

Simple Stresses

Simple stresses are expressed as the ratio of the applied force divided by the resisting area or

$$\sigma = \text{Force} / \text{Area}.$$

It is the expression of force per unit area to structural members that are subjected to external forces and/or induced forces. Stress is the lead to accurately describe and predict the elastic deformation of a body.

Simple stress can be classified as normal stress, shear stress, and bearing stress.

Normal stress develops when a force is applied perpendicular to the cross-sectional area of the material. If the force is going to pull the material, the stress is said to be **tensile stress** and **compressive stress** develops when the material is being compressed by two opposing forces. **Shear stress** is developed if the applied force is parallel to the resisting area. Example is the bolt that holds the tension rod in its anchor. Another condition of shearing is when we twist a bar along its longitudinal axis. This type of shearing is called torsion and covered in Chapter 3. Another type of simple stress is the **bearing stress**, it is the contact pressure between two bodies.

Suspension bridges are good example of structures that carry these stresses. The weight of the vehicle is carried by the bridge deck and passes the force to the stringers (vertical cables), which in turn, supported by the main suspension cables. The suspension cables then transferred the force into bridge towers.



Normal Stress

Stress

Stress is the expression of force applied to a unit area of surface. It is measured in psi (English unit) or in MPa (SI unit). Another unit of stress which is not commonly used is the dynes (cgs unit). Stress is the ratio of force over area.

$$\text{stress} = \text{force} / \text{area}$$

Simple Stresses

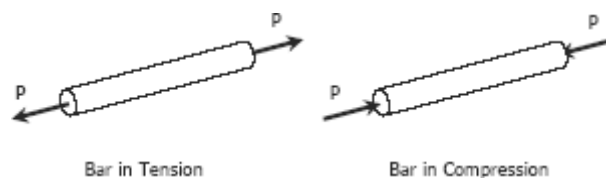
There are three types of simple stress namely; normal stress, shearing stress, and bearing stress.

Normal Stress

The resisting area is perpendicular to the applied force, thus normal. There are two types of normal stresses; tensile stress and compressive stress. Tensile stress applied to bar tends the bar to elongate while compressive stress tend to shorten the bar.

$$\sigma = \frac{P}{A}$$

where P is the applied normal load in Newton and A is the area in mm². The maximum stress in tension or compression occurs over a section normal to the load.



SOLVED PROBLEMS IN NORMAL STRESS

Problem 104

A hollow steel tube with an inside diameter of 100 mm must carry a tensile load of 400 kN. Determine the outside diameter of the tube if the stress is limited to 120 MN/m².

Solution 104

$$P = \sigma A$$

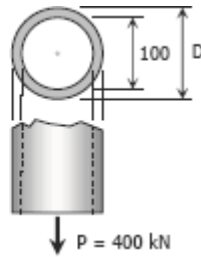
where:

$$P = 400 \text{ kN} = 400\,000 \text{ N}$$

$$\sigma = 120 \text{ MPa}$$

$$A = \frac{1}{4}\pi D^2 - \frac{1}{4}\pi(100^2)$$

$$= \frac{1}{4}\pi(D^2 - 10\,000)$$



thus,

$$400\,000 = 120 \left[\frac{1}{4}\pi(D^2 - 10\,000) \right]$$

$$400\,000 = 30\pi D^2 - 300\,000\pi$$

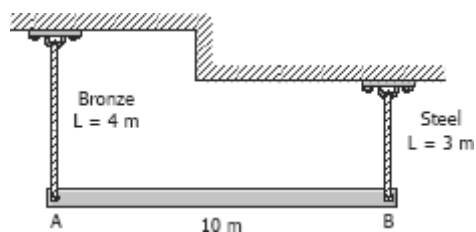
$$D^2 = \frac{400\,000 + 300\,000\pi}{30\pi}$$

$$D = 119.35 \text{ mm}$$

Problem 105

A homogeneous 800 kg bar AB is supported at either end by a cable as shown in Fig. P-105. Calculate the smallest area of each cable if the stress is not to exceed 90 MPa in bronze and 120 MPa in steel.

Figure P-105



Solution 105

By symmetry:

$$P_{br} = P_{st} = \frac{1}{2}(7848) \\ = 3924 \text{ N}$$

For bronze cable:

$$P_{br} = \sigma_{br} A_{br}$$

$$3924 = 90 A_{br}$$

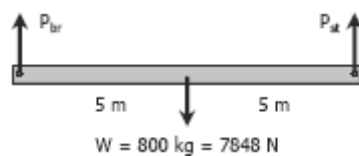
$$A_{br} = 43.6 \text{ mm}^2$$

For steel cable:

$$P_{st} = \sigma_{st} A_{st}$$

$$3924 = 120 A_{st}$$

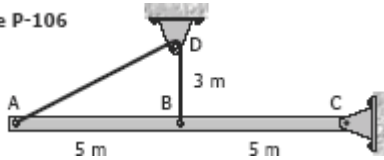
$$A_{st} = 32.7 \text{ mm}^2$$



Problem 106

The homogeneous bar shown in Fig. P-106 is supported by a smooth pin at C and a cable that runs from A to B around the smooth peg at D. Find the stress in the cable if its diameter is 0.6 inch and the bar weighs 6000 lb.

Figure P-106



Solution 106

$$\begin{aligned}\Sigma M_C &= 0 \\ 5T + 10\left(\frac{3}{\sqrt{34}}T\right) &= 5(6000) \\ T &= 2957.13 \text{ lb} \\ T &= \sigma A \\ 2957.13 &= \sigma\left[\frac{1}{4}\pi(0.6^2)\right] \\ \sigma &= 10\,458.72 \text{ psi}\end{aligned}$$

Problem 107

A rod is composed of an aluminum section rigidly attached between steel and bronze sections, as shown in Fig. P-107. Axial loads are applied at the positions indicated. If $P = 3000 \text{ lb}$ and the cross sectional area of the rod is 0.5 in^2 , determine the stress in each section.

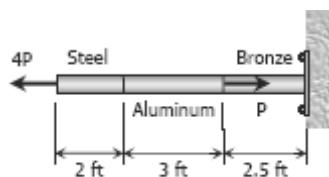


Figure P-107

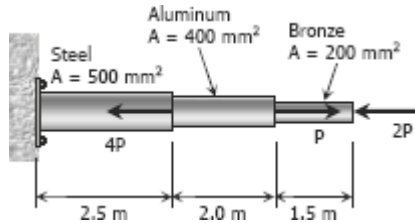
Solution 107

$$\begin{aligned}\text{For steel:} \quad & \sigma_{st} A_{st} = P_{st} \\ & \sigma_{st}(0.5) = 12 \\ & \sigma_{st} = 24 \text{ ksi} \\ \text{For aluminum:} \quad & \sigma_{al} A_{al} = P_{al} \\ & \sigma_{al}(0.5) = 12 \\ & \sigma_{al} = 24 \text{ ksi} \\ \text{For bronze:} \quad & \sigma_{br} A_{br} = P_{br} \\ & \sigma_{br}(0.5) = 9 \\ & \sigma_{br} = 18 \text{ ksi}\end{aligned}$$

Problem 108

An aluminum rod is rigidly attached between a steel rod and a bronze rod as shown in Fig. P-108. Axial loads are applied at the positions indicated. Find the maximum value of P that will not exceed a stress in steel of 140 MPa, in aluminum of 90 MPa, or in bronze of 100 MPa.

Figure P-108



Solution 108

For bronze:

$$\sigma_{br} A_{br} = 2P$$

$$100(200) = 2P$$

$$P = 10\,000 \text{ N}$$

For aluminum:

$$\sigma_{al} A_{al} = P$$

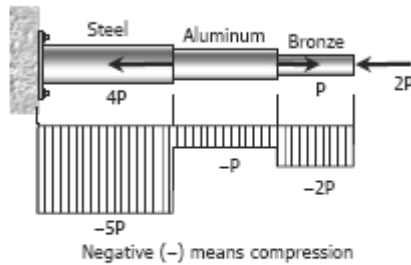
$$90(400) = P$$

$$P = 36\,000 \text{ N}$$

For Steel:

$$\sigma_{st} A_{st} = 5P$$

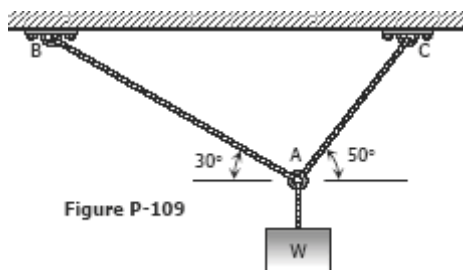
$$P = 14\,000 \text{ N}$$



For safe P , use $P = 10\,000 \text{ N} = 10 \text{ kN}$

Problem 109

Determine the largest weight W that can be supported by two wires shown in Fig. P-109. The stress in either wire is not to exceed 30 ksi. The cross-sectional areas of wires AB and AC are 0.4 in^2 and 0.5 in^2 , respectively.



Solution 109

For wire AB:

By sine law (from the force polygon):

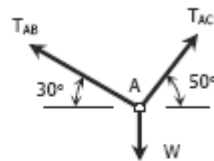
$$\frac{T_{AB}}{\sin 40^\circ} = \frac{W}{\sin 80^\circ}$$

$$T_{AB} = 0.6527W$$

$$\sigma_{AB}A_{AB} = 0.6527W$$

$$30(0.4) = 0.6527W$$

$$W = 18.4 \text{ kips}$$



FBD of knot A

For wire AC:

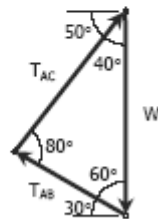
$$\frac{T_{AC}}{\sin 60^\circ} = \frac{W}{\sin 80^\circ}$$

$$T_{AC} = 0.8794W$$

$$T_{AC} = \sigma_{AC}A_{AC}$$

$$0.8794W = 30(0.5)$$

$$W = 17.1 \text{ kips}$$



Force polygon of forces on knot A

Safe load $W = 17.1 \text{ kips}$

Problem 110

A 12-inches square steel bearing plate lies between an 8-inches diameter wooden post and a concrete footing as shown in Fig. P-110. Determine the maximum value of the load P if the stress in wood is limited to 1800 psi and that in concrete to 650 psi.

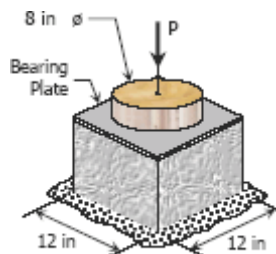


Figure P-110

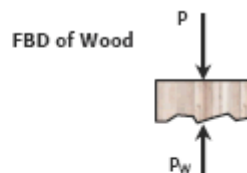
Solution 110

For wood:

$$\begin{aligned} P_w &= \sigma_w A_w \\ &= 1800 \left[\frac{1}{4} \pi (8^2) \right] \\ &= 90\,477.9 \text{ lb} \end{aligned}$$

From FBD of Wood:

$$P = P_w = 90\,477.9 \text{ lb}$$

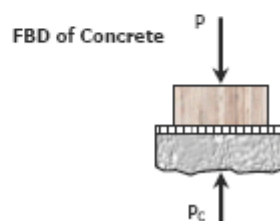


For concrete:

$$\begin{aligned} P_c &= \sigma_c A_c \\ &= 650(12^2) \\ &= 93\,600 \text{ lb} \end{aligned}$$

From FBD of Concrete:

$$P = P_c = 93\,600 \text{ lb}$$



Safe load $P = 90\,478 \text{ lb}$

Problem 111

For the truss shown in Fig. P-111, calculate the stresses in members CE, DE, and DF.

The cross-sectional area of each member is 1.8 in^2 . Indicate tension (T) or compression (C).

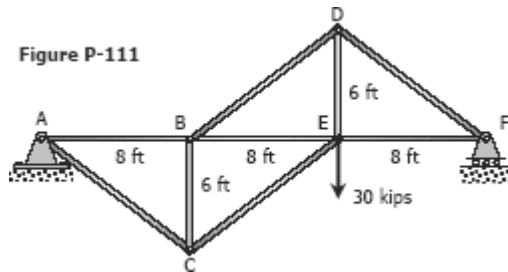
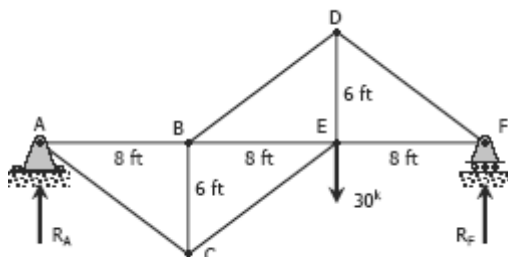


Figure P-111

Solution 111



Free Body Diagram of Truss

From the *FBD* of the truss:

$$\Sigma M_A = 0$$

$$24R_B = 16(30)$$

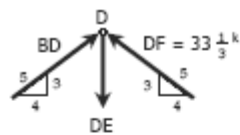
$$R_F = 20^k$$

At joint F:

$$\Sigma F_v = 0$$

$$\frac{3}{8} DF = 20$$

$$DF = 33 \frac{1}{3} \text{ k (C)}$$



Joint D

At joint D : (by symmetry)

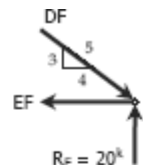
$$BD = DF = 33\frac{1}{3} \text{ k (C)}$$

$$\Sigma F_v = 0$$

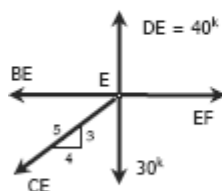
$$DE = \frac{3}{5}BD + \frac{3}{5}DF$$

$$= \frac{3}{5} (33 \frac{1}{3}) + \frac{3}{5} (33 \frac{1}{3})$$

$$= 40^{\circ}\text{C} \text{ (T)}$$



Joint F



Joint E

At joint E:

$$\Sigma F_v = 0$$

$$\frac{1}{20} CE + 30 = 40$$

$$CE = 16 \frac{2}{3} \text{ k (T)}$$

Stresses:

$$\text{Stress} = \text{Force} / \text{Area}$$

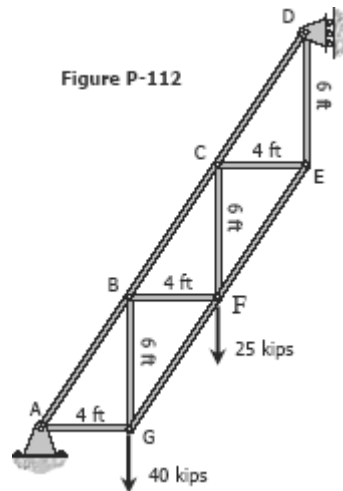
$$\sigma_{CE} = \frac{16^{\frac{2}{3}}}{1.8} = 9.26 \text{ ksi (T)}$$

$$\sigma_{DE} = \frac{40}{1.8} = 22.22 \text{ ksi (T)}$$

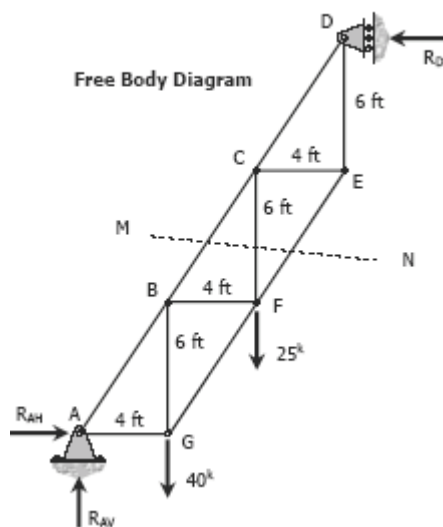
$$\sigma_{DF} = \frac{33\frac{1}{3}}{1.8} = 18.52 \text{ ksi (C)}$$

Problem 112

Determine the crosssectional areas of members AG, BC, and CE for the truss shown in Fig. P-112 above. The stresses are not to exceed 20 ksi in tension and 14 ksi in compression. A reduced stress in compression is specified to reduce the danger of buckling.



