MRK

Math 2101

Mathematics III

Differential Equation

$\frac{dy}{dx} + y = 0 \longrightarrow$ differential equation

$\frac{d}{dx} \rightarrow$ differential co-efficient

$\frac{d}{dx}(y) \longrightarrow$ dependent

$\downarrow$

independent

Differential equation : An equation involving derivatives of one or more dependent variable(s) with respect to one or more independent variable(s) is called differential equation.

Example : $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$

$\frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0$

Ordinary differential equation : An equation involving ordinary derivatives of single independent variable is called ordinary differential equation.

Example : $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x$

Partial differential equation : An equation involving partial derivatives of more than one independent variables is called partial differential equation.

Example : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Order : The order of a differential equation is the number of highest ~~order~~ derivative involved in a differential equation.

Degree : The degree of a differential equation is the ~~number~~ ~~of~~ power of its highest derivative after the equation has been made rational and integral in all of its derivatives.

$$\frac{d^3y}{dx^3} + x\left(\frac{dy}{dx}\right)^5 = y$$

Here, order is 3 and degree is 1

$$\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{2}} + x\frac{dy}{dx} = y$$

Here, order is 2 and degree is 1

- Order and degree can not be fractional number

⊞ Properties of linear and non-linear equation

(i) If two or more dependent variables are in product form the equation is non-linear.

(ii) If the dependent variables ~~are~~ can be expressed in infinite series. the equation is non-linear

~~(iii)~~

- $\left(\frac{d^2y}{dx^2}\right)^2 + y = x \longrightarrow$ non-linear

- $y\frac{d^2y}{dx^2} + x = 1 \longrightarrow$ non-linear

- $\frac{d^2y}{dx^2} + y^2 = x \longrightarrow$ non-linear

- $\frac{d^2y}{dx^2} + e^y = x \longrightarrow$ non-linear

- $\frac{d^2y}{dx^2} + e^x = x^2 \longrightarrow$ linear

- $\frac{d^2y}{dx^2} + e^x = y \longrightarrow$ linear

- $\frac{d^2y}{dx^2} + e^x = y^2 \longrightarrow$ non-linear

- $\frac{d^2y}{dx^2} + xy = 1 \longrightarrow$ linear

- $\left(\frac{dy}{dx}\right)^2 + \sin x = e^x \longrightarrow$ non-linear

Converting normal equation to ordinary differential equation.

Problem 1
Form an ordinary differential equation by eliminating arbitrary constants from $y = ax + bx^2$

Solution :
Given that $y = ax + bx^2$ ——— ①
Now differentiating equation ① with respect to x two times we get,

$$\frac{dy}{dx} = a + 2bx \quad \text{———— (ii)}$$

and $\frac{d^2y}{dx^2} = 2b$

$$\therefore b = \frac{1}{2} \times \frac{d^2y}{dx^2} \quad \text{———— (iii)}$$

Putting the value of b in equation (ii) we get

$$\frac{dy}{dx} = a + 2 \times \frac{1}{2} \times \frac{d^2y}{dx^2}\, x$$

$$\Rightarrow \frac{dy}{dx} = a + x\frac{d^2y}{dx^2}$$

$$\therefore a = \frac{dy}{dx} - x\frac{d^2y}{dx^2} \quad \text{———— (iv)}$$

Now, putting the value of a and b in equation (i) we get

$$y = \left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)x + \frac{1}{2} \times \frac{d^2y}{dx^2}\, x^2$$

$$\Rightarrow y = x\frac{dy}{dx} - x^2\frac{d^2y}{dx^2} + \frac{x^2}{2}\frac{d^2y}{dx^2}$$

$$\therefore y = x\frac{dy}{dx} - \frac{1}{2}x^2\frac{d^2y}{dx^2}$$

(Ans).

Problem 2

Form an ordinary differential equation by eliminating arbitrary constants from

$c(y+c)^2 = x^3$.

Solution

Given that $c(y+c)^2 = x^3$ ——— (i)

Now differentiating equation (i) with respect to x we get

$$2c(y+c)\frac{dy}{dx} = 3x^2 \quad \text{——— (ii)}$$

By doing (i) ÷ (ii) we get

$$\frac{y+c}{2\frac{dy}{dx}} = \frac{x}{3}$$

$$\Rightarrow y+c = \frac{2}{3}x\frac{dy}{dx}$$

$$\therefore c = \frac{2}{3}x\frac{dy}{dx} - y$$

Putting the value of c in equation (i) we get,

$$\left(\frac{2}{3}x\frac{dy}{dx} - y\right)\left(y + \frac{2}{3}x\frac{dy}{dx} - y\right)^2 = x^3$$

$$\Rightarrow \left(\frac{2}{3}x\frac{dy}{dx} - y\right) \times \frac{4}{9}x^2\left(\frac{dy}{dx}\right)^2 = x^3$$

$$\Rightarrow \frac{8}{27}x^3\frac{dy}{dx} - \frac{4}{9}x^2y\left(\frac{dy}{dx}\right)^2 = x^3$$

(Ans).

Problem 3

Find the differential equation of all circles passing through the origin and having their centers on x-axis.

Solution

We know the equation of circles is

$x^2 + y^2 + 2gx + 2fy + c = 0$

the circle passing through the origin so, $c=0$

and in x-axis $f = 0$

$\therefore$ the equation is

$$x^2 + y^2 + 2gx = 0 \quad \text{---} \quad (1)$$

Now, differentiating equation (1) with respect to x we get

$$2x + 2y\frac{dy}{dx} + 2g = 0$$

$$\therefore g = -x - y\frac{dy}{dx}$$

Putting the value of g in equation (1) we get

$$x^2 + y^2 + 2\left(-x - y\frac{dy}{dx}\right)x = 0$$

$$\Rightarrow x^2 + y^2 - 2x^2 - 2xy\frac{dy}{dx} = 0$$

$$\therefore y^2 - x^2 - 2xy\frac{dy}{dx} = 0$$

(Ans).

Solution of 1st order 1st degree differential equation:

$$\frac{dy}{dx} = f(x, y)$$

1. Separation of variables method:

Problem 1

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\Rightarrow \frac{1+x^2}{dx} = \frac{1+y^2}{dy}$$

$$\Rightarrow \frac{dy}{1+y^2} = \frac{dx}{1+x^2} \quad \text{——} \ (1)$$

By integrating equation (1) we get.

$$\tan^{-1}y = \tan^{-1}x + \tan^{-1}c$$

$$\Rightarrow \tan^{-1}y - \tan^{-1}x = \tan^{-1}c$$

$$\Rightarrow \tan^{-1}\frac{y-x}{1+xy} = \tan^{-1}c$$

$$\Rightarrow \frac{y-x}{1+xy} = c$$

$$\therefore y - x = c(1+xy)$$

Problem 2

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\Rightarrow \frac{dy}{dx} = e^{-y}(e^x + x^2)$$

$$\Rightarrow \frac{dy}{e^{-y}} = (e^x + x^2)\,dx$$

$$\Rightarrow e^y\,dy = (e^x + x^2)\,dx \quad \text{——} \ (1)$$

By integrating equation (1) we get

$$e^y = e^x + \frac{x^3}{3} + c$$

$$\Rightarrow 3(e^y - e^x) = x^3 + 3c$$

Problem 3

Find the solution of $\frac{dy}{dx} = e^{x+y}$ It is given that $y=1$ when $x=1$. Find y when $x=-1$

$$\frac{dy}{dx} = e^{x+y}$$

$$\Rightarrow e^{-y}\frac{dy}{dx} = e^x$$

$$\Rightarrow e^{-y}dy = e^x dx \longrightarrow (1)$$

By integrating equation (1) we get

$$-e^{-y} = e^x + c$$

$$\therefore e^{-y} = -e^x - c \longrightarrow (11)$$

By putting $y=1$ when $x=1$ in equation (11)

$$e^{-1} = -e^1 - c$$

$$\Rightarrow c = -e - e^{-1}$$

$\therefore$ From equation (11)

$$e^{-y} = -e^x + e + e^{-1}$$

when $x=-1$ then

$$e^{-y} = -e^{-1} + e + e^{-1}$$

$$\Rightarrow e^{-y} = e \quad \therefore y = -1$$

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$\boxed{4}$ $\dfrac{dy}{dx} = f(ax+by+c)$ —— ①

let $ax+by+c = v$

$\Rightarrow a + b\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\Rightarrow b\dfrac{dy}{dx} = \dfrac{dv}{dx} - a$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{b}\left(\dfrac{dv}{dx} - a\right)$

Now. $\dfrac{1}{b}\left(\dfrac{dv}{dx} - a\right) = f(v)$

$\Rightarrow \dfrac{dv}{dx} - a = b f(v)$

$\Rightarrow \dfrac{dv}{dx} = bf(v) + a$

$\Rightarrow \dfrac{dv}{bf(v)+a} = dx$

$\Rightarrow \displaystyle\int \frac{dv}{bf(v)+a} = x + c$

Problem

$\dfrac{dy}{dx} = (4x+y+1)^2$ —— ①

let $4x+y+1 = v$

$\Rightarrow 4 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore \dfrac{dy}{dx} = \dfrac{dv}{dx} - 4$

Now $\frac{dv}{dx} - 4 = v^2$

$\Rightarrow \frac{dv}{dx} = 4 + v^2$

$\Rightarrow \frac{dv}{4+v^2} = dx$

By integrating the equation.

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$\Rightarrow \frac{1}{2}\tan^{-1}\frac{v}{2} = x + c$

$\Rightarrow \tan^{-1}\frac{v}{2} = 2x + 2c$

$\Rightarrow \frac{v}{2} = \tan(2x+2c)$

$\Rightarrow v = 2\tan(2x+2c)$

$\Rightarrow \therefore 4x+y+1 = 2\tan(2x+2c)$

Problem

$(x-y)^2 \frac{dy}{dx} = a^2$ ——— (1)

let $x - y = v$

$\Rightarrow 1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dy}{dx} = 1 - \frac{dv}{dx}$

Now $1 - \frac{dv}{dx} = \cancel{a^2} \frac{a^2}{v^2}$

$\Rightarrow \frac{dv}{dx} = 1 - \frac{a^2}{v^2}$

$\Rightarrow \cancel{\frac{dv}{1-v^2} = dx}$

By integrating ~~the~~ equation

$$\Rightarrow \frac{v^2 - a^2}{v^2} = \frac{dv}{dx}$$

$$\Rightarrow \frac{v^2\,dv}{v^2 - a^2} = \cancel{b}\,dx$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \frac{x-a}{x+a} + c$$

$$\Rightarrow \frac{v^2 - a^2 + a^2}{v^2 - a^2}\,dv = dx$$

$$\Rightarrow \cancel{b}\left(1 + \frac{a^2}{v^2 - a^2}\right)dv = dx$$

$$\Rightarrow v + \frac{a^2}{2a} \ln \frac{v-a}{v+a} = x + c$$ [By integration]

$$\Rightarrow x - y + \frac{a^2}{2a} \ln \frac{x-y-a}{x-y+a} = x + c$$

$$\Rightarrow y = \frac{a^2}{2a} \ln \frac{x-y-a}{x-y+a} + C$$

1st order 1st degree homogeneous differential equation

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \text{ or } f\left(\frac{x}{y}\right)$$

$$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$$

[লব ও হরের ঘাত সমান হলে তাকে Homogeneous D.E]

$$\frac{y}{x} = v \Rightarrow y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = f(v)$$

$$\Rightarrow x\frac{dv}{dx} = f(v) - v$$

$$\Rightarrow \frac{dv}{f(v)-v} = \frac{1}{x}dx$$

$$\rightarrow \int \frac{dv}{f(v)-v} = \ln x + C$$

Problem

$$(x^2+y^2)dx + 2xydy = 0 \text{ ——— } (1)$$

let $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{[By integrating]}$$

From equation (1) we get

$$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$$

$$= \frac{-(x^2+v^2x^2}{2x.vx}$$

$$= \frac{-\cancel{x^2}(-1+v^2)}{2\cancel{x^2}v}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1+v^2}{2v}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1+v^2+2v^2}{2v} = -\frac{1+3v^2}{2v}$$

$$\Rightarrow \left(\frac{2v}{1+3v^2}\right) dv = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{3}\ln(1+3v^2) = -\ln x + \ln c$$

$$\Rightarrow \ln(1+3v^2) = -3\ln x + 3\ln c$$

$$\Rightarrow \ln(1+3v^2) = \ln x^{-3} + \ln c^3$$

$$\Rightarrow \ln(1+3v^2) = \ln(x^{-3}c^3)$$

$$\Rightarrow 1+3v^2 = x^{-3}c^3$$

$$\Rightarrow 1 + 3\frac{y^2}{x^2} = \frac{c^3}{x^3}$$

$$\Rightarrow \frac{x^2+3y^2}{x^2} = \frac{c^3}{x^3}$$

$$\Rightarrow x^2 + 3y^2 = \frac{c^3}{x}$$

$$\therefore x^2 + 3y^2 = \frac{c^3}{x}$$

☐ 1st order 1st degree linear differential equation.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Integrating Factor (I.F) $= e^{\int P dx}$

