

10E.06
2028 * solve by using Laplace transform

$$y'' - 3y' + 2y = 2e^t ; y(0) = 2, y'(0) = -1$$

Solⁿ: The given differential equation is

$$y'' - 3y' + 2y = 2e^t \text{ ---- (1)}$$

Taking the Laplace transform of both sides of (1), we get

$$L\{y''\} - 3L\{y'\} + 2L\{y\} = 2L\{e^t\}$$

$$\text{or, } s^2 Y(s) - sY(0) - Y'(0) - 3\{sY(s) - Y(0)\} + 2Y(s) = \frac{2}{s+1}$$

$$\text{or, } s^2 Y(s) - 2s + 1 - 3sY(s) + 6 + 2Y(s) = \frac{2}{s+1}$$

$$\text{or, } s^2 Y(s) - 3sY(s) + 2Y(s) = 2s - 7 + \frac{2}{s+1}$$

$$\text{or, } (s^2 - 3s + 2) Y(s) = 2s - 7 + \frac{2}{s+1}$$

$$\text{or, } (s-1)(s-2) Y(s) = 2s - 7 + \frac{2}{s+1}$$

$$\therefore Y(s) = \frac{2s}{(s-1)(s-2)} - \frac{7}{(s-1)(s-2)} + \frac{2}{(s-1)(s-2)(s+1)}$$

$$= \frac{2s(s+1) - 7(s+1) + 2}{(s+1)(s-1)(s-2)}$$

$$= \frac{2s^2 + 2s - 7s - 7 + 2}{(s+1)(s-1)(s-2)}$$

$$= \frac{2s^2 - 5s - 5}{(s+1)(s-1)(s-2)}$$

$$\text{Now, } \frac{2s^2 - 5s - 5}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$$

$$\therefore 2s^2 - 5s - 5 = A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1) \text{ ---- (2)}$$

putting $s = 1$, in (2), we get

$$2 - 5 - 5 = B \cdot 2 \cdot (-1) = -2B$$

$$\Rightarrow -8 = -2B \therefore B = 4$$

putting, $s = -1$, in (2) we get

$$2 + 5 - 5 = A \cdot (-2) \cdot (-3) = 6A$$

$$\therefore A = \frac{1}{3}$$

Again, putting, $s = 2$ in (2) we get

$$6 - 10 - 5 = C \cdot 3 \cdot 1$$

$$\Rightarrow -7 = 3c \quad \therefore c = -7/3$$

$$\therefore Y(s) = \frac{1/3}{s+1} + \frac{4}{s-1} + \frac{-7/3}{s-2} \quad \dots \textcircled{1}$$

→ Taking the inverse Laplace transform of both sides of $\textcircled{1}$ we get

$$L^{-1}\{Y(s)\} = \frac{1}{3} L^{-1}\left\{\frac{1}{s+1}\right\} + 4 L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{7}{3} L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\text{or, } y = y(t) = \frac{1}{3} e^{-t} + 4 e^t - \frac{7}{3} e^{2t}$$

which is the required solution.

* Solve the following differential equation using Laplace transform $ty'' + (1-2t)y' - 2y = 0$; $y(0) = 1$, $y'(0) = 2$.

Solⁿ: The given DE can be written as

$$ty'' + y' - 2ty' - 2y = 0 \quad \text{--- (1)}$$

Taking the Laplace transform of both sides of (1) we get

$$L\{ty''\} + L\{y'\} - 2L\{ty'\} - 2L\{y\} = 0$$

$$\Rightarrow -\frac{d}{ds} \{s\tilde{Y}(s) - sY(0) - Y'(0)\} + sY(s) - Y(0) + 2\frac{d}{ds} \{sY(s) - Y(0)\} - 2Y(s) = 0$$

$$\Rightarrow -\frac{d}{ds} \{s\tilde{Y}(s) - s - 2\} + sY(s) - 1 + 2\frac{d}{ds} \{sY(s) - 1\} - 2Y(s) = 0$$

$$\Rightarrow -s\frac{dY}{ds} - 2sY + 1 + sY - 1 + 2s\frac{dY}{ds} + 2Y - 2Y(s) = 0$$

