frac{1}{10} = \frac{t}{1500} \ln (\frac{1}{2})$$

$$\Rightarrow t = \frac{1500 ln10}{ln2}$$

$$\therefore t = 4985\ years$$

(Ans.)

Problem 7:

A radioactive material has an initial mass 100mg. After two years it is left to 75mg. Find the amount of the material at any time. What is the period of its half-life?

Solution:

We measure the amount of the material present at any time t.

We have

$$\frac{dy}{dt} = ky$$

and we have the initial condition

$$y(t_0) = y_0$$

$$y(0) = 100$$

thus, we can find the arbitrary constant c

$$y = Ce^{kt}$$

$$\Rightarrow y(0) = ce^{0}$$

$$\Rightarrow 100 = c.$$

$$\therefore y = 100e^{kt}$$

Now we will find the additional constant k by using the additional condition

$$y(2) = 75$$

$$75 = 100e^{2k}$$

$$k = \frac{ln0.75}{2} = -0.1438$$

Thus we have,

$$y(t) = 100e^{-0.1438t}$$

from the latest equation we will find the Half-life of the material which is the time when

$y = 50mg.$

$$50 = 100e^{-0.1438t}$$

$$\Rightarrow e^{-0.1438t} = 0.5$$

$$\Rightarrow t = \frac{ln0.5}{-0.1438}$$

$$\therefore t = 4.82\ years$$

(Ans.)

Problem 8:

A zircon sample contains 4000 atoms of the radioactive element 235U. Given that 235U has a half -life of 700 million years, how long would it take to decay to 125 atoms?

Solution:

$$y = 4000(\frac{1}{2})^{\frac{t}{700}}$$

$$\Rightarrow 125 = 4000(\frac{1}{2})^{\frac{t}{700}}$$

$$\Rightarrow ln\frac{125}{4000} = \ln(\frac{1}{2})^{\frac{t}{700}}$$

$\therefore$ t= 3500 million years

before zircon sample degrades to 125 atoms.

(Ans.)

Problem 9:

Radium decomposes at a rate proportional to the amount present. If half of the original amount disappears in 1600 years, find the percentage lost in 100 years.

Solution:

$x = x_0 e^{kt}$, where $x(0) = x_0$

when t =1600, $x = \frac{1}{2}x_0$

$\frac{1}{2}x_0 = x_0 e^{1600k}$,

$$e^k = (\frac{1}{2})^{1/1600}$$

Thus

$$x = x_0(\frac{1}{2})^{t/1600} \text{ in t years.}$$

The amount present after 100 years will be

$$x = x_0 = (\frac{1}{2})^{\frac{1}{16}} = 0.958x_0$$

The percentage decrease from the initial amount x_0 is $\frac{x_0 - 0.958x_0}{x_0} \times 100 = 4.2\%$

(Ans.)

Problem 10:

A radioactive substance has a half-life of 1620 years.

(a) If its mass is now 4 g (grams), how much will be left 810 years from now?

(b) Find the time t_1 when 1.5 g of the substance remain.

Solution:

(a)

$$t_0 = 0, x_0 = 4$$

$$x = 4e^{-kt}$$

$$k = \frac{ln2}{t} = \frac{ln2}{1620}$$

$$Q = 4e^{-\frac{t\,In\,2}{1620}}$$

Hence the mass left after 810 years will be

$$Q(810) = 4e^{-\frac{810\,In\,2}{1620}}$$

$$= 2.83$$

(b)

Setting $t = t_1$

$$Q(t_1) = 1.5 \text{ yields}$$

$$\frac{3}{2} = 4e^{-\frac{t_1 In\,2}{1620}}$$

Dividing by 4 and taking logarithms yield

$$ln\frac{3}{8} = -\frac{t_1 ln2}{1620}$$

$$t_1 = 2292.4\ years.$$

(Ans.)

Frictional Forces & Falling Body Problems

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Frictional Forces:

If a body moves on a rough surface, it will encounter not only air resistance but also another resistance force due to the roughness of the surface. This additional force is called friction. It is shown in physics that the friction is given by μN, where

1. p is a constant of proportionality called the coefficient of friction, which depends upon the roughness of the given surface; and

2. N is the normal (that is, perpendicular) force which the surface exerts on the body.

We now apply Newton's second law to a problem in which friction is involved.