Solution 112



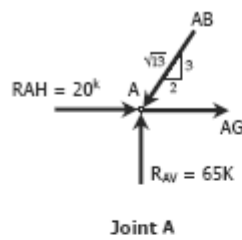
$$\begin{aligned}\sum F_V &= 0 \\ R_{AV} &= 40 + 25 \\ &= 65^k\end{aligned}$$

$$\begin{aligned}\sum M_A &= 0 \\ 18R_D &= 8(25) + 4(40) \\ R_D &= 20^k\end{aligned}$$

$$\begin{aligned}\sum F_H &= 0 \\ R_{AH} &= R_D = 20^k\end{aligned}$$

Check:

$$\begin{aligned}\sum M_D &= 0 \\ 12R_{AV} &= 18(R_{AH}) + 4(25) + 8(40) \\ 12(65) &= 18(20) + 4(25) + 8(40) \\ 780 \text{ ft}\cdot\text{kip} &= 780 \text{ ft}\cdot\text{kip} \text{ (OK!)}\end{aligned}$$



For member AG:

At joint A:

$$\sum F_V = 0$$

$$\frac{3}{\sqrt{13}} AB = 65$$

$$AB = \frac{65\sqrt{13}}{3} = 78.12^k$$

$$\sum F_H = 0$$

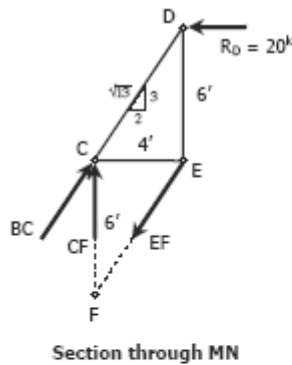
$$AG + 20 = \frac{2}{\sqrt{13}} AB$$

$$AG = \frac{2}{\sqrt{13}} (78.12) - 20 = 20.33^k \text{ Tension}$$

$$AG = \sigma_{\text{tension}} A_{AG}$$

$$20.33 = 20 A_{AG}$$

$$A_{AG} = 1.17 \text{ in}^2$$



For member BC:

At section through MN

$$\sum M_F = 0$$

$$6\left(\frac{2}{\sqrt{13}} BC\right) = 12(20)$$

$$BC = 20\sqrt{13} = 72.11^k \text{ Compression}$$

$$BC = \sigma_{\text{compression}} A_{BC}$$

$$72.11 = 14 A_{BC}$$

$$A_{BC} = 5.15 \text{ in}^2$$

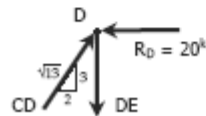
For member CE:

At joint D:

$$\sum F_H = 0$$

$$\frac{2}{\sqrt{13}} CD = 20$$

$$CD = 10\sqrt{13} = 36.06^k$$



Joint D

$$\sum F_V = 0$$

$$DE = \frac{3}{\sqrt{13}} CD$$

$$= \frac{3}{\sqrt{13}} (36.06)$$

$$= 30^k$$

At joint E:

$$\sum F_V = 0$$

$$\frac{3}{\sqrt{13}} EF = 30$$

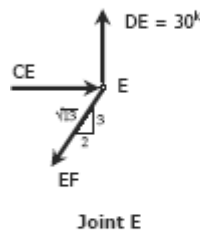
$$EF = 10\sqrt{13} = 36.06^k$$

$$\sum F_H = 0$$

$$CE = \frac{2}{\sqrt{13}} EF$$

$$= \frac{2}{\sqrt{13}} (36.06)$$

$$= 20^k \text{ Compression}$$



$$CE = \sigma_{\text{compression}} A_{CE}$$

$$20 = 14 A_{CE}$$

$$A_{CE} = 1.43 \text{ in}^2$$

Problem 113

Find the stresses in members BC, BD, and CF for the truss shown in Fig. P-113. Indicate the tension or compression. The cross sectional area of each member is 1600 mm^2 .

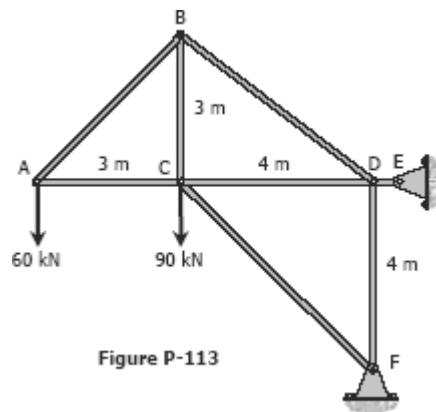
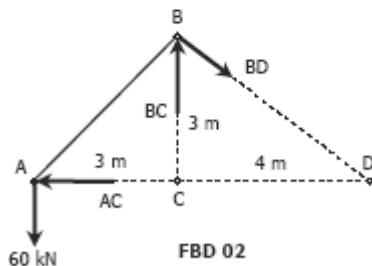
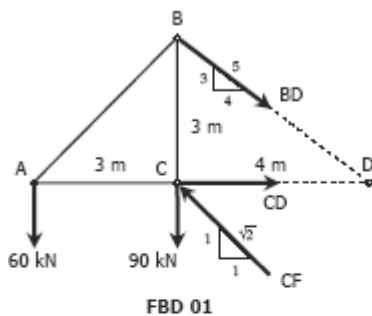


Figure P-113

Solution 113



For member BD : (See FBD 01)

$$\Sigma M_c = 0$$

$$3\left(\frac{4}{5}BD\right) = 3(60)$$

 $BD = 75 \text{ kN Tension}$

$$BD = \sigma_{BD} A$$

$$75(1000) = \sigma_{BD}(1600)$$

$$\sigma_{BD} = 46.875 \text{ MPa (Tension)}$$

For member CF: (See FBD 01)

$$\Sigma M_D = 0$$

$$4\left(\frac{1}{\sqrt{2}} CF\right) = 4(90) + 7(60)$$

$$CF = 195\sqrt{2}$$

$$= 275.77 \text{ kN Compression}$$

$$CF = \sigma_{CF} A$$

$$275.77(1000) = \sigma_{CF}(1600)$$

$\sigma_{CF} = 172.357 \text{ MPa (Compression)}$

For member BC: (See FBD 02)

$$\Sigma M_D = 0$$

$$4BC = 7(60)$$

 $BC = 105 \text{ kN Compression}$

$$BC = \sigma_{BC} A$$

$$105(1000) = \sigma_{BC}(1600)$$

$\sigma_{BC} = 65.625 \text{ MPa (Compression)}$

Problem 114

The homogeneous bar ABCD shown in Fig. P-114 is supported by a cable that runs from A to B around the smooth peg at E, a vertical cable at C, and a smooth inclined surface at D. Determine the mass of the heaviest bar that can be supported if the stress in each cable is limited to 100 MPa. The area of the cable AB is 250 mm^2 and that of the cable at C is 300 mm^2 .

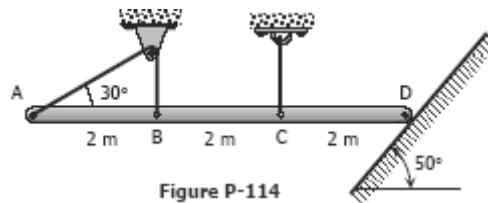
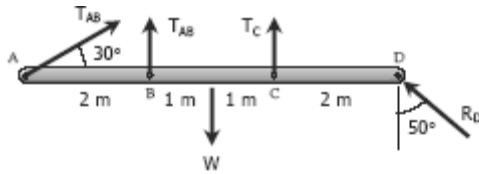


Figure P-114

Solution 114



$$\Sigma F_H = 0$$

$$T_{AB} \cos 30^\circ = R_D \sin 50^\circ$$

$$R_D = 1.1305 T_{AB}$$

$$\Sigma F_V = 0$$

$$T_{AB} \sin 30^\circ + T_{AB} + T_C + R_D \cos 50^\circ = W$$

$$T_{AB} \sin 30^\circ + T_{AB} + T_C + (1.1305 T_{AB}) \cos 50^\circ = W$$

$$2.2267 T_{AB} + T_C = W$$

$$T_C = W - 2.2267 T_{AB}$$

$$\Sigma M_D = 0$$

$$6(T_{AB} \sin 30^\circ) + 4T_{AB} + 2T_C = 3W$$

$$7T_{AB} + 2(W - 2.2267 T_{AB}) = 3W$$

$$2.5466 T_{AB} = W$$

$$T_{AB} = 0.3927 W$$

$$T_C = W - 2.2267 T_{AB}$$

$$= W - 2.2267(0.3927 W)$$

$$= 0.1256 W$$

Based on cable AB:

$$T_{AB} = \sigma_{AB} A_{AB}$$

$$0.3927 W = 100(250)$$

$$W = 63\,661.83 \text{ N}$$

Based on cable at C:

$$T_C = \sigma_C A_C$$

$$0.1256 W = 100(300)$$

$$W = 238\,853.50 \text{ N}$$

Safe weight $W = 63\,669.92 \text{ N}$

$$W = mg$$

$$63\,669.92 = m(9.81)$$

$$m = 6\,490 \text{ kg}$$

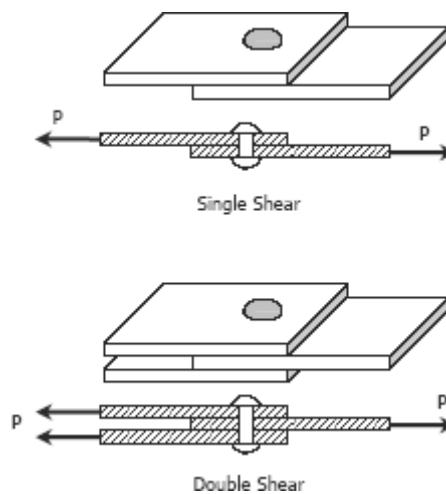
$$= 6.49 \text{ Mg}$$

Shearing Stress

Forces parallel to the area resisting the force cause shearing stress. It differs to tensile and compressive stresses, which are caused by forces perpendicular to the area on which they act. Shearing stress is also known as tangential stress.

$$\tau = \frac{V}{A}$$

where V is the resultant shearing force which passes through the centroid of the area A being sheared.

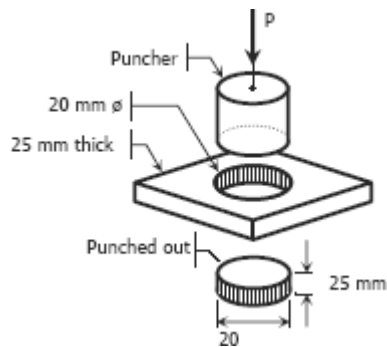


SOLVED PROBLEMS IN SHEARING STRESS

Problem 115

What force is required to punch a 20-mm-diameter hole in a plate that is 25 mm thick? The shear strength is 350 MN/m^2 .

Solution 115



The resisting area is the shaded area along the perimeter and the shear force V is equal to the punching force P .

$$\begin{aligned} V &= \tau A \\ P &= 350[\pi(20)(25)] \\ &= 549\,778.7 \text{ N} \\ &= 549.8 \text{ kN} \end{aligned}$$

Problem 116

As in Fig. 1-11c, a hole is to be punched out of a plate having a shearing strength of 40 ksi. The compressive stress in the punch is limited to 50 ksi. **(a)** Compute the maximum thickness of plate in which a hole 2.5 inches in diameter can be punched. **(b)** If the plate is 0.25 inch thick, determine the diameter of the smallest hole that can be punched.

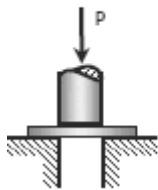


Figure 1-11c

(a) Maximum thickness of plate:

Based on puncher strength:

$$\begin{aligned} P &= \sigma A \\ &= 50 \left[\frac{1}{4} \pi (2.5^2) \right] \\ &= 78.125\pi \text{ kips} \rightarrow \text{Equivalent shear force of the plate} \end{aligned}$$

Based on shear strength of plate:

$$\begin{aligned} V &= \tau A \quad \rightarrow V = P \\ 78.125\pi &= 40 [\pi (2.5t)] \\ t &= 0.781 \text{ inch} \end{aligned}$$

(b) Diameter of smallest hole:

Based on compression of puncher:

$$\begin{aligned} P &= \sigma A \\ &= 50 \left(\frac{1}{4} \pi d^2 \right) \\ &= 12.5\pi d^2 \quad \rightarrow \text{Equivalent shear force for plate} \end{aligned}$$

Based on shearing of plate:

$$\begin{aligned} V &= \tau A \quad \rightarrow V = P \\ 12.5\pi d^2 &= 40 [\pi d (0.25)] \\ d &= 0.8 \text{ in} \end{aligned}$$

Problem 117

Find the smallest diameter bolt that can be used in the clevis shown in Fig. 1-11b if $P = 400 \text{ kN}$. The shearing strength of the bolt is 300 MPa.

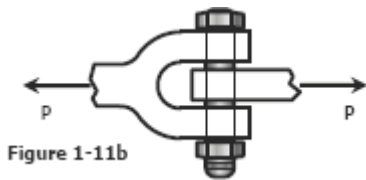


Figure 1-11b

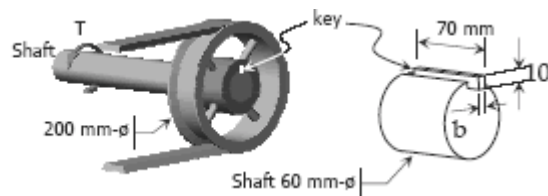
The bolt is subject to double shear.

$$\begin{aligned} V &= \tau A \\ 400(1000) &= 300 \left[2 \left(\frac{1}{4} \pi d^2 \right) \right] \\ d &= 29.13 \text{ mm} \end{aligned}$$

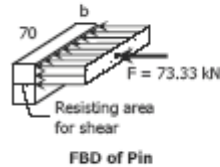
Problem 118

A 200-mm-diameter pulley is prevented from rotating relative to 60-mm-diameter shaft by a 70-mm-long key, as shown in Fig. P-118. If a torque $T = 2.2 \text{ kN}\cdot\text{m}$ is applied to the shaft, determine the width b if the allowable shearing stress in the key is 60 MPa.

Figure P-118



Solution 118



$$T = 0.03F$$

$$2.2 = 0.03F$$

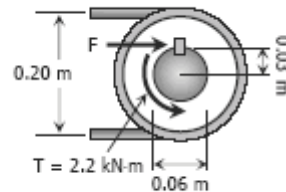
$$F = 73.33 \text{ kN}$$

$$V = \tau A$$

Where: $V = F = 73.33 \text{ kN}$
 $A = 70b$; $\tau = 60 \text{ MPa}$

$$73.33(1000) = 60(70b)$$

$$b = 17.46 \text{ mm}$$



Problem 119

Compute the shearing stress in the pin at B for the member supported as shown in Fig. P-119. The pin diameter is 20 mm.