Now,

$$e^{\int Pdx}\frac{dy}{dx} + Py e^{\int Pdx} = Q e^{\int Pdx}$$

$$\Rightarrow \frac{d}{dx}\left(y e^{\int Pdx}\right) = Q e^{\int Pdx}$$

$$\therefore y e^{\int Pdx} = \int Q e^{\int Pdx} dx + c$$

When a linear differential equation is multiplied by a factor $e^{\int Pdx}$ and the eqn become readily integrable. Such a factor is called Integrating factor.

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Problem

$$(1-x^2)\frac{dy}{dx} - xy = 1$$

$$\Rightarrow \frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}$$

Hene $P(x) = -\frac{x}{1-x^2}$

$Q(x) = \frac{1}{1-x^2}$

$$I.F = e^{\int -\frac{x}{1-x^2}dx} = e^{\frac{1}{2}\ln(1-x^2)} = e^{\ln(1-x^2)^{\frac{1}{2}}} = \sqrt{1-x^2}$$

Now,

$$\sqrt{1-x^2}\frac{dy}{dx} - \frac{x}{1-x^2}\sqrt{1-x^2}\,y = \frac{\sqrt{1-x^2}}{1-x^2}$$

$$\Rightarrow \frac{d}{dx}\left(y\sqrt{1-x^2}\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y\sqrt{1-x^2} = \int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c$$

$$\therefore y = \frac{\sin^{-1}x}{\sqrt{1-x^2}} + \frac{c}{\sqrt{1-x^2}}$$

Problem

$$x\frac{dy}{dx} + 2y = x^2\log x$$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = x\log x$$

Here, $P(x) = \frac{2}{x}$

$Q(x) = x\log x$

$$I.F = e^{\int\frac{2}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$$

Now,

$$x^2 \frac{dy}{dx} + \frac{2}{x} \cdot x^2 y = x^2 \cdot x \log x$$

$$\Rightarrow \frac{d}{dx}(y \cdot x^2) = x^3 \log x$$

$$\Rightarrow yx^2 = \int x^3 \log x \, dx$$

$$= \log x \int x^3 dx - \int \left\{ \frac{d}{dx} \log x \int x^3 dx \right\} dx$$

$$= \frac{x^4}{4} \log x - \int \frac{x^3}{4} dx$$

$$= \frac{x^4}{4} \log x - \frac{x^4}{16} + c$$

$$\therefore yx^2 = \frac{1}{4} x^4 \log x - \frac{1}{16} x^4 + c$$

Bernoulli's differential Equation (Non-linear)

$$\frac{dy}{dx} + P(x) y = Q(x) y^n \quad ; n \neq 0, 1$$

$$\Rightarrow y^{-n} \frac{dy}{dx} + P(x) y^{1-n} = Q(x) \quad \text{——— (1)}$$

let $y^{1-n} = v$

$$\Rightarrow (1-n) y^{-n} \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dv}{dx}$$

From equation (1)

$$\frac{1}{1-n} \cdot \frac{dv}{dx} + P(x) v = Q(x)$$

$$\Rightarrow \frac{dv}{dx} + (1-n) P(x) v = (1-n) Q(x)$$

Problem

$$\frac{dy}{dx} = x^3y^3 - xy$$

$$\Rightarrow y^{-3}\frac{dy}{dx} = x^3 - xy^{-2}$$

$$\Rightarrow y^{-3}\frac{dy}{dx} + xy^{-2} = x^3 \quad \text{——} (1)$$

Let $y^{-2} = v$

$$\Rightarrow -2y^{-3}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$$

From equation (1)

$$-\frac{1}{2}\frac{dv}{dx} + xv = x^3$$

$$\Rightarrow \frac{dv}{dx} - 2xv = -2x^3$$

$$I.F = e^{-\int 2x\,dx} = e^{-x^2}$$

Now,

$$e^{-x^2}\frac{dv}{dx} - 2xe^{-x^2}v = -2x^3e^{-x^2}$$

$$\Rightarrow \frac{d}{dx}(ve^{-x^2}) = -2x^3e^{-x^2}$$

$$\Rightarrow ve^{-x^2} = -\int 2x^3e^{-x^2}\,dx$$

$$= -\int x^2e^{-x^2}(2x\,dx)$$

Let $x^2 = z$
$2x\,dx = dz$

$$= -\int ze^{-z}\,dz$$

$$= -\left[z(-e^{-z}) - e^{-z}\right] + c$$

$$\Rightarrow y^{-2} e^{-x^2} = -\left[x^2 (-e^{-x^2}) - e^{-x^2} \right] + c$$

$$\Rightarrow y^{-2} e^{-x^2} = x^2 e^{-x^2} + e^{-x^2} + c$$

$$\Rightarrow \frac{e^{-x^2}}{y^{2}} = x^2 e^{-x^2} + e^{-x^2} + c$$

$$\Rightarrow \frac{1}{y^2} = x^2 + 1 + \frac{c}{e^{-x^2}}$$

$$\therefore \frac{1}{y^2} = x^2 + 1 + c e^{x^2} \quad \text{(Ans)}.$$

Problem

$$x \frac{dy}{dx} - 2y = x y^4$$

$$\Rightarrow y^{-4} x \frac{dy}{dx} - 2y^{-3} = x$$

$$\Rightarrow y^{-4} \frac{dy}{dx} - \frac{2}{x} y^{-3} = 1 \quad \text{——— (1)}$$

Let $y^{-3} = v$

$$\Rightarrow -3y^{-4} \frac{dy}{dx} = \frac{dv}{dx}$$

~~From~~

$$\Rightarrow y^{-4} \frac{dy}{dx} = -\frac{1}{3} \frac{dv}{dx}$$

From equation (1)

$$-\frac{1}{3} \frac{dv}{dx} - \frac{2}{x} v = 1$$

$$\Rightarrow \frac{dv}{dx} + \frac{6}{x} v = -3$$

$$I.F = e^{\int \frac{6}{x} dx} = e^{6 \ln x} = e^{\ln x^6} = x^6$$

Now, $x^6 \frac{dv}{dx} + \frac{6}{x} x^6 v = -3x^6$

$$\Rightarrow \frac{d}{dx}(x^6 v) = -3x^6$$

$$\Rightarrow x^6 v = -\int 3x^6$$

$$\Rightarrow x^6 v = -3\frac{x^7}{7} + c$$

$$\Rightarrow x^6 y^{-3} = -\frac{3}{7}x^7 + c$$

$$\Rightarrow \frac{1}{y^3} = -\frac{3x}{7}$$

Exact differential equation

$M\,dx + N\,dy = 0$ is said to be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$y\,dx + x\,dy = 0$

$\Rightarrow d(xy) = 0$

$\Rightarrow xy = c$

$M = y$

$\Rightarrow \frac{\partial M}{\partial y} = 1$

$N = x$

$\Rightarrow \frac{\partial N}{\partial x} = 1$

$M\,dx + N\,dy = 0$

Integrating M with respect to x treating y as a constant.

In N terms free from x then integrate with respect to y

Then add both the terms.

Problem

$$(y^4 + 4x^3y + 3x)\,dx + (x^4 + 4xy^3 + y + 1)\,dy = 0 \quad \text{——} \; (1)$$

Here $M = y^4 + 4x^3y + 3x$

$$\Rightarrow \frac{\partial M}{\partial y} = 4y^3 + 4x^3$$

$$N = x^4 + 4xy^3 + y + 1$$

$$\Rightarrow \frac{\partial N}{\partial x} = 4x^3 + 4y^3$$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, therefore equation (1) is an exact differential equation.

Now, integrating M with respect to x treating as a constant

$$\text{i.e} = \cancel{4}\frac{x^4}{\cancel{4}}y +$$

$$\text{i.e} = xy^4 + x^4y + \frac{3}{2}x^2$$

In N terms free from x is $y+1$. Now integrating with respect to y and is equal to $\frac{y^2}{2} + y$

Then adding we have

$$xy^4 + x^4y + \frac{3}{2}x^2 + \frac{y^2}{2} + y = c \quad \text{(Ans).}$$

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Problem

$x(x^2+y^2-a^2)dx + y(x^2-y^2-b^2)dy = 0$ ——— ①

Here, $M = x^3 + xy^2 - xa^2$

$\Rightarrow \frac{\partial M}{\partial y} = 2xy$

$N = x^2y - y^3 - yb^2$

$\Rightarrow \frac{\partial N}{\partial x} = 2xy$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ therefore equation ① is an exact differential equation.

Now integrating M with respect to x creating y as a constant.

i.e $= \cancel{3x^2} + \frac{x^4}{4} + \frac{x^2y^2}{2} - \frac{x^2a^2}{2}$

In N terms free from x is $-y^3-yb^2$. Now integrating with respect to y and is equal to $-\frac{y^4}{4} - \frac{y^2b^2}{2}$

Then adding we have

$\frac{x^4}{4} + \frac{x^2y^2}{2} - \frac{x^2a^2}{2} - \frac{y^4}{4} - \frac{y^2b^2}{2} = c$

$\Rightarrow x^4 + 2x^2y^2 - 2x^2a^2 - y^4 - 2y^2b^2 = 4c$

$\therefore x^4 - y^4 + 2x^2y^2 - 2a^2x^2 - 2b^2y^2 = c_1$ (Ans).

$Mdx + Ndy = 0$ but $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Rule I :

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ a function of x only then $e^{\int f(x)\,dx}$ is an integrating factor.

Rule II :

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$, a function of y only then $e^{-\int g(y)\,dy}$ is an integrating factor

Problem

$(x^2+y^2+x)\,dx + xy\,dy = 0$ —— ①

Here $M = x^2+y^2+x$

$\Rightarrow \frac{\partial M}{\partial y} = 2y$

$N = xy$

$\Rightarrow \frac{\partial N}{\partial x} = y$

since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then equation ① is not exact.

Now, $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{2y - y}{xy} = \dfrac{1}{x} = f(x)$

$\therefore$ I.F $= e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

Multiplying equation ① by I.F

$x(x^2+y^2+x)\,dx + x^2y\,dy = 0$ ——— ②

Here $M = x^3 + xy^2 + x^2$

$\Rightarrow \frac{\partial M}{\partial y} = 2xy$

$N = x^2y$

$\Rightarrow \frac{\partial N}{\partial x} = 2xy$

since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ therefore equation ② is an exact differential equation.