Problem 1:

An object weighing 48 lb is released from rest at the top of a plane metal slide that is inclined 30° to the horizontal. Air resistance (in pounds) is numerically equal to one-half the velocity (in feet per second), and the coefficient of friction is one-quarter.

A. What is the velocity of the object 2 sec after it is released?

B. If the slide is 24 ft long, what is the velocity when the object reaches the bottom?

Formulation:

The line of motion is along the slide we choose the origin at the top and the positive x direction down the slide. If we temporarily neglect the friction and air resistance, the forces acting upon the A are:

1. Its weight, $48\ lb$, which acts vertically downward; and
2. The normal force, N, exerted by the slide which acts in an upward direction perpendicular to the slide. (See figure 1)

The components of weight parallel and perpendicular to the slide have magnitude

$$48 \sin 30° = 24$$

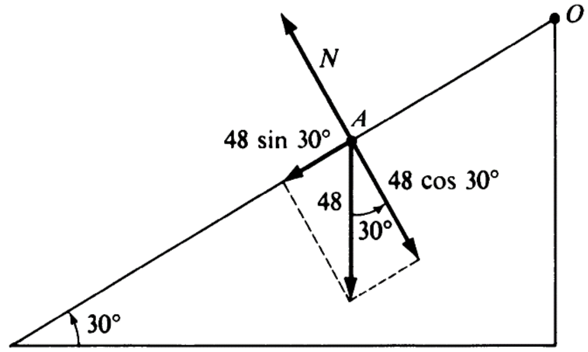

Figure: 1

And

$$48 \cos 30° = 24\sqrt{3}$$

Respectively, the components perpendicular to the slide are in equilibrium and hence the normal force N has magnitude, $24\sqrt{3}$.

Now, taking into consideration the friction and air resistance, we see that the forces are acting in the object as it moves along the slide are the following,

1. F_1, the component of the weight parallel to the plane, having numerical value 24.Since this force acts in the positive (downward) direction along the slide, we have

$$F_1 = 24$$

2. F_2, the frictional force, having numerical value $\mu N = \frac{1}{4}\left(24\sqrt{3}\right)$.Since this act in the negative (upward) direction along the side, we have

$$F_2 = -6\sqrt{3}$$

3. F_3, the air resistance, having numerical value $\frac{1}{2}v$.Since $v > 0$ and this also acts in the negative direction, we have

$$F_3 = -\frac{1}{2}v$$

We apply Newton's second law $F = ma$.

Here

$$F = F_1 + F_2 + F_3 = 24 - 6\sqrt{3} - \frac{1}{2}v$$

And

$$m = \frac{w}{g} = \frac{48}{32} = \frac{3}{2}$$

Thus, we have the differential equation

$$\frac{3}{2}\frac{dy}{dx} = 24 - 6\sqrt{3} - \frac{1}{2}v \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since the object is released from the rest, the initial condition is,

$$v(0) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Solution:

Equation (1) is separable; separating variables we have

$$\frac{dv}{48-12\sqrt{3}-v} = \frac{dt}{3}$$

Integrating and simplifying, we find

$$v = 48 - 12\sqrt{3} - C_1 e^{-\frac{t}{3}}$$

The condition (2) gives

$$C_1 = 48 - 12\sqrt{3}$$

Thus, we obtain

$$v = \left(48 - 12\sqrt{3}\right)\left(1 - e^{-\frac{t}{3}}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Question A is thus answered by letting $t = 2$ in Equation (3).

We find

$$v(2) = (48 - 12\sqrt{3})(1 - e^{-\frac{2}{3}}) \approx 13.2(ft/sec)$$

In order to answer question B, we integrate (3.39) to obtain

$$x = (48 - 12\sqrt{3})\,(t + 3\,e^{-\frac{t}{3}}) + C_2.$$

Since, $x(0) = 0$,

$$C_2 = -(48 - 12\sqrt{3})(3)$$

Thus, the distance covered at time t is given by,

$$x = (48 - 12\sqrt{3})\,(t + 3e^{-\frac{T}{3}} - 3).$$

Since the slide is 24 ft long, the object reaches the bottom at the time T determined from the transcendental equation

$$24 = (48 - 12\sqrt{3})\,(T + 3\,e^{-\frac{T}{3}} - 3)$$

which may be written as

$$3\,e^{-\frac{T}{3}} = \frac{47 + 2\sqrt{3}}{13} - T$$

The value of T that satisfies this equation is approximately 2.6. Thus from Equation (3.39) the velocity of the object when it reaches the bottom is given approximately by

$$(48 - 12\sqrt{3})(1 - e^{-0.9}) \approx 16.2\ (ft/sec).$$

(Ans.)

