GRADIENT, DIVERGENCE And CURL

GRADIENT

let $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e ϕ defines a differentiable scalar field). Then the gradient of ϕ, written $\nabla\phi$ or grad ϕ is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

// The component of $\nabla\phi$ in the direction of a unit vector a is given by $\nabla\phi.a$ and is called the directional derivative of ϕ in the direction a. Physically, this is the rate of change of ϕ at (x,y,z) in the direction a //

Problem-01

Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2i - j - 2k$

Soln

$$\nabla\phi = \nabla(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{i} + x^2z\hat{j} + (x^2y + 8xz)\hat{k}$$

$$= 8\hat{i} - \hat{j} - 10\hat{k} \quad \text{at } (1,-2,-1)$$

The unit vector in the direction of $2i - j - 2k$ is

$$a = \frac{2i - j - 2k}{\sqrt{2^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

Then the directional derivative is

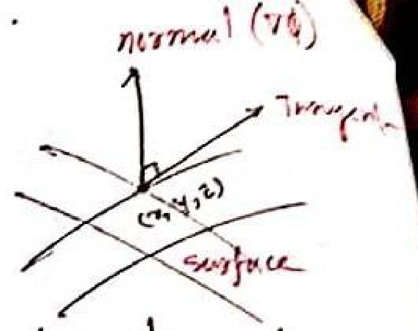
$$\nabla\phi.a = (8\hat{i} - \hat{j} - 10\hat{k}) \times \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= \frac{16}{3} + \frac{1}{3} + \frac{20}{3}$$

$$= \frac{37}{3}$$

[Since this is positive, ϕ is increasing in this direction

Problem-02: Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z)=c$ where c is a constant.



Soln

Let $r = x\hat{i} + y\hat{j} + z\hat{k}$ be a position vector to any point $P(x,y,z)$ on the surface. Then

$d r = dx\hat{i} + dy\hat{j} + dz\hat{k}$ lies in the tangent plane to the surface at P.

Again,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = 0$$

$$\Rightarrow \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot \left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right) = 0$$

$$\Rightarrow \nabla\phi \cdot dr = 0$$

that is $\nabla\phi \cdot dr = 0$ so that $\nabla\phi$ is perpendicular to dr. and therefore to the surface.

Problem-03

Find the unit normal to the surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$

Soln

$$\nabla\phi = \nabla(x^2y + 2xz)$$
$$= (2xy + 2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$$
$$= -2\hat{i} + 4\hat{j} + 4\hat{k}$$

* যেকোন position vector কে derivative করলে তার অবস্থান থাকে tangent বরাবর এবং $\nabla\phi$ grad ϕ থাকে normal বরাবর।

unit normal to the surface $= \dfrac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{(-2)^2 + 4^2 + 4^2}}$

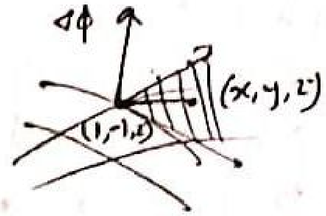
$$= -\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Another unit normal is $\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}$ **Ans.**

am-04

Find an equation for the tangent plane to the surface $2xz^2-3xy-4x=7$ at the point $(1,-1,2)$

Soln



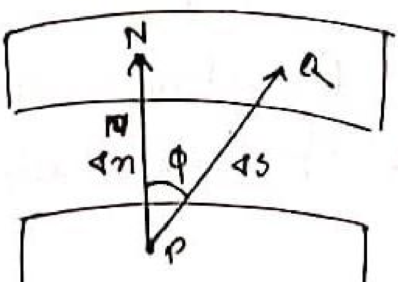
$$\nabla(2xz^2-3xy-4x)$$
$$=(2z^2-3y-4)\hat{i}-3y\hat{j}+4xz\hat{k}$$
$$=7\hat{i}-3\hat{j}+8\hat{k} \quad (\text{at point } 1,-1,2)$$

The equation of a plane passing through a point whose position vector is r_0 and which is perpendicular to the normal N is $\boxed{(\bar{r}-r_0)N=0}$

$$\left\{(x\hat{i}+y\hat{j}+z\hat{k})-(\hat{i}-\hat{j}+2\hat{k})\right\}\cdot(7\hat{i}-3\hat{j}+8\hat{k})=0$$
$$\Rightarrow 7(x-1)-3(y+1)+8(z-2)=0 \quad \text{Ans.}$$

Geometrical Interpretation

let $f(x,y,z)=K$
and $f(x,y,z)=K+\Delta K$
be neighbouring level surfaces.



let PN be the normal of the surface $f(x,y,z)=K$ in the direction of increasing f.
let Pa be the other line inclined at an angle ϕ to PN

$$\Delta s=\Delta s\cos\phi$$

$$\lim_{\Delta s\to 0}\frac{\Delta f}{\Delta s}=\lim_{\Delta n\to 0}\frac{\Delta f}{\Delta n}\lim_{\Delta s\to 0}\frac{\Delta n}{\Delta s}$$

$$\boxed{\text{grad} f=\frac{\partial f}{\partial n}n}$$

$$\frac{\partial f}{\partial s}=\frac{\partial f}{\partial n}\cos\phi$$
$$=|\text{grad} f|\cos\phi$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial n}\cos\phi$$

$$= \cancel{\cos}\,|\text{grad } f|\cos\phi \quad \text{——— (1)}$$

when ϕ is 0 then the right hand side of (1) attain maximum value. And PQ coincides with the normal PN. Thus, we can say that the gradient of a scalar point function $f(x,y,z)$ is a vector acting along the normal to the level surface $f(x,y,z)=k$ at (x,y,z) and in the direction of increasing f whose magnitude is the greatest rate of change of f.

Problem-05

(a) In what direction for the point $(2,1,-1)$ is the directional derivative of $\phi = x^2yz^3$ a maximum.

(b) what is the magnitude of this maximum?

$$\nabla\phi = \nabla(x^2yz^3) = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$$

$$= -4\hat{i} - 4\hat{j} + 12\hat{k} \quad \text{at } (2,1,-1)$$

$\therefore$ The directional derivative $-4\hat{i} - 4\hat{j} + 12\hat{k}$

the magnitude of this maximum is $\nabla\phi = \sqrt{(-4)^2 + (-4)^2 + (12)^2}$

$$= 4\sqrt{11}$$

dy =

Problem-06:

Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Soln (The angle between the surfaces at the point is the angle between the normals to the surfaces at the point)

A normal to $x^2+y^2+z^2=9$ at $(2,-1,2)$ is

$$\nabla\phi_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$
$$= 4\hat{i}-2\hat{j}+4\hat{k}$$

A normal to $x^2+y^2-z=3$ at $(2,-1,2)$ is

$$\nabla\phi_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$$
$$= 4\hat{i}-2\hat{j}-\hat{k}$$

$$(\nabla\phi_1).(\nabla\phi_2) = |\nabla\phi_1||\nabla\phi_2|\cos\theta$$

$$(4\hat{i}-2\hat{j}+4\hat{k}).(4\hat{i}-2\hat{j}-\hat{k}) = |4\hat{i}-2\hat{j}+4\hat{k}||4\hat{i}-2\hat{j}-\hat{k}|\cos\theta$$

$$16+4-4 = \sqrt{4^2+(-2)^2+4^2}.\sqrt{4^2+(-2)^2+(-1)^2}\cos\theta$$

$$\cos\theta = \frac{16}{6\sqrt{21}} = 0.5819$$

$$\theta = 54^\circ 25' \quad \text{Ans.}$$

Divergence

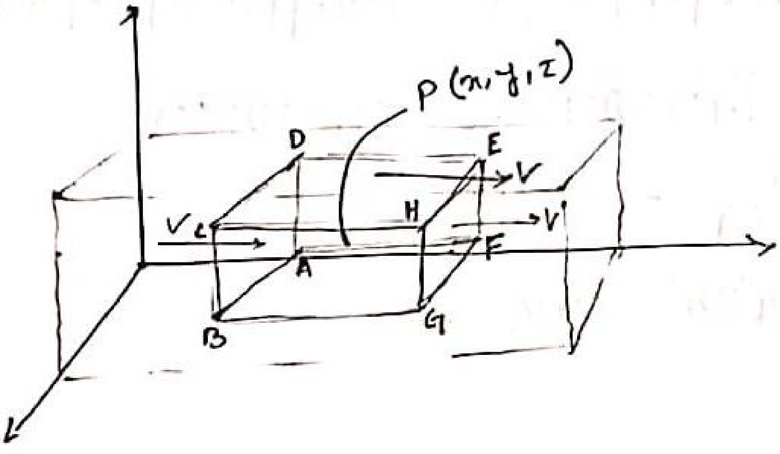
Let $V(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space (i.e. V defines a differential vector field) then the divergence of V, written $\nabla.V$ or div V is defined by

$$\nabla.V = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right).\left(V_1 i + V_2 j + V_3 k\right)$$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Problem