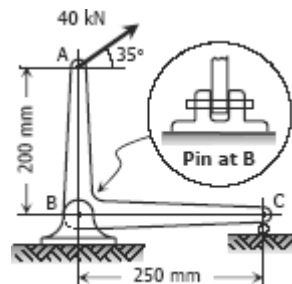


Figure P-119

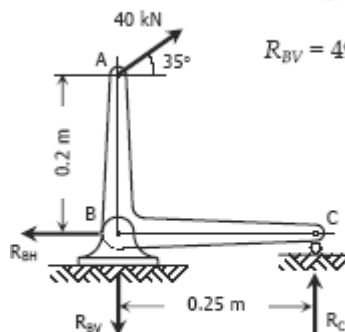
Solution 119

From the FBD:

$$\sum M_C = 0$$

$$0.25R_{BV} = 0.25(40 \sin 35^\circ) + 0.2(40 \cos 35^\circ)$$

$$R_{BV} = 49.156 \text{ kN}$$



Free Body Diagram

$$\sum F_H = 0$$

$$R_{BH} = 40 \cos 35^\circ$$

$$= 32.766 \text{ kN}$$

$$R_B = \sqrt{R_{BH}^2 + R_{BV}^2}$$

$$= \sqrt{32.766^2 + 49.156^2}$$

$$= 59.076 \text{ kN} \rightarrow \text{shear force of pin at B}$$

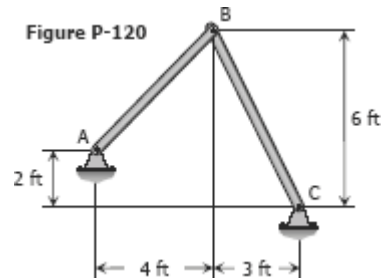
$$V_B = \tau_B A \rightarrow \text{double shear}$$

$$59.076 (1000) = \tau_B [2 \left[\frac{1}{4} \pi (20^2) \right]]$$

$$\tau_B = 94.02 \text{ MPa}$$

Problem 120

The members of the structure in Fig. P-120 weigh 200 lb/ft. Determine the smallest diameter pin that can be used at A if the shearing stress is limited to 5000 psi. Assume single shear.

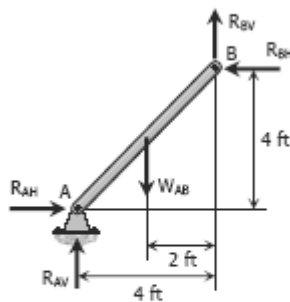


Solution 120

For member AB:

$$\text{Length, } L_{AB} = \sqrt{4^2 + 4^2} = 5.66 \text{ ft}$$

$$\text{Weight, } W_{AB} = 5.66(200) = 1132 \text{ lb}$$



FBD of member

$$\sum M_A = 0$$

$$4R_{BH} + 4R_{BV} = 2W_{AB}$$

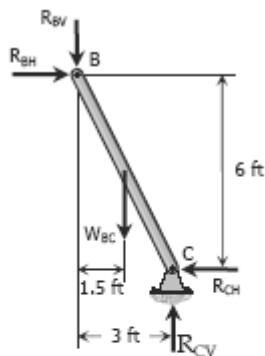
$$4R_{BH} + 4R_{BV} = 2(1132)$$

$$R_{BH} + R_{BV} = 566 \rightarrow (1)$$

For member BC:

$$\text{Length, } L_{BC} = \sqrt{3^2 + 6^2} = 6.71 \text{ ft}$$

$$\text{Weight, } W_{BC} = 6.71(200) = 1342 \text{ lb}$$



FBD of member BC

$$\sum M_C = 0$$

$$6R_{BH} = 1.5W_{BC} + 3R_{BV}$$

$$6R_{BH} - 3R_{BV} = 1.5(1342)$$

$$2R_{BH} - R_{BV} = 671 \rightarrow (2)$$

Add equations (1) and (2)

$$R_{BH} + R_{BV} = 566 \rightarrow (1)$$

$$2R_{BH} - R_{BV} = 671 \rightarrow (2)$$

$$\frac{3R_{BH}}{3} = \frac{1237}{3}$$

$$R_{BH} = 412.33 \text{ lb}$$

From equation (1):

$$412.33 + R_{BV} = 566$$

$$R_{BV} = 153.67 \text{ lb}$$

From the FBD of member AB

$$\sum F_H = 0$$

$$R_{AH} = R_{BH} = 412.33 \text{ lb}$$

$$\sum F_V = 0$$

$$R_{AV} + R_{BV} = W_{AB}$$

$$R_{AV} + 153.67 = 1132$$

$$R_{AV} = 978.33 \text{ lb}$$

$$\begin{aligned}
 R_A &= \sqrt{R_{AH}^2 + R_{AV}^2} \\
 &= \sqrt{412.33^2 + 978.33^2} \\
 &= 1061.67 \text{ lb} \rightarrow \text{shear force of pin at A}
 \end{aligned}$$

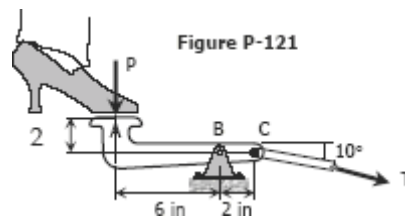
$$V = \tau A$$

$$1061.67 = 5000 \left(\frac{1}{4} \pi d^2 \right)$$

$$d = 0.520 \text{ in}$$

Problem 121

Referring to Fig. P-121, compute the maximum force P that can be applied by the machine operator, if the shearing stress in the pin at B and the axial stress in the control rod at C are limited to 4000 psi and 5000 psi, respectively. The diameters are 0.25 inch for the pin, and 0.5 inch for the control rod. Assume single shear for the pin at B.

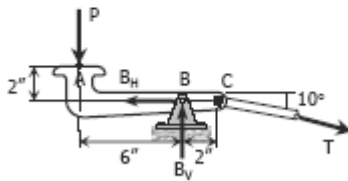


Solution 121

$$[\Sigma M_B = 0] \quad 6P = 2T \sin 10^\circ \rightarrow (1)$$

$$\begin{aligned}
 [\Sigma F_H = 0] \quad B_H &= T \cos 10^\circ \rightarrow \text{from (1), } T = 3P / \sin 10^\circ \\
 B_H &= (3P / \sin 10^\circ) \cos 10^\circ \\
 B_H &= 3 \cot 10^\circ P
 \end{aligned}$$

$$\begin{aligned}
 [\Sigma F_V = 0] \quad B_V &= T \sin 10^\circ + P \rightarrow \text{from (1), } T \sin 10^\circ = 3P \\
 B_V &= 3P + P \\
 B_V &= 4P
 \end{aligned}$$



$$\begin{aligned}
 R_B^2 &= B_H^2 + B_V^2 \\
 R_B^2 &= (3 \cot 10^\circ P)^2 + (4P)^2 \\
 R_B^2 &= 305.47P^2 \\
 R_B &= 17.48P \\
 P &= R_B / 17.48 \rightarrow (2)
 \end{aligned}$$

Based on tension of rod (equation 1):

$$P = \frac{1}{3} T \sin 10^\circ$$

$$P = \frac{1}{3} [5000 \times \frac{1}{4} \pi (0.5)^2] \sin 10^\circ$$

$$P = 56.83 \text{ lb}$$

Based on shear of rivet (equation 2):

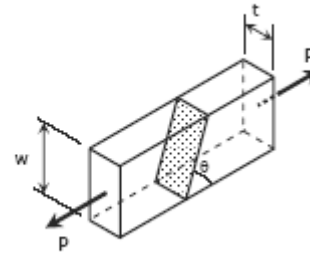
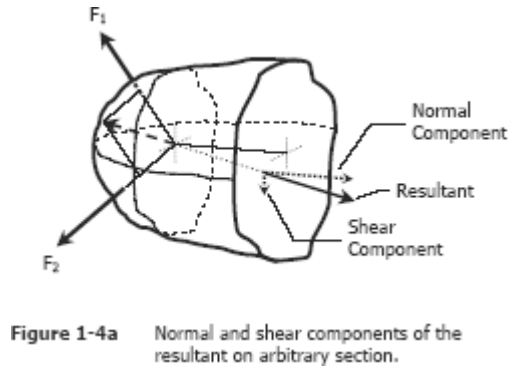
$$P = 4000 \times \frac{1}{4} \pi (0.25)^2 / 17.48$$

$$P = 11.23 \text{ lb}$$

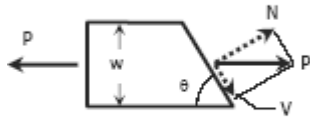
Safe load $P = 11.23 \text{ lb}$

Problem 122

Two blocks of wood, width w and thickness t , are glued together along the joint inclined at the angle θ as shown in Fig. P-122. Using the free-body diagram concept in Fig. 1-4a, show that the shearing stress on the glued joint is $\tau = P \sin 2\theta / 2A$, where A is the cross-sectional area.



Solution 122



$$\begin{aligned} \text{Shear area, } A_{\text{shear}} &= t (w \csc \theta) \\ &= tw \csc \theta \\ &= A \csc \theta \end{aligned}$$

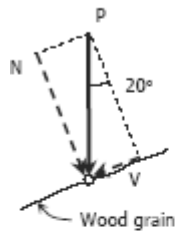
$$\text{Shear force, } V = P \cos \theta$$

$$\begin{aligned} V &= \tau A_{\text{shear}} \\ P \cos \theta &= \tau (A \csc \theta) \\ \tau &= P \sin \theta \cos \theta / A \\ &= P (2 \sin \theta \cos \theta) / 2A \\ &= P \sin 2\theta / 2A \quad (\text{ok!}) \end{aligned}$$

Problem 123

A rectangular piece of wood, 50 mm by 100 mm in cross section, is used as a compression block shown in Fig. P-123. Determine the axial force P that can be safely applied to the block if the compressive stress in wood is limited to 20 MN/m^2 and the shearing stress parallel to the grain is limited to 5 MN/m^2 . The grain makes an angle of 20° with the horizontal, as shown. (Hint: Use the results in Problem 122.)

Solution 123



Based on maximum compressive stress:

Normal force:

$$N = P \cos 20^\circ$$

Normal area:

$$\begin{aligned} A_N &= 50 (100 \sec 20^\circ) \\ &= 5320.89 \text{ mm}^2 \end{aligned}$$

$$N = \sigma A_N$$

$$P \cos 20^\circ = 20 (5320.89)$$

$$P = 113\,247 \text{ N}$$

$$= 113.25 \text{ kN}$$

Based on maximum shearing stress:

Shear force:

$$V = P \sin 20^\circ$$

Shear area:

$$\begin{aligned} A_V &= A_N \\ &= 5320.89 \text{ mm}^2 \end{aligned}$$

$$V = \tau A_V$$

$$P \sin 20^\circ = 5 (5320.89)$$

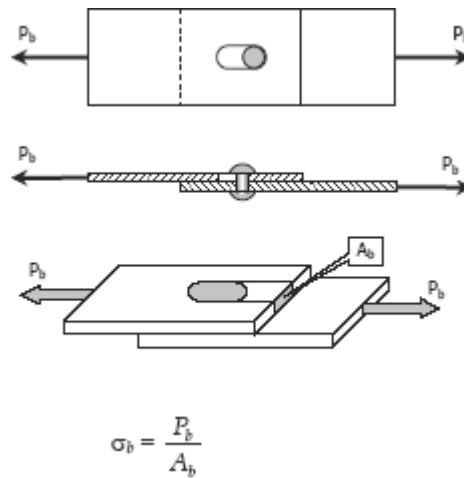
$$P = 77\,786 \text{ N}$$

$$= 77.79 \text{ kN}$$

For safe compressive force, use $P = 77.79 \text{ kN}$

Bearing Stress

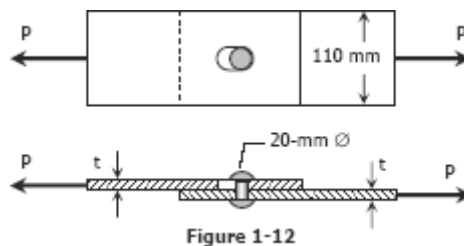
Bearing stress is the contact pressure between the separate bodies. It differs from compressive stress, as it is an internal stress caused by compressive forces.



SOLVED PROBLEMS IN BEARING STRESS

Problem 125

In Fig. 1-12, assume that a 20-mm-diameter rivet joins the plates that are each 110 mm wide. The allowable stresses are 120 MPa for bearing in the plate material and 60 MPa for shearing of rivet. Determine (a) the minimum thickness of each plate; and (b) the largest average tensile stress in the plates.



Solution 125

(a) From shearing of rivet:

$$\begin{aligned} P &= \tau A_{\text{rivets}} \\ &= 60 \left[\frac{1}{4} \pi (20^2) \right] \\ &= 6000\pi \text{ N} \end{aligned}$$

From bearing of plate material:

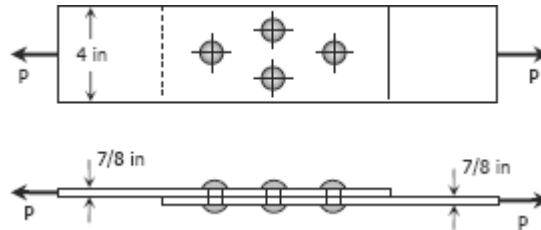
$$\begin{aligned} P &= \sigma_b A_b \\ 6000\pi &= 120(20t) \\ t &= 7.85 \text{ mm} \end{aligned}$$

(b) Largest average tensile stress in the plate:

$$\begin{aligned} P &= \sigma A \\ 6000\pi &= \sigma [7.85(110 - 20)] \\ \sigma &= 26.67 \text{ MPa} \end{aligned}$$

Problem 126

The lap joint shown in Fig. P-126 is fastened by four $\frac{3}{4}$ -in.-diameter rivets. Calculate the maximum safe load P that can be applied if the shearing stress in the rivets is limited to 14 ksi and the bearing stress in the plates is limited to 18 ksi. Assume the applied load is uniformly distributed among the four rivets.



Solution 126

Based on shearing of rivets:

$$P = \tau A$$

$$P = 14 \left[4 \left(\frac{1}{4} \pi \right) \left(\frac{3}{4} \right)^2 \right]$$

$$P = 24.74 \text{ kips}$$

Based on bearing of plates:

$$P = \sigma_b A_b$$

$$P = 18 \left[4 \left(\frac{3}{4} \right) \left(\frac{7}{8} \right) \right]$$

$$P = 47.25 \text{ kips}$$

Safe load $P = 24.74 \text{ kips}$

Problem 127

In the clevis shown in Fig. 1-11b, find the minimum bolt diameter and the minimum thickness of each yoke that will support a load $P = 14 \text{ kips}$ without exceeding a shearing stress of 12 ksi and a bearing stress of 20 ksi.

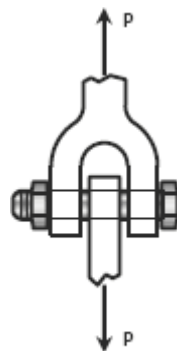
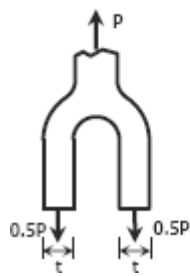


Figure 1-11b

Solution 127



For shearing of rivets (double shear)

$$P = \tau A$$

$$14 = 12[2(\frac{1}{2}\pi d^2)]$$

$$d = 0.8618 \text{ in} \rightarrow \text{diameter of bolt}$$

For bearing of yoke:

$$P = \sigma_b A_b$$

$$14 = 20[2(0.8618t)]$$

$$t = 0.4061 \text{ in} \rightarrow \text{thickness of yoke}$$

Problem 128

A W18 $\times$ 86 beam is riveted to a W24 $\times$ 117 girder by a connection similar to that in Fig. 1-13. The diameter of the rivets is 7/8 in., and the angles are each 4 $\times$ 3 1/2 $\times$ 3/8 in. For each rivet, assume that the allowable stresses are $\tau = 15$ ksi and $\sigma_b = 32$ ksi.

Find the allowable

load on the connection.