Problem 2:

Suppose that the object of mass 1 slug is thrown downward with an initial velocity of 2 ft/s and that the object is attached to a parachute, increasing this resistance so that it is given by u. Find the velocity at any time t and determine the limiting velocity of the object.

Solution:

This situation is modeled by the initial value problem

$$\frac{du}{dt} = 32 - u^2 where\ u(0) = 2$$

we solve the differential equation by separating the variables and using partial fractions.

$$\frac{1}{32-u^2}du = dt$$

$$\frac{1}{(4\sqrt{2}+u)(4\sqrt{2}-u)}du = dt$$

$$\frac{1}{8\sqrt{2}}\left(\frac{1}{u+4\sqrt{2}} - \frac{1}{u-4\sqrt{2}}\right)du = dt$$

$$\ln\left[u + 4\sqrt{2}\right] - \ln\left[u - 4\sqrt{2}\right] = \sqrt{2t8} + C_1$$

$$\ln\left[\frac{u+4\sqrt{2}}{u-4\sqrt{2}}\right] = 8\sqrt{2}t + C_1$$

$$\left[\frac{u+4\sqrt{2}}{u-4\sqrt{2}}\right] = C_2 e^{8\sqrt{2}t}$$

Solving for,

$$u + 4\sqrt{2} = C_3 e^{8t\sqrt{2}}$$

$$\left(1 - C_3 e^{t8\sqrt{2}}\right)u = -4\sqrt{2}\, C_3 e^{8t\sqrt{2}} + 1$$

So,

$$u = -4\sqrt{2} * \frac{C_3 e^{8t\sqrt{2}}+1}{1-C_3 e^{8t\sqrt{2}}}$$

u is an application of the initial condition yield.

$$C_3 = \frac{1+2\sqrt{2}}{1-2\sqrt{2}}$$

The limiting velocity of the object is found with L' Hospitals' rule to be, Limit tends to infinity

$$u(t) = -\frac{4\sqrt{2}C_3}{-C_3} = \frac{4\sqrt{2}ft}{s}.$$

(Ans.)

Falling Body Problem:

We shall now consider some examples of a body falling through air toward the earth. In such a circumstance the body encounters air resistance as it falls. The amount of air resistance depends upon the velocity of the body, but no general law exactly expressing this dependence is known. In some instances, the law $R = kv$ appears to be quite satisfactory, while in others $R = kv^2$ appears to be more exact. In any case, the constant of proportionality k on turn depends on several circumstances. In the examples that follow we shall assume certain reasonable resistance laws in each case. Thus we shall actually be dealing with idealized problems in which the true resistance law is approximated and in which certain comparatively negligible factors are disregarded.

Example 3:

A body weighting 8 lb falls from rest toward the earth from a great height. As it falls, air resistance acts upon it, and we shall assume that this resistance (in feet per second). Find the velocity and resistance fallen at time t seconds.

Formulation:

We choose the positive x axis vertically downward along the path of the body B the origin at the point from which the body fell. The forces acting on the body are:

1. F_1 it's weight, 8 lb, which acts downward and hence is positive.

2. F_2 , the air resistance numerically equal to 2v, which acts upward and hence is the negatively quantity -2v

Newton's second law, $F = ma$, becomes

$$m\frac{dv}{dt} = F_1 + F_2$$

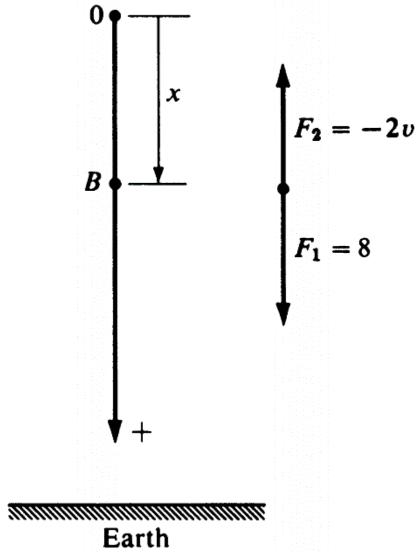

Figure: 1

or,

taking $g = 32$ and using

$$m = \frac{w}{g} = \frac{8}{32} = \frac{1}{4}$$

$$\frac{1}{4}\frac{dv}{dt} = 8 - 2v \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Since the body was initially at rest, we have the initial condition

$$v(0) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Solution:

Equation (1) is separable: separating variables,

we have

$$\frac{dv}{8 - 2v} = 4\,dt$$

Integrating we find

$$-0.5\ln(8 - 2v) = 4t + C_0$$

Which reduces to

$$8 - 2v = C_1 e^{-8t}$$

Applying the condition (2) we find $C_1 = 8$.

Thus, the velocity at time t is given by

$$v = 4(1 - e^{-8t}) \dots \dots \dots \dots \dots \dots \dots \dots \dots (3)$$

Now to determine the distance fallen at time t, we write (3) in the form

$$\frac{dx}{dt} = 4(\,1 - e^{-8t})$$

And note that $x(0) = 0$.

Integrating the above equation, we obtain

$$x = 4\left(t + \frac{1}{8}e^{-8t}\right) + C_2$$

Since $x = 0$ when $t = 0$, we find $c_2 = \frac{-1}{2}$ and hence the distance fallen is given by

$$x = 4(t + \tfrac{1}{8}e^{-8t} - \tfrac{1}{8}).$$

(Ans.)

Problem 4:

A skydiver equipped with parachute and other essential equipment falls from rest toward the earth. The total weight of the man plus the equipment is $160lb$.Before parachutes opens, the air resistance (in pounds) is numerically equal to $\frac{1}{2}v$, where v is the velocity (in feet per second). The parachute opens 5 sec after the fall begins; after it opens, the air resistance (in pound) is numerically equal to $\frac{5}{8}v^2$, where v is the velocity (in feet per second). Find the velocity of the skydiver (A) before the parachute opens, and (B) after the parachute opens.

Solution:

Let F_1, the weight, the $160lb$, which acts downward, hence is positive.

F_2, the air resistance, numerically equal to $\frac{1}{2}v$, which acts upward and hence is the negative quantity $-\frac{1}{2}v$.

From Newton's second law $F = ma$, where $F=F_1+F_2$, let $m = \frac{w}{g}$ and $g = 32$

$5\frac{dv}{dt} = 160 - \frac{1}{2}v$.

Since the skydiver was initially at rest, $v = 0$ when $t = 0$

we shall first consider the problem (A), We find a one-parameter family of solution of

$$5\frac{dv}{dt} = 160 - \frac{1}{2}v$$

Separating variables, we obtain

$$\frac{dv}{v-320} = -\frac{1}{10}dt$$

by integrating this,

$$\ln(v-320) = -\frac{1}{10}t + c_0$$

$$v = 320 + Ce^{-t/10}.$$

when $v = 0, t = 0$, we get $C = -320$

$$v = 320\,(1 - e^{-\frac{t}{10}}).$$

which is valid for $0 \le t \le 5$. In particular, where $t = 5$

$$v_1 = 320(1 - e^{-\frac{1}{2}}) = 126$$

which is the velocity when parachute opens.

For problem (B),

$$5\frac{dv}{dt} = 160 - \frac{5}{8}v^2$$

$$\Rightarrow \frac{dv}{v^2-256} = -\frac{1}{8}dt$$

By integrating this,

$$ln\frac{v-16}{v+16} = -4t + c_1$$

$$\Rightarrow \frac{v-16}{v+16} = ce^{-4t}$$

$$\therefore v = \frac{16(ce^{-4t}+1)}{1-ce^{-4t}}$$

Here $v_1 = v = 126$

$$\therefore v = \frac{16(\frac{110}{142}e^{20-4t}+1)}{1-(\frac{110}{142}e^{20-4t})}$$

which is valid $t \geq 5$.

(Ans.)

Newton’s Law of Cooling Problems

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Newton's Law of Cooling Problems:

It states that the rate at which the temperature $T = T(t)$ changes in a cooling body is proportional to the difference between the temperature T of the body and the constant temperature t_0 of the surrounding medium.

i.e.,

$$\frac{dT}{dt} = -k\,(T - T_0), \text{ where k > 0.}$$

Negative sign is taken due to the reduction in the temperature of the hot body, when it cools. Note that $T > T_0$.