21 A fluid moves so that its velocity at any point is $V(x,y,z)$. Show that the gain of fluid per unit volumn per unit time in a small parallelepiped having centre at $P(x,y,z)$ and edges parallel to the co-ordinate axes and having magnitude $\Delta x \Delta y \Delta z$ respectively, is given approximately by div $V = \nabla.V$



x component of velocity v at $P = V_1$

x " of v at centre of face AFED $= V_1 - \frac{1}{2}\frac{\partial V_1}{\partial x}\Delta x$

x " of v at " " " GHCB $= V_1 + \frac{1}{2}\frac{\partial V_1}{\partial x}\Delta x$

Volumn of fluid accossing AEFD per unit time

$$= \left(V_1 - \frac{1}{2}\frac{\partial V_1}{\partial x}\Delta x\right)\Delta y \Delta z$$

volumn of " " GHCB per unit time

$$= \left(V_1 + \frac{1}{2}\frac{\partial V_1}{\partial x}\Delta x\right)\Delta y \Delta z$$

similarly,

gain in volumn per unit time in x direction

$$\left(v_1 + \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x\right)\Delta y \Delta z - \left(v_1 - \frac{1}{2}\frac{\partial v_1}{\partial x}\Delta x\right)\Delta y \Delta z$$

$$\Rightarrow \frac{\partial v_1}{\partial x}\Delta x \Delta y \Delta z$$

similarly

gain in volumn per unit time in y direction

$$= \frac{\partial v_2}{\partial y}\Delta x \Delta y \Delta z$$

gain in volumn per unit time in z direction

$$= \frac{\partial v_3}{\partial z}\Delta x \Delta y \Delta z$$

$\therefore$ Total gain in volume per unit volume per unit time

$$= \frac{\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)\Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z}$$

$$= \text{div } V$$

$$= \nabla . V$$

* Divergence is zero sometimes called solenoidal.

<u>Problem-22:</u> Determine the constant a so that the vector $V = (x+3y)i + (y-2z)j + (x+az)\hat{k}$ is solinoidal

$$\vec{\nabla}.\vec{V} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y+2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$\Rightarrow 1 + 1 + a = 0$$

$$\Rightarrow a = -2 \quad \underline{\text{Ans}}.$$

CURL

If $V(x,y,z)$ is a differentiable vector field then the curl or rotation of V, written $\nabla \times V$.

Curl V or rot V is defined by

$$\nabla \times V = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times \left(V_1 i + V_2 j + V_3 k\right)$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} \quad \begin{array}{l} = 0 \text{ irrotational} \\ \neq 0 \text{ rotational} \end{array}$$

Analysis

$$F = y\hat{i} - x\hat{j}$$

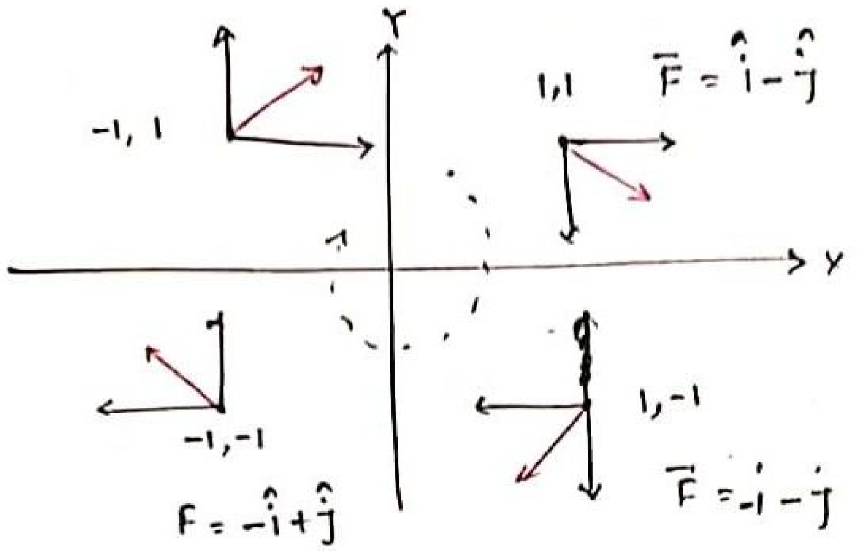
$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}$$

$$= i(0-0) + j(0-0) + k(-1-1)$$

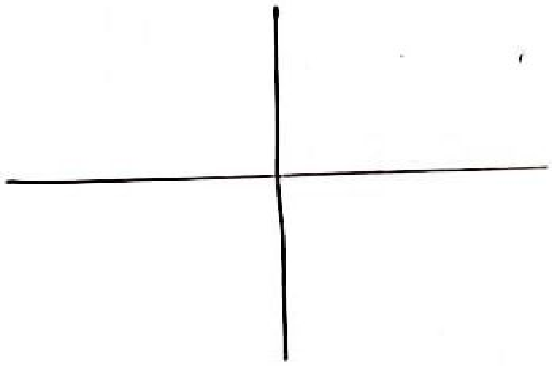
$$= -2k \neq 0$$

Since $\nabla \times F \neq 0$ therefore F is irrotational

$F = y\hat{i} - x\hat{j}$



For any vector $\vec{F} = y\hat{i} + x\hat{j}$



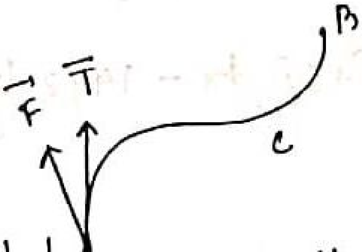
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ & & \\ & & \end{vmatrix}$$

$= 0$

Vector Integration

Line Integration

Definition: let $\vec{r} = \vec{F}(s)$. where s is the arc length measured from a fixed point, be the equation of the curve c and let A and B be two points on the curve. If $\vec{T}$ be the unit tangent to the curve at a point, then at that point on the ~~that~~ curve $\vec{F}.\vec{T} = (F\cos\theta)$ is the component of $\vec{F}$ in the in the direction of $\vec{T}$.



let us subdivided ~~A to B~~ the arc from A to B into n-no elements. Then the length $\Delta s \to 0$.

$$\therefore \lim_{\Delta s \to 0} \sum F.T\,\Delta s$$

$$= \int_c \vec{F}.\vec{T}\,ds$$

$$= \int_c \vec{F}.\frac{\frac{d\vec{r}}{dt}}{ds/dt}\,ds \qquad \vec{T} = \frac{dr/dt}{|dr/dt|} = \frac{dr/dt}{\frac{ds}{dt}}$$

$$= \int_c F.dr$$

[বইয়ের- 7, 8, 9 Number problem করাতে হবে]

$\int^{b} \int^{12} \partial M \, dy \, dx$ 1

Problem-01:

If $A = (3x^2+6y)i - 14yz\,j + 20xz^2 k$, evalute $\int_c A \cdot dr$ from (0,0,0) to (1,1,1)
along the following paths c

(a) $x = t,\ y = t^2,\ z = t^3$

(b) the straight lines from (0,0,0) to (1,0,0) then to (1,1,0) and then to (1,1,1)

(c) the straight line joining (0,0,0), (1,1,1)

$$\int_c A \cdot dr = \int_c \left((3x^2+6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \int_c (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

(a)

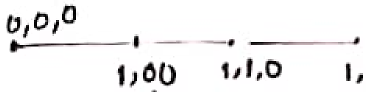
$$\int_0^1 (3t^2+6t^2)\,dt - 14(t^2)t^3\,dt^2 + 20t(t^3)^2\,dt^3$$

$$= \int_0^1 9t^2\,dt - 28t^6 + 60t^9\,dt$$

$$= \left[3t^3 - 4t^7 + 6t^{10}\right]_0^1$$

$$= 5 \text{ Am.}$$

(b)



$$\int_{x=0}^1 (3x^2+6(0))\,dx - 14(0)(0) + 20\cdot(0)^2\cdot x$$

$$= \int_{x=0}^1 3x^2$$

$$= \left[x^3\right]_0^1 = 1$$

[Along the straight line from (0,0,0) to (1,0,0) $y=0,\ z=0$ $dy=0,\ dz=0$ while x varies from 0 to 1]

$$\int_{y=0}^1 (3(1)^2+6y)\cdot 0 - 14y\cdot(0)\cdot(dy) + 20\cdot(1)\cdot(0)dz)$$

$$= 0$$

[Along the straight line from (1,0,0) to (1,1,0) $x=1,\ z=0$ $dx=0,\ dz=0$ while y varies from 0 to 1] along the straight line from (1,1,0) to (1,1,1) where $x=1$; $y=1$

$z=0$ to $z=1$ $\quad dx=0 \quad dy=0$

$$\int_{z=0}^{1} \{3.(1)^2+6(0)\}.0 - \{14.(1).(0)\}.0 + 20.(1).z^2 dz$$

$$= \int_0^1 20z^2 dz$$

$$= \cancel{40}\ \frac{20}{3} \left| z^3 \right|_0^1$$

$$= \frac{20}{3}$$

$\therefore$ Adding $\int_c A\,dr$

$$= 1\cancel{8}-10+\frac{20}{3}$$

$$= \frac{23}{3} \quad \text{Ans.}$$

Conservative field

The force F is a conservative force if the amount of work done in displacing a unit test body from one position to another is dependent of the path and depends on the end point only.