Now integrating M with respect to x treating y as a constat

i.e $= \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3}$

In N terms free from x is 0, ~~an~~

Now. integrating with respect to y and is equal to c

Then adding we have

$\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + c = C$

$\Rightarrow 3x^4 + 6x^2y^2 + 4x^3 + 12c = 12C$

$\therefore 3x^4 + 6x^2y^2 + 4x^3 = c_1$ (Ans).

Higher order differential equation with constant co-efficient (Linear) :

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + P_n y = X \quad [n>1]$$

where $P_1, P_2, P_3, \ldots, P_n$ are constant and X is a function of x

Trial solution

$$y = e^{mx}$$

$$\frac{dy}{dx} = m e^{mx}$$

$$\vdots$$

$$\frac{d^n y}{dx^n} = m^n e^{mx}$$

$$\therefore m^n e^{mx} + P_1 m^{n-1} e^{mx} + P_2 m^{n-2} e^{mx} + \ldots + P_n e^{mx} = 0$$

$$\Rightarrow (m^n + P_1 m^{n-1} + P_2 m^{n-2} + \ldots + P_n) e^{mx} = 0$$

$$\therefore m^n + P_1 m^{n-1} + P_2 m^{n-2} + \ldots + P_n = 0 \quad [\because e^{mx} \neq 0]$$

$$m = m_1, m_2, \ldots, m_n$$

General solution

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$$

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Problem

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

Let $y = e^{mx}$ be a trial solution

$$\therefore \frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2 e^{mx}$$

$$\therefore m^2 e^{mx} + 5me^{mx} + 6e^{mx} = 0$$

$$\Rightarrow (m^2 + 5m + 6)e^{mx} = 0$$

$$\therefore m^2 + 5m + 6 = 0$$

$$\Rightarrow m^2 + 3m + 2m + 6 = 0$$

$$\Rightarrow m(m+3) + 2(m+3) = 0$$

$$\Rightarrow (m+2)(m+3) = 0$$

$$\therefore m = -2, -3$$

$$\therefore y = c_1 e^{-2x} + c_2 e^{-3x}$$ (Ans).

Problem

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

Let $y = e^{mx}$ be a trial solution

$$\therefore \frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2 e^{mx}$$

$$\therefore m^2 e^{mx} + 4me^{mx} + 4e^{mx} = 0$$

$$\Rightarrow (m^2 + 4m + 4)\,e^{mx} = 0$$

$$\Rightarrow m^2 + 4m + 4 = 0$$

$$\Rightarrow m^2 + 2 \cdot m \cdot 2 + 2^2 = 0$$

$$\Rightarrow (m+2)^2 = 0$$

$$\therefore m = -2, -2$$

$$\therefore y = (c_1 + c_2 x)\,e^{-2x}$$

Problem

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Let $y = e^{mx}$ be a trial solution

$$\therefore \frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2 e^{mx}$$

$\therefore m^2 e^{mx} - 2me^{mx} + e^{mx} = 0$

$\Rightarrow (m^2 - 2m + 1)e^{mx} = 0$

$\Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\therefore m = 1, 1$

$\therefore y = (c_1 + c_2 x)e^x$

Problem

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{——— (1)}$$

Let $y = e^{mx}$ be a trial solution of (1)

Then $\frac{dy}{dx} = me^{mx}$

$\therefore \frac{d^2y}{dx^2} = m^2 e^{mx}$

$\therefore$ Auxiliary equation of (1) is

$m^2 e^{mx} + e^{mx} = 0$

$\Rightarrow m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm\sqrt{-1}$

$\Rightarrow m = \pm i$

If $m = a \pm ib$

$$\boxed{y = e^{ax}(c_1 \cos bx + c_2 \sin bx)}$$

$\therefore y = c_1 \cos x + c_2 \sin x$

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Problem

$(D^3 - 4D^2 + 5D - 2)y = 0$

where $D = \frac{d}{dx}$

$\therefore \left(\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2\right)y = 0$

Let $y = e^{mx}$ be a trial solution

$\therefore \frac{dy}{dx} = me^{mx}$

$\frac{d^2y}{dx^2} = m^2e^{mx}$

$\frac{d^3y}{dx^3} = m^3e^{mx}$

Auxiliary equation of (1) is

$(m^3e^{mx} - 4m^2e^{mx} + 5me^{mx} - 2)e^{mx} = 0$

$\Rightarrow m^3 - 4m^2 + 5m - 2 = 0$

$\therefore m = 1, 1, 2$

$\therefore y = (c_1 + c_2x)e^x + c_3e^{2x}$

Problem

$(D^3 - 4D^2 + 5D - 2)\, y = e^{3x}$

Auxiliary equation is $m^3 - 4m^2 + 5m - 2 = 0$

$\therefore m = 1, 1, 2$

$\therefore y_c = (c_1 + c_2 x)e^{x} + c_3 e^{2x}$

$f(D)y = e^{ax}$

$f(P) = \frac{1}{f(D)} e^{ax}$

$= \frac{1}{f(a)} e^{ax}$

when $f(a) \neq 0$

$$y_p = \frac{1}{D^3 - 4D^2 + 5D - 2} e^{3x}$$

$$= \frac{1}{3^3 - 4\times 3^2 + 5\times 3 - 2} e^{3x}$$

$$= \frac{1}{4} e^{3x}$$

The complete solution is $y = y_c + y_p$

$$\therefore y = (c_1 + c_2 x)e^{x} + c_3 e^{2x} + \frac{e^{3x}}{4}$$

Problem

$(D^2 - 3D + 2)y = e^{x}$

Auxiliary equation is $m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\therefore m = 1, 2$

$\therefore y_c = c_1 e^{x} + c_2 e^{2x}$

$$y_p = \frac{1}{D^2-3D+2} e^x$$

$$= \frac{1}{\cancel{1^2 - 3\times 1 + 1}} e^x$$

$$= \frac{1}{(D-2)(D-1)} e^x$$

$$= \frac{1}{D-1}\left(\frac{1}{D-2} e^x\right)$$

$$= \frac{-1}{D-1} e^x$$

$$= \frac{-x}{1!} e^x$$

$$= -xe^x$$

$$f(p) = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{(D-a)^n} e^{ax}$$

$$= \frac{x^n}{n!} e^{ax}$$

when $f(a)=0$

$\therefore$ The complete solution is $y = y_c + y_p$

$\therefore y = c_1 e^x + c_2 e^{2x} - xe^x$

Problem

$(D^2-1)y = e^{2x}$

Auxiliany equation $m^2 - 1 = 0$

$\Rightarrow m^2 = 1$

$\therefore m = -1, 1$

$\therefore y_c = c_1 e^{-x} + c_2 e^x$

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$$y_p = \frac{1}{D^2-1} e^{2x}$$

$$= \frac{1}{3} e^{2x}$$

Problem

$(D^2-1)y = e^{-x}$

Auxiliary equation $\quad m^2-1=0$

$\rightarrow m^2 = 1$

$m = -1, 1$

$$y_c = c_1 e^{-x} + c_2 e^{x}$$

$$y_p = \frac{1}{D^2-1} e^{-x}$$

$$= \frac{1}{(D+1)(D-1)} e^{-x}$$

$$= \frac{1}{D+1}\left[\frac{1}{D-1} e^{-x}\right]$$

$$= \frac{1}{D+1}\left[-\frac{1}{2} e^{-x}\right]$$

$$= -\frac{1}{2}\frac{1}{D+1} e^{-x} = -\frac{1}{2}\frac{x}{1!} e^{-x}$$

$$= -\frac{1}{2} x e^{-x}$$

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Problem

$(D^2-1)y = x^2$

Auxiliary equation is $m^2 - 1 = 0$

$m = \pm 1$

$\therefore y_c = c_1 e^{-x} + c_2 e^{x}$

$$y_p = \frac{1}{D^2-1} x^2$$

$$= \frac{1}{-(1-D^2)} x^2$$

$$= -(1-D^2)^{-1} x^2$$

$$= -(1+D^2+D^4+D^6+\cdots) x^2$$

$$= -(x^2+2)$$

$$= -x^2-2 \quad \text{Ans}$$

$\therefore y = y_c + y_p$

Problem

$(D^2-2D+1)y = x+1$

Auxiliary equation is $m^2-2m+1=0$

$\therefore m = 1, 1$

$\therefore y_c = (c_1 + c_2 x)e^x$

$$y_p = \frac{1}{D^2-2D+1}(x+1)$$

$$= \frac{1}{(D-1)^2}(x+1)$$

$$= \frac{1}{(1-D)^2}(x+1)$$

$$= (1-D)^{-2}(x+1)$$

$$= [1 + 2D + 3D^2 + \cdots](x+1)$$

$$= (x+1+2) = x+3$$

$$= \cancel{\frac{1}{1+(D^2-2D)}(x+1)}$$

$$= \cancel{[1+(D^2-2D)]^{-1}(x+1)}$$

$$= \cancel{[1-(D^2-2D)+(D^2-2D)^2 - \cdots](x+1)}$$

Problem

[sin and cos এর বেলায় সূত্র same হবে]

$(D^2-1)y = \sin 2x$

Auxiliary equation is $m^2 - 1) = 0$

$\Rightarrow m^2 = 1$

$\therefore m = \pm 1$

$\therefore y_c = c_1 e^{x} + c_2 e^{-x}$

$y_p = \dfrac{1}{f(D^2)} \sin ax$ or $\cos ax$

$= \dfrac{1}{f(-a^2)}$ when $f(-a^2) \neq 0$

[D^2 এর পরিবর্তে $-a^2$ বসিয়ে কাজ করতে হবে]

$y_p = \dfrac{1}{D^2-1} \sin 2x$

$= \dfrac{1}{(-2)^2 - 1}$ ~~sin~~ $\sin 2x$

$= -\dfrac{1}{5} \sin 2x$

$\therefore y = y_c + y_p = c_1 e^{x} + c_2 e^{-x} - \dfrac{1}{5} \sin 2x$

Problem

$(D^2 - 2D + 1)y = \sin 2x$

Auxiliary equation is $m^2 - 2m + 1 = 0$

$\therefore m = 1, 1$

$\therefore y_c = (c_1 + c_2 x) e^{x}$

$$y_p = \frac{1}{D^2 - 2D + 1} \sin 2x$$

$$= \frac{1}{-2^2 - 2D + 1} \sin 2x$$

$$= \frac{1}{-3 - 2D} \sin 2x$$

$$= \frac{-3 + 2D}{(-3 + 2D)(-3 - 2D)} \sin 2x$$

$$= \frac{-3 + 2D}{9 - 4D^2} \sin 2x$$

$$= \frac{-3 + 2D}{9 + 4 \times 2^2} \sin 2x$$

$$= \frac{-3 + 2D}{25} \sin 2x$$

$$= \frac{1}{25} \left(-3 \sin 2x + 4 \cos 2x \right)$$

$$\therefore y = y_c + y_p$$

$$= (c_1 + c_2 x) e^x + \frac{1}{25} \left(-3 \sin 2x + 4 \cos 2x \right)$$

Problem : $(D^2+1)y = \sin x$

Auxiliary equation is $m^2+1=0$

$m^2 = -1$

$\therefore m = \pm i$

$\therefore y_c = c_1\cos x + c_2\sin x$

$y_p = \frac{1}{D^2+1}\sin x$

$=$

Problem

$(D^2-9)y = xe^{3x}$

Auxiliary equation is $m^2-9=0$

$\Rightarrow m^2 = 9$

$\therefore m = \pm 3$

$y_c = c_1 e^{3x} + c_2 e^{-3x}$

$y_p = \frac{1}{f(D)} ve^{ax}$

where v is any function $= e^{ax} \frac{1}{f(D+a)} v$

$$y_p = \frac{xe^{3x}}{D^2-9}$$

$$= \frac{1}{D^2-9} xe^{3x}$$

$$= e^{3x} \frac{1}{(D+3)^2-9} x$$

$$= e^{3x} \frac{1}{D^2+6D} x$$

$$= e^{3x} \frac{1}{D(D+6)} x$$

$$= e^{3x} \frac{1}{D} \left[\frac{1}{D+6} x\right]$$

$$= e^{3x} \frac{1}{D} \cdot \frac{1}{6} \left[1+\frac{D}{6}\right]^{-1} x$$

$$= \frac{e^{3x}}{6} \frac{1}{D} \left[1 - \frac{D}{6} + \frac{D^2}{36} - \cdots\right] x$$