$$\Rightarrow -s\frac{dY}{ds} - sY + 2s\frac{dY}{ds} = 0$$

$$\Rightarrow s\frac{dY}{ds} + 2\frac{dY}{ds} + Y = 0$$

$$\Rightarrow (s+2)\frac{dY}{ds} + Y = 0$$

$$\Rightarrow \frac{dY}{Y} + \frac{ds}{s+2} = 0$$

Integrating it, we get

$$\log Y + \log(s+2) = \log c$$

$$\Rightarrow \log Y = \log c - \log(s+2) = \log \frac{c}{s+2}$$

$$\therefore Y = \frac{c}{s+2}$$

Taking inverse Laplace transformation, of both sides, we get

$$L^{-1}\{Y\} = L^{-1}\left\{\frac{c}{s+2}\right\}$$

$$\therefore y(t) = ce^{-2t}$$

$$\text{Here } y(0) = c = 1 \quad \therefore c = 1.$$

$$\therefore y = e^{-2t}.$$

Q.E, 09 * Solve the partial differential equation $\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}$
 $u(0,t)=0, u(3,t)=0, u(x,0)=10 \sin 2\pi x - 6 \sin 4\pi x$ by
 Laplace transformation.

Soln: Given PDE is $\frac{\partial u}{\partial t} = 4 \frac{\partial^2 u}{\partial x^2}$ --- (1)

Taking Laplace transform of the PDE (1), we have

$$L\left\{\frac{\partial u}{\partial t}\right\} = 4 L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\text{or, } sU(x,s) - u(x,0) = 4 \frac{d^2 U}{dx^2}, \text{ where, } u = u(x,s) = L\{u(x,t)\}$$

$$\text{or, } 4 \frac{d^2 U}{dx^2} - sU = -u(x,0)$$

$$\text{or, } \frac{d^2 U}{dx^2} - \frac{s}{4} U = -\frac{1}{4} u(x,0) = -\frac{1}{4} (10 \sin 2\pi x - 6 \sin 4\pi x)$$

Auxiliary equation is $m^2 - \frac{s}{4} = 0 \therefore m = \pm \sqrt{\frac{s}{4}}$

$$\therefore \text{C.F} = u_c = c_1 e^{\sqrt{\frac{s}{4}} \cdot x} + c_2 e^{-\sqrt{\frac{s}{4}} \cdot x}$$

particular integral is

$$\begin{aligned} u_p &= \frac{1}{D^2 - \frac{s}{4}} \left\{ -\frac{1}{4} (10 \sin 2\pi x - 6 \sin 4\pi x) \right\} \\ &= \frac{-5/2}{-4\pi^2 - \frac{s}{4}} \cdot \sin 2\pi x + \frac{3/2}{-16\pi^2 - \frac{s}{4}} \cdot \sin 4\pi x \\ &= \frac{10 \sin 2\pi x}{s + 16\pi^2} + \frac{6 \sin 4\pi x}{s + 64\pi^2} \end{aligned}$$

Hence the general solution is

$$\begin{aligned} u = u_c + u_p &= c_1 e^{\sqrt{\frac{s}{4}} \cdot x} + c_2 e^{-\sqrt{\frac{s}{4}} \cdot x} \\ &\quad + \frac{10 \sin 2\pi x}{s + 16\pi^2} + \frac{6 \sin 4\pi x}{s + 64\pi^2} \end{aligned} \quad \text{--- (2)}$$

Taking the Laplace transform of those boundary conditions which involve t , we have

$$L\{u(0,t)\} = u(0,s) = 0$$

$$L\{u(3,t)\} = u(3,s) = 0$$

Using the first condition $u(0,s) = 0$ in (2), we have

$$0 = c_1 e^0 + c_2 e^0 \text{ or, } c_1 + c_2 = 0 \quad \text{--- (3)}$$

Again, using the second condition $u(3, s) = 0$, we have

$$0 = c_1 e^{\frac{\sqrt{3}}{4} \cdot 3} + c_2 e^{-\frac{3\sqrt{3}}{4}} + 0$$

$$\text{or, } c_1 e^{\frac{3\sqrt{3}}{4}} + c_2 e^{-\frac{3\sqrt{3}}{4}} = 0 \dots \text{--- (4)}$$

Solving the above two equations (3) and (4) we get

$$c_1 \neq 0 \text{ and } c_2 = 0$$

Hence (3) becomes

$$u = \frac{10 \sin 2\pi x}{s + 16\pi^2} - \frac{6 \sin 4\pi x}{s + 64\pi^2}$$

Now taking inverse Laplace transform we get

$$\begin{aligned} L^{-1}\{u\} &= 10 \sin 2\pi x L^{-1}\left\{\frac{1}{s + 16\pi^2}\right\} - 6 \sin 4\pi x L^{-1}\left\{\frac{1}{s + 64\pi^2}\right\} \\ &= 10 \sin 2\pi x e^{-16\pi^2 t} - 6 \sin 4\pi x e^{-64\pi^2 t} \end{aligned}$$

$$\text{Since } L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$\therefore u(x, t) = 10 \sin 2\pi x e^{-16\pi^2 t} - 6 \sin 4\pi x e^{-64\pi^2 t}$$

which is the required solution.