Problem:

(a) If $F=\nabla\phi$, where ϕ is single valued and has continuous partial derivatives. Show that (the conservative field) work done in moving a particle from one point $P_1=(x_1,y_1,z_1)$ in this field to another point $P_2 \equiv (x_2,y_2,z_2)$ is dependent of the path joining the two points)

(b) Conversely, if $\int_c F.dr$ is independent of the path C joining any

(a) Work done $= \int_C F \cdot dr$

$$= \int_{P_1}^{P_2} \nabla\phi \, dr$$

$$= \int_{P_1}^{P_2} \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right)\left(dx\, i + dy\, j + dz\, \hat{k}\right)$$

$$= \int_{P_1}^{P_2} \frac{\partial\phi}{\partial x} \cdot dx + \frac{\partial\phi}{\partial y} \cdot dy + \frac{\partial\phi}{\partial z} \cdot dz$$

$$= \int_{P_1}^{P_2} d\phi$$

$$= \phi(P_2) - \phi(P_1)$$

$$= \phi(x_2, y_2, z_2) - \phi(x_1, y_1, z_1)$$

Then the integral depends only on points P_1 and P_2 and not on the path joining them. This is true of course only if $\phi(x, y, z)$ is single valued at all points P_2 and P_2

(b) let $F = F_1 i + F_2 j + F_3 k$.
let the two points (x_1, y_1, z_1) and (x, y, z) respectively

$$\phi(x, y, z) = \int_{x_1, y_1, z_1}^{x, y, z} F \cdot dr$$

$$= \int_{(x_1, y_1, z_1)}^{(x, y, z)} (F_1 i + F_2 j + F_3 k)(dx\, i + dy\, j + dz\, k)$$

$$= \int_{x_1, y_1, z_1}^{x, y, z} F_1 dx + F_2 dy + F_3 dz$$ Is independent of the path joining (x_1, y_1, z_1) and (x, y, z). Thus.

$$\phi(x+\Delta x, y, z) - \phi(x,y,z) = \int_{x,y,z}^{x+\Delta x, y, z} F_1\,dx + F_2\,dy + F_3\,dz$$

$$- \int_{x,y,z}^{x_1,y_1,z_1} F_1\,dx + F_2\,dy + F_3\,dz$$

$$= \int_{x_1,y_1,z_1}^{x,y,z} (\quad) + \int_{x,y,z}^{x+\Delta x, y, z} (\quad)$$

$$= \int_{x_1,y_1,z_1}^{x+\Delta x, y, z} F_1\,dx + F_2\,dy + F_3\,dz$$

$$\therefore \frac{\phi(x+\Delta x, y, z) - \phi(x,y,z)}{\Delta x} = \frac{1}{\Delta x}\int_{x_1,y_1,z_1}^{x+\Delta x, y, z} F_1\,dx$$

let $\Delta x \longrightarrow 0$

$$\frac{\partial \phi}{\partial x} = F_1$$

smillarly, $\dfrac{\partial \phi}{\partial y} = F_2$

$$\frac{\partial \phi}{\partial z} = F_3$$

$$\therefore F = F_1 i + F_2 j + F_3 k$$

$$= \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)\phi$$

$$= \nabla \phi$$

Problem-11

(a) If F is a conservative field, prove that curl $F = \nabla \times F = 0$

(b) Conversely if $\nabla \times F = 0$ (i.e F is irrotational). Prove that F is conservative.

Soln

(a) For conservative $F = \nabla\phi$

$$\therefore \text{curl } F = \nabla \times \nabla F$$

$$= \nabla \times \left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right)$$

$$= \begin{vmatrix} i & j & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\cdot\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial y}\right] i + \left[\frac{\partial}{\partial x}\cdot\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\cdot\frac{\partial\phi}{\partial x}\right] j$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}\cdot\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\cdot\frac{\partial\phi}{\partial x}\right]$$

$$= \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right) i + \left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right) j$$
$$+ \left(\frac{\partial^2\phi}{\partial x\cdot\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) k$$

$$= 0$$

(b) if $\nabla \times F = 0$, then
$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$F = F_1 i + F_2 j + F_3 k$

Surface Integration

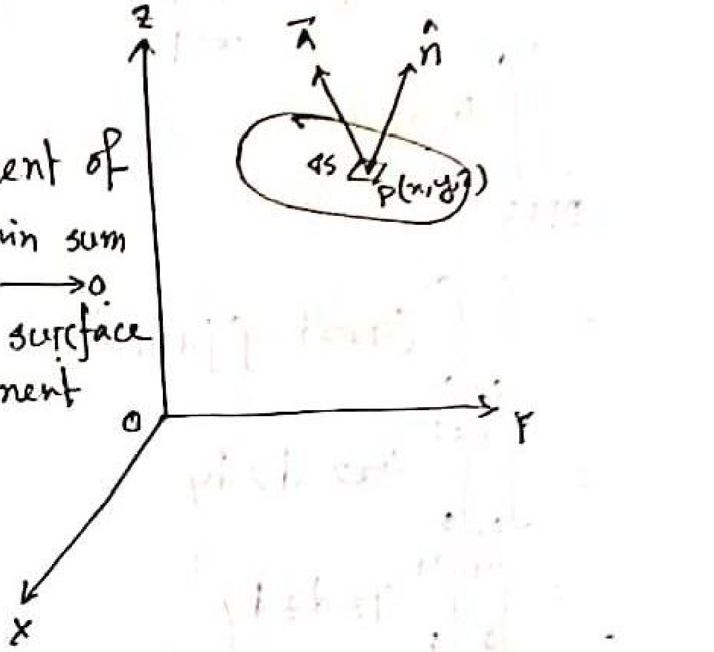
\# Given a definition of $\iint_S \vec{A}\cdot\hat{n}\,ds$ over a surface s in terms of limit of a sum.

Soln: Subdivided the area s into N elements of area Δs. Choose any point $P(x,y,z)$ on Δs. Define $\vec{A}$ and $\hat{n}$ be the unit positive normal to s at P. Form the sum

$$\sum \vec{A}\cdot\hat{n}\,\Delta s$$

where $\vec{A}\cdot\hat{n}$ is the normal component of $\vec{A}$ at P. Now take the limit of this sum as $N \to \infty$: then the area $\Delta s \to 0$. This limit if it exists is called surface integration of the normal component of $\vec{A}$ over s and is denoted