$$= \frac{e^{3x}}{6} \frac{1}{D}\left(x - \frac{1}{6}\right)$$ বুঝিনি

$\frac{1}{D} \rightarrow$ Integration

$$= \frac{e^{3x}}{6}\left(\frac{x^2}{2} - \frac{x}{6}\right)$$

$$= \frac{1}{12} x^2 e^{3x} - \frac{1}{36} x e^{3x}$$

$\therefore y = y_c + y_p$

Problem

$(D^2 - 2D + 4)\, y = e^x \cos x$

Auxiliary equation is $m^2 - 2m + 4 = 0$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4\times4\times1}}{2\times1}$$

$$= \frac{2 \pm \sqrt{4-16}}{2}$$

$$= \frac{3 \pm i\sqrt{12}}{2}$$

$$= \frac{3}{2} \pm i\sqrt{3}$$

$$\therefore y_c = e^{\frac{3}{2}x}\left(c_1 \cos\sqrt{3}x \pm c_2 \sin\sqrt{3}x\right)$$

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$$y_p = \frac{e^x \cos x}{e^x\, D^2 - 2D + 4}$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^2 + \cancel{2D} + 1 - \cancel{2D} - 2 + 4} \cos x$$

$$= e^x \frac{1}{D^2 + 3} \cos x$$

$$= e^x \frac{1}{-1^2 + 3} \cos x$$

$$= \frac{1}{2} e^x \cos x$$

$$\therefore y = y_c + y_p = e^{\frac{3}{2}x}\left(c_1 \cos\sqrt{3}x \pm c_2 \sin\sqrt{3}x\right) + \frac{1}{2} e^x \cos x$$

[$f(D)\,y = xv$

$\therefore y_p = \frac{1}{f(D)} xv$

v = cosine on sine function on exponential function]

Problem

$(D^2 + 4)\,y = x \sin x$

Auxilialy equation is $m^2 + 4 = 0$

$\Rightarrow m^2 = -4$

$\Rightarrow m^2 = 4i^2$

$\therefore m = \pm 2i$

$$y_c = (c_1 \cos 2x \pm c_2 \sin 2x)$$

$$\left[xe^{ix} = x\cos x \pm ix\sin x\right]$$

बुझिनाति

$$y_p = \frac{1}{D^2+4}\, x\sin x$$

$$= \text{Imaginary part of } \frac{1}{D^2+4}\, x e^{ix}$$

$$= \text{I.P of } e^{ix}\frac{1}{(D+i)^2+4}\, x$$

$$= \text{I.P of } e^{ix}\frac{1}{D^2+2iD-1+4}\, x$$

$$= \text{I.P of } e^{ix}\frac{1}{D^2+2iD+3}\, x$$

$$= \text{I.P of } e^{ix}\frac{1}{3\left(1+\frac{2}{3}iD+\frac{D^2}{3}\right)}\, x$$

$$= \text{I.P of } \frac{e^{ix}}{3}\left[1+\frac{2iD}{3}+\frac{D^2}{3}\right]^{-1} x$$

$$= \text{I.P of } \frac{e^{ix}}{3}\left[1-\left(\frac{2iD}{3}+\frac{D^2}{3}\right)+\left(\frac{2iD}{3}+\frac{D^2}{3}\right)^2 - \cdots\cdots\right] x$$

$$= \text{I.P of } \frac{e^{ix}}{3}\left(x-\frac{2i}{3}\right)$$

$$= \text{I.P of } \frac{1}{3}(\cos x + i\sin x)\left(x-\frac{2i}{3}\right)$$

$$= \frac{1}{3}\left(-\frac{2}{3}\cos x + x\sin x\right)$$

$$= -\frac{2}{9}\cos x + \frac{1}{3}x\sin x$$

Problem

$(D^2-1)\,y = x^2 \cos x$

Auxiliary equation is $m^2 - 1 = 0$

$\therefore m = \pm 1$

$y_c = c_1 e^{-x} + c_2 e^{x}$

$$y_p = \frac{1}{D^2-1} x^2 \cos x$$

$$= \text{Real part of } e^{ix} \frac{1}{D^2-1} x^2$$

$$= \text{Real part of } e^{ix} \frac{1}{(D+i)^2-1} x^2$$

$$= \text{R.P of } e^{ix} \frac{1}{D^2+2iD-2} x^2$$

$$= \text{R.P of } e^{ix} \frac{1}{2\left(-1+iD+\frac{D^2}{2}\right)} x^2$$

$$= \text{R.P of } e^{ix} \frac{1}{2\left(\frac{D^2+2iD}{2}-1\right)} x^2$$

$$= \text{R.P of } -\frac{e^{ix}}{2}\left[1-\frac{D^2+2iD}{2}\right]^{-1} x^2$$

$$= \text{R.P of } -\frac{e^{ix}}{2}\left[1+\frac{D^2+2iD}{2}+\left(\frac{D^2+2iD}{2}\right)^2+\cdots\right] x^2$$

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Problem :

$(D^2-4D+4)y = 3x^2 e^{2x} \sin 2x$

Auxiliary equation is $m^2-4m+4 = 0$

$\Rightarrow m^2 - 2.m.2 + 2^2 = 0$

$\Rightarrow (m-2)^2 = 0$

$\therefore m = 2, 2$

$\therefore y_c = (c_1 + c_2 x) e^{2x}$

$$y_p = \frac{1}{D^2-4D+4} 3x^2 e^{2x} \sin 2x$$

$$= 3 \frac{1}{(D-2)^2} x^2 e^{2x} \sin 2x$$

$$= 3e^{2x} \frac{1}{(D+2-2)^2} x^2 \sin 2x$$

$$= 3e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

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$$= 3e^{2x}\frac{1}{D}\left[-\frac{x^2}{2}\cos 2x + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right]$$

$$= 3e^{2x}\left[-\frac{x^2\sin 2x}{4} - \frac{x\cos 2x}{4} + \frac{\sin 2x}{4} - \frac{x\cos 2x}{4} + \frac{\sin 2x}{8} + \frac{\sin 2x}{8}\right]$$

$$= \frac{3}{4}e^{2x}\left[-x^2\sin 2x - x\cos 2x + \sin 2x - x\cos 2x + \frac{\sin 2x}{2} + \frac{\sin 2x}{2}\right]$$

$$= \frac{3}{4}e^{2x}\left[-x^2\sin 2x - x\cos 2x + \sin 2x - x\cos 2x + \sin 2x\right]$$

$$= -\frac{3}{4}e^{2x}\left[2x\cos 2x + (x^2-2)\sin 2x\right]$$

$\therefore\ y = y_c + y_p$

$$= (c_1 + c_2 x)e^{2x} - \frac{3}{4}e^{2x}\left[2x\cos 2x + (x^2-2)\sin 2x\right]$$

Solution of higher order DE with variable co-efficient

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y$$

Let $x = e^z$

$\Rightarrow \log x = z$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$$

$$\therefore x \frac{dy}{dx} = \frac{dy}{dz}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{d\cancel{z}}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{d^2y}{dz^2}\frac{1}{x} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x^2}\left\{\frac{d^2y}{dz^2} - \frac{dy}{dz}\right\}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Homogeneous linear equation

$$x^2 \frac{d^2y}{dx^2} + y = 3x^2 \quad \text{——} (1)$$

Solution:

Let $x = e^z$

$\log x = z$ —— (II)

$$x \frac{dy}{dx} = \frac{dy}{dz} \quad \text{——} (III)$$

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dx}$$

using equation (III) and (IV) in (1) we get

$$\frac{d^2y}{dz^2} - \frac{dy}{dx} + y = 3e^{2z} \quad \text{——} (V)$$

Let $y = e^{mz}$ be the trial solution of

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + y = 0$$

$\therefore$ Auxiliary equation is $m^2 - m + 1 = 0$

$$\Rightarrow m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4\times1\times1}}{2\times1}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

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$\therefore y_c = e^{\frac{1}{2}z}\left(c_1 \cos \frac{\sqrt{3}}{2}z \pm c_2 \sin \frac{\sqrt{3}}{2}z\right)$

$y_p = \frac{1}{D^2 - D + 1} 3e^{2z}$

$\Rightarrow 3 \frac{1}{2^2 - 2 + 1} e^{2z}$

$= e^{2z}$

$\therefore y = y_c + y_c$

$=$

Problem:

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4 \quad \text{——— (i)}$$

Solution: Let $x = e^z$

$$\log x = z$$

$$\therefore \quad x\frac{dy}{dx} = \frac{dy}{dz} \quad \text{——— (ii)}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \quad \text{——— (iii)}$$

using equation (ii) and (iii) in (i) we get

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} - 2\frac{dy}{dz} - 4y = x^4$$

$$\Rightarrow \frac{d^2y}{dz^2} - \frac{3dy}{dz} - 4y = x^4 \quad \text{——— (IV)}$$

Let $y = e^{mz}$ be the trial solution of equation (IV)

$\therefore$ Auxiliary equation is $m^2 - 3m - 4 = 0$

$$\Rightarrow m^2 - 4m + m - 4 = 0$$

$$\Rightarrow m(m-4) + 1(m-4) = 0$$

$$\Rightarrow (m-4)(m+1) = 0$$

$$\therefore \quad m = (4, -1)$$

$$\therefore \quad y_c = c_1 e^{4z} + c_2 e^{-z}$$

$$y_p = \frac{1}{D^2 - 3D - 4} e^{4z}$$

$$= \cancel{(D-4)^2 - 3(D-4) - 4}$$

$$= \frac{1}{(D-4)(D+1)} e^{4z}$$

$$= \frac{1}{D-4} \left[\frac{e^{4z}}{D+1} \right]$$

$$= \frac{1}{D-4} \frac{e^{4z}}{5}$$

$$= \frac{1}{5} \frac{e^{4z}}{D-4}$$

$$= \frac{1}{5} e^{4z} \frac{z}{1!}$$

$$= \frac{z e^{4z}}{5}$$

$$= \frac{1}{5} \log x \cdot x^4$$

$$= \frac{1}{5} x^4 \log x$$

$$y = y_c + y_p$$

Differential Equation when dependent variable absent :

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \frac{d^{n-2} y}{dx^{n-2}}, \text{------}, \frac{dy}{dx}, \cancel{x}\right) = 0 \quad \text{—— (i)}$$

Let us put, $p = \frac{dy}{dx}$, $\frac{d^2y}{dx^2} = \frac{dp}{dx}$ .... etc.