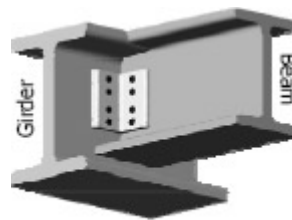


Figure 1-13

Solution 128

Note: Textbook is Strength of Materials 4th edition by Pytel and Singer

Relevant data from the table (Appendix B of textbook): *Properties of Wide-Flange Sections (W shapes): U.S. Customary Units*

Designation	Web thickness
W18 × 86	0.480 in
W24 × 117	0.550 in

Shearing strength of rivets:

There are 8 single-shear rivets in the girder and 4 double-shear (equivalent to 8 single-shear) in the beam, thus, the shear strength of rivets in girder and beam are equal.

$$V = \tau A = 15 \left[\frac{1}{4} \pi \left(\frac{7}{8} \right)^2 (8) \right]$$

$$V = 72.16 \text{ kips}$$

Bearing strength on the girder:

The thickness of girder W24 × 117 is 0.550" while that of the angle clip $L4 \times 3\frac{1}{2} \times \frac{3}{8}$ is $\frac{3}{8}$ " or 0.375", thus, the critical in bearing is the clip.

$$P = \sigma_b A_b = 32 \left[\frac{7}{8} (0.375)(8) \right]$$

$$P = 84 \text{ kips}$$

Bearing strength on the beam:

The thickness of beam W18 × 86 is 0.480" while that of the clip angle is $2 \times 0.375" = 0.75"$ (clip angles are on both sides of the beam), thus, the critical in bearing is the beam.

$$P = \sigma_b A_b = 32 \left[\frac{7}{8} (0.480)(4) \right]$$

$$P = 53.76 \text{ kips}$$

The allowable load on the connection is $P = 53.76 \text{ kips}$

Problem 129

A 7/8-in.-diameter bolt, having a diameter at the root of the threads of 0.731 in., is used to fasten two timbers together as shown in Fig. P-129. The nut is tightened to cause a tensile stress of 18 ksi in the bolt. Compute the shearing stress in the head of the bolt and in the threads. Also, determine the outside diameter of the washers if their inside diameter is 9/8 in. and the bearing stress is limited to 800 psi.

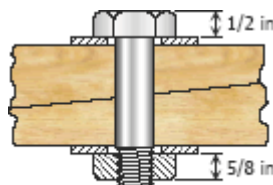
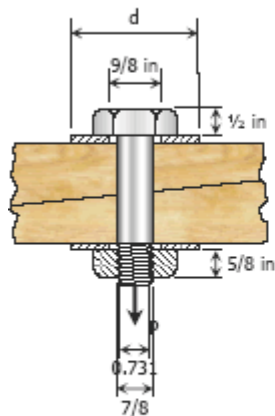


Figure P-127

Solution 129



Tensile force on the bolt:

$$P = \sigma A = 18 \left[\frac{1}{4} \pi \left(\frac{7}{8} \right)^2 \right]$$

$$P = 10.82 \text{ kips}$$

Shearing stress in the head of the bolt:

$$\tau = \frac{P}{A} = \frac{10.82}{\pi \left(\frac{7}{8} \right) \left(\frac{1}{2} \right)}$$

$$\tau = 7.872 \text{ ksi}$$

Shearing stress in the threads:

$$\tau = \frac{P}{A} = \frac{10.82}{\pi (0.731) \left(\frac{5}{8} \right)}$$

$$\tau = 7.538 \text{ ksi}$$

Outside diameter of washer

$$P = \sigma_b A_b$$

$$10.82(1000) = 800 \left[\frac{1}{4} \pi \left[d^2 - \left(\frac{9}{8} \right)^2 \right] \right]$$

$$d = 4.3 \text{ in}$$



Washer

Problem 130

Figure P-130 shows a roof truss and the detail of the riveted connection at joint B. Using allowable stresses of $\tau = 70 \text{ MPa}$ and $\sigma_b = 140 \text{ MPa}$, how many 19-mm diameter rivets are required to fasten member BC to the gusset plate? Member BE? What is the largest average tensile or compressive stress in BC and BE?

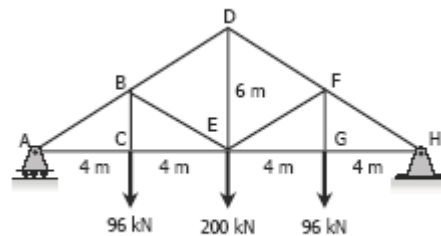
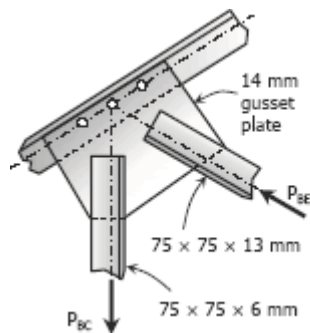
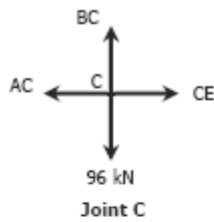


Figure P-130 and P-131

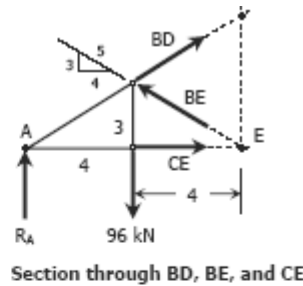
Solution 130



At Joint C:
 $\Sigma F_V = 0$
 $BC = 96 \text{ kN}$ (Tension)

Consider the section through member BD, BE, and CE:

$\Sigma M_A = 0$
 $8(\frac{3}{5} BE) = 4(96)$
 $BE = 80 \text{ kN}$ (Compression)



For Member BC:

Based on shearing of rivets:

$$BC = \tau A$$

Where A = area of 1 rivet $\times$ number of rivets, n

$$96\,000 = 70[\frac{1}{4} \pi (19^2)n]$$

$$n = 4.8 \text{ say } 5 \text{ rivets}$$

Based on bearing of member:

$$BC = \sigma_b A_b$$

Where A_b = diameter of rivet $\times$ thickness of BC $\times$ number of rivets, n

$$96\,000 = 140[19(6)n]$$

$$n = 6.02 \text{ say } 7 \text{ rivets}$$

use 7 rivets for member BC

For member BE:

Based on shearing of rivets:

$$BE = \tau A$$

Where A = area of 1 rivet $\times$ number of rivets, n

$$80\,000 = 70[\frac{1}{4} \pi (19^2)n]$$

$$n = 4.03 \text{ say } 5 \text{ rivets}$$

Based on bearing of member:

$$BE = \sigma_b A_b$$

Where A_b = diameter of rivet $\times$ thickness of BE $\times$ number of rivets, n

$$80\,000 = 140[19(13)n]$$

$$n = 2.3 \text{ say } 3 \text{ rivets}$$

use 5 rivets for member BE

Relevant data from the table (Appendix B of textbook): *Properties of Equal Angle Sections: SI Units*

Designation	Area
L75 $\times$ 75 $\times$ 6	864 mm ²
L75 $\times$ 75 $\times$ 13	1780 mm ²

Tensile stress of member BC (L75 $\times$ 75 $\times$ 6):

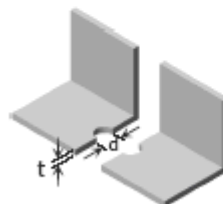
$$\sigma = \frac{P}{A} = \frac{96(1000)}{864 - 19(6)}$$

$$\sigma = 128 \text{ Mpa}$$

Compressive stress of member BE (L75 $\times$ 75 $\times$ 13):

$$\sigma = \frac{P}{A} = \frac{80(1000)}{1780}$$

$$\sigma = 44.94 \text{ Mpa}$$



t = thickness of member
 d = diameter of rivet hole

Note:

$$A = \text{Area} - dt$$

Problem 131

Repeat Problem 130 if the rivet diameter is 22 mm and all other data remain unchanged.

Solution 131

For member BC:

$$P = 96 \text{ kN (Tension)}$$

Based on shearing of rivets:

$$P = \tau A$$

$$96\,000 = 70 \left[\frac{1}{4} \pi (22^2) n \right]$$

$$n = 3.6 \text{ say } 4 \text{ rivets}$$

Based on bearing of member:

$$P = \sigma_b A_b$$

$$96\,000 = 140[22(6)n]$$

$$n = 5.2 \text{ say } 6 \text{ rivets}$$

Use 6 rivets for member BC

Tensile stress:

$$\sigma = \frac{P}{A} = \frac{96(1000)}{864 - 22(6)}$$

$$\sigma = 131.15 \text{ MPa}$$

For member BE:

$$P = 80 \text{ kN (Compression)}$$

Based on shearing of rivets:

$$P = \tau A$$

$$80\,000 = 70 \left[\frac{1}{4} \pi (22^2) n \right]$$

$$n = 3.01 \text{ say } 4 \text{ rivets}$$

Based on bearing of member:

$$P = \sigma_b A_b$$

$$80\,000 = 140[22(13)n]$$

$$n = 1.998 \text{ say } 2 \text{ rivets}$$

use 4 rivets for member BE

Compressive stress:

$$\sigma = \frac{P}{A} = \frac{80(1000)}{1780}$$

$$\sigma = 44.94 \text{ MPa}$$

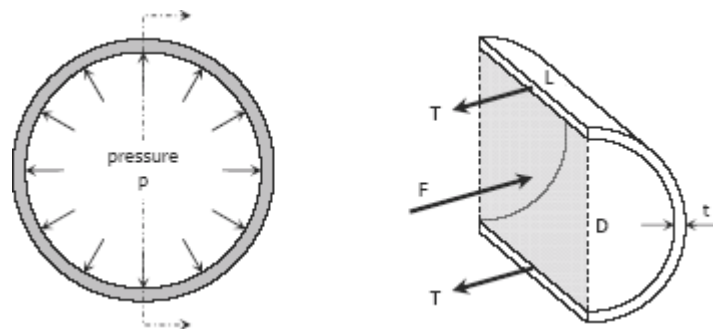
Thin-Walled Pressure Vessels

A tank or pipe carrying a fluid or gas under a pressure is subjected to tensile forces, which resist bursting, developed across longitudinal and transverse sections.

TANGENTIAL STRESS

(Circumferential Stress)

Consider the tank shown being subjected to an internal pressure p . The length of the tank is L and the wall thickness is t . Isolating the right half of the tank:



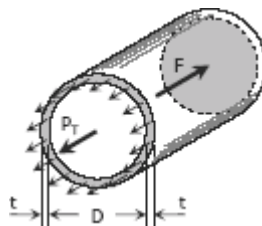
$$\begin{aligned}
 F &= pA = pDL \\
 T &= \sigma_t A_{\text{wall}} = \sigma_t tL \\
 [\Sigma F_H = 0] \\
 F &= 2T \\
 pDL &= 2(\sigma_t tL) \\
 \sigma_t &= \frac{pD}{2t}
 \end{aligned}$$

If there exist an external pressure p_o and an internal pressure p_i , the formula may be expressed as:

$$\sigma_t = \frac{(p_i - p_o)D}{2t}$$

LONGITUDINAL STRESS, σ_L

Consider the free body diagram in the transverse section of the tank:



The total force acting at the rear of the tank F must equal to the total longitudinal stress on the wall $P_T = \sigma_L A_{\text{wall}}$. Since t is so small compared to D , the area of the wall is close to πDt

$$F = pA = p \frac{\pi D^2}{4}$$

$$P_T = \sigma_L \pi D t$$

$$[\Sigma F_H = 0]$$

$$P_T = F$$

$$\sigma_L \pi D t = p \frac{\pi D^2}{4}$$

$$\sigma_L = \frac{pD}{4t}$$

If there exist an external pressure p_o and an internal pressure p_i , the formula may be expressed as:

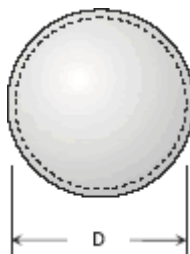
$$\sigma_L = \frac{(p_i - p_o)D}{4t}$$

It can be observed that the tangential stress is twice that of the longitudinal stress.

$$\sigma_t = 2 \sigma_L$$

SPHERICAL SHELL

If a spherical tank of diameter D and thickness t contains gas under a pressure of p , the stress at the wall can be expressed as:



$$\sigma_L = \frac{(p_i - p_o)D}{4t}$$

SOLVED PROBLEMS IN THIN WALLED PREASSURE VESSELS

Problem 133

A cylindrical steel pressure vessel 400 mm in diameter with a wall thickness of 20 mm, is subjected to an internal pressure of 4.5 MN/m². (a) Calculate the tangential and longitudinal stresses in the steel. (b) To what value may the internal pressure be increased if the stress in the steel is limited to 120 MN/m²? (c) If the internal pressure were increased until the vessel burst, sketch the type of fracture that would occur.

Solution 133

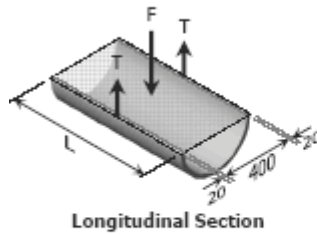
(a) Tangential stress (longitudinal section):

$$F = 2T$$

$$pDL = 2(\sigma_t tL)$$

$$\sigma_t = \frac{pD}{2t} = \frac{4.5(400)}{2(20)}$$

$$\sigma_t = 45 \text{ MPa}$$



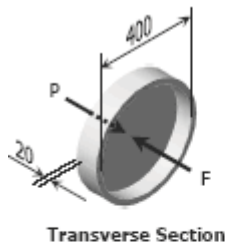
Longitudinal Stress (transverse section):

$$F = P$$

$$\frac{1}{4} \pi D^2 p = \sigma_l (\pi D t)$$

$$\sigma_l = \frac{pD}{4t} = \frac{4.5(400)}{4(20)}$$

$$\sigma_l = 22.5 \text{ MPa}$$



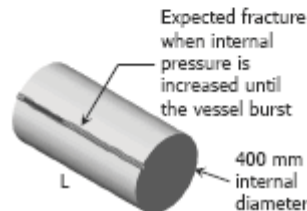
(b) From (a), $\sigma_t = \frac{pD}{2t}$ and $\sigma_l = \frac{pD}{4t}$ thus, $\sigma_t = 2\sigma_l$, this shows that tangential stress is the critical.

$$\sigma_t = \frac{pD}{2t}$$

$$120 = \frac{p(400)}{2(20)}$$

$$P = 12 \text{ MPa}$$

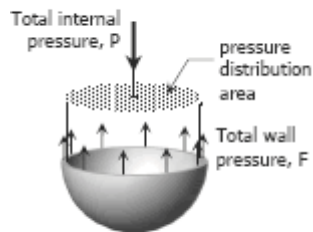
(c) The bursting force will cause a stress on the longitudinal section that is twice to that of the transverse section. Thus, fracture is expected as shown.



Problem 134

The wall thickness of a 4-ft-diameter spherical tank is 5/16 in. Calculate the allowable internal pressure if the stress is limited to 8000 psi.

Solution 134



Total internal pressure:

$$P = p \left(\frac{1}{4} \pi D^2 \right)$$

Resisting wall:

$$F = P$$

$$\sigma A = p \left(\frac{1}{4} \pi D^2 \right)$$

$$\sigma (\pi D t) = p \left(\frac{1}{4} \pi D^2 \right)$$

$$\sigma = \frac{p D}{4 t}$$

$$8000 = \frac{p (4 \times 12)}{4 \left(\frac{5}{16} \right)}$$

$$p = 208.33 \text{ psi}$$

Problem 135

Calculate the minimum wall thickness for a cylindrical vessel that is to carry a gas at a pressure of 1400 psi. The diameter of the vessel is 2 ft, and the stress is limited to 12 ksi.

Solution 135

The critical stress is the tangential stress

$$\sigma_t = \frac{p D}{2 t}$$

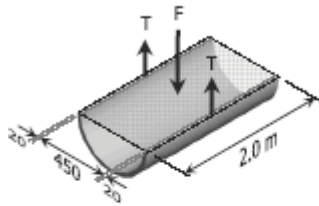
$$12\,000 = \frac{1\,400 (2 \times 12)}{2 t}$$

$$t = 1.4 \text{ in}$$

Problem 136

A cylindrical pressure vessel is fabricated from steel plating that has a thickness of 20 mm. The diameter of the pressure vessel is 450 mm and its length is 2.0 m. Determine the maximum internal pressure that can be applied if the longitudinal stress is limited to 140 MPa, and the circumferential stress is limited to 60 MPa.