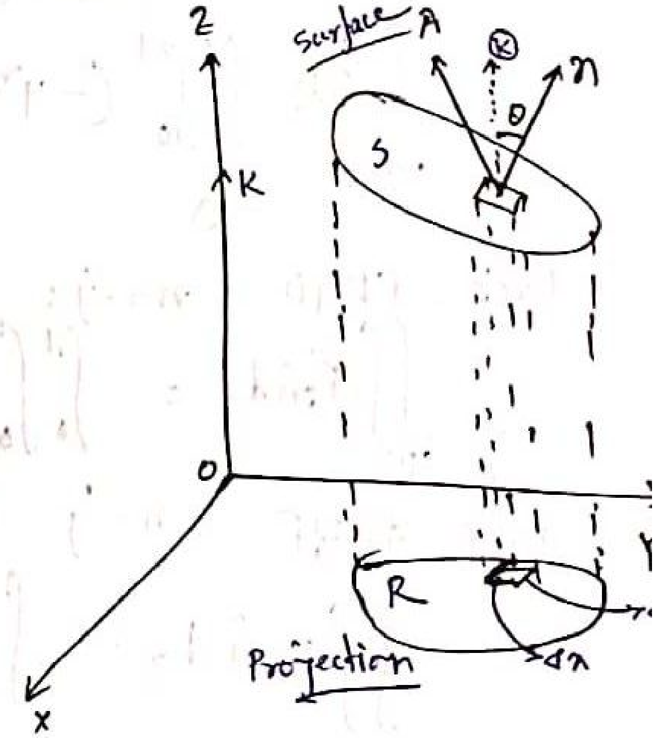
$$\iint_S \vec{A}\cdot\hat{n}\,ds$$



Problem: Suppose that the surface s has projection R on the xy plane. Show that

$$\boxed{\iint_S A\cdot n\,ds = \iint_R A\cdot n\,\frac{dx\,dy}{|n\cdot k|}}$$

Soln

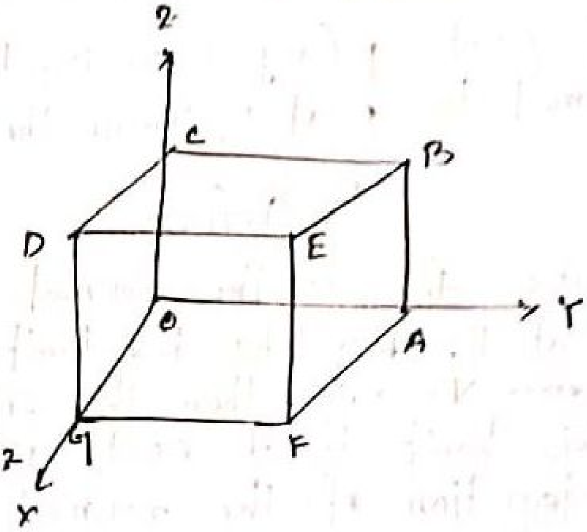


$$\Delta x\,\Delta y = \Delta s\cos\theta$$
$$= \Delta s\,|\hat{n}\cdot k|$$
$$\therefore \Delta s = \frac{\Delta x\,\Delta y}{|n\cdot k|}$$
$$\therefore ds = \frac{dx\,dy}{|n\cdot k|}$$
$$\therefore \iint_S A\cdot n\,ds = \iint_R A\cdot n\,\frac{dx\,dy}{|n\cdot k|}$$

Problem-23

IF $F = 4xzi - y^2j + yzk$, evalute $\iint_S F.n\,ds$ where S is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$

Soln



Face DEFG $\quad n=i \quad \underline{\theta=0^\circ}$

$x=1$

$$\iint_{DEFG} F.n\,ds.$$

$$= \int_0^1\int_0^1 (4xzi - y^2j + yzk).i\,\cancel{dx}\,dy\,dz$$

$$= \int_0^1\int_0^1 4xz\,dz\,dy$$

$$= \int_0^1\int_0^1 4z\,dz\,dy$$

$$= \overset{2}{\cancel{4}} \times \frac{1}{\cancel{2}} \times 1$$

$$= 2$$

Face ABCO $\quad n=-i$

$x=0$

$$\iint_{ABCO} F.n\,ds = \int_0^1\int_0^1 (-y^2j + yzk).(-i)\,dy\,dz$$

$$= 0$$

face CDGO $\quad n=-j; \quad y=0$

$$\iint_{CDGO} \overline{F}.\hat{n}\,ds = \int_0^1\int_0^1 (4xz\,i)(-j)\,dx\,dz = 0$$

Face ABEF $\quad n=j \quad y=1$

$$\iint_{ABEF} F.\hat{n}\,ds = \int_0^1\int_0^1 (4xzi - j + zk).j\,dx\,dz = -1$$

Face AOGF $z=0$ $n=-\hat{k}$

$$\iint_{AOGF} F.\hat{n}\,ds = \int_0^1\int_0^1 (-y^2 j).(-k)\,dx\,dy = 0$$

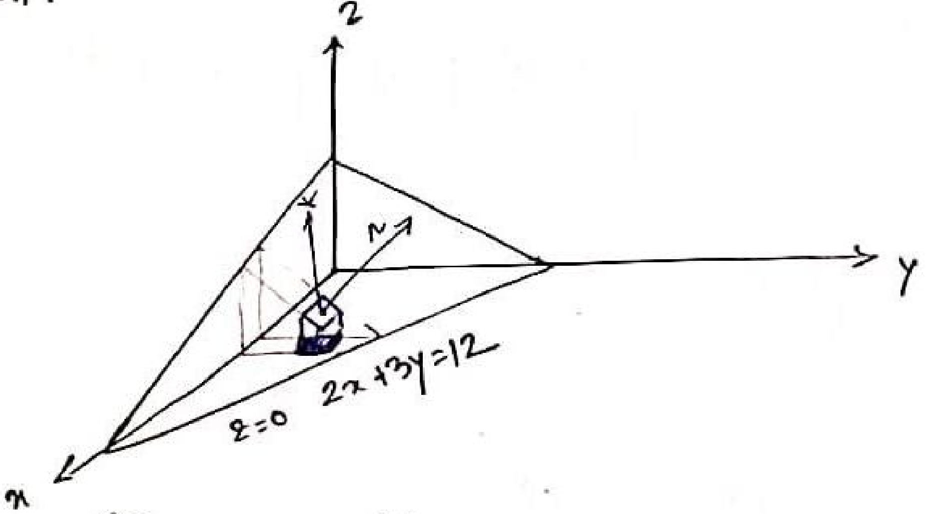
Face BCDE $z=1$ $\hat{n}=k$

$$\iint_{BCDE} F.\hat{n}\,ds = \int_0^1\int_0^1 (4x i - y^2 j + yk).\hat{k}\,dx\,dy = \frac{1}{2}$$

Adding $\iint_S F n\,ds = 2+0+0+(-1)+\frac{1}{2}+0 = \frac{3}{2}$ Ans.

Problem-19

Evaluate $\iint_S A.n\,ds$, where $A = 18zi - 12j + 3yk$ and S is the part of plane $2x+3y+6z=12$ which is located in the first octant.



$$\iint_S A.n\,ds = \iint_R A.n\,\frac{dx.dy}{|n.k|}$$

$$\hat{n} = \frac{\text{Grand } \phi}{|\text{Grad } \phi|}$$

$$= \frac{\left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)(2x+3y+6z)}{|n|} = \frac{2i+3j+6k}{\sqrt{2^2+3^2+6^2}}$$

$$= \frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$$

$$\hat{n}\cdot k| = \left(\frac{2}{7}i + \frac{3}{7}j + \frac{6}{7}k\right)\cdot k$$

$$= \frac{6}{7}$$

$$\therefore |\hat{n}\cdot k| = \frac{6}{7}$$

$$\therefore A\cdot\hat{n} = (18zi - 12j + 3yk)\left(\frac{2i}{7} + \frac{3j}{7} + \frac{6k}{7}\right)$$

$$= \frac{36z - 36 + 18y}{7}$$

$$= \frac{6(3y + 6z) - 36}{7}$$

$$= \frac{6(12 - 2x) - 36}{7}$$

$$\bar{A}\cdot\hat{n} = \frac{36 - 12x}{7}$$

Now

$$\iint_S \bar{A}\hat{n}\, ds = \iint_R \bar{A}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \iint_R \left(\frac{36 - 12x}{7}\right) \times \frac{7}{6}\, dx\, dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} 6 - 2x\, dx\, dy$$

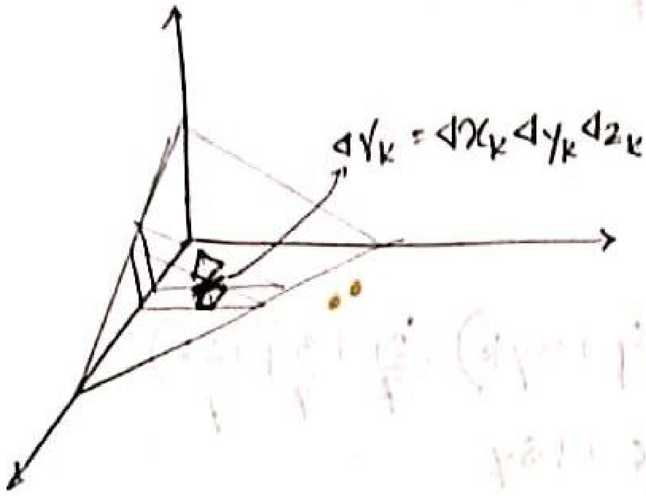
$$= \int_{x=0}^{6} \left[6y - 2xy\right]_0^{\frac{12-2x}{3}} dx$$

$$= \int_{x=0}^{6} 24 - 4x - 8x + \frac{4x^2}{3}\, dx$$

$$= \left[24x - 6x^2 - \frac{4x^3}{9}\right]_0^6 = 24 \text{ Ans.}$$

(If we had the positive unit normal vector $\hat{n}$ opposite to that in the figure above we get -24)

Volume Integration



let there be a region v bounded a close surface s, the scalar function ϕ of x, y, z difined by the volume. let us divide the region v into N sub-regions Δv. Consider the sum

$$\sum \phi \Delta v$$

taken over all posible sub-region or cubes in the region V, If the limit of the sum as $N \longrightarrow \infty$ i.e. each of the sub regions $\Delta v \longrightarrow 0$ and if it exists is called volume integration and is denoted by

$$\iiint_V \phi \, dv \quad \text{or} \quad \iiint_V \phi \, dx \, dy \, dz$$

$\phi = 45x^2y$ and let v denote the closed region bounded by the planes $4x+2y+z=8$; $x=0, y=0, z=0$. Evalute $\iiint_v \phi\, dv$