Then (i) becomes

$$f\left(\frac{d^{n-1}p}{dx^{n-1}}, \frac{d^{n-2}p}{dx^{n-2}}, \text{------}, p, x\right) = 0 \quad \text{—— (ii)}$$

$\therefore p = F(x)$

$\Rightarrow \frac{dy}{dx} = F(x)$

$\therefore y = \int F(x)\,dx + c$

which is required solution.

[p = only first order]

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \frac{d^{n-2} y}{dx^{n-2}}, \text{----}, \frac{d^k y}{dx^k}, x\right) = 0 \quad \text{—— (i)}$$

Let us put $q = \frac{d^k y}{dx^k}$, $\frac{d^2y}{dx^2} = \frac{d}{dx}$ .... etc.

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Problem:

$$2\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 + 4 = 0 \quad \text{------} \quad (1)$$

Solution:

since the equation is free from y then

put $\frac{dy}{dx} = p$

$$\frac{d^2y}{dx^2} = \frac{dp}{dx}$$

$$\therefore 2\frac{dp}{dx} - p^2 + 4 = 0$$

$$\Rightarrow 2\frac{dp}{dx} = p^2 - 4$$

$$\Rightarrow \frac{2dp}{p^2-4} = dx$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{p-2} - \frac{1}{p+2}\right) dp = dx$$

$$\Rightarrow \frac{1}{2}\ln(p-2) - \frac{1}{2}\frac{1}{p+2}\, dp = dx$$

$$\Rightarrow \frac{1}{2}\ln(p-2) - \frac{1}{2}\ln(p+2) = x + \ln c_1$$

$$\Rightarrow \frac{1}{2}\ln\frac{p-2}{p+2} = x + \ln c_1$$

$$\Rightarrow \ln\frac{p-2}{p+2} = 2x + \ln c_1^2$$

$$\Rightarrow \ln\frac{p-2}{p+2} = \ln e^{2x} + \ln c_1^2$$

$$\Rightarrow \ln\frac{p-2}{p+2} = \ln e^{2x}. c_1^2$$

$$\Rightarrow \frac{p-2}{p+2} = e^{2x} c_1^2$$

$\Rightarrow p-2 = (p+2)\, e^{2x} c_1^2$

$\Rightarrow p - pe^{2x}c_1^2 = 2 + 2e^{2x}c_1^2$

$$\Rightarrow p = \frac{2(1+e^{2x}+c_1^2)}{(1-e^{2x}c_1^2)} = \frac{dy}{dx}$$

$$= 2\left(1 + \frac{2e^{2x}c_1^2}{1-e^{2x}c_1^2}\right)$$

$$= 2x + 2\int \frac{2c_1^2 e^{2x}}{1-e^{2x}c_1^2}\,dx$$

$$\therefore y = 2x - 2\ln(1-e^{2x}c_1^2) + c_2$$

~~Problem~~

Lowest coefficient এর order $\neq 1$ হলে

$$f\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots\ldots, \frac{d^k y}{dx^k}, x\right) = 0, \quad k \neq 1$$

Put $\frac{d^k y}{dx^k} = q$

* $\frac{d^4 y}{dx^4} - \cot x \frac{d^3 y}{dx^3} = 0$ ------ (1)

Solution:

Since (1) doesn't contain y directly and the lowest differential coefficient is $\frac{d^3y}{dx^3}$, then

put $\frac{d^3 y}{dx^3} = q$

$\Rightarrow \frac{d^4 y}{dx^4} = \frac{dq}{dx}$ in (1)

$\therefore \frac{dq}{dx} - \cot x \cdot q = 0$

$\Rightarrow \frac{dq}{q} - \cot x \, dx = 0$

$\Rightarrow \ln q - \ln \sin x = \ln c_1$

$\Rightarrow \ln q = \ln(\sin x \cdot c_1)$

$\therefore q = c_1 \sin x$

$\Rightarrow \frac{d^3y}{dx^3} = c_1 \sin x$

$\Rightarrow \frac{d^2y}{dx^2} = -c_1 \cos x + c_2$

$\Rightarrow \frac{dy}{dx} = -c_1 \sin x + c_2 x + c_3$

$\therefore y = c_1 \cos x + c_2 \frac{x^2}{2} + c_3 x$

When independent variable absent

$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \ldots ; \frac{dy}{dx}, y\right) = 0$

put,

$\frac{dy}{dx} = p$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dy} \cdot \frac{dy}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = p \cdot \frac{dp}{dy}$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 = 0 \quad \text{———} \quad (1)$$

Solution:

Since (1) does not contain x directly and the lowest differential coefficient is $\frac{dy}{dx}$, then put

$$\frac{dy}{dx} = p$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dp}{dy} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = p \cdot \frac{dp}{dy} \quad \text{in } (1)$$

$$\therefore \; p\frac{dp}{dy} + p + p^3 = 0$$

$$\Rightarrow \frac{dp}{dy} + 1 + p^2 = 0$$

$$\Rightarrow \frac{dp}{1+p^2} + dy = 0$$

$$\Rightarrow \tan^{-1}p + y = c_1$$

$$\Rightarrow \tan^{-1}p = c_1 - y$$

$$\Rightarrow p = \tan(c_1 - y)$$

$$\Rightarrow \frac{dy}{dx} = \tan(c_1 - y)$$

$$\Rightarrow \frac{dy}{\tan(c_1 - y)} = dx$$

$$\Rightarrow \cot(c_1 - y)\,dy = dx$$

$$\Rightarrow -\log\sin(c_1 - y) = x + \log c_2$$

$$\Rightarrow \log\sin(c_1 - y) = -x - \log c_2$$

$$\Rightarrow \log\sin(c_1 - y) = \log e^{-x} - \log c_2$$

$$= \log\frac{e^{-x}}{c_2}$$

$$\Rightarrow \cancel{\log}\sin(c_1 - y) = \frac{e^{-x}}{c_2}$$

$$\therefore c_2\sin(c_1 - y) = e^{-x}$$

Solution of 1st order DE but not of 1st degree

The differential eqn of the form

$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots + P_n = 0$ ——— (1)

where $P_1, P_2, P_3, \ldots$ are fns of x and y and $p = \frac{dy}{dx}$.

Factorize (1)

$[p - f_1(x)][p - f_2(x)] \ldots\ldots [p - f_n(x)] = 0$

Now,

$p - f_1(x) = 0$

$\Rightarrow p = f_1(x)$

$\Rightarrow \frac{dy}{dx} = f_1(x)$

$\Rightarrow y = \int f_1(x)\,dx + c_1$ or $\left\{ \begin{array}{l} \Rightarrow y = F_1(x) + c_1 \\ \Rightarrow (y - F_1(x) - c_1) = 0 \end{array} \right.$

$\Rightarrow y = \phi(x, c_1)$

$\therefore y - Q_1 = 0$

$p = f_2(x)$

$\Rightarrow y = \phi(x, c_2)$

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$p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$ ----- (1)

Solution: On factorization, the given eqn becomes

$p(p+2x)(p-y^2) = 0$ ——— (ii)

Then from (ii) we have,

$p = 0$

$\Rightarrow \frac{dy}{dx} = 0$

$\Rightarrow y = c$

$p + 2x = 0$

$\Rightarrow \frac{dy}{dx} + 2x = 0$

$\Rightarrow dy + 2x\,dx = 0$

$\Rightarrow y + x^2 = c$

$\therefore y + x^2 - c = 0$

$p - y^2 = 0$

$\Rightarrow \frac{dy}{dx} = y^2$

$\Rightarrow \frac{dy}{y^2} = dx$

$\Rightarrow -\frac{1}{y} = x + c$

$\Rightarrow -1 = yx + yc$

$\therefore yx + yc + 1 = 0$

Then the general solution of (1) is,

$(y-c)(y+x^2-c)(yx+yc+1) = 0$

$xy(p^2+1) = (x^2+y^2)p$

Solution: $xy(p^2+1) = (x^2+y^2)p$

$\Rightarrow p^2xy + xy = px^2 + py^2$

$\Rightarrow p^2xy + xy - px^2 - py^2 = 0$ ——— (1)

$\Rightarrow p^2xy - px^2 + xy - py^2 = 0$

$\Rightarrow xp(yp - x) + y(x - yp) = 0$

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$\Rightarrow xp(yp-x) - y(yp-x) = 0$

$\Rightarrow (yp-x)(xp-y) = 0$ ——— (ii)

Then from (ii) we have,

$yp - x = 0$

$y \cdot \frac{dy}{dx} - x = 0$

$\Rightarrow y \cdot dy - x dx = 0$

$\Rightarrow \frac{y^2}{2} - \frac{x^2}{2} = c_1$

$\Rightarrow y^2 - x^2 = 2c_1$

$\Rightarrow y^2 - x^2 = c$

$\therefore y^2 - x^2 - c = 0$

$xp - y = 0$

$\Rightarrow x\frac{dy}{dx} - y = 0$

$\Rightarrow x dy = y dx$

$\Rightarrow \frac{x}{dx} = \frac{y}{dy}$

$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$

$\Rightarrow \cancel{\ln x + c = \ln y}$

$\Rightarrow \ln x + \ln c = \ln y$

$\Rightarrow y = xc$

$\therefore y - cx = 0$

Then the general solution of (1) is,

$(y^2 - x^2 - c)(y - cx) = 0$

Equation solvable for y

$y = f(x, p)$ ——— (i)

Differentiate (i) with respect to x

$\Rightarrow \frac{dy}{dx} = p$

$\Rightarrow \frac{dy}{dx} = p = f\left(x, p, \frac{dp}{dx}\right)$ ——— (ii)

Suppose, solution of (ii) is,

$\phi(x, p, c) = 0$

then $p = \phi(x, c)$

☐ $y = 2px + p^4 x^2$ ——— (i)

Diff (i) w.r. to x

$p = 2p + 2x\frac{dp}{dx} + 4p^3x^2\frac{dp}{dx} + 2xp^4$

$\Rightarrow p - 2p - 2xp^4 = 2x\frac{dp}{dx} + 4p^3x^2\frac{dp}{dx}$

$\Rightarrow -p(1 + 2xp^3) = 2x(1 + 2xp^3)\frac{dp}{dx}$

$\Rightarrow -p = 2x\frac{dp}{dx}$

$\Rightarrow \frac{dx}{2x} = -\frac{dp}{p}$

$\Rightarrow \frac{dx}{x} = -\frac{2dp}{p}$

$\Rightarrow \ln x = -2\ln p + \ln c$

$\ln\left(\frac{x}{c}\right) = \ln\frac{1}{p^2}$

$\therefore \frac{x}{c} = \frac{1}{p^2}$

$\therefore p^2 = \frac{c}{x}$

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Then ① becomes,

$y = 2px + \frac{c^2}{x^2} x^2$

$\Rightarrow y - c^2 = 2px$

$\Rightarrow (y - c^2)^2 = 4x^2p^2$

$\Rightarrow (y - c^2)^2 = 4x^2 \cdot \frac{c}{x}$

$\therefore (y - c^2)^2 = 4cx$ (Ans).