Solution 136



Based on circumferential stress (tangential):

$$\Sigma F_V = 0$$

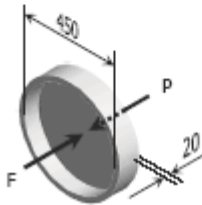
$$F = 2T$$

$$p(DL) = 2(\sigma_t Lt)$$

$$\sigma_t = \frac{pD}{2t}$$

$$60 = \frac{p(450)}{2(20)}$$

$$p = 5.33 \text{ MPa}$$



Based on longitudinal stress:

$$\Sigma F_H = 0$$

$$F = P$$

$$p\left(\frac{1}{4}\pi D^2\right) = \sigma_t(\pi Dt)$$

$$\sigma_t = \frac{pD}{4t}$$

$$140 = \frac{p(450)}{4(20)}$$

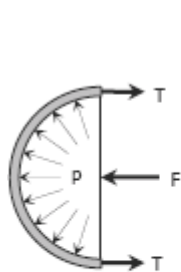
$$p = 24.89 \text{ MPa}$$

Use $p = 5.33 \text{ MPa}$

Problem 137

A water tank, 22 ft in diameter, is made from steel plates that are 1/2 in. thick. Find the maximum height to which the tank may be filled if the circumferential stress is limited to 6000 psi. The specific weight of water is 62.4 lb/ft³.

Solution 137



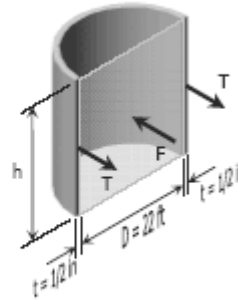
$$\begin{aligned}\sigma_t &= 6000 \text{ psi} \\ \sigma_t &= \frac{6000 \text{ lb}}{\text{in}^2} \left(\frac{12 \text{ in}}{\text{ft}} \right)^2 \\ \sigma_t &= 864\,000 \text{ lb/ft}^2\end{aligned}$$

Assuming pressure distribution to be uniform:

$$\begin{aligned}p &= \gamma h = 62.4h \\ F &= pA = 62.4h(Dh) \\ F &= 62.4(22)h^2 \\ F &= 1372.8h^2\end{aligned}$$

$$\begin{aligned}T &= \sigma_t A_t = 864\,000(th) \\ T &= 864\,000 \left(\frac{1}{2} \times \frac{1}{12} \right) h \\ T &= 36\,000h\end{aligned}$$

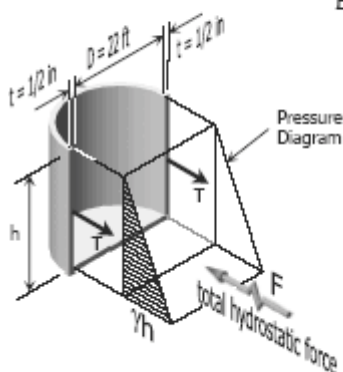
$$\begin{aligned}\Sigma F &= 0 \\ F &= 2T \\ 1372.8h^2 &= 2(36\,000h) \\ h &= 52.45 \text{ ft}\end{aligned}$$



Comment:

Given a free surface of water, the actual pressure distribution on the vessel is not uniform. It varies linearly from 0 at the free surface to γh at the bottom (see figure below). Using this actual pressure

distribution, the total hydrostatic pressure is reduced by 50%. This reduction of force will take our design into critical situation; giving us a maximum height of 200% more than the h above.

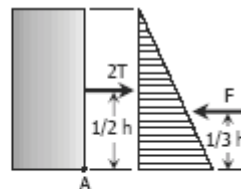


Based on actual pressure distribution:

$$\begin{aligned}\text{Total hydrostatic force, } F &: \\ F &= \text{volume of pressure diagram} \\ F &= \frac{1}{2} (\gamma h^2) D = \frac{1}{2} (62.4h^2)(22) \\ F &= 686.4h^2\end{aligned}$$

$$\begin{aligned}\Sigma M_A &= 0 \\ 2T \left(\frac{1}{2} h \right) - F \left(\frac{1}{3} h \right) &= 0 \\ T &= \frac{1}{3} F\end{aligned}$$

$$\begin{aligned}\sigma_t (ht) &= \frac{1}{3} (686.4h^2) \\ h &= \frac{3\sigma_t t}{686.4} = \frac{3(864\,000) \left(\frac{1}{2} \times \frac{1}{12} \right)}{686.4} \\ h &= 157.34 \text{ ft}\end{aligned}$$



Problem 138

The strength of longitudinal joint in Fig. 1-17 is 33 kips/ft, whereas for the girth is 16 kips/ft. Calculate the maximum diameter of the cylinder tank if the internal pressure is 150 psi.

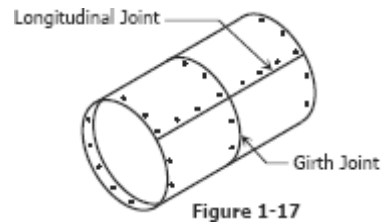


Figure 1-17

Solution 138

Internal pressure, p :

$$p = 150 \text{ psi} = \frac{150 \text{ lb}}{\text{in}^2} \left(\frac{12 \text{ in}}{\text{ft}} \right)^2$$

$$p = 21\,600 \text{ lb/ft}^2$$

For longitudinal joint (tangential stress):

Consider 1 ft length

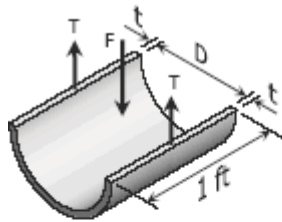
$$F = 2T$$

$$pD = 2\sigma_t t$$

$$\sigma_t = \frac{pD}{2t}$$

$$\frac{33\,000}{t} = \frac{21\,600 D}{2t}$$

$$D = 3.06 \text{ ft} = 36.67 \text{ in}$$



For girth joint (longitudinal stress):

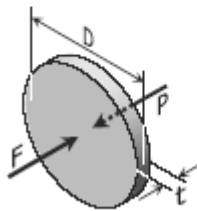
$$F = P$$

$$p \left(\frac{1}{4} \pi D^2 \right) = \sigma_l (\pi D t)$$

$$\sigma_l = \frac{pD}{4t}$$

$$\frac{16\,000}{t} = \frac{21\,600 D}{4t}$$

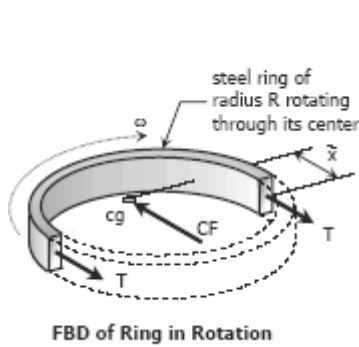
$$D = 2.96 \text{ ft} = 35.56 \text{ in}$$



Use the smaller diameter, $D = 35.56 \text{ in}$.

Problem 139

Find the limiting peripheral velocity of a rotating steel ring if the allowable stress is 20 ksi and steel weighs 490 lb/ft³. At what revolutions per minute (rpm) will the stress reach 30 ksi if the mean radius is 10 in.?

Solution 139

Centrifugal Force, CF:

$$CF = M\omega^2 \bar{x}$$

$$\text{where: } M = \frac{W}{g} = \frac{\gamma V}{g} = \frac{\gamma \pi R A}{g}$$

$$\omega = v/R$$

$$\bar{x} = 2R/\pi$$

$$CF = \frac{\gamma \pi R A}{g} \left(\frac{v}{R} \right)^2 \left(\frac{2R}{\pi} \right)$$

$$CF = \frac{2\gamma A v^2}{g}$$

$$2T = CF$$

$$2\sigma A = \frac{2\gamma A v^2}{g}$$

$$\sigma = \frac{\gamma v^2}{g}$$

From the given data:

$$\sigma = 20 \text{ ksi} = (20\,000 \text{ lb/in}^2)(12 \text{ in/ft})^2$$

$$\sigma = 2\,880\,000 \text{ lb/ft}^2$$

$$\gamma = 490 \text{ lb/ft}^3$$

$$2\,880\,000 = \frac{490 v^2}{32.2}$$

$$v = 435.04 \text{ ft/sec}$$

When $\sigma = 30 \text{ ksi}$, and $R = 10 \text{ in}$

$$\sigma = \frac{\gamma v^2}{g}$$

$$30\,000(12^2) = \frac{490 v^2}{32.2}$$

$$v = 532.81 \text{ ft/sec}$$

$$\omega = v/R = \frac{532.81}{10/12}$$

$$\omega = 639.37 \text{ rad/sec}$$

$$\omega = \frac{639.37 \text{ rad}}{\text{sec}} \times \frac{1 \text{ rev}}{2\pi \text{ rad}} \times \frac{60 \text{ sec}}{1 \text{ min}}$$

$$\omega = 6,105.54 \text{ rpm}$$

Problem 140

At what angular velocity will the stress of the rotating steel ring equal 150 MPa if its mean radius is 220 mm? The density of steel 7.85 Mg/m³.

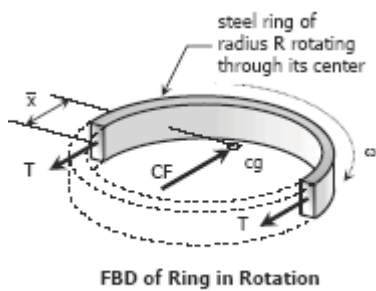
Solution 140

$$CF = M\omega^2 \bar{x}$$

Where: $M = \rho V = \rho A \pi R$
 $\bar{x} = 2R/\pi$

$$CF = \rho A \pi R \omega^2 (2R/\pi)$$

$$CF = 2\rho A R^2 \omega^2$$



$$2T = CF$$

$$2\sigma A = 2\rho A R^2 \omega^2$$

$$\sigma = \rho R^2 \omega^2$$

From the given (Note: 1 N = 1 kg-m/sec²):

$$\sigma = 150 \text{ MPa}$$

$$= 150\,000\,000 \text{ kg-m/sec}^2\text{-m}^2$$

$$= 150\,000\,000 \text{ kg/m-sec}^2$$

$$\rho = 7.85 \text{ Mg/m}^3 = 7850 \text{ kg/m}^3$$

$$R = 220 \text{ mm} = 0.22 \text{ m}$$

$$150\,000\,000 = 7850(0.22)^2 \omega^2$$

$$\omega = 628.33 \text{ rad/sec}$$

Problem 141

The tank shown in Fig. P-141 is fabricated from 1/8-in steel plate. Calculate the maximum longitudinal and circumferential stress caused by an internal pressure of 125 psi.

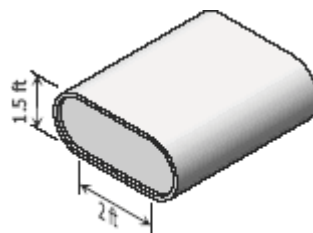
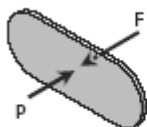


Figure P-141

Solution 141



See dimensions in Fig. P-141,
thickness, $t = 1/8$ in.

Longitudinal Stress:

$$F = pA = 125[1.5(2) + \frac{1}{2}\pi(1.5)^2](12)^2$$

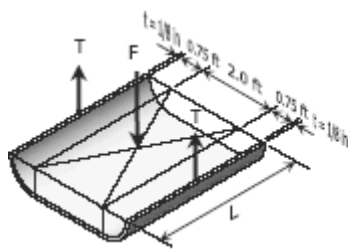
$$F = 85\,808.62 \text{ lbs}$$

$$P = F$$

$$\sigma_l [2(2 \times 12)(\frac{1}{8}) + \pi(1.5 \times 12)(\frac{1}{8})] = 85\,808.62$$

$$\sigma_l = 6\,566.02 \text{ psi}$$

$$\sigma_l = 6.57 \text{ ksi}$$



Circumferential Stress:

$$F = pA = 125[(2 \times 12)L + 2(0.75 \times 12)L]$$

$$F = 5250L \text{ lbs}$$

$$2T = F$$

$$2[\sigma_t(\frac{1}{8})L] = 5250L$$

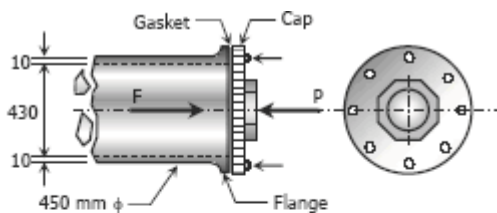
$$\sigma_t = 21\,000 \text{ psi}$$

$$\sigma_t = 21 \text{ ksi}$$

Problem 142

A pipe carrying steam at 3.5 MPa has an outside diameter of 450 mm and a wall thickness of 10 mm. A gasket is inserted between the flange at one end of the pipe and a flat plate used to cap the end. How many 40-mm-diameter bolts must be used to hold the cap on if the allowable stress in the bolts is 80 MPa, of which 55 MPa is the initial stress? What circumferential stress is developed in the pipe? Why is it necessary to tighten the bolt initially, and what will happen if the steam pressure should cause the stress in the bolts to be twice the value of the initial stress?

Solution 142



$$F = \sigma A$$

$$= 3.5[\frac{1}{4}\pi(430^2)]$$

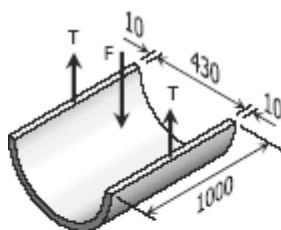
$$= 508\,270.42 \text{ N}$$

$$P = F$$

$$(\sigma_{\text{bolt}}A)n = 508\,270.42 \text{ N}$$

$$(80 - 55)[\frac{1}{4}\pi(40^2)]n = 508\,270.42$$

$$n = 16.19 \text{ say } 17 \text{ bolts}$$



Circumferential stress (consider 1-m strip):

$$F = pA = 3.5[430(1000)]$$

$$F = 1\,505\,000 \text{ N}$$

$$2T = F$$

$$2[\sigma_t(1000)(10)] = 1\,505\,000$$

$$\sigma_t = 75.25 \text{ MPa}$$

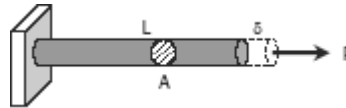
Discussion:

It is necessary to tighten the bolts initially to press the gasket to the flange, to avoid leakage of steam. If the pressure will cause 110 MPa of stress to each bolt causing it to fail, leakage will occur. If this is sudden, the cap may blow.

Strain

Simple Strain

Also known as unit deformation, strain is the ratio of the change in length caused by the applied force, to the original length.



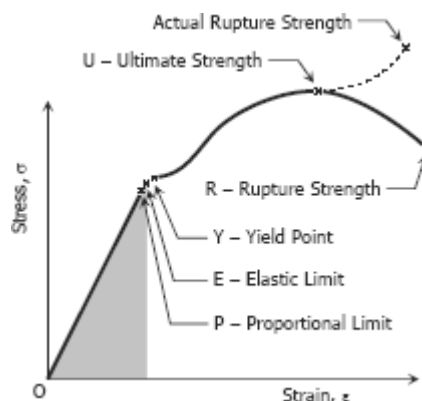
$$\varepsilon = \frac{\delta}{L}$$

where δ is the deformation and L is the original length, thus ε is dimensionless.

Stress-Strain Diagram

Suppose that a metal specimen be placed in tension-compression testing machine. As the axial load is gradually increased in increments, the total elongation over the gage length is measured at each increment of the load and this is continued until failure of the specimen takes place. Knowing the original cross-sectional area and length of the specimen, the normal stress σ and the strain ε can be obtained. The graph of these quantities with the stress σ along the y-axis and the strain ε along the x-axis is called the stress-strain diagram. The stress-strain diagram differs in form for various materials. The diagram shown below is that for a medium carbon structural steel.

Metallic engineering materials are classified as either ductile or brittle materials. A ductile material is one having relatively large tensile strains up to the point of rupture like structural steel and aluminum, whereas brittle materials has a relatively small strain up to the point of rupture like cast iron and concrete. An arbitrary strain of 0.05 mm/mm is frequently taken as the dividing line between these two classes.