Problem 1:

A body whose temperature is initially $100^0\ C$ is allowed to cool in air, whose temperature remains at a constant $20^0\ C$. It is given that after 10 minutes, the body has cooled to $40^0\ C$. Find the temperature of the body after half an hour.

Solution:

If T denoted the instantaneous temperature of the body in degree Celsius and t denotes the time in minutes when the body begins to cool, the rate of cooling is,

$$\frac{dT}{dt} = -k\,(T - 20), \text{ where k > 0}$$

$$\therefore \frac{dT}{T-20} = -kdt$$

$$\Rightarrow log\,(t - 20) = -kt + log\,C$$

$$\Rightarrow T - 20 = Ce^{-kt}$$

$$\Rightarrow T = 20 + Ce^{-kt}$$

At $t = 0, T = 100$ and so $C = 80$

Thus,

$$T = 20 + 80e^{-kt}$$

is the temperature of the body at time t. It is given that when

$$t = 10, T = 40.$$

(Ans.)

Problem 2:

A metal bar at a temperature of 100^0 F is placed in a room at a constant temperature of 0^0 F. If after 20 minutes the temperature of the bar is half. Find an expression for the temperature of the bar at any time.

Solution:

Here, $T_0 = 0$, therefore

$$\frac{dT}{dt} = -kT \text{ implies } T = Ce^{-kt}.$$

At $t = 0$, we are given $T = 100$, and so $C = 100$

$$\therefore T = 100e^{-kt}$$

When $t = 20$, we are given $T = 50$

$$\therefore 50 = 100e^{-20k}$$

$$\Rightarrow -20k = \log(\tfrac{1}{2})$$

$$\Rightarrow -20k = -0.693$$

$$\Rightarrow k = 0.035$$

Hence, $T = 100\ exp\,(-0.035\ t)$.

(Ans.)

Problem 3:

According to Newton's Law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is $290\ K$ and the substance cools from $370\ K$ to $330\ K$ in 10 minutes. Find when the temperature will be $295\ K$.

Solution:

Let T be the temperature of the substance at time t minutes. Then

$$\frac{dT}{dt} = -k\,(T - 290)$$

$$\Rightarrow \frac{dT}{(T-290)} = -k\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Integrating (1) between the limits $t = 0\ (T = 370)$ and $t = 10\ (T = 330)$, we obtain

$$\int_{370}^{330} \frac{dT}{(T-290)} = -k\int_0^{10} dt$$

$$\Rightarrow log\ 40 - log\ 80 = -10\,k$$

$$\Rightarrow 10\,k = log\ 2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Integrating (1) between the limits $t = 0\ (T = 370)$ and $t = t\ (T = 295)$,

$$\int_{370}^{295} \frac{dT}{(T-290)} = -k\int_0^{t} dt$$

$$\Rightarrow \log 5 - \log 80 = -kt$$

$$\Rightarrow kt = \log 16$$

$$\Rightarrow \log 16 = \log 2^4$$

$$\Rightarrow kt = 4\log 2 = 4\,(10\,k),\ \text{using (2).}$$

∴ t = 40 minutes. Hence the temperature will be 295 K in 40 minutes.

(Ans.)

Problem 4:

According to Newton's Law of cooling, the rate at which a substance cools in air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is $30^0\ C$ and the substance cools from $100^0\ C$ to $70^0\ C$ in 15 minutes. Find when the temperature will be $40^0\ C$.

Solution:

Let T be the temperature of the substance at time t minutes.

Then

$$\frac{dT}{dt} = -k\ (T - 30)$$

$$\Rightarrow \frac{dT}{(T-30)} = -k\ dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Integrating (1) between the limits $t = 0\ (T = 100)$ and $t = 15\ (T = 70)$, we obtain

$$\int_{100}^{70} \frac{dT}{(T-30)} = -k \int_0^{15} dt$$

$$\Rightarrow log\ 40 - log\ 70 = -15\ k$$

$$\Rightarrow -15\ k = \log \frac{7}{4}$$

$$\Rightarrow -15\ k = 0.56$$

Integrating (1) between the limits $t = 0\ (T = 100)$ and $t = t\ (T = 40)$,

$$\int_{100}^{40} \frac{dT}{(T-30)} = -k \int_0^{t} dt$$

$$\Rightarrow \log 10 - \log 70 = -kt$$

$$\Rightarrow 15\ kt = 15 \log 7\ t = \frac{15 \log 7}{0.56} = 32 \text{ minutes.}$$

(Ans.)