Ex $y = 3px + 6p^2y^2$ (solvable for x)

Solution:

$y = 3px + 6p^2y^2$

$\Rightarrow 3px = y - 6p^2y^2$

$\Rightarrow x = \frac{y}{3}\left(\frac{y}{p} - 6py^2\right)$

Diff w. r. to y

$\frac{dx}{dy} = \frac{1}{3}\left(\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 12py - 6y^2\frac{dp}{dy}\right)$

$\therefore \frac{1}{p} = \frac{1}{3}\left(\frac{1}{p} - \frac{y^2}{p^2}\frac{dp}{dy} - 13py - 6y^2\frac{dp}{dy}\right)$

Clairaut's form

$y = px + f(p)$

If solvable for y,

$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p) \cdot \frac{dp}{dx}$

$\Rightarrow p = p + x + f(p)\frac{dp}{dx}$

$\Rightarrow \frac{dp}{dx} = 0$ | $y = c_1 x + f(c)$

$\therefore p = c$

Ex $y = px + p^2$

Since this is clairaut's. eqn. then we have,

$p = c$

$\therefore$ General solution is $y = cx + c^2$

Ex $y = 3px + 6p^2y^2$ (solving for x)

Solution:

$y = 3px + 6p^2y^2$

$\Rightarrow 3px = y - 6p^2y^2$

$\Rightarrow 3x = \frac{y}{p} - 6py^2$ ——— (ii)

Differentiating (ii) w. r. to y,

$$3\frac{dx}{dy} = \frac{1}{p}\cdot 1 + y(-1)\cdot\frac{1}{p^2}\cdot\frac{dp}{dy} - 12py - 6y^2\frac{dp}{dy}$$

$$\Rightarrow \frac{3}{p} = \frac{1}{p} - \frac{y}{p^2}\cdot\frac{dp}{dy} - 12py - 6y^2\frac{dp}{dy}$$

$$\Rightarrow \frac{3}{p} - \frac{1}{p} + 12py = -\frac{y}{p^2}\cdot\frac{dy}{dy} - 6y^2\frac{dp}{dy}$$

$$\Rightarrow \frac{2}{p} + 12py = -\left(\frac{y}{p^2} + 6y^2\right)\frac{dp}{dy}$$

$$\Rightarrow \frac{2+12p^2y}{p} = -\left(\frac{y}{p^2} + 6y^2\right)\frac{dp}{dy}$$

$$\Rightarrow -\left(\frac{-2-12p^2y}{p}\right) = -\left(\frac{y}{p^2} + 6y^2\right)\frac{dp}{dy}$$

$$\Rightarrow \left(\frac{y}{p^2} + 6y^2\right)\frac{dp}{dy} = \frac{-2-12p^2y}{p}$$

$$\Rightarrow \frac{1}{p^2}\left(y + 6p^2y^2\right)\frac{dp}{dy} = \frac{-2-12p^2y}{p}$$

$$\Rightarrow \frac{1}{p}\cdot y(1+6p^2y)\cdot\frac{dp}{dy} = -2(1+6p^2y)$$

$$\Rightarrow \frac{y}{p}\left(\frac{dp}{dy}\right) = -2$$

$$\Rightarrow \frac{dp}{p} = -2\frac{dy}{y}$$

$$\Rightarrow \ln p = -2\ln y + \ln c$$

$$\Rightarrow \ln\left(\frac{p}{c}\right) = \ln y^{-2}$$

$$\Rightarrow \frac{p}{c} = \frac{1}{y^2}$$

$$\therefore p = \frac{c}{y^2}$$

$\frac{dp}{dy}$

Putting the value of p in eqn ①, we get, $y = 3 \cdot \frac{c}{y^2} \cdot x + 6 \cdot \frac{c^2}{y^4} \cdot y^2$

$$\Rightarrow y = \frac{3cx}{y^2} + \frac{6c^2}{y^2}$$

$$\Rightarrow y = \frac{1}{y^2}(3cx + 6c^2)$$

$$\therefore y^3 = 3cx + 6c^2$$

This is the required solution.

Series solution

<u>Power series</u>: A power series is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c)^1 + a_2 (x-c)^2 + \ldots$$

where a_n represents the coefficient of the nth term and c is a constant

<u>Analytic function</u>: A function $f(x)$ is said to be analytic if its derivative exist all points in its domain.

<u>Ex</u>: x, $\sin x$, $\cos x$ etc.

Ordinary point: ~~আদের~~ x এর মানের জন্য যেসব $f(x)$ valid

Singular point: x এর মানের জন্য যেসব $f(x)$ invalid

A point $x = x_0$ is called an ordinary point of the equation

$$y'' + P(x)y' + Q(x)y = 0$$

if both the functions $P(x)$ and $Q(x)$ are analytic at $x = x_0$

If the point $x = x_0$ is not an ordinary point of the differential equation (1), then it is called a singular point of the differential equation of (1). There are two types of singular points:

(i) regular singular points

and (ii) irregular singular points.

A singular point $x = x_0$ of the differential equation (1) is called a regular singular point of the differential equation (1) if both $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at $x = x_0$

A singular point which is not regular is called an irregular singular point.

Ex $y'' + xy' + x^2 y = 0$

For ordinary point

Ex **Example:**

Find ~~A~~ regular singular points of the differential equation.

$$2x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (x^2-4)\,y = 0 \qquad \text{———— (1)}$$

Solution:

$$\frac{d^2y}{dx^2} + \frac{3}{2x}\frac{dy}{dx} + \frac{x^2-4}{2x^2}\,y = 0$$

$P_1 = \frac{3}{2x}$ and $P_2 = \frac{x^2-4}{2x^2}$

$Q_1 = x \cdot P_1 = x\frac{3}{2x} = \frac{3}{2}$ $\quad Q_2 = x^2 P_2 = x^2 \frac{x^2-4}{2x^2} = \frac{1}{2}(x^2-4)$

Since both P_1 and P_2 are not analytic $(P_1 = \infty,\ P_2 = \infty)$ at $x = 0$ therefore $x = 0$ is a singular point of (1). Moreover both Q_1 and Q_2 are analytic $(Q_1 \neq \infty,\ Q_2 \neq \infty)$ at $x = 0$.

Hence $x = 0$ is a regular singular point of (1)

Example:

$$x^2(x-2)^2 y'' + 2(x-2)y' + (x+3)y = 0$$

Solution:

$$P_1 = \frac{2(x-2)}{x^2(x-2)^2} = \frac{2}{x^2(x-2)} \quad \text{and} \quad P_2 = \frac{x+3}{x^2(x-2)^2}$$

P_1 and P_2 are not analytic ($P_1 = \infty$, $P_2 = \infty$) at $x=0$ and $x=2$. Hence both these points are singular points of (1)

(i) At $x=0$

$$Q_1 = x.P_1 = \frac{2}{x(x-2)}$$

$$Q_2 = x^2.P_2 = x^2 \cdot \frac{x+3}{x^2(x-2)^2} = \frac{x+3}{(x-2)^2}$$

Since Q_1 is not analytic ($Q_1 = \infty$) at $x=0$, is an irregular singular point.

(ii) At $x=2$

$$Q_1 = (x-2)P_1 = (x-2) \cdot \frac{2(x-2)}{x^2(x-2)^2} = \frac{2}{x^2}$$

$$Q_2 = (x-2)^2 P_2 = (x-2)^2 \frac{x+3}{x^2(x-2)^2} = \frac{x+3}{x^2}$$

Since both Q_1 and Q_2 are analytic ($Q_1 \neq \infty$, $Q_2 \neq \infty$) at $x=2$, so $x=2$ is a regular singular point.

Frobenius method

If $x=0$ is a regular singularity of the equation.

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)\,y = 0 \qquad \text{——— (1)}$$

Then the series solution is

$$y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) = \sum_{k=0}^{\infty} a_k x^{m+k}$$

The value of m will be determined by substituting the expressions for $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1) we get the identity.

On equating the coefficient of lowest power of x in the identity to zero, a quadratic equation in m (~~i.e~~ indicial eqn) is obtained

Thus, we will get two values of m. The series solution of (1) will depend on the nature of the roots of the indicial eqn.

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(i) Case 1: When roots m_1, m_2 are distinct and not differing by an integer, $m_1 - m_2 \neq 0$ or a positive integer. e.g. $m_1 = \frac{1}{2}$ $m_2 = 2$

The complete solution is $y = c_1 (y)_{m_1} + c_2 (y)_{m_2}$

(ii) Case 2: When roots m_1, m_2 are equal

$m_1 = m_2$

$$y = c_1 (y)_{m_1} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m_1}$$

(iii) Case 3: When roots m_1, m_2 are distinct and differ by an integer $(m_1 < m_2)$

$m_1 = \frac{3}{2}$, $m_2 = \frac{5}{2}$ or $m_1 = 2$, $m_2 = 4$

If some of the coefficients of y series become infinite when $m = m_1$, to overcome this difficulty, replace a_0 by $b_0 (m - m_1)$. We get a solution which is only a constant multiple of the first solution.

$$y = c_1 (y)_{m_1} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m_2}$$

→ not necessary

(IV) Case 4: Roots are distinct and differing by an integer, making some coefficients Indeterminate.

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}$$

if the coefficients do not become infinite when $m = m_2$

Case 1

Example: Find solution in generalized series form about $x = 0$ of the differential eqn.

$$3x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \quad \text{--- (1)}$$

Solution:

Since $x = 0$ is a regular singular point, we assume the solution in the form

$$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

such that $$\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k-2}$$

Substituting for y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation (1) we get.

$$3\sum a_k(m+k)(m+k-1)x^{m+k-1} + 2\sum a_k(m+k)x^{m+k-1} + \sum a_k x^{m+k} = 0$$

$$\Rightarrow \sum a_k\left[3(m+k)(m+k-1) + 2(m+k)\right]x^{m+k-1} + \sum a_k x^{m+k} = 0 \quad \text{——} ②$$

The coefficient of the lowest degree term x^{m-1} in the identity ② is obtained by putting $k=0$ in first summation only and equating it to zero. Then the indicial equn is.

$$a_0\left[3m(m-1) + 2m\right] = 0$$

$$\Rightarrow a_0\left[3m^2 - m\right] = 0$$

$$\Rightarrow a_0 m(3m-1) = 0$$

$\therefore a_0 \neq 0 \quad m = 0$ or $\frac{1}{3}$

The coefficient of next lowest degree term x^m in the identity ② is obtained by putting $k=1$ in first summation and $k=0$ in the second summation and equating it to zero.

$$a_1\left[3(m+1)m + 2(m+1)\right] + a_0 = 0$$

$$\Rightarrow a_1\left[3m^2 + 3m + 2m + 2\right] + a_0 = 0$$

$$\Rightarrow a_1\left[3m^2 + 5m + 2\right] + a_0 = 0$$

$$\Rightarrow a_1(3m+2)(m+1) + a_0 = 0$$

$$\Rightarrow a_1 = -\frac{1}{(3m+2)(m+1)}\,a_0$$

~~Equating to zero the coefficient of~~ $\cancel{x^{m+k}}$

The coefficient of next lowest degree term x^{m+1} in the identity (2) is obtained by putting $k = 2$ in first summation and $k = 1$ in the second summation and equating it to zero.

$$a_2\left[3(m+2)(m+1) + 2(m+2)\right] + a_1 = 0$$

$$\Rightarrow a_2\left[(3m+6)(m+1) + 2m + 4\right] + a_1 = 0$$

$$\Rightarrow a_2\left[3m^2 + 6m + 3m + 6 + 2m + 4\right) + a_1 = 0$$

$$\Rightarrow a_2\left[3m^2 + 11m + 10\right] + a_1 = 0$$

$$\Rightarrow a_2\left[3m^2 + 5m + 6m + 10\right] + a_1 = 0$$

$$\Rightarrow a_2\left[m(3m+5) + 2(3m+5)\right] + a_1 = 0$$

$$\Rightarrow a_2 = \frac{-1}{(3m+5)(m+2)}\,a_1$$

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Equating to zero the coefficient of x^{m+k} the recurrence relation is given by

$$a_{k+1}\left[3(m+k+1)(m+k)+2(m+k+1)\right]+a_k=0$$

$$\Rightarrow a_{k+1}\left[(3m+3k+2)(m+k+1)\right]+a_k=0$$

$$\therefore a_{k+1} = \frac{-a_k}{(3m+3k+2)(m+k+1)}$$

This gives

For $k=0$, $a_1 = \dfrac{-1}{(3m+2)(m+1)} a_0$

For $k=1$, $a_2 = \dfrac{-1}{(3m+5)(m+2)} a_1 = \dfrac{a_0}{(m+1)(m+2)(3m+2)(3m+5)}$