Taking the Laplace transform of the system and using the initial conditions, we find

$$-5i_1 - \{si_1 - I_1(0)\} + 2\{si_2 - I_2(0)\} + 10i_2 = 0$$

$$\{si_1 - I_1(0)\} + 20i_1 + 15i_2 = 55/s$$

or

$$(s+5)i_1 - (2s+10)i_2 = 0$$

$$(s+20)i_1 + 15i_2 = 55/s$$

From the first equation, $i_1 = 2i_2$, so that the second equation yields

$$(2s+55)i_2 = \frac{55}{s} \quad \text{or} \quad i_2 = \frac{55}{s(2s+55)} = \frac{1}{s} - \frac{2}{2s+55}$$

Then

$$I_2 = 1 - e^{-55t/2}$$

$$I_1 = 2I_2 = 2 - 2e^{-55t/2}$$

$$I = I_1 + I_2 = 3 - 3e^{-55t/2}$$

APPLICATIONS TO BEAMS

18. A beam which is hinged at its ends $x=0$ and $x=l$ [see Fig. 3-13] carries a uniform load W_0 per unit length. Find the deflection at any point.

The differential equation and boundary conditions are

$$\frac{d^4 Y}{dx^4} = \frac{W_0}{EI} \quad 0 < x < l \quad (1)$$

$$Y(0) = 0, \quad Y''(0) = 0, \quad Y(l) = 0, \quad Y''(l) = 0 \quad (2)$$

Taking Laplace transforms of both sides of (1), we have, if $y = y(s) = \mathcal{L}\{Y(x)\}$,

$$s^4 y - s^3 Y(0) - s^2 Y'(0) - s Y''(0) - Y'''(0) = \frac{W_0}{EI s} \quad (3)$$

Using the first two conditions in (2) and the unknown conditions $Y'(0) = c_1$, $Y'''(0) = c_2$, we find

$$y = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{W_0}{EI s^5}$$

Then inverting,

$$Y(x) = c_1 x + \frac{c_2 x^3}{3!} + \frac{W_0 x^4}{EI 4!} = c_1 x + \frac{c_2 x^3}{6} + \frac{W_0 x^4}{24EI}$$

From the last two conditions in (2), we find

$$c_1 = \frac{W_0 l^3}{24EI}, \quad c_2 = -\frac{W_0 l}{2EI}$$

Thus the required deflection is

$$Y(x) = \frac{W_0}{24EI} (l^3 x - 2lx^3 + x^4) = \frac{W_0}{24EI} x(l-x)(l^2 + lx - x^2)$$

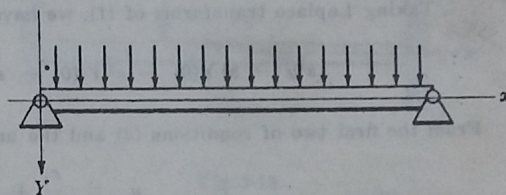


Fig. 3-13

19. A cantilever beam [Fig. 3-14] is clamped at the end $x=0$ and is free at the end $x=l$. It carries a load per unit length given by

$$W(x) = \begin{cases} W_0 & 0 < x < l/2 \\ 0 & l/2 < x < l \end{cases}$$

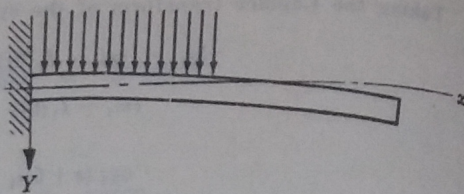


Fig. 3-14

Find the deflection.