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Problem 5:

The rate at which a body cools is proportional to the difference between the temperature of the body and that of the surrounding air. If a body in air at 25°C will cool from 100° to 75° in one minute, find its temperature at the end of three minutes.

Solution:

Let temperature of the body be T°C.

$$\frac{dT}{dt} = k\,(T - 25)$$

$$\Rightarrow \frac{dT}{T-25} = k\,dt$$

$$\Rightarrow log\,(T - 25) = kt + log\,A$$

$$\Rightarrow \log\frac{T-25}{A} = kt$$

$$\Rightarrow T - 25 = Ae^{kt}$$

When, $t = 0$, then $T = 100$, from (1) $A = 75$

When $t = 1$, then $T = 75$ and $A = 75$, From (1) $\frac{2}{3} = e^{k}$

∴ (1) becomes

$$T = 25 + 75e^{kt}$$

When $t = 3$, then $T = 25 + 75e^{3k} = 25 + 75 * \frac{8}{27} = 47.22$

(Ans.)

Problem 6:

The rate at which the ice melts is proportional to the amount of ice at the instant. Find the amount of ice left after 2 hours if half the quantity melts in 30 minutes.

Solution:

Let m be the amount of ice at any time t.

$$\frac{dm}{dt} = km$$

$$\Rightarrow \frac{dm}{m} = k\,dt$$

$$\Rightarrow \int \frac{dm}{m} = k \int dt + C$$

$$\Rightarrow \log m = kt + C \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

At, $t = 0, m = M$

$$Log\, M = 0 + C$$

$$C = log\, M$$

On putting the value of C, (1) becomes

$$Log\, m = kt + log\, M \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$m = \frac{M}{2}$ when $t = \frac{1}{2}$ hour

$$\log \frac{M}{2} = \frac{k}{2} + \log M$$

$$\Rightarrow \log \frac{M}{2M} = \frac{k}{2}$$

$$\Rightarrow \log \frac{1}{2} = \frac{k}{2}$$

$$\therefore k = 2 \log \frac{1}{2}$$

On putting the value of k in (2), we have

$$\log m = \left(2\log\frac{1}{2}\right)t + \log M \ldots\ldots\ldots\ldots\ldots (3)$$

On putting t = 2 hours in (3), we have

$$\log m = \left(4\log\frac{1}{2}\right) + \log M$$

$$\Rightarrow \log\frac{m}{M} = \log\left(\frac{1}{2}\right)^4$$

$$\Rightarrow \frac{m}{M} = \frac{1}{16}$$

$$\Rightarrow m = \frac{M}{16}$$

After 2 hours, amount of ice left = $\frac{1}{16}$ of the amount of ice at the beginning.

(Ans.)

Problem 7:

If the temperature of the air is $300\ K$ and the substance cools from $370\ K$ to $340\ K$ in 15 minutes. Find when the temperature will be $310\ K$.

Solution:

Let T be the temperature of the substance at time t minutes. Then

$$\frac{dT}{dt} = -k\,(T - 300)$$

$$\Rightarrow \frac{dT}{(T-300)} = -k\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Integrating (1) between the limits $t = 0\ (T = 370)$ and $t = 15\ (T = 340)$, we obtain

$$\int_{370}^{340} \frac{dT}{(T-3000)} = -k\int_0^{15} dt$$

$$\Rightarrow log\,40 - log\,70 = -15\,k$$

$$\Rightarrow 15k = log\,\frac{7}{4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Integrating (1) between the limits $t = 0$ ($T = 370$) and $t = t$ ($T = 310$),

$$\int_{370}^{310} \frac{dT}{(T-300)} = -k \int_0^t dt$$

$$\Rightarrow \log 10 - \log 70 = -kt$$

$$\Rightarrow kt = \log 7$$

⇒ t = 52.158 minutes, using (2).

∴ t = 52.158 minutes. Hence the temperature will be 295 K in 52.158 minutes.

(Ans.)

Problem 8:

A body at temperature 72^0F is taken outdoors, where the temperature is 20^0F. After 5 minutes the temperature of the body is 55^0F. How long will it take the body to reach a temperature of 32^0F.