For $k=2$, $a_3 = \dfrac{-1}{(3m+8)(m+3)} a_2 = \dfrac{-a_0}{(m+1)(m+2)(3m+2)(3m+5)(m+3)(3m+8)}$

For $m=0$

$a_1 = \dfrac{-1}{2} a_0$, $a_2 = \dfrac{1}{20} a_0$, $a_3 = \dfrac{-1}{480} a_0$

Hence $y_1 = a_0\left(1 - \frac{1}{2}x + \frac{1}{20}x^2 - \frac{1}{480}x^3 + \dots\right)$

For $m = \frac{1}{3}$

$a_1 = -\dfrac{1}{4}a_0$, $a_2 = \dfrac{1}{56}a_0$, $a_3 = \dfrac{-1}{1680}a_0$

Hence $y_2 = a_0\left(x^{\frac{1}{3}} - \frac{1}{4}x^{\frac{4}{3}} + \frac{1}{56}x^{\frac{7}{3}} - \frac{1}{1680}x^{\frac{10}{3}} + \dots\right)$

Thus the complete solution is

$$y = Ay_1 + By_2$$

$$y = a_0\left(1 - \frac{x}{2} + \frac{x^2}{20} - \frac{x^3}{480} + \dots\right) + b_0 x^{\frac{1}{3}}\left(1 - \frac{x}{4} + \frac{x^2}{56} - \frac{x^3}{1680} + \dots\right)$$

(Ans).

Case 2

Example: Solve $x(x-1)y'' + (3x-1)y' + y = 0$

Solution:

Since $x=0$ is a regular singular point, we assume the solution in the form

$$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

such that $\frac{dy}{dx} = \sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1}$

$$\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k(m+k)(m+k+1)x^{m+k-2}$$

Substituting the expressions for $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in ① we have.

$$\sum x(x-1)a_k(m+k)(m+k+1)x^{m+k-2} + (3x-1)a_k(m+k)x^{m+k-1} + a_k x^{m+k} = 0$$

$\Rightarrow \sum (x-1)\, a_k (m+k)(m+k-1)\, x^{m+k-1} + \sum (3x-1)\, a_k (m+k)\, x^{m+k-1}$
$+ \sum a_k\, x^{m+k} = 0$

$\Rightarrow \sum a_k (m+k)(m+k-1)\, x^{m+k} - \sum a_k (m+k)(m+k-1)\, x^{m+k-1}$
$+ \sum 3a_k (m+k)\, x^{m+k} - \sum a_k (m+k)\, x^{m+k-1} + \sum a_k x^{m+k} = 0$

$\Rightarrow \sum a_k \left[(m+k)(m+k-1) + 3(m+k) + 1\right] x^{m+k}$
$- \sum a_k \left[(m+k)(m+k-1) + (m+k)\right] x^{m+k-1} = 0$

$\Rightarrow \sum a_k \left[\{(m+k)(m+k-1+3) + 1\}\right] x^{m+k}$
$- \sum a_k \left[(m+k)(m+k-\cancel{1}+\cancel{1})\right] x^{m+k-1} = 0$

$\Rightarrow \sum a_k \left[(m+k)(m+k+2) + 1\right] x^{m+k} - \sum a_k (m+k)^2 x^{m+k-1}$
$= 0$

—— ②

The coefficient of lowest degree term x^{m-1} in ② is obtained by putting $k=0$ in second sumation only of ② and equating it to zero. Then the indicial equation is

$a_0 m^2 = 0$

$\therefore a_0 \neq 0, \quad m = 0, 0$

The coefficient of the next lowest degree term x^m in (2) is obtained by putting $k=0$ in 1st summation and $k=1$ in 2nd. summation only of (2) and equating it to zero we get,

$$a_0[m(m+2)+1] - a_1(m+1)^2 = 0$$

$$\Rightarrow a_0[m^2+2m+1] - a_1[m^2+2m+1] = 0$$

$$\therefore a_0 = a_1$$

Equating the coefficient of x^{m+k} to zero, the recurrence relation is given by

$$a_k[(m+k)(m+k+2)+1] - a_{k+1}(m+k+1)^2 = 0$$

$$\Rightarrow a_k[m^2+km+km+k^2+2m+2k+1] - a_{k+1}(m+k+1)^2 = 0$$

$$\Rightarrow a_k(m+k+1)^2 - a_{k+1}(m+k+1)^2 = 0$$

Hence $a_{k+1} = a_k$

$$\therefore \cancel{y = a_0 x^m} \quad y = a_0 x^m[1+x+x^2+x^3+\dots]$$

when $m = 0, 0,$ this gives only one solution istead of two.

Second solution is given by

$\left(\frac{\partial y}{\partial m}\right)_{m=0}$ and $y_1 = a_0(1+x+x^2+x^3+\cdots)$

$\frac{\partial y}{\partial m} =$

Case 3:

Example: Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-4)y = 0$ —— ①

Solution

Let $y = \sum a_k x^{m+k}$

$\frac{dy}{dx} = \sum a_k (m+k) x^{m+k-1}$

$\frac{d^2y}{dx^2} = \sum a_k (m+k)(m+k-1) x^{m+k-2}$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}, y$ in ① we get.

$x^2 \sum a_k (m+k)(m+k-1) x^{m+k-2} + x \sum a_k (m+k) x^{m+k-1} + (x^2-4)\sum a_k x^{m+k} = 0$

$\Rightarrow \sum a_k (m+k)(m+k-1) x^{m+k} + \sum a_k (m+k) x^{m+k} - 4\sum a_k x^{m+k} + \sum a_k x^{m+k+2} = 0$

$\Rightarrow \sum a_k \left[(m+k)(m+k-1) + (m+k) - 4\right] x^{m+k} + \sum a_k x^{m+k+2} = 0$

$\Rightarrow \sum a_k \left[(m+k)(m+k\cancel{-1}\cancel{+1}) - 4\right] x^{m+k} + \sum a_k x^{m+k+2} = 0$

~~$\Rightarrow \sum a_k (m^2+2km+k^2-4) x^{m+k} + \sum a_k x^{m+k+2} = 0$~~

$\Rightarrow \sum a_k \left[(m+k)^2 - 2^2\right] x^{m+k} + \sum a_k x^{m+k+2} = 0$

$\Rightarrow \sum a_k (m+k+2)(m+k-2) x^{m+k} + \sum a_k x^{m+k+2} = 0$ —— ②

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$k \geq 0$

The coefficient of lowest degree term x^m in ② is obtained by putting $k=0$ in first summation only and equating it to zero. Then the indicial equation is,

$$a_0(m+2)(m-2) = 0 \quad (a_0 \neq 0)$$

$\therefore m = -2, 2$

The coefficient of next lowest term x^{m+1} in ② is obtained by putting $k=1$ in first summation only and equating it to zero

$$a_1(m+3)(m-1) = 0$$

$\therefore a_1 = 0$

equating the coefficient of x^{m+k+2}

$$a_{k+2}(m+k+4)(m+k) + a_k = 0$$

$$\Rightarrow a_{k+2} = -\frac{1}{(m+k+4)(m+k)} a_k \qquad a_1 = a_3 = a_5 = \dots = 0$$

For, $k=0$, $a_2 = \frac{-1}{m(m+4)} a_0$

For, $k=2$ $a_4 = \frac{-1}{(m+6)(m+2)} a_2 = \frac{1}{m(m+6)(m+2)(m+4)} a_0$

For, $k=4$, $a_6 = \frac{-1}{(m+8)(m+4)} a_4 = \frac{a_0}{m(m+2)(m+4)^2(m+6)(m+8)}$

$m = -2$

$\Rightarrow (m+2) = 0 \longrightarrow$ হরগুলোতে

Hence, $y = a_0 x^m \left[1 - \frac{x^2}{m(m+4)} + \frac{x^4}{m(m+2)(m+4)(m+6)} - \frac{x^6}{m(m+2)(m+4)^2(m+6)(m+8)} + \cdots \right]$

Putting $m = 2$ in ③ we get

$$y_1 = a_0 x^2 \left[1 - \frac{x^2}{12} + \frac{x^4}{384} - \frac{x^6}{23040} + \cdots \right]$$

Coefficient of x^4, x^6 etc. in ③ becomes infinite on putting $m = -2$. To overcome this difficualty we put

$a_0 = b_0(m+2)$ in ③ aend we get

$$y = b_0 x^m \left[(m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \frac{x^6}{m(m+4)^2(m+6)(m+8)} + \cdots \right]$$

On differentiating ⑤ w. r. to 'm' we get

$$\frac{\partial y}{\partial m} = b_0$$

$m = -2$

$\Rightarrow (m+2) = 0 \longrightarrow$ हरनात्मक-

Hence, $y = a_0 x^m \left[1 - \frac{x^2}{m(m+4)} + \frac{x^4}{m(m+2)(m+4)(m+6)} - \frac{x^6}{m(m+2)(m+4)^2(m+6)(m+8)} + \dots \right]$

Putting $m = 2$ in ③ we get

$$y_1 = a_0 x^2 \left[1 - \frac{x^2}{12} + \frac{x^4}{384} - \frac{x^6}{23040} + \dots \right]$$

Coefficient of x^4, x^6 etc. in ③ becomes infinite on putting $m = -2$. To overcome this difficulty we put

$a_0 = b_0(m+2)$ in ③ and we get

$$y = b_0 x^m \left[(m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \frac{x^6}{m(m+4)^2(m+6)(m+8)} + \dots \right]$$

On differentiating ⑤ w. r. to 'm' we get

$$\frac{\partial y}{\partial m} = b_0$$

$$\Rightarrow \sum a_r (m+r)(m+r-1) x^{m+r} + \sum a_r (m+r) x^{m+r} + \sum a_r x^{m+r+2} - n^2 \sum a_r x^{m+r} = 0$$

$$\Rightarrow \sum a_r \left[(m+r)(m+r-1) + (m+r) - n^2\right] x^{m+r} + \sum a_r x^{m+r+2} = 0$$

$$\Rightarrow \sum a_r \left[(m+r)(m+r) - n^2\right] x^{m+r} + \sum a_r x^{m+r+2} = 0$$

$$\Rightarrow \sum a_r \left[(m+r)^2 - n^2\right] x^{m+r} + \sum a_r x^{m+r+2} = 0$$

Equating the coefficient of x^m to zero, we get,

$$a_0 (m^2 - n^2) = 0$$

$\therefore a_0 \neq 0$, $m^2 = n^2$ $\therefore m = n$

Equating the coefficient of x^{m+1}

$$a_1 \left[(m+1)^2 - n^2\right] = 0$$

$\therefore a_1 = 0$, $(m+1)^2 - n^2 \neq 0$

Equating the coefficient of x^{m+r+2} to zero, to find relation in successive coefficients, we get,

$$a_{r+2}\left[(m+r+2)^2 - n^2\right] + a_r = 0$$

$$\therefore a_{r+2} = \frac{-1}{(m+r+2)^2 - n^2} a_r$$

Therefore $a_1 = a_3 = a_5 = \dots = 0$

If $r = 0$, $a_2 = \frac{-1}{(m+2)^2 - n^2} a_0$.