Given that

$$\phi = 45x^2y$$

$$\therefore \iiint_v \phi\, dv = \iiint_v \phi\, dx\,dy\,dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45x^2y\, dz\,dy\,dx$$

$$= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y\,(8-4x-2y)\, dy\,dx$$

$$= 90\int_{x=0}^{2}\int_{y=0}^{4-2x} (4x^2y - 2x^3y - x^2y^2)\, dy\,dx$$

$$= 90\int_{x=0}^{2}\left[2x^2y^2 - x^3y^2 - \frac{x^2y^3}{3}\right]_0^{4-2x} dx$$

$$= 90\int_{x=0}^{2}\left\{2x^2 - x^3 - \frac{x^2}{3}(4-2x)\right\}\{4-2x\}^2\, dx$$

$$= 90\int_{x=0}^{2} x^2\left(\frac{2}{3} - \frac{1}{3}x\right)(4-2x)^2\, dx$$

$$= 45\int_{x=0}^{2}\frac{x^3}{3}(4-2x)^3\, dx$$

$$= 128 \text{ Ans}$$

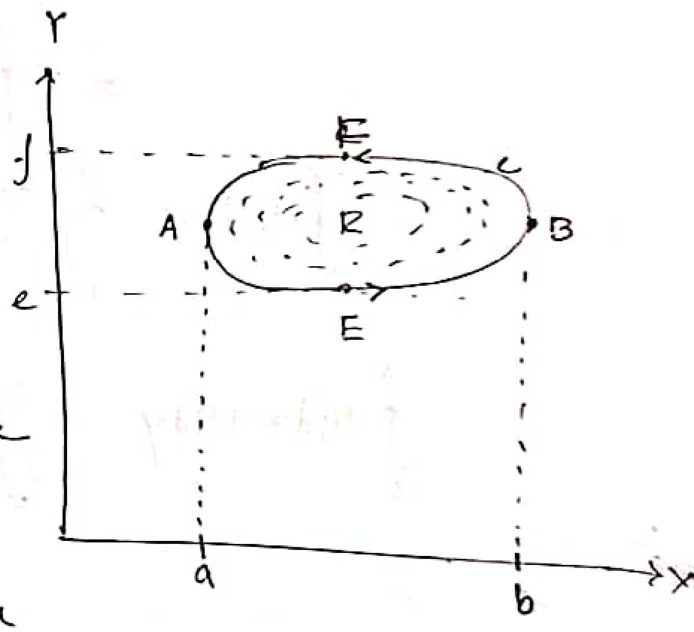
Green's Theorem

If 'R' is a closed region bounded by a simple closed curve c and 'M' and 'N' are two continuous functions having continuous derivatives than we get,

$$\boxed{\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

where c is moving in positive (counter clockwise) direction.

Proof



let the equation of the curves AEB and AFB be $y = Y_1(x)$ and $y = Y_2(x)$. If R is the region bounded by c, we have

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b}\int_{Y_1(x)}^{Y_2(x)} \frac{\partial M}{\partial y}\,dy\,dx$$

$$= \int_{x=a}^{b} \Big[M(x,y)\Big]_{Y_1(x)}^{Y_2(x)} dx$$

$$= \int_{x=a}^{b} M(x, Y_2(x))\,dx - \int_{x=a}^{b} M(x, Y_1(x))\,dx$$

$$= -\int_{b}^{a} M(x, Y_2(x))\,dx - \int_{a}^{b} M(x, Y(x))\,dx$$

$$= -\oint M\,dx$$

$$\therefore \oint M\,dx = \iint \frac{\partial M}{\partial y}\,dx\,dy \quad \text{———} (1)$$

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_e^f \int_{x_1(y)}^{x_2(y)} \frac{\partial N}{\partial x}\,dx\,dy$$

$$= \int_e^f \left[N(x,y)\right]_{x_1(y)}^{x_2(y)} dy$$

$$= \int_e^f N(x, x_2(y))\,dy - \int_e^f N(x, x_1(y))\,dy$$

$$= \int_e^f N(x, (x_2(y))\,dy + \int_f^e N(x, x_1(y))\,dy$$

$$= \oint N\,dy$$

Similarly the equation curve $EAF = x_1(y)$ and $EBF = x_2(y)$

$$\therefore \oint N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy$$

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Proved

Verify Green's theorem in the plane for $\oint_c (xy+y^2)dx + x^2dy$ where c is the closed curve of the region bounded by $y=x$ and $y=x^2$

solution

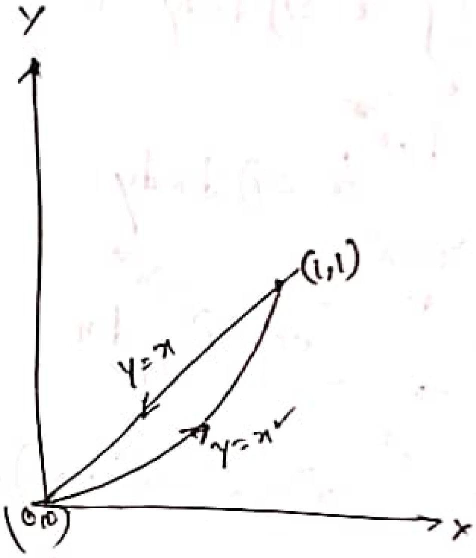
$M = xy + y^2$

$N = x^2$

$y = x$	$y = x^2$
$dy = dx$	$dy = 2x\,dx$

$\oint_c (xy+y^2)dx + x^2dy$

$\oint_{x=1}^{0} (x^2+x^2)dx + x^2dx$ [Anti clockwise



$\Rightarrow \oint_{x=1}^{0} \{2x^2dx + x^2dx\}$

$= \left[\frac{2x^3}{3} + \frac{x^3}{3}\right]_1^0$

$= -\left(\frac{2}{3} + \frac{1}{3}\right)$

$= -1$ ——— (1)

$\oint_0^1 (xy+y^2)dx + x^2dy$

$= \int_0^1 (x^3+x^4)dx + x^2 \cdot 2x\,dx$

$= \int_0^1 \{(x^3+x^4)dx + 2x^3dx\}$

$= \left[\frac{x^4}{4} + \frac{x^5}{5} + \frac{x^4}{4} \times 2\right]_0^1$

$= \frac{1}{4} + \frac{1}{5} + \frac{1}{2}$

$= \frac{19}{20}$

$\therefore \oint_c (xy+y^2)dx = \frac{19}{20} - 1$

$= -\frac{1}{20}$

$$\frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} = x + 2y$$

$$\therefore \iint_R (2x - x - 2y)\, dx\, dy$$

$$= \iint_R (x - 2y)\, dx\, dy$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\, dx\, dy$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 (x^2 - x^2 - x^3 + x^4)\, dx$$

$$= \int_0^1 (x^4 - x^3)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4}$$

$$= -\frac{1}{20}$$

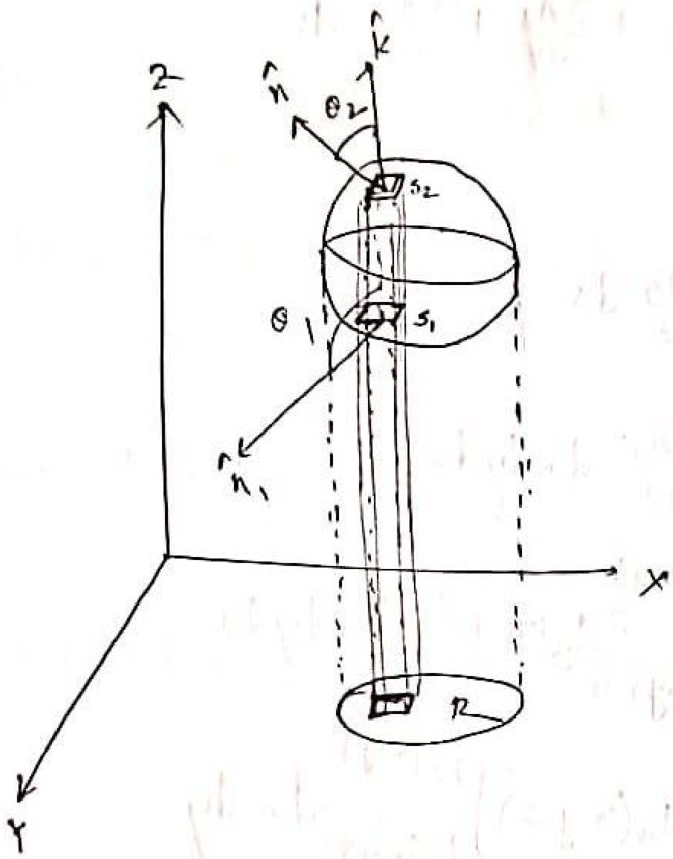
So that the theorem is verified.