PROPORTIONAL LIMIT (HOOKE'S LAW)

From the origin O to the point called proportional limit, the stress-strain curve is a straight line. This linear relation between elongation and the axial force causing was first noticed by Sir Robert Hooke in 1678 and is called Hooke's Law that within the proportional limit, the stress is directly proportional to strain or



Robert Hooke

$$\sigma \propto \epsilon \text{ or } \sigma = k \epsilon$$

The constant of proportionality k is called the Modulus of Elasticity E or Young's Modulus and is equal to the slope of the stress-strain diagram from O to P. Then

$$\sigma = E \epsilon$$

ELASTIC LIMIT

The elastic limit is the limit beyond which the material will no longer go back to its original shape when the load is removed, or it is the maximum stress that may be developed such that there is no permanent or residual deformation when the load is entirely removed.

ELASTIC AND PLASTIC RANGES

The region in stress-strain diagram from O to P is called the elastic range. The region from P to R is called the plastic range.

YIELD POINT

Yield point is the point at which the material will have an appreciable elongation or yielding without any increase in load.

ULTIMATE STRENGTH

The maximum ordinate in the stress-strain diagram is the ultimate strength or tensile strength.

RAPTURE STRENGTH

Rapture strength is the strength of the material at rupture. This is also known as the breaking strength.

MODULUS OF RESILIENCE

Modulus of resilience is the work done on a unit volume of material as the force is gradually increased from O to P, in Nm/m^3 . This may be calculated as the area under the stress-strain curve from the origin O to up to the elastic limit E (the shaded area in the figure). The resilience of the material is its ability to absorb energy without creating a permanent distortion.

MODULUS OF TOUGHNESS

Modulus of toughness is the work done on a unit volume of material as the force is gradually increased from O to R, in Nm/m^3 . This may be calculated as the area under the entire stress-strain curve (from O to R). The toughness of a material is its ability to absorb energy without causing it to break.

WORKING STRESS, ALLOWABLE STRESS, AND FACTOR OF SAFETY

Working stress is defined as the actual stress of a material under a given loading. The maximum safe stress that a material can carry is termed as the allowable stress. The allowable stress should be limited to values not exceeding the proportional limit. However, since proportional limit is difficult to determine accurately, the allowable stress is taken as either the yield point or ultimate strength divided by a factor of safety. The ratio of this strength (ultimate or yield strength) to allowable strength is called the factor of safety.

AXIAL DEFORMATION

In the linear portion of the stress-strain diagram, the stress is proportional to strain and is given by

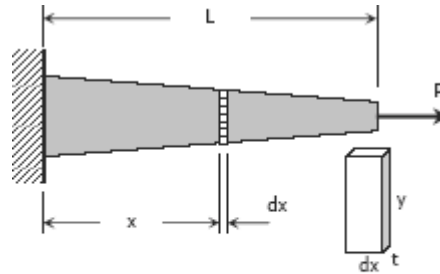
$$\sigma = E\varepsilon$$

since $\sigma = P / A$ and $\varepsilon = \delta / L$, then $P / A = E \delta / L$. Solving for δ ,

$$\delta = \frac{PL}{AE} = \frac{\sigma L}{E}$$

To use this formula, the load must be axial, the bar must have a uniform cross-sectional area, and the stress must not exceed the proportional limit. If however, the cross-sectional area is not uniform, the axial deformation can be determined by considering a differential length and applying integration.

If however, the cross-sectional area is not uniform, the axial deformation can be determined by considering a differential length and applying integration.



$$\delta = \frac{P}{E} \int_0^L \frac{dx}{A}$$

where $A = ty$ and y and t , if variable, must be expressed in terms of x .

For a rod of unit mass ρ suspended vertically from one end, the total elongation due to its own weight is

$$\delta = \frac{\rho g L^2}{2E} = \frac{MgL}{2AE}$$

where ρ is in kg/m^3 , L is the length of the rod in mm, M is the total mass of the rod in kg, A is the cross-sectional area of the rod in mm^2 , and $g = 9.81 \text{ m/s}^2$.

STIFFNESS, k

Stiffness is the ratio of the steady force acting on an elastic body to the resulting displacement. It has the unit of N/mm.

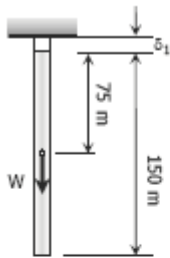
$$k = P / \delta$$

SOLVED PROBLEMS IN AXIAL DEFORMATION

Problem 206

A steel rod having a cross-sectional area of 300 mm^2 and a length of 150 m is suspended vertically from one end. It supports a tensile load of 20 kN at the lower end. If the unit mass of steel is 7850 kg/m^3 and $E = 200 \times 10^3 \text{ MN/m}^2$, find the total elongation of the rod.

Solution 206



Let δ = total elongation

δ_1 = elongation due to its own weight

δ_2 = elongation due to applied load

$$\delta = \delta_1 + \delta_2$$

$$\delta_1 = \frac{PL}{AE}$$

$$\text{Where: } P = W = 7850(1/1000)^3(9.81)[300(150)(1000)]$$

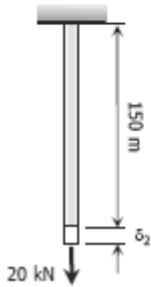
$$P = 3465.3825 \text{ N}$$

$$L = 75(1000) = 75\,000 \text{ mm}$$

$$A = 300 \text{ mm}^2$$

$$E = 200\,000 \text{ MPa}$$

$$\delta_1 = \frac{3465.3825(75000)}{300(200\,000)} = 4.33 \text{ mm}$$



$$\delta_2 = \frac{PL}{AE}$$

$$\text{Where: } P = 20 \text{ kN} = 20\,000 \text{ N}$$

$$L = 150 \text{ m} = 150\,000 \text{ mm}$$

$$A = 300 \text{ mm}^2$$

$$E = 200\,000 \text{ MPa}$$

$$\delta_2 = \frac{20000(150000)}{300(200000)} = 50 \text{ mm}$$

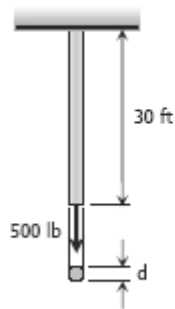
Total elongation:

$$\delta = 4.33 + 50 = 54.33 \text{ mm}$$

Problem 207

A steel wire 30 ft long, hanging vertically, supports a load of 500 lb . Neglecting the weight of the wire, determine the required diameter if the stress is not to exceed 20 ksi and the total elongation is not to exceed 0.20 in . Assume $E = 29 \times 10^6 \text{ psi}$.

Solution 207



Based on maximum allowable stress:

$$\sigma = \frac{P}{A}$$

$$20\,000 = \frac{500}{\frac{1}{4}\pi d^2}$$

$$d = 0.0318 \text{ in}$$

Based on maximum allowable deformation:

$$\delta = \frac{PL}{AE}$$

$$0.20 = \frac{500(30 \times 12)}{\frac{1}{4}\pi d^2 (29 \times 10^6)}$$

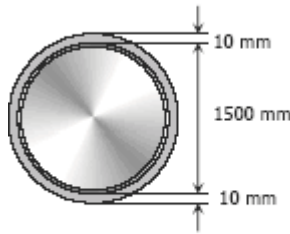
$$d = 0.0395 \text{ in}$$

Use the bigger diameter, $d = 0.0395 \text{ in}$

Problem 208

A steel tire, 10 mm thick, 80 mm wide, and 1500.0 mm inside diameter, is heated and shrunk onto a steel wheel 1500.5 mm in diameter. If the coefficient of static friction is 0.30, what torque is required to twist the tire relative to the wheel? Neglect the deformation of the wheel. Use $E = 200 \text{ GPa}$.

Solution 208



$$\delta = \frac{PL}{AE}$$

Where: $\delta = \pi(1500.5 - 1500) = 0.5\pi \text{ mm}$
 $P = T$
 $L = 1500\pi \text{ mm}$
 $A = 10(80) = 800 \text{ mm}^2$
 $E = 200\,000 \text{ MPa}$

$$0.5\pi = \frac{T(1500\pi)}{800(200\,000)}$$

$$T = 53\,333.33 \text{ N}$$

$$F = 2T$$

$$p(1500)(80) = 2(53\,333.33)$$

$$p = 0.8889 \text{ MPa} \rightarrow \text{internal pressure}$$

Total normal force, N :

$$N = p \times \text{contact area between tire and wheel}$$

$$N = 0.8889 \times \pi(1500.5)(80)$$

$$N = 335\,214.92 \text{ N}$$

Friction resistance, f :

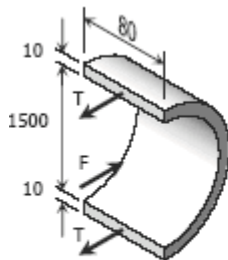
$$f = \mu N = 0.30(335\,214.92)$$

$$f = 100\,564.48 \text{ N} = 100.56 \text{ kN}$$

$$\text{Torque} = f \times \frac{1}{2} (\text{diameter of wheel})$$

$$\text{Torque} = 100.56 \times 0.75025$$

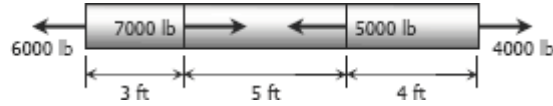
$$\text{Torque} = 75.44 \text{ kN}\cdot\text{m}$$



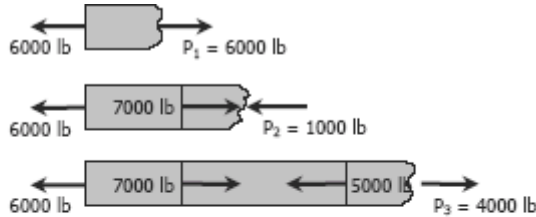
Problem 209

An aluminum bar having a cross-sectional area of 0.5 in^2 carries the axial loads applied at the positions shown in Fig. P-209. Compute the total change in length of the bar if $E = 10 \times 10^6 \text{ psi}$. Assume the bar is suitably braced to prevent lateral buckling.

Figure P-209 and P-210



Solution 209



$P_1 = 6000 \text{ lb tension}$

$P_2 = 1000 \text{ lb compression}$

$P_3 = 4000 \text{ lb tension}$

$$\delta = \frac{PL}{AE}$$

$$\delta = \delta_1 - \delta_2 + \delta_3$$

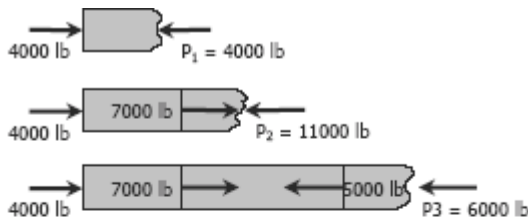
$$\delta = \frac{6000(3 \times 12)}{0.5(10 \times 10^6)} - \frac{1000(5 \times 12)}{0.5(10 \times 10^6)} + \frac{4000(4 \times 12)}{0.5(10 \times 10^6)}$$

$$\delta = 0.0696 \text{ in (lengthening)}$$

Problem 210

Solve Prob. 209 if the points of application of the 6000-lb and the 4000-lb forces are interchanged.

Solution 210



$P_1 = 4000 \text{ lb compression}$

$P_2 = 11000 \text{ lb compression}$

$P_3 = 6000 \text{ lb compression}$

$$\delta = \frac{PL}{AE}$$

$$\delta = -\delta_1 - \delta_2 - \delta_3$$

$$\delta = -\frac{4000(3 \times 12)}{0.5(10 \times 10^6)} - \frac{11000(5 \times 12)}{0.5(10 \times 10^6)} - \frac{6000(4 \times 12)}{0.5(10 \times 10^6)}$$

$$\delta = -0.19248 \text{ in} = 0.19248 \text{ in (shortening)}$$

Problem 211

A bronze bar is fastened between a steel bar and an aluminum bar as shown in Fig. P-211. Axial loads are applied at the positions indicated. Find the largest value of P that will not exceed an overall deformation of 3.0 mm, or the following stresses: 140 MPa in the steel, 120 MPa in the bronze, and 80 MPa in the aluminum. Assume that the assembly is suitably braced to prevent buckling. Use $E_{st} = 200$ GPa, $E_{al} = 70$ GPa, and $E_{br} = 83$ GPa.

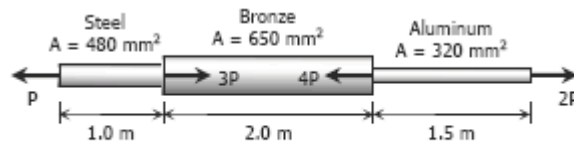


Figure P -211

Solution 211

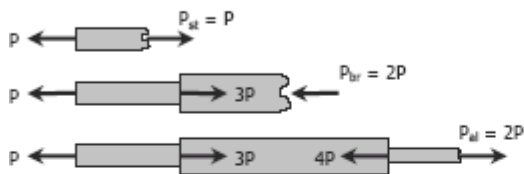
Based on allowable stresses:

Steel:

$$P_{st} = \sigma_{st} A_{st}$$

$$P = 140(480) = 67\,200 \text{ N}$$

$$P = 67.2 \text{ kN}$$



Bronze:

$$P_{br} = \sigma_{br} A_{br}$$

$$2P = 120(650) = 78\,000$$

$$P = 39\,000 \text{ N} = 39 \text{ kN}$$

Aluminum:

$$P_{al} = \sigma_{al} A_{al}$$

$$2P = 80(320) = 25\,600 \text{ N}$$

$$P = 12\,800 \text{ N} = 12.8 \text{ kN}$$

Based on allowable deformation:

(steel and aluminum lengthens, bronze shortens)

$$\delta = \delta_{st} - \delta_{br} + \delta_{al}$$

$$3 = \frac{P(1000)}{480(200\,000)} - \frac{2P(2000)}{650(70\,000)} + \frac{2P(1500)}{320(83\,000)}$$

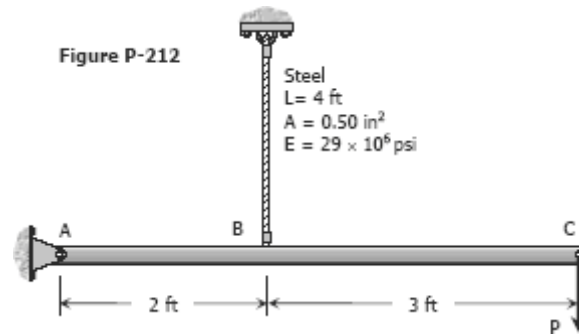
$$3 = \left(\frac{1}{96\,000} - \frac{1}{113.75} + \frac{3}{26\,560} \right) P$$

$$P = 84\,610.99 \text{ N} = 84.61 \text{ kN}$$

Use the smallest value of P , $P = 12.8 \text{ kN}$

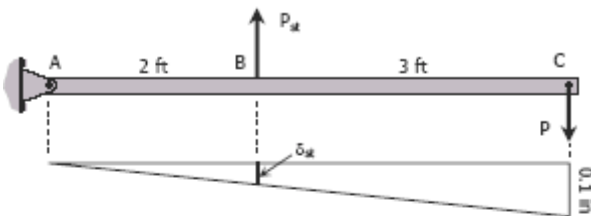
Problem 212

The rigid bar ABC shown in Fig. P-212 is hinged at A and supported by a steel rod at B. Determine the largest load P that can be applied at C if the stress in the steel rod is limited to 30 ksi and the vertical movement of end C must not exceed 0.10 in.