Solution:

Let T be the temperature of the substance at time t minutes.

Then

$$\frac{dT}{dt} = -k(T-20)$$

$$\Rightarrow \frac{dT}{(T-20)} = -k\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Integrating (1) between the limits $t = 0$ ($T = 72$) and $t = 5$ ($T = 55$), we obtain

$$\int_{72}^{55} \frac{dT}{(T-20)} = -k \int_0^5 dt$$

$$\Rightarrow log\,35 - log\,52 = -5k$$

$$\Rightarrow -5k = \log \frac{35}{52}$$

$$\Rightarrow 5k = \log \frac{52}{35} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Integrating (1) between the limits $t = 0\ (T = 72)$ and $t = t\ (T = 32)$,

$$\int_{72}^{32} \frac{dT}{(T-20)} = -k \int_0^t dt$$

$$\Rightarrow \log 12 - \log 52 = -kt$$

$$\Rightarrow \log \frac{13}{3} = 18.519 \text{ minutes.}$$

(Ans.)

Problem 9:

A body of temperature 80^0F is placed at time t = 0 in a medium, the temperature of which is maintained at 50^0F. At the end of 5 minutes, the body has cooled to a temperature of 70^0F. Determine the temperature of the body as a function of time for t>0. In particular answer the following questions.

(a) What is the temperature of the body at the end of 10 minutes?

(b) When will the temperature of the body be 60^0F?

(c) After how many minutes will the temperature of the body be within 1^0F of the constant 50^0F temperature of the room.

Solution:

(a)

Let T be the temperature of the substance at time t minutes.

Then

$$\frac{dT}{dt} = -k\,(T-50)$$

$$\Rightarrow \frac{dT}{(T-50)} = -k\,dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Integrating (1) between the limits $t = 0\ (T = 80)$ and $t = 5\ (T = 70)$, we obtain

$$\int_{80}^{70} \frac{dT}{(T-50)} = -k \int_{0}^{5} dt$$

$$\Rightarrow log\ 20 - log\ 30 = -5\ k$$

$$\Rightarrow -5\ k = \log\frac{2}{3}$$

$$\Rightarrow 5\ k = \log\frac{3}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Integrating (1) between the limits $t = 0\ (T = 72)$ and $t = t\ (T = T_1)$,

$$\int_{72}^{T_1} \frac{dT}{(T-50)} = -k \int_{0}^{t} dt$$

$$\Rightarrow \log(T_1 - 50) - \log 30 = -10k$$

$$\Rightarrow \log(T_1 - 50) = \log 30 - 10.\frac{\log\frac{3}{2}}{5}$$

$$\Rightarrow T_1 - 50 = 13.33$$

$$\Rightarrow T_1 = 63.33^0 F.$$

(Ans.)

(b)

Let T be the temperature of the substance at time t minutes.

Then

$$\frac{dT}{dt} = -k\ (T - 50)$$

$$\Rightarrow \frac{dT}{(T-50)} = -k\ dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Integrating (1) between the limits $t = 0\ (T = 80)$ and $t = 5\ (T = 70)$, we obtain

$$\int_{80}^{70} \frac{dT}{(T-50)} = -k \int_{0}^{5} dt$$

$$\Rightarrow log\ 20 - log\ 30 = -5\ k$$

$$\Rightarrow -5\ k = \log\frac{2}{3}$$

$$\Rightarrow 5\,k = \log\frac{3}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Integrating (1) between the limits $t = 0$ $(T = 80)$ and $t = t$ $(T = 60)$,

$$\int_{80}^{60} \frac{dT}{(T-50)} = -k\int_{0}^{t} dt$$

$$\Rightarrow \log 10 - \log 30 = -kt$$

$$\Rightarrow \log 3 = kt$$

$$\therefore t = 13.55 \text{ minutes}$$

(Ans.)

(c)

Thus we seek the time when the temperature x is 51. Thus lettering x = 51 in

$$x = 50 + 30(\frac{2}{3})^{\frac{t}{5}}$$

We quickly find,

$$(\frac{2}{3})^{\frac{t}{5}} = \frac{1}{30}$$

From which

$$t = 5\frac{log\,\frac{1}{30}}{log\,\frac{2}{3}}$$

$$\therefore t \approx 41.94 \text{ minutes.}$$

(Ans.)

THANK YOU