Divergence Theorem

If 'V' is the volume bounded by a closed surface S and $\vec{A}$ is a vector function of position with continuous derivatives, the we get

$$\iiint_V \bar{\nabla} \cdot \vec{A}\, dv = \iint_S \vec{A} \cdot \hat{n}\, ds$$

Proof



let S_1 be the lower portion and S_2 be the upper portion of the surface S.

let $z = f_1(x,y)$ and $z = f_2(x,y)$ be the equation of the surface S_1 and S_2 respectively. and R is the projection of the surface on the plane xy.

$$\iiint_V (\vec{\nabla} \cdot A)\, dv$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right) dv$$

$$= \iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dv$$

Consider first

$$= \iiint_V \frac{\partial A_3}{\partial z}\, dv$$

$$= \iiint_V \frac{\partial A_3}{\partial z}\, dx\, dy\, dz$$

$$= \iint_R \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial}{\partial z} A_3(x,y,z)\, dx,\, dy\, dz$$

$$= \iint_R \left[A_3(x,y,z)\right]_{f_1(x,y)}^{f_2(x,y)} dx\, dy$$

$$= \iint_R A_3\left(x,y,f_2(x,y)\right) dx\, dy - \iint_R A_3\left(x,y,f_1(x,y)\right) dx\, dy \quad \text{—— (1)}$$

For upper surface S_2

$$dx \cdot dy = \cos\theta_2\, ds_2$$

$$dx \cdot dy = \hat{n} \cdot \hat{k}\, ds_2$$

For lower surface s_1

$$dxdy = \cos\theta_1 \, ds_1$$

$$dxdy = -\hat{n}\hat{k}\, ds_1$$

$$\cancel{\iint_R A_3(x,y,f_2)\,\hat{n}.\hat{k}\,ds_2}$$

$$\iint_R A_3(x,y,f_2)\,dx\,dy = \iint_R A_3(x,y,f_2)\,\hat{n}.\hat{k}\,ds_2 \text{ ——— (II)}$$

$$\iint_R A_3(x,y,f_1)\,dxdy = -\iint_R A_3(x,y,f_1)\,\hat{n}.\hat{k}\,ds_1 \text{ ——— (III)}$$

$\therefore$ (II) $-$ (III)

$$\iint_R A_3(x,y,f_2(x,y))\,dx\,dy - \iint_R A_3(x,y,f_1)\,dxdy$$

$$\Rightarrow \iint_R A_3(x,y,f_2)\,\hat{n}.\hat{k}\,ds_2 + \iint_R A_3(x,y,f_1)\,\hat{n}.\hat{k}\,ds_1$$

$$\iiint_V \frac{\partial A_3}{\partial z}\,dv = \iint_S A_3\,\hat{k}.\hat{n}\,ds \text{ ——— (IV)}$$

again

$$\iiint_V \frac{\partial A_2}{\partial y}\,dv = \iint_S A_2\,\hat{j}\,\hat{n}\,ds \text{ ——— (V)}$$

and

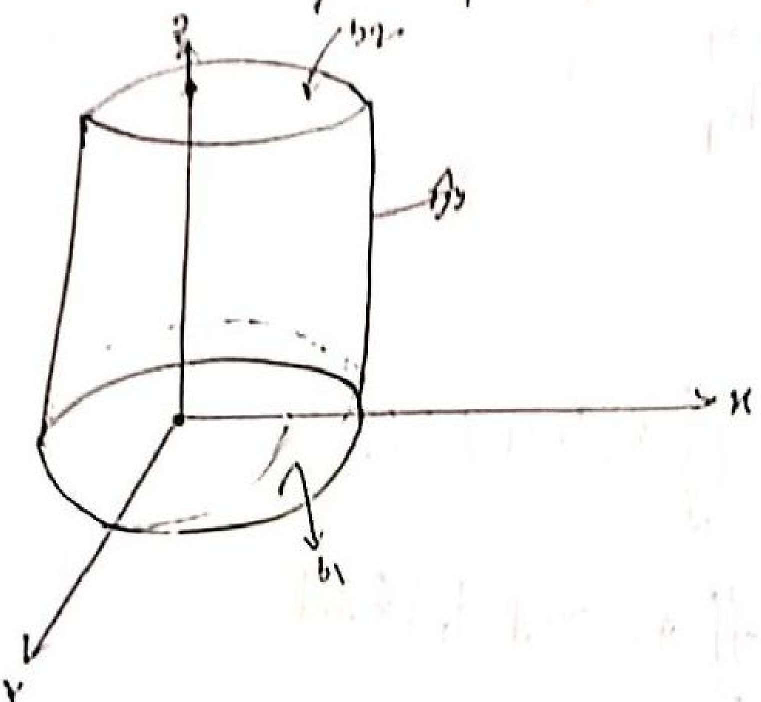
$$\iiint_V \frac{\partial A_1}{\partial x}\,dv = \iint_S A_1\hat{i}.\hat{n}\,ds \text{ ——— (VI)}$$

(IV) + (V) + (VI)

$$\therefore \iiint(\vec{\nabla}.A)\,dv = \iint_S (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\,\hat{n}\,ds = \iint_S \vec{A}.\hat{n}\,ds$$

181 Verify the divergence theorem for $A = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$

Soln



Volume integral

$$\iiint_V \vec{\nabla}\cdot\vec{A}\, dv$$

$$= \iiint_V \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(4x\hat{i} + -2y^2\hat{j} + z^2\hat{k}\right) dv$$

$$= \iiint_V \left(\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right) dv$$

$$= \iiint_V (4 - 4y + 2z)\, dv$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4 - 4y + 2z)\, dz\, dy\, dx$$

$$= 84\pi$$

__surface integral__

$$\iint_{S_1} A\cdot n\, ds_1$$

$z=0$
$n=-k$
$A = 4xi - 2y^2 j + \cancel{z^2 k}^{\to 0}$

$\therefore \iint_{S_1} (4xi - 2y^2 j)\cdot(-\hat{k})\, ds_1$
$= 0$

$$\iint_{S_2} A\cdot n\, ds_1$$

$z=3$
$n=\hat{k}$

$$\iint_{S} (4x\hat{i} - 2y^2 j + 9\hat{k})\hat{k}\, ds_1$$

$$= \iint_{S} 9\, ds_2$$

$$= 9\iint_{S} ds_2$$

$$= 9\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, dx$$

$$= 9\int_{-2}^{2} [y]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\, dx$$

$$= 9\int_{-2}^{2} (\sqrt{4-x^2} + \sqrt{4-x^2})\, dx$$

$$= 9\int_{-2}^{2} 2\sqrt{4-x^2}\, dx$$

$$= 18\int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$= 18 \times 6.283$

$= 36\pi$

$4 - x^2 = z$
$\Rightarrow z$
$-2x\,dx = dz$

Surface - 3:

$$\iint_S \vec{A}.n \, ds_3$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} \text{ or } \frac{\nabla\phi}{|n|}$$

$$= \frac{\nabla(x^2+y^2)}{|n|}$$

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$$

$$= \frac{x\hat{i}+y\hat{j}}{2} \qquad [\because x^2+y^2=4]$$

$$\therefore \iint_{S_3} (4x\hat{i}+2y^2\hat{j}+z^2\hat{k}) \cdot \left(\frac{x\hat{i}+y\hat{j}}{2}\right) ds_3$$

$$= \iint_S 2x^2-y^3 \, ds_3$$

$$= \int_{\theta=0}^{2\pi}\int_0^3 (2\times 4\cos^2\theta - 8\sin^3\theta) \, 2\,dz\,d\theta$$