If $r = 2$, $a_4 = \frac{-1}{(m+4)^2 - n^2} a_2 = \frac{1}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} a_0$

and so on

On subsituting the values of the coefficient in ② we have

$$y = a_0 x^m - \frac{a_0}{(m+2)^2 - n^2} x^{m+2} + \frac{a_0}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} x^{m+4} + \dots$$

$$\therefore y = a_0 x^m \left[1 - \frac{1}{(m+2)^2 - n^2} x^2 + \frac{1}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} x^4 + \dots\right]$$

For $m = n$

$$y = a_0 x^n \left[1 - \frac{1}{(n+2)^2 - n^2} x^2 + \frac{1}{[(m+2)^2 - n^2][(n+4)^2 - n^2]} x^4 + \dots\right]$$

~~where~~ a_0 ~~is an arbitrary constant~~

~~For $m = -n$~~

$\cancel{y = a_0 x^{-n}}\left[\cancel{1} -\right.$

$$\therefore y = a_0 x^n \left[1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2 \cdot 2!(n+1)(n+2)} x^4 - \dots\right]$$

where a_0 is an arbitrary constant.

For $m = -n$

$$y = a_0 x^{-n}\left[1 - \frac{1}{4(-n+1)}x^2 + \frac{1}{4^2 \cdot 2!(-n+1)(-n+2)}x^4 - \cdots\right]$$

Bessel Functions $J_n(x)$

The Bessel's equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

Solution of ① is

$$y = a_0 x^n\left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} - \cdots + (-1)^r\frac{x^{2r}}{(2^r.r!)\,2^r(n+1)(n+2)\cdots(n+r)} + \cdots\right]$$

$$= a_0 x^n \sum_{k=0}^{\infty}(-1)^r \frac{x^{2r}}{2^{2r}.r!(n+1)(n+2)\cdots(n+r)}$$

where a_0 is an arbitrary constant

If $a_0 = \frac{1}{2^n\Gamma(n+1)}$

The above solution is called Bessel's function denoted by $J_n(x)$.

Thus $J_n(x) = \frac{1}{2^n\Gamma(n+1)}\sum(-1)^r\frac{\quad}{2^{2r}.r!(n+1)(n+2)\cdots(n+r)}$

$(\Gamma n+1 = n!)$

$$G(n+1) = nG(n)$$

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{ \frac{1}{\Gamma(n+1)} - \frac{1}{\lfloor 1 \, \Gamma(n+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{\lfloor 2 \, \Gamma(n+3)}\left(\frac{x}{2}\right)^4 - \frac{1}{\lfloor 3 \, \Gamma(n+4)}\left(\frac{x}{2}\right)^6 + \cdots \right\}$$

$$\therefore J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r} \quad \text{——— (2)}$$

If $n=0$, $J_0(x) = \sum \frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

If $n=1$, $J_1(x) = \cancel{\sum (-1)} \; \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \cdots$

Example:

Prove that

(a) $J_{\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \sin x$ (b) $J_{-\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \cos x$

Solution:

We know that

$$J_n(x) = \frac{x^n}{2^n \, \Gamma(n+1)}\left[1 - \frac{x^2}{2 \cdot 2(n+1)} + \frac{x^4}{2 \cdot 4 \cdot 2^2 (n+1)(n+2)} - \cdots\right] \quad \text{——— (1)}$$

(a) Substituting $n = \frac{1}{2}$ in (1) we obtain

$$J_{\frac{1}{2}}(x) = \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}} \, \Gamma\frac{3}{2}}\left[1 - \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4 \cdot 5} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2} \, \Gamma\frac{3}{2}} \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2} \cdot \frac{1}{2}\sqrt{\frac{1}{2}}} \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]$$

$$= \frac{1}{\sqrt{2x}\,\frac{1}{2}\sqrt{\pi}} \sin x \qquad \left[\sqrt{\tfrac{1}{2}} = \sqrt{\pi}\right)$$

$$= \sqrt{\left(\frac{2}{\pi x}\right)} \sin x$$

(b) Again substituting $n = -\frac{1}{2}$ in (1) we have

Example

Prove that

$J_{-n}(x) = (-1)^n J_n(x)$

where n is a positive integer.

Solution:

We know, $J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(-n+r)!} \left(\frac{x}{2}\right)^{-n+2r}$

$$= \sum_{r=0}^{n-1} \frac{(-1)^r}{r!\,(-n+r)!} \left(\frac{x}{2}\right)^{-n+2r} + \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,(-n+r)!} \left(\frac{x}{2}\right)^{-n+2r}$$

$$= 0 + \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,(-n+r)!} \left(\frac{x}{2}\right)^{-n+2r} = \sum_{r=n}^{\infty} \frac{(-1)^r}{r!\,(-n+r)!} \left(\frac{x}{2}\right)^{-n+2r} \quad \text{——— (1)}$$

Put $r = n+k$ in (1) we have

$$= \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{(n+k)!\,(k)!} \left(\frac{x}{2}\right)^{n+2k} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{n+2k}$$

$$= (-1)^n J_n(x)$$

Recurrence Formulae:

Formula I. $xJ_n' = nJ_n - xJ_{n+1}$

Proof. We know that

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

Differentiating with respect to x we get

$$J_n' = \sum \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

$$\Rightarrow x J_n' = n \sum \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} + x \frac{(-1)^r \cdot 2r}{2 \cdot r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n + x \sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= nJ_n + x \cdot \sum_{s=0}^{\infty} \frac{(-1)^{s+r}}{s!\,\Gamma(n+s+2)} \left(\frac{x}{2}\right)^{n+2s+1}$$

Putting $r-1 = s$, $r = s+1$

$$= nJ_n - x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma[(n+1)+s+1]} \left(\frac{x}{2}\right)^{n+1+2s}$$

$\therefore xJ_n' = nJ_n - xJ_{n+1}$

Formula II: $xJ_n' = -nJ_n + xJ_{n-1}$

Proof. We have $J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Differentiating w.r. to x we get

$$J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2}$$

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$$\Rightarrow x J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r [(2n+2r)-n]}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} - n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2(n+r)}{r!\,(n+r)!} \left(\frac{x}{2}\right)^{n+2r} - n J_n$$

$$= x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n-1+r)!} \left(\frac{x}{2}\right)^{n-1+2r} - n J_n$$

$$\therefore x J_n' = x J_{n-1} - n J_n$$

Formula III. $2 J_n' = J_{n-1} - J_{n+1}$

Proof. We know that

$$x J_n' = n J_n - x J_{n+1} \quad \text{——— (1)}$$

$$x J_n' = x J_{n-1} - n J_n \quad \text{——— (2)}$$

Adding eqn (1) and (2) we get

$$2x J_n' = - x J_{n+1} + x J_{n-1}$$

$$2x J_n' = J_{n-1} - J_{n+1}$$

$$\therefore 2 J_n' = J_{n-1} - J_{n+1}$$

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Formula IV: $2nJ_n = x(J_{n-1} + J_{n+1})$

Proof. We know that

$$xJ_n' = nJ_n - xJ_{n-1} \quad \text{—— (1)}$$

$$xJ_n' = -nJ_n + xJ_{n+1} \quad \text{—— (2)}$$

Substracting (2) from (1) we get

$$0 = -2nJ_n + xJ_{n+1} + xJ_{n-1}$$

$$= 2nJ_n - xJ_{n+1} - xJ_{n-1}$$

$$\Rightarrow 2nJ_n = x(J_{n-1} + J_{n+1})$$

Formula V: $\frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1}$

Proof. We know that

$$xJ_n' = nJ_n - xJ_{n+1} \quad \text{—— (1)}$$

Multiplying by x^{-n-1} we obtain

$$x^{-n}J_n' = nx^{-n-1}J_n - x^{-n}J_{n+1}$$

$$\Rightarrow x^{-n}J_n' - nx^{-n-1}J_n = -x^{-n}J_{n+1}$$

$$\Rightarrow \frac{d}{dx}(x^{-n}J_n) = -x^{-n}J_{n+1}$$

Formula VI: $\frac{d}{dx}(x^n J_n) = x^n J_{n-1}$

Proof: We know that

$$x J_n' = -n J_n + x J_{n-1} \quad \text{——— (11)}$$

Multiplying by x^{n-1} we have

$$x^n J_n' = -n x^{n-1} J_n + x^n J_{n-1}$$

$$\Rightarrow x^n J_n' + n x^{n-1} J_n = x^n J_{n-1}$$

$$\Rightarrow \frac{d}{dx}(x^n J_n) = x^n J_{n-1}$$

Example: Determine the fractional values of $J_n(x)$ i.e.. $J_{\frac{3}{2}}, J_{-\frac{3}{2}}, J_{\frac{5}{2}}, J_{-\frac{5}{2}}$

We know,

$$2n J_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right) \quad \text{——— (1)}$$

Put $n = \frac{1}{2}$ in (1)

$$2 \cdot \frac{1}{2} J_{\frac{1}{2}}(x) = x\left(J_{\frac{1}{2}-1}(x) + J_{\frac{1}{2}+1}(x)\right)$$

$$\Rightarrow J_{\frac{1}{2}}(x) = x\left(J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x)\right)$$

$$\Rightarrow J_{\frac{3}{2}}(x) = \frac{1}{x} \cdot J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}} \sin x - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

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Orthogonality of bessel Functions

$$\int_0^1 x J_n(\alpha x) \cdot J_n(\beta x)\, dx = 0$$

where α and β are the roots of $J_n(x) = 0$

Proof. We know that

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (\alpha^2 x^2 - n^2) y = 0 \quad \text{——— (i)}$$

$$x^2 \frac{d^2 z}{dx^2} + x\frac{dz}{dx} + (\beta^2 x^2 - n^2) z = 0 \quad \text{——— (ii)}$$

Solutions of (i) and (ii) are $y = J_n(\alpha x)$.
$z = J_n(\beta x)$

Multiplying (i) by $\frac{z}{x}$ and (ii) by $-\frac{y}{x}$ and add we get.

$$x\left(z\frac{d^2 y}{dx^2} - y\frac{d^2 z}{dx^2}\right) + z\frac{dy}{dx} - y\frac{dz}{dx}\Big) + (\alpha^2 - \beta^2) xyz = 0$$

$$\frac{d}{dx}\left[x\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right] + (\alpha^2 - \beta^2) xyz = 0 \quad \text{——— (iii)}$$

Integrating (iii) w. r. to x between the limits 0 and 1 we get.

$$\left[x\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right]_0^1 + (\alpha^2 - \beta^2)\int_0^1 xyz\, dx = 0$$

$\Rightarrow (\beta^2-\alpha^2)\int_0^1 xyz\,dx = \left[x\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right]_0^1$

$= z\frac{dy}{dx} - y\frac{dz}{dx}$ ——— (IV)

Putting the values of $y = J_n(\alpha x)$, $\frac{dy}{dx} = \alpha . J_n'(\alpha x)$
$z = J_n(\beta x)$, $\frac{dz}{dx} = \beta J_n'(\beta x)$ in (IV) we get

$(\beta^2-\alpha^2)\int_0^1 x J_n(\alpha x) . J_n(\beta x)\,dx =$

$= \left[\alpha J_n'(\alpha x) J_n(\beta x) - \beta J_n'(\beta x) J_n(\alpha x)\right]_{x=1}$

$= \alpha J_n'(\alpha) J_n(\beta) - \beta J_n'(\beta) J_n(\alpha)$ ——— (V)

Since α, β are the roots of $J_n(x) = 0$

so, $J_n(\alpha) = J_n(\beta) = 0$

Putting the values of $J_n(\alpha)$ & $J_n(\beta) = 0$
in (V) we get

$(\beta^2-\alpha^2)\int_0^1 x J_n(\alpha x) . J_n(\beta x)\,dx = 0$