The differential equation and boundary conditions are

$$\frac{d^4 Y}{dx^4} = \frac{W(x)}{EI} \quad 0 < x < l \quad (1)$$

$$Y(0) = 0, \quad Y'(0) = 0, \quad Y''(l) = 0, \quad Y'''(l) = 0 \quad (2)$$

In order to apply Laplace transforms, we extend the definition of $W(x)$ as follows:

$$W(x) = \begin{cases} W_0 & 0 < x < l/2 \\ 0 & x > l/2 \end{cases} \quad (3)$$

This can be written in terms of Heaviside's unit function as

$$W(x) = W_0 \{u(x) - u(x - l/2)\} \quad (4)$$

Taking Laplace transforms of (1), we have, if $y = y(s) = \mathcal{L}\{Y(x)\}$,

$$s^4 y - s^3 Y(0) - s^2 Y'(0) - s Y''(0) - Y'''(0) = \frac{W_0}{EI} \left\{ \frac{1 - e^{-sl/2}}{s} \right\}$$

From the first two of conditions (2) and the unknown conditions $Y''(0) = c_1$, $Y'''(0) = c_2$, we find

$$y = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{W_0}{EI s^5} \{1 - e^{-sl/2}\}$$

Inverting, we find

$$Y(x) = \frac{c_1 x^2}{2!} + \frac{c_2 x^3}{3!} + \frac{W_0 x^4}{EI 4!} - \frac{W_0 (x - l/2)^4}{EI 4!} u(x - l/2)$$

This is equivalent to

$$Y(x) = \begin{cases} \frac{c_1 x^2}{2!} + \frac{1}{6} c_2 x^3 + \frac{W_0}{24EI} x^4 & 0 < x < l/2 \\ \frac{c_1 x^2}{2!} + \frac{1}{6} c_2 x^3 + \frac{W_0}{24EI} x^4 - \frac{W_0}{24EI} (x - l/2)^4 & x > l/2 \end{cases}$$

We now use the conditions $Y''(l) = 0$, $Y'''(l) = 0$ to find

$$c_1 = \frac{W_0 l^2}{8EI}, \quad c_2 = -\frac{W_0 l}{2EI}$$

Thus the required deflection is

$$Y(x) = \frac{W_0 l^2}{16EI} x^2 - \frac{W_0 l}{12EI} x^3 + \frac{W_0}{24EI} x^4 - \frac{W_0}{24EI} (x - l/2)^4 u(x - l/2)$$

or

$$Y(x) = \begin{cases} \frac{W_0 l^2}{16EI} x^2 - \frac{W_0 l}{12EI} x^3 + \frac{W_0}{24EI} x^4 & 0 < x < l/2 \\ \frac{W_0 l^2}{16EI} x^2 - \frac{W_0 l}{12EI} x^3 + \frac{W_0}{24EI} x^4 - \frac{W_0}{24EI} (x - l/2)^4 & l/2 < x < l \end{cases}$$

20. A beam has a concentrated load P_0 acting at the point $x = a$. Show that we can represent this loading by $W(x) = P_0 \delta(x - a)$ where δ is the Dirac delta function or impulse function.

Consider a uniform loading W_0 per unit length over the portion of the beam between a and $a + \epsilon$ [see Fig. 3-15]. Then the total loading on this portion is

$$W_0 [a + \epsilon - a] = W_0 \epsilon$$

Since this total loading is to equal P_0 , we must have

$$W(x) = \begin{cases} P_0/\epsilon & a < x < a + \epsilon \\ 0 & \text{otherwise} \end{cases}$$

But we have already agreed to represent this in the limit as $\epsilon \rightarrow 0$ by

$$W(x) = P_0 \delta(x - a)$$

Thus the required result is demonstrated.

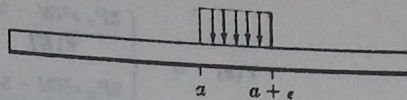


Fig. 3-15

21. A beam has its ends clamped at $x = 0$ and $x = l$ [see Fig. 3-16]. A concentrated load P_0 acts vertically downward at the point $x = l/3$. Find the resulting deflection.

By Problem 20, the concentrated load at $x = l/3$ can be represented by $P_0 \delta(x - l/3)$ where δ is the Dirac delta function or impulse function. Then the differential equation for the deflection and the associated boundary conditions are given by

$$\frac{d^4 Y}{dx^4} = \frac{P_0}{EI} \delta(x - l/3) \quad (1)$$

$$Y(0) = 0, \quad Y'(0) = 0, \quad Y(l) = 0, \quad Y'(l) = 0 \quad (2)$$

Taking Laplace transforms, we have, if $y = \mathcal{L}\{Y(x)\}$,

$$s^4 y - s^3 Y(0) - s^2 Y'(0) - s Y''(0) - Y'''(0) = \frac{P_0}{EI} e^{-ls/3} \quad (3)$$

Using the first two conditions in (2) and calling $Y''(0) = c_1$, $Y'''(0) = c_2$, we find

$$y = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{P_0}{EI} \frac{e^{-ls/3}}{s^4} \quad (4)$$