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$$= \int_{\theta=0}^{2\pi} (48\cos^2\theta - 48\sin^3\theta)\, d\theta$$

$$= 48\pi$$

$$\therefore S_1+S_2+S_3 = 84\pi$$

$x^2+y^2=4$

$\left(\frac{x}{2}\right)^2+\left(\frac{y}{2}\right)^2=1$

$\left(\frac{x}{2}\right)^2+\left(\frac{y}{2}\right)^2=\sin^2\theta+$

$x=2\cos\theta$

$y=2\sin\theta$

parametric equation

$x=r\cos$

$x=r\sin$

Stroke's Theorem

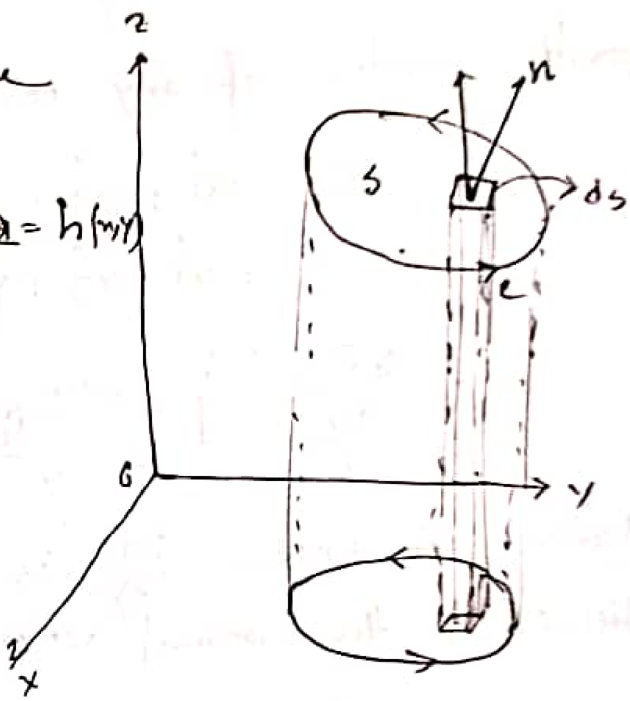
If 's' is an open two-sided surface bounded by a single closed curve 'c' and if $\bar{A}$ has continuous derivatives then we get,

$$\boxed{\oint_c \bar{A}\cdot d\bar{r} = \iint_S (\bar{\nabla}\times\bar{A})\cdot\hat{n}\, ds}$$

Proof

Assume the surface 's' has the following equation

$z = f(x,y)$; $y = g(x,z)$ & $x = h(y,z)$



We have to prove that

$$\oint \vec{A}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{A})\,\hat{n}\, ds$$

$$= \iint_S \left\{\vec{\nabla}\times\left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right)\right\}\hat{n}\, ds$$

Considering first

$$\iint_S (\vec{\nabla}\times A_1\hat{i})\,\hat{n}\, ds$$

Now, $\vec{\nabla}\times A_1\hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix} = ~~\frac{\partial A_1}{\partial y}~~ \frac{\partial A_1}{\partial z}\hat{j} - \frac{\partial A_1}{\partial y}\hat{k}$

$$\therefore \iint_S (\bar{\nabla} \times A_1\hat{i}).\hat{n}\,ds$$

$$= \iint_S \left(\frac{\partial A_1}{\partial z}\hat{j} - \frac{\partial A_1}{\partial y}\hat{k}\right).\hat{n}\,ds$$

$$= \iint_S \left(\frac{\partial A_1}{\partial z}\hat{j}.\hat{n} - \frac{\partial A_1}{\partial y}\hat{k}.n\right)ds$$

position vector of any point on the surface is

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \Big| \; z = f(x,y)$$

$$= x\hat{i} + y\hat{j} + \{f(x,y)\}\hat{k}$$

$$\therefore \frac{\partial \bar{r}}{\partial y} = \hat{j} + \frac{\partial f(x,y)}{\partial y}\hat{k}$$

Again ~~set~~ since $\frac{\partial \bar{r}}{\partial y}$ is a tangent vector and perpendicular to the normal vector $\hat{n}$, then

$$\hat{n}.\frac{\partial \bar{r}}{\partial y} = 0$$

$$\hat{n}.\left(\hat{j} + \frac{\partial f}{\partial y}\hat{k}\right) = 0$$

$$\therefore \hat{j}.\hat{n} = -\frac{\partial f}{\partial y}\hat{k}.\hat{n}$$

$$\therefore \iint_S (\bar{\nabla} \times A_1\hat{i}).\hat{n}\,ds = -\iint_S \left(\frac{\partial A_1}{\partial z}.\frac{\partial f}{\partial y} + \frac{\partial A_1}{\partial y}\right)\hat{k}.\hat{n}.\,ds \quad \text{(iv)}$$

$$A_1(x,y,z) = A_1\{x, y, f(x,y)\} = F(x,y)$$

Hence,

$$\frac{\partial A_1}{\partial y} + \frac{\partial A_1}{\partial z} \cdot \frac{\partial f}{\partial y} = \frac{\partial F}{\partial y} \quad \text{(সুবিধা নাই)} \quad \text{——— (iii)}$$

From eq^n (ii) and (iii)

$$\iint_S (\vec{\nabla} \times A_1 \hat{i}) \cdot \hat{n}\, ds = -\iint_S \frac{\partial F}{\partial y}\, \hat{k} \cdot \hat{n}\, ds \quad \text{——— (iv)}$$

from surface integration we know

$$\iint_S \bar{A} \cdot \hat{n}\, ds = \iint_R \bar{A} \cdot \hat{n}\, \frac{dx\, dy}{|\hat{n} \cdot \hat{k}|}$$

$$\therefore \iint_S \frac{\partial F}{\partial y}\, \hat{k} \cdot \hat{n}\, ds = \iint_{R_z} \frac{\partial F}{\partial y} \cdot \hat{k} \cdot \hat{n}\, \frac{dx\, dy}{|\hat{n} \cdot \hat{k}|}$$

$$= \iint_{R_z} \frac{\partial F}{\partial y}\, dx\, dy$$

equation (iv) becomes

$$\iint_S (\vec{\nabla} \times A_1 \hat{i}) \cdot \hat{n} \cdot ds = -\iint_{R_z} \frac{\partial F}{\partial y}\, dx\, dy$$

$$= \oint_{C_z} F\, dx$$

$$= \oint A_1\, dx \quad \text{——— (vii)}$$

From Green's Theorem

let

$N = 0$

$M = F$

Simillarly

$$\iint_S (\overline{\nabla} \times A_2 \hat{j}) \cdot \hat{n}\, ds = \oint_c A_2\, dy \qquad \text{(VIII)}$$

$$\iint_S (\overline{\nabla} \times A_3 \hat{k}) \cdot \hat{n}\, ds = \oint_c A_3\, dz \qquad \text{(IX)}$$

(VII) + (VIII) + IX

$$\iint \left\{ (\overline{\nabla} \times A_1 i) + (\overline{\nabla} \times (A_2 j)) + (\overline{\nabla} \times A_3 \hat{k}) \right\} \hat{n} \cdot ds$$

$$= \oint A_1\, dx + A_2\, dy + A_3\, dz$$

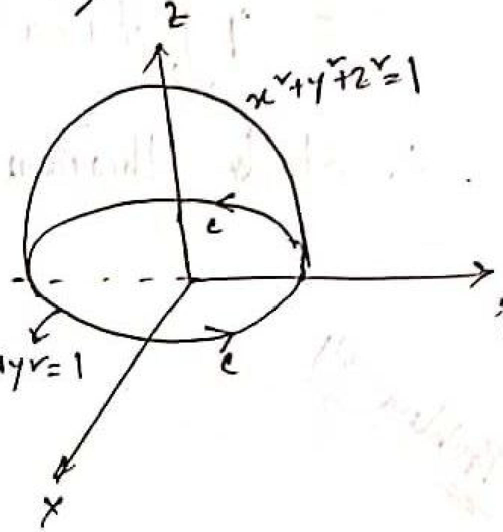
$$= \iint_S (\overline{\nabla} \times A) \cdot \hat{n}\, ds = \oint_c \overline{A} \cdot dr \quad \underline{\text{Ans}}$$

32. verify stoke's theorem for $A = (2x-y)\,i - y^2z^2\,j - y^2z\,\hat{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and c is its boundary.