Solution 212

Free body and deformation diagrams:



Based on maximum stress of steel rod:

$$\begin{aligned}\sum M_A &= 0 \\ 5P &= 2P_{st} \\ P &= 0.4P_{st} \\ P &= 0.4\sigma_{st}A_{st} \\ P &= 0.4[30(0.50)] \\ P &= 6 \text{ kips}\end{aligned}$$

Based on movement at C:

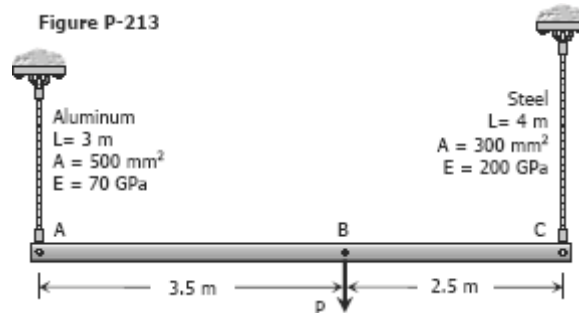
$$\begin{aligned}\frac{\delta_{st}}{2} &= \frac{0.1}{5} \\ \delta_{st} &= 0.04 \text{ in} \\ \frac{P_{st}L}{AE} &= 0.04 \\ \frac{P_{st}(4 \times 12)}{0.50(29 \times 10^6)} &= 0.04 \\ P_{st} &= 12\,083.33 \text{ lb}\end{aligned}$$

$$\begin{aligned}\sum M_A &= 0 \\ 5P &= 2P_{st} \\ P &= 0.4P_{st} \\ P &= 0.4(12\,083.33) \\ P &= 4833.33 \text{ lb} = 4.83 \text{ kips}\end{aligned}$$

Use the smaller value, $P = 4.83 \text{ kips}$

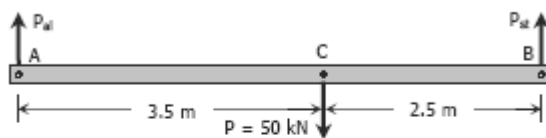
Problem 213

The rigid bar AB, attached to two vertical rods as shown in Fig. P-213, is horizontal before the load P is applied. Determine the vertical movement of P if its magnitude is 50 kN.



Solution 213

Free body diagram:



For aluminum:

$$[\Sigma M_B = 0] \quad 6P_{al} = 2.5(50)$$

$$P_{al} = 20.83 \text{ kN}$$

$$\left[\delta = \frac{PL}{AE} \right]_{al} \quad \delta_{al} = \frac{20.83(3)1000^2}{500(70000)}$$

$$\delta_{al} = 1.78 \text{ mm}$$

For steel:

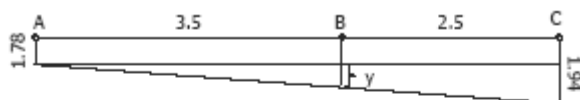
$$[\Sigma M_A = 0] \quad 6P_{st} = 3.5(50)$$

$$P_{st} = 29.17 \text{ kN}$$

$$\left[\delta = \frac{PL}{AE} \right]_{st} \quad \delta_{st} = \frac{29.17(4)1000^2}{300(200000)}$$

$$\delta_{st} = 1.94 \text{ mm}$$

Movement diagram:



$$\frac{y}{3.5} = \frac{1.94 - 1.78}{6}$$

$$y = 0.09 \text{ mm}$$

$$\delta_B = \text{vertical movement of } P$$

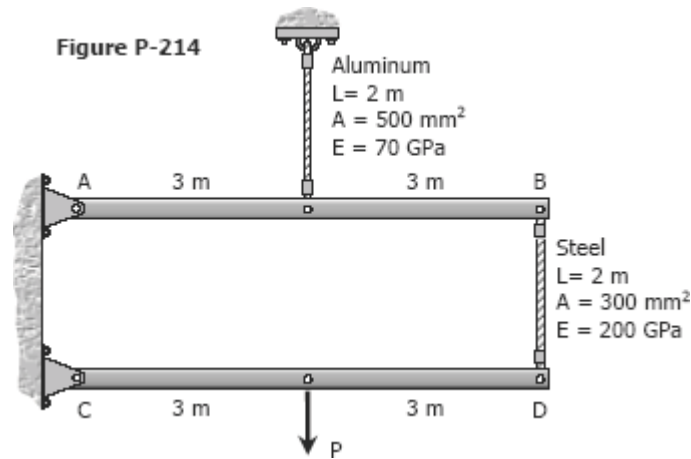
$$\delta_B = 1.78 + y = 1.78 + 0.09$$

$$\delta_B = 1.87 \text{ mm}$$

Problem 214

The rigid bars AB and CD shown in Fig. P-214 are supported by pins at A and C and the two rods. Determine the maximum force P that can be applied as shown if its vertical movement is limited to 5 mm. Neglect the weights of all members.

Figure P-214



Solution 214

$$[\Sigma M_A = 0] \quad \begin{aligned} 3P_{al} &= 6P_{st} \\ P_{al} &= 2P_{st} \end{aligned}$$

By ratio and proportion:

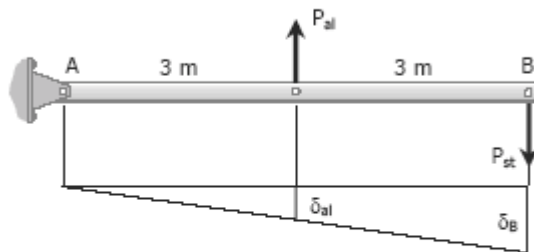
$$\frac{\delta_B}{6} = \frac{\delta_{al}}{3}$$

$$\delta_B = 2\delta_{al} = 2 \left[\frac{PL}{AE} \right]_{al}$$

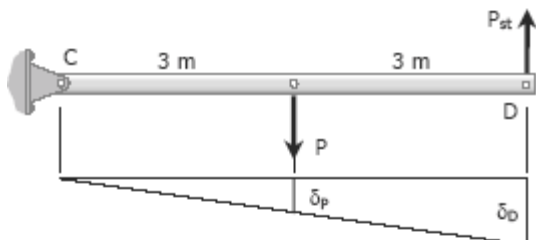
$$\delta_B = 2 \left[\frac{P_{al}(2000)}{500(70000)} \right]$$

$$\delta_B = \frac{1}{8750} P_{al} = \frac{1}{8750} (2P_{st})$$

$$\delta_B = \frac{1}{4375} P_{st} \rightarrow \text{movement of B}$$



FBD and movement diagram of bar AB



FBD and movement diagram of bar CD

Movement of D:

$$\delta_D = \delta_{st} + \delta_B = \left[\frac{PL}{AE} \right]_{st} + \frac{1}{4375} P_{st}$$

$$\delta_D = \frac{P_{st}(2000)}{300(200000)} + \frac{1}{4375} P_{st}$$

$$\delta_D = \frac{11}{42000} P_{st}$$

$$[\Sigma M_C = 0] \quad \begin{aligned} 6P_{st} &= 3P \\ P_{st} &= \frac{1}{2} P \end{aligned}$$

By ratio and proportion:

$$\frac{\delta_P}{3} = \frac{\delta_D}{6}$$

$$\delta_P = \frac{1}{2} \delta_D = \frac{1}{2} \left(\frac{11}{42000} P_{st} \right)$$

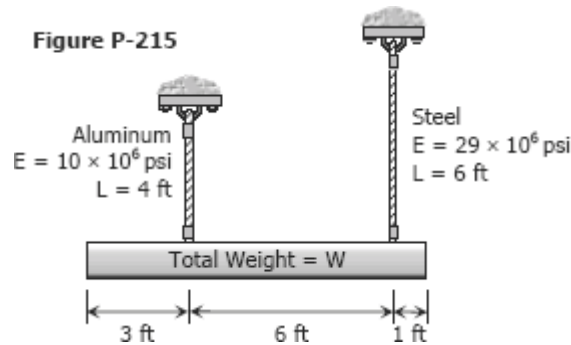
$$\delta_P = \frac{11}{84000} P_{st}$$

$$5 = \frac{11}{84000} \left(\frac{1}{2} P \right)$$

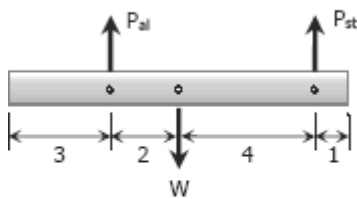
$$P = 76\,363.64 \text{ N} = 76.4 \text{ kN}$$

Problem 215

A uniform concrete slab of total weight W is to be attached, as shown in Fig. P-215, to two rods whose lower ends are on the same level. Determine the ratio of the areas of the rods so that the slab will remain level.



Solution 215



$$[\Sigma M_{al} = 0] \quad 6P_{st} = 2W$$

$$P_{st} = \frac{1}{3}W$$

$$[\Sigma M_{st} = 0] \quad 6P_{al} = 4W$$

$$P_{al} = \frac{2}{3}W$$

$$\delta_{st} = \delta_{al}$$

$$\left[\frac{PL}{AE} \right]_{st} = \left[\frac{PL}{AE} \right]_{al}$$

$$\frac{\frac{1}{3}W(6 \times 12)}{A_{st}(29 \times 10^6)} = \frac{\frac{2}{3}W(4 \times 12)}{A_{al}(10 \times 10^6)}$$

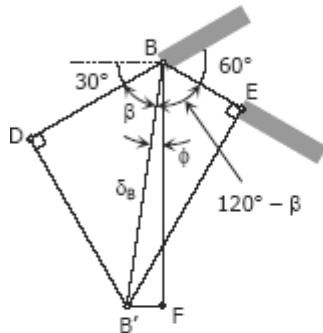
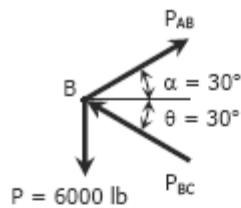
$$\frac{A_{al}}{A_{st}} = \frac{\frac{2}{3}W(4 \times 12)(29 \times 10^6)}{\frac{1}{3}W(6 \times 12)(10 \times 10^6)}$$

$$A_{al}/A_{st} = 3.867$$

Problem 216

As shown in Fig. P-216, two aluminum rods AB and BC, hinged to rigid supports, are pinned together at B to carry a vertical load $P = 6000 \text{ lb}$. If each rod has a crosssectional area of 0.60 in^2 and $E = 10 \times 10^6 \text{ psi}$, compute the elongation of each rod and the horizontal and vertical displacements of point B. Assume $\alpha = 30^\circ$ and $\theta = 30^\circ$.

Solution 216



Movement of B

$$\begin{aligned}
 [\Sigma F_H = 0] \quad & P_{AB} \cos 30^\circ = P_{BC} \cos 30^\circ \\
 & P_{AB} = P_{BC} \\
 [\Sigma F_V = 0] \quad & P_{AB} \sin 30^\circ + P_{BC} \sin 30^\circ = 6000 \\
 & P_{AB} (0.5) + P_{AB} (0.5) = 6000 \\
 & P_{AB} = 6000 \text{ lb tension} \\
 & P_{BC} = 6000 \text{ lb compression}
 \end{aligned}$$

$$\delta = \frac{PL}{AE}$$

$$\delta_{AB} = \frac{6000(10 \times 12)}{0.6(10 \times 10^6)} = 0.12 \text{ in. lengthening}$$

$$\delta_{BC} = \frac{6000(6 \times 12)}{0.6(10 \times 10^6)} = 0.072 \text{ in. shortening}$$

$$DB = \delta_{AB} = 0.12 \text{ in}$$

$$BE = \delta_{BC} = 0.072 \text{ in}$$

$$\delta_B = BB' = \text{displacement of B}$$

$$B' = \text{final position of B after elongation}$$

Triangle BDB':

$$\cos \beta = \frac{0.12}{\delta_B}$$

$$\delta_B = \frac{0.12}{\cos \beta}$$

Triangle BEB':

$$\cos (120^\circ - \beta) = \frac{0.072}{\delta_B}$$

$$\delta_B = \frac{0.072}{\cos (120^\circ - \beta)}$$

$$\delta_B = \delta_B$$

$$\frac{0.12}{\cos \beta} = \frac{0.072}{\cos (120^\circ - \beta)}$$

$$\frac{\cos 120^\circ \cos \beta + \sin 120^\circ \sin \beta}{\cos \beta} = 0.6$$

$$-0.5 + \sin 120^\circ \tan \beta = 0.6$$

$$\tan \beta = 1.1 / \sin 120^\circ; \beta = 51.79^\circ$$

$$\phi = 90 - (30^\circ + \beta) = 90^\circ - (30^\circ + 51.79^\circ)$$

$$\phi = 8.21^\circ$$

$$\delta_B = \frac{0.12}{\cos 51.79^\circ}$$

$$\delta_B = 0.194 \text{ in}$$

Triangle BFB' :

$$\delta_h = B'F = \delta_B \sin \phi = 0.194 \sin 8.21^\circ$$

$$\delta_h = 0.0277 \text{ in}$$

$$\delta_h = \mathbf{0.0023 \text{ ft}} \rightarrow \text{horizontal displacement of B}$$

$$\delta_v = BF = \delta_B \cos \phi = 0.194 \cos 8.21^\circ$$

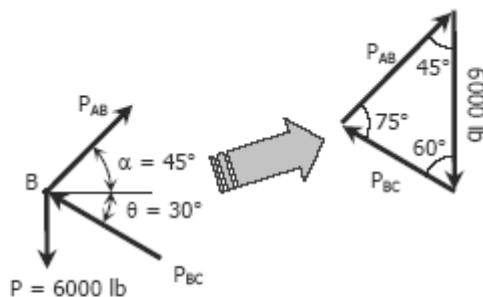
$$\delta_v = 0.192 \text{ in}$$

$$\delta_v = \mathbf{0.016 \text{ ft}} \rightarrow \text{vertical displacement of B}$$

Problem 217

Solve Prob. 216 if rod AB is of steel, with $E = 29 \times 10^6$ psi. Assume $\alpha = 45^\circ$ and $\theta = 30^\circ$; all other data remain unchanged.

Solution 217



By Sine Law

$$\frac{P_{AB}}{\sin 60^\circ} = \frac{6000}{\sin 75^\circ}$$

$$P_{AB} = 5379.45 \text{ lb (Tension)}$$

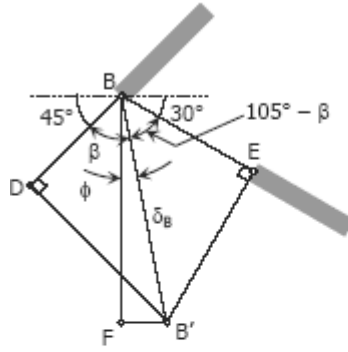
$$\frac{P_{BC}}{\sin 45^\circ} = \frac{6000}{\sin 75^\circ}$$

$$P_{BC} = 4392.30 \text{ lb (Compression)}$$

$$\delta = \frac{PL}{AE}$$

$$\delta_{AB} = \frac{5379.45(10 \times 12)}{0.6(29 \times 10^6)} = \mathbf{0.0371 \text{ in. (lengthening)}}$$

$$\delta_{BC} = \frac{4392.30(6 \times 12)}{0.6(10 \times 10^6)} = 0.0527 \text{ in. (shortening)}$$



Movement of B

$$DB = \delta_{AB} = 0.0371 \text{ in}$$

$$BE = \delta_{BE} = 0.0527 \text{ in}$$

$$\delta_B = BB' = \text{displacement of } B$$

$$B' = \text{final position of } B \text{ after deformation}$$

Triangle BDB' :

$$\cos \beta = \frac{0.0371}{\delta_B}$$

$$\delta_B = \frac{0.0371}{\cos \beta}$$

Triangle BEB' :

$$\cos (105^\circ - \beta) = \frac{0.0527}{\delta_B}$$

$$\delta_B = \frac{0.0527}{\cos (105^\circ - \beta)}$$

$$\delta_B = \delta_B$$

$$\frac{0.0371}{\cos \beta} = \frac{0.0527}{\cos (105^\circ - \beta)}$$

$$\frac{\cos 105^\circ \cos \beta + \sin 105^\circ \sin \beta}{\cos \beta} = 1.4205$$

$$-0.2588 + 0.9659 \tan \beta = 1.4205$$

$$\tan \beta = \frac{1.4205 + 0.2588}{0.9659}$$

$$\tan \beta = 1.7386$$

$$\beta = 60.1^\circ$$

$$\delta_B = \frac{0.0371}{\cos 60.1^\circ}$$

$$\delta_B = 0.0744 \text{ in}$$

$$\begin{aligned} \phi &= (45^\circ + \beta) - 90^\circ \\ &= (45^\circ + 60.1^\circ) - 90^\circ \\ &= 15.1^\circ \end{aligned}$$

Triangle BFB' :

$$\delta_h = FB' = \delta_B \sin \phi = 0.0744 \sin 15.1^\circ$$

$$\delta_h = 0.0194 \text{ in}$$

$$\delta_h = 0.00162 \text{ ft} \rightarrow \text{horizontal displacement of } B$$

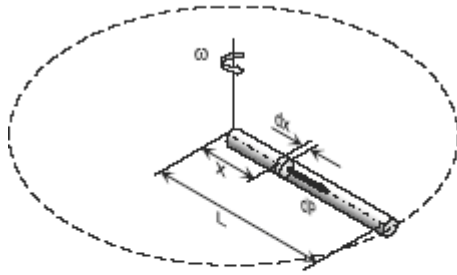
$$\delta_v = BF = \delta_B \cos \phi = 0.0744 \cos 15.1^\circ$$

$$\delta_v = 0.07183 \text{ in}$$

$$\delta_v = 0.00598 \text{ ft} \rightarrow \text{vertical displacement of } B$$

Problem 218

A uniform slender rod of length L and cross sectional area A is rotating in a horizontal plane about a vertical axis through one end. If the unit mass of the rod is ρ , and it is rotating at a constant angular velocity of ω rad/sec, show that the total elongation of the rod is $\rho\omega^2 L^3/3E$.