Inverting, we obtain

$$Y(x) = \frac{c_1 x^2}{2!} + \frac{c_2 x^3}{3!} + \frac{P_0}{EI} \frac{(x - l/3)^3}{3!} u(x - l/3) \quad (5)$$

or equivalently,

$$Y(x) = \begin{cases} \frac{1}{2} c_1 x^2 + \frac{1}{6} c_2 x^3 & 0 < x < l/3 \\ \frac{1}{2} c_1 x^2 + \frac{1}{6} c_2 x^3 + \frac{P_0}{6EI} (x - l/3)^3 & l/3 < x < l \end{cases}$$

From the last two conditions in (2), we find

$$c_1 = \frac{4P_0 l}{27EI}, \quad c_2 = \frac{-20P_0}{27EI}$$

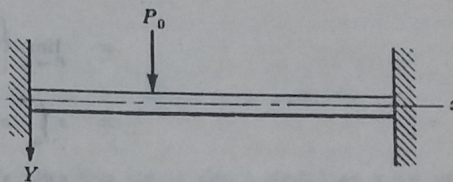


Fig. 3-16

Then the required deflection is

$$Y(x) = \frac{2P_0 lx^2}{27EI} - \frac{10P_0 x^3}{81EI} + \frac{P_0}{6EI} (x - l/3)^3 u(x - l/3)$$

or

$$Y(x) = \begin{cases} \frac{2P_0 x^2(3l - 5x)}{81EI} & 0 < x < l/3 \\ \frac{2P_0 x^2(3l - 5x)}{81EI} + \frac{P_0}{6EI} (x - l/3)^3 & l/3 < x < l \end{cases}$$

PARTIAL DIFFERENTIAL EQUATIONS

22. Given the function $U(x, t)$ defined for $a \leq x \leq b$, $t > 0$. Find

$$(a) \quad \mathcal{L}\left\{\frac{\partial U}{\partial t}\right\} = \int_0^\infty e^{-st} \frac{\partial U}{\partial t} dt, \quad (b) \quad \mathcal{L}\left\{\frac{\partial U}{\partial x}\right\} = \int_0^\infty e^{-st} \frac{\partial U}{\partial x} dt$$

assuming suitable restrictions on $U = U(x, t)$.

(a) Integrating by parts, we have

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial U}{\partial t}\right\} &= \int_0^\infty e^{-st} \frac{\partial U}{\partial t} dt = \lim_{P \rightarrow \infty} \int_0^P e^{-st} \frac{\partial U}{\partial t} dt \\ &= \lim_{P \rightarrow \infty} \left\{ e^{-st} U(x, t) \Big|_0^P + s \int_0^P e^{-st} U(x, t) dt \right\} \\ &= s \int_0^\infty e^{-st} U(x, t) dt - U(x, 0) \\ &= s u(x, s) - U(x, 0) = s u - U(x, 0) \end{aligned}$$

where $u = u(x, s) = \mathcal{L}\{U(x, t)\}$.

We have assumed that $U(x, t)$ satisfies the restrictions of Theorem 1-1, Page 2, when regarded as a function of t .

(b) We have, using Leibnitz's rule for differentiating under the integral sign,

$$\mathcal{L}\left\{\frac{\partial U}{\partial x}\right\} = \int_0^\infty e^{-st} \frac{\partial U}{\partial x} dt = \frac{d}{dx} \int_0^\infty e^{-st} U dt = \frac{du}{dx}$$

Referring to Problem 22, show that

$$(a) \quad \mathcal{L}\left\{\frac{\partial^2 U}{\partial t^2}\right\} = s^2 u(x, s) - s U(x, 0) - U_t(x, 0)$$

$$(b) \quad \mathcal{L}\left\{\frac{\partial^2 U}{\partial x^2}\right\} = \frac{d^2 u}{dx^2}$$

where $U_t(x, 0) = \frac{\partial U}{\partial t} \Big|_{t=0}$ and $u = u(x, s) = \mathcal{L}\{U(x, t)\}$.

Let $V = \partial U / \partial t$. Then as in part (a) of Problem 22, we have

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial^2 U}{\partial t^2}\right\} &= \mathcal{L}\left\{\frac{\partial V}{\partial t}\right\} = s \mathcal{L}\{V\} - V(x, 0) \\ &= s [s \mathcal{L}\{U\} - U(x, 0)] - U_t(x, 0) \\ &= s^2 u - s U(x, 0) - U_t(x, 0) \end{aligned}$$

Note the similarity of the results of this problem and part (a) of Problem 22 with Theorems 1-6 and 1-9, Page 4. Extensions are easily made.