$$\oint \bar{A}.\,dr$$

$$= \oint_c \{(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}\}\,(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_c (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$



let

$x = \cos\theta$ $y = \sin\theta$ $z=0$ be the parametric

$$= \oint_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta\, d\theta)$$

$$= \int_0^{2\pi} (\sin^2\theta - \sin 2\theta)\,d\theta$$

$$= \int_0^{2\pi} \left\{\frac{1}{2}(1-\cos 2\theta) - \sin 2\theta\right\} d\theta$$

$$= \pi$$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = K$$

$$\therefore \iint_S (\bar{\nabla}\times A).n\,ds = \iint_S K.n\,ds = \iint_R K.n\,\frac{dx\,dy}{|K.n|} = \iint_R dx\,dy$$

$$\int_{x=1}^{1}\int_{y=\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx$$

$$= 4\int_0^1\int_0^{\sqrt{1-x^2}} dy\, dx$$

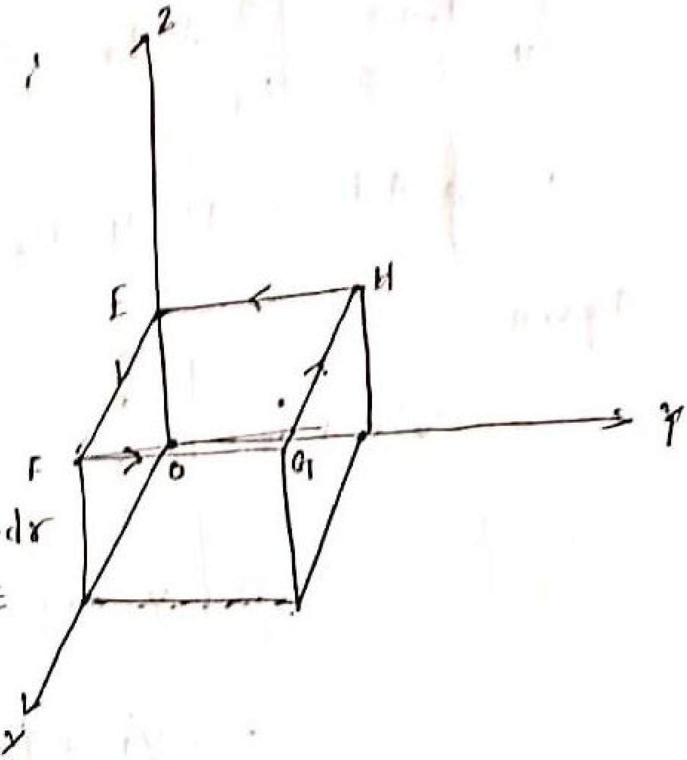
$$= 4\int_0^1 \sqrt{1-x^2} = \pi$$

$\therefore$ stocks theorem is verified $\underline{\text{h}}$.

Problem-34

Problem - 63

$E = (0,0,2)$, F $(2,0,2)$,
$G = (2,2,2)$ & $H = (0,2,2)$



$$\oint_E \vec{A} \cdot dr$$
$$= \int_{EF} A \cdot dr + \int_{FG} A \cdot dr + \int_{GH} A \cdot dr + \int_{HE} A \cdot dr$$

$A \cdot dr$
$$= \{(y-z+2)\hat{i} + (yz+4)\hat{j} - xz\hat{k}\}\{dx\hat{i} + dy\hat{j} + dz\hat{k}\}$$
$$= (y-z+2)dx + (yz+4)dy - xz\,dz$$

On EF x varies 0 to 2 $y=0$; $z=2$
FG y varies 0 to 2 $x=2$; $z=2$
GH x varies 2 to 0 $y=2$; $z=2$
HE y varies 2 to 0 $x=0$; $z=2$

$$\therefore \int_{EF} A \cdot dr = \int_E^F (0-2+2)dx + 0 - 0 = 0$$
$$\int_{FG} A \cdot dr = \int_F^G (2y+4)dy = \int_0^2 (2y+4)dy = 12$$
$$\int_{GH} A \cdot dr = \int_G^H 2dx = -4$$

$$\int_H^E A.dr = \int_H^E A.dr = \int_2^0 (2y+4)\,dy = -12$$

$$\therefore \oint A\,dr = 12-4-12 = -4$$

Again

$$\nabla \times A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= -yi + (-1+z)j - k$$

$$\iint_S (\nabla \times A).n\,ds$$ $\boxed{n = k}$

$$= \iint_S \{-yi + (-1+z)j - k\}.n\,ds$$

$$\Rightarrow \iint_S -1\,ds$$

$$= -\iint_S dx\,dy$$

$$= -\int_{x=0}^{2}\int_{y=0}^{2} dy\,dx$$

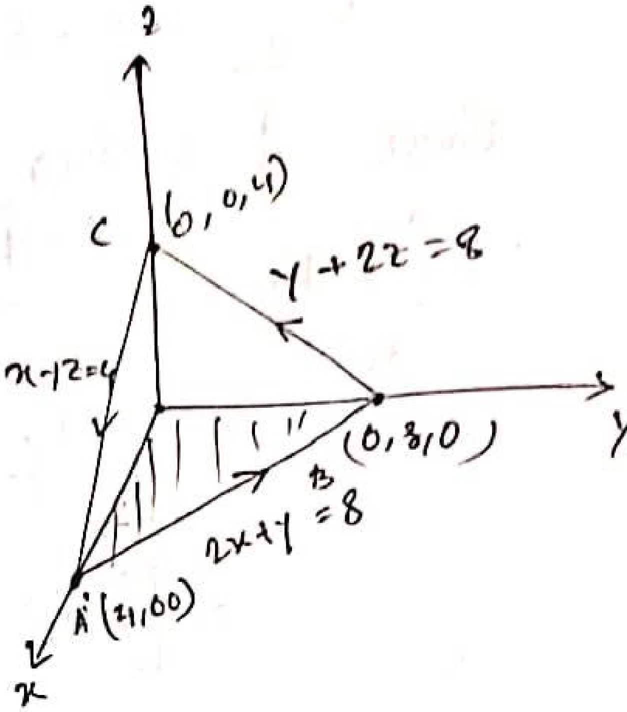
$$= -4$$

[Proved]

Problem-64

Verify stoke's theorem for $F = xz\,i - y\,j + x^2 y\,k$. where S is t
surface of the region bounded by $x=0$, $y=0$, $z=0$, $2x+y+2z=8$ which is not inclined in the xz plane.

Sol: $\oint F.dr = \iint_S (\nabla \times F).n\,ds$



Hence

$$\oint F.dr = \int_{AB} F.dr + \int_{BC} F.dr + \int_{CA} F.dr$$

$F.dr = xz\,dx - y\,dy + x^2 y\,dz$

the eqation AB is

$$2x + y = 8$$

$$x + \frac{y}{2} = \cancel{8}\,4$$

$$x - 4 = \frac{y}{-2} = t$$

$\therefore x = t+4,\ y = -2t \quad z = 0$

where t varies 0 to -4

[x=0 হবে:
y=8 যখন

$$\therefore \int_{AB} \bar{F}.dr = \int_0^{-4} xz\,dx - y\,dy + x^2 y\,dz$$

$$= \int_0^{-4} 2t(-2)\,dt$$

$$= \int_0^{4} -4t\,dt$$

$$= -32$$

Equation BC

$$y + 2z = 8$$

$$\frac{y-8}{2} = -z = t$$

where t varies 0 to -4 and $x = 0$ $y = 2t + 8$

$z = -t$

$$\therefore \int_{BC} F\,dr = \int_0^{-4} -(2t+8)(2dt)$$

$$= \int_0^{-4} (-4t - 16)\,dt$$

$$= \left[-2t^2 - 16t\right]_0^{-4}$$

$$= 32$$

Equation CA

$$2x + 2z = 8$$

$$x = 4 - z$$

$$-x = z - 4 = t$$

where t varies 0 to -4

$$\therefore \int_0^{-4} -t(t+4)(dt)$$

$$= \int_0^{-4} (t^2 + 4t)\,dt$$

$$= \frac{32}{3}$$

$$\therefore \oint F\,dr = -32 + 32 + \frac{32}{3}$$

$$= \frac{32}{3}$$

$$\therefore \iint_S \overline{\nabla} \times F \cdot \hat{n}\, ds$$

unit normal $\hat{n} = \dfrac{\text{Grand } \phi}{|n|}$

$$= \frac{\nabla(2x+y+2z)}{|n|}$$

$$= \frac{2i+j+2k}{\sqrt{2^2+1^2+4}}$$

$$= \frac{2}{3}i + \frac{1}{3}j + \frac{2}{3}\hat{k}$$

$$A = \overline{\nabla} \times \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -y & x^2y \end{vmatrix}$$

$$= (x^2-0)i - (2xy-x)j + \cancel{k(2xy)}$$

$$= x^2\hat{i} - (2xy-x)\hat{j} + \cancel{2xy\hat{k}} + 0\hat{k}$$

$$\iint_S (\nabla \times F) \cdot n\, ds$$

$$\Rightarrow \iint_S A \cdot n\, ds = \iint_R A \cdot n \cdot \frac{dx\,dy}{|n \cdot k|} \quad \Bigg| \quad \hat{n} \cdot \hat{k} = 2/3$$

$$= \iint \left(\frac{2}{3}x^2 + (x-2xy) \cdot \frac{1}{3}\right) \frac{dx\,dy}{2/3}$$

$$= \int_{x=0}^{4}\int_{y=0}^{8-2x} \left\{ \tfrac{2}{3}x^2 + (x - 2xy)\cdot\tfrac{1}{3} \right\} \frac{dx\,dy}{2/3}$$

$$= \frac{1}{2}\int_{x=0}^{4} \left(2x^2y + xy - xy^2\right)\Big|_{0}^{8-2x} dx$$

$$\vdots$$

$$= \frac{32}{3}$$

Hence the stokes theorem is varified.