Solution 218

$$\delta = \frac{PL}{AE}$$

from the frigure:

$$d\delta = \frac{dP x}{AE}$$

Where:

dP = centrifugal force of differential mass

$dP = dM \omega^2 x = (\rho A dx) \omega^2 x$

$dP = \rho A \omega^2 x dx$

$$d\delta = \frac{(\rho A \omega^2 x dx) x}{AE}$$

$$\delta = \frac{\rho \omega^2}{E} \int_0^L x^2 dx = \frac{\rho \omega^2}{E} \left[\frac{x^3}{3} \right]_0^L$$

$$\delta = \frac{\rho \omega^2}{3E} [L^3 - 0^3]$$

$$\delta = \rho \omega^2 L^3 / 3E \quad \text{ok!}$$

Problem 219

A round bar of length L , which tapers uniformly from a diameter D at one end to a smaller diameter d at the other, is suspended vertically from the large end. If w is the weight per unit volume, find the elongation of the rod caused by its own weight. Use this result to determine the elongation of a cone suspended from its base.

Solution 219

$$\delta = \frac{PL}{AE}$$

For the differential strip shown:

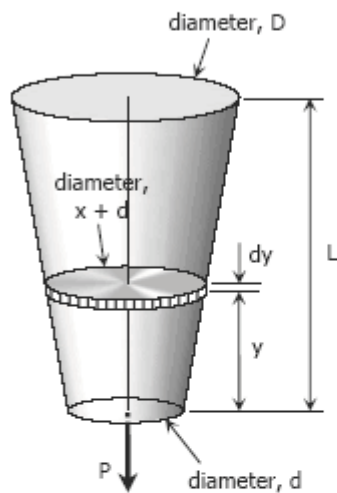
$$\delta = d\delta$$

P = weight carried by the strip

= weight of segment y

$$L = dy$$

A = area of the strip



For weight of segment y (Frustum of a cone):

$$P = wV_y$$

From section along the axis

$$\frac{x}{y} = \frac{D-d}{L}$$

$$x = \frac{D-d}{L}y$$

Volume for frustum of cone

$$V = \frac{1}{3}\pi h (R^2 + r^2 + Rr)$$

$$V_y = \frac{1}{3}\pi h \left[\frac{1}{4}(x+d)^2 + \frac{1}{4}d^2 + \frac{1}{2}(x+d)\left(\frac{1}{2}d\right) \right]$$

$$V_y = \frac{1}{12}\pi y [(x+d)^2 + d^2 + (x+d)d]$$

$$P = \frac{1}{12}\pi w [(x+d)^2 + d^2 + (x+d)d] y$$

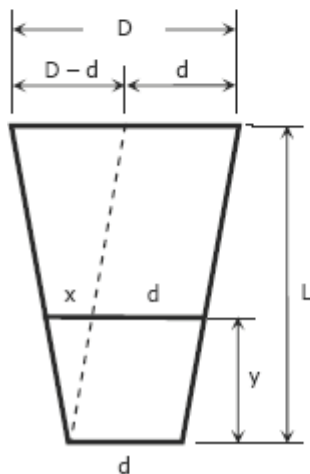
$$P = \frac{1}{12}\pi w [x^2 + 2xd + d^2 + d^2 + xd + d^2] y$$

$$P = \frac{1}{12}\pi w [x^2 + 3xd + 3d^2] y$$

$$P = \frac{\pi w}{12} \left[\frac{(D-d)^2}{L^2} y^2 + \frac{3d(D-d)}{L} y + 3d^2 \right] y$$

Area of the strip:

$$A = \frac{1}{4}\pi(x+d)^2 = \frac{\pi}{4} \left(\frac{D-d}{L} y + d \right)^2$$



Section along the axis of the bar

Thus,

$$\delta = \frac{PL}{AE}$$

$$d\delta = \frac{\frac{\pi w}{12} \left[\frac{(D-d)^2}{L^2} y^2 + \frac{3d(D-d)}{L} y + 3d^2 \right] y \, dy}{\frac{\pi}{4} \left(\frac{D-d}{L} y + d \right)^2 E}$$

$$d\delta = \frac{4w}{12E} \left[\frac{\frac{(D-d)^2}{L^2} y^2 + \frac{3d(D-d)}{L} y + 3d^2}{\frac{(D-d)^2}{L^2} y^2 + \frac{2d(D-d)}{L} y + d^2} \right] y \, dy$$

$$d\delta = \frac{w}{3E} \left[\frac{\frac{(D-d)^2 y^2 + 3Ld(D-d)y + 3L^2 d^2}{L^2}}{\frac{(D-d)^2 y^2 + 2Ld(D-d)y + L^2 d^2}{L^2}} \right] y \, dy$$

$$d\delta = \frac{w}{3E} \left[\frac{(D-d)^2 y^2 + 3Ld(D-d)y + 3L^2 d^2}{(D-d)^2 y^2 + 2Ld(D-d)y + L^2 d^2} \right] y \, dy$$

Let: $a = D - d$; $b = Ld$

$$d\delta = \frac{w}{3E} \left[\frac{a^2 y^2 + 3aby + 3b^2}{a^2 y^2 + 2aby + b^2} \right] y \, dy$$

$$d\delta = \frac{w}{3E} \left[\frac{a^2 y^2 + 3aby + 3b^2}{(ay)^2 + 2(ay)b + b^2} \times \frac{a}{a} \right] y \, dy$$

$$d\delta = \frac{w}{3aE} \left[\frac{a^3 y^3 + 3(a^2 y^2)b + 3(ay)b^2}{(ay + b)^2} \right] dy$$

$$d\delta = \frac{w}{3aE} \left\{ \frac{[(ay)^3 + 3(ay)^2 b + 3(ay)b^2 + b^3] - b^3}{(ay + b)^2} \right\} dy$$

The quantity $(ay)^3 + 3(ay)^2 b + 3(ay)b^2 + b^3$ is the expansion of $(ay + b)^3$

$$d\delta = \frac{w}{3aE} \left[\frac{(ay + b)^3 - b^3}{(ay + b)^2} \right] dy$$

$$d\delta = \frac{w}{3aE} \left[\frac{(ay + b)^3}{(ay + b)^2} - \frac{b^3}{(ay + b)^2} \right] dy$$

$$\begin{aligned}
d\delta &= \frac{w}{3aE} [(ay+b) - b^3(ay+b)^{-2}] dy \\
\delta &= \frac{w}{3aE} \int_0^L [(ay+b) - b^3(ay+b)^{-2}] dy \\
\delta &= \frac{w}{3aE} \left[\frac{(ay+b)^2}{2a} - \frac{b^3(ay+b)^{-1}}{-a} \right]_0^L \\
\delta &= \frac{w}{3a^2E} \left[\frac{(ay+b)^2}{2} + \frac{b^3}{ay+b} \right]_0^L \\
\delta &= \frac{w}{3a^2E} \left\{ \left[\frac{1}{2}(aL+b)^2 + \frac{b^3}{aL+b} \right] - \left[\frac{1}{2}b^2 + \frac{b^3}{b} \right] \right\} \\
\delta &= \frac{w}{3a^2E} \left\{ \frac{1}{2}(aL+b)^2 + \frac{b^3}{aL+b} - \frac{3}{2}b^2 \right\} \\
\delta &= \frac{w}{3a^2E} \left[\frac{(aL+b)^3 + 2b^3 - 3b^2(aL+b)}{2(aL+b)} \right] \\
\delta &= \frac{w}{6a^2E} \left[\frac{(aL)^3 + 3(aL)^2b + 3(aL)b^2 + b^3 + 2b^3 - 3ab^2L - 3b^3}{aL+b} \right] \\
\delta &= \frac{w}{6a^2E} \left[\frac{a^3L^3 + 3a^2bL^2}{aL+b} \right]; \text{Note: } a = D - d \text{ \& } b = Ld \\
\delta &= \frac{w}{6(D-d)^2E} \left[\frac{(D-d)^3L^3 + 3(D-d)^2(Ld)L^2}{(D-d)L + (Ld)} \right] \\
\delta &= \frac{w}{6(D-d)^2E} \left\{ \frac{(D-d)L^3[(D-d)^2 + 3d(D-d)]}{LD - Ld + Ld} \right\} \\
\delta &= \frac{wL^3}{6(D-d)E} \left[\frac{(D-d)^2 + 3d(D-d)}{LD} \right] \\
\delta &= \frac{wL^3}{6(D-d)E} \left[\frac{D^2 - 2Dd + d^2 + 3Dd - 3d^2}{LD} \right] \\
\delta &= \frac{wL^3}{6(D-d)E} \left[\frac{D^2 + Dd - 2d^2}{LD} \right] \\
\delta &= \frac{wL^3}{6(D-d)E} \left[\frac{D(D+d) - 2d^2}{LD} \right] \\
\delta &= \frac{wL^3}{6(D-d)E} \left[\frac{D(D+d)}{LD} \right] - \frac{wL^3}{6(D-d)E} \left[\frac{2d^2}{LD} \right] \\
\delta &= \frac{wL^2(D+d)}{6E(D-d)} - \frac{wL^2d^2}{3ED(D-d)}
\end{aligned}$$

For a cone:

$$D = D \text{ and } d = 0$$

$$\delta = \frac{wL^2(D+0)}{6E(D-0)} - \frac{wL^2(0)^2}{3ED(D-0)}$$

$$\delta = \frac{wL^2}{6E}$$

SOLVED PROBLEMS IN STRAIN AND AXIAL DEFORMATION

Problem 203

The following data were recorded during the tensile test of a 14-mm-diameter mild steel rod. The gage length was 50 mm.

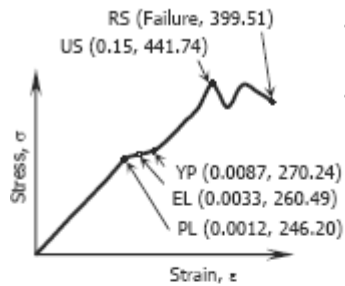
Load (N)	Elongation (mm)	Load (N)	Elongation (mm)
0	0	46200	1.25
6310	0.010	52400	2.50
12600	0.020	58500	4.50
18800	0.030	68000	7.50
25100	0.040	59000	12.50
31300	0.050	67800	15.50
37900	0.060	65000	20.00
40100	0.163	61500	Fracture
41600	0.433		

Plot the stress-strain diagram and determine the following mechanical properties: (a) proportional limits; (b) modulus of elasticity; (c) yield point; (d) ultimate strength; and (e) rupture strength.

Solution 203

$$\text{Area, } A = \frac{1}{4} \pi (14)^2 = 49\pi \text{ mm}^2; \quad \text{Length, } L = 50 \text{ mm}$$

$$\text{Strain} = \text{Elongation}/\text{Length}; \quad \text{Stress} = \text{Load}/\text{Area}$$



Stress-Strain Diagram
(not drawn to scale)

PL = Proportional Limit
EL = Elastic Limit
YP = Yield Point
US = Ultimate Strength
RS = Rupture Strength

Load (N)	Elongation (mm)	Strain (mm/mm)	Stress (MPa)
0	0	0	0
6310	0.010	0.0002	40.99
12600	0.020	0.0004	81.85
18800	0.030	0.0006	122.13
25100	0.040	0.0008	163.05
31300	0.050	0.001	203.33
37900	0.060	0.0012	246.20
40100	0.163	0.0033	260.49
41600	0.433	0.0087	270.24
46200	1.250	0.025	300.12
52400	2.500	0.05	340.40
58500	4.500	0.09	380.02
68000	7.500	0.15	441.74
59000	12.500	0.25	383.27
67800	15.500	0.31	440.44
65000	20.000	0.4	422.25
61500	Failure		399.51

From stress-strain diagram:

- Proportional Limit = 246.20 MPa
- Modulus of Elasticity
 $E = \text{slope of stress-strain diagram within proportional limit}$
 $E = \frac{246.20}{0.0012} = 205\,166.67 \text{ MPa}$
 $E = 205.2 \text{ GPa}$
- Yield Point = 270.24 MPa
- Ultimate Strength = 441.74 MPa
- Rupture Strength = 399.51 MPa

Problem 204

The following data were obtained during a tension test of an aluminum alloy. The initial diameter of the test specimen was 0.505 in. and the gage length was 2.0 in.

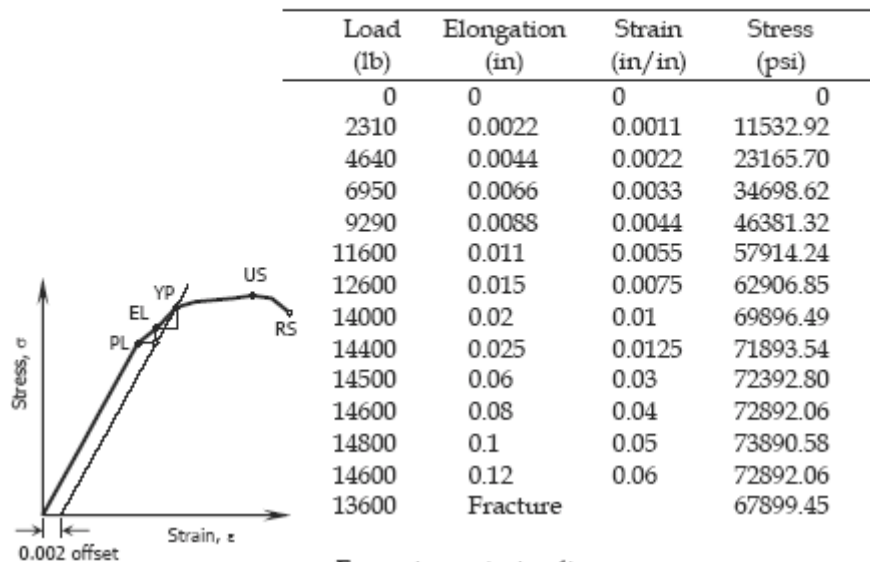
Load (lb)	Elongation (in.)	Load (lb)	Elongation (in.)
0	0	14 000	0.020
2 310	0.00220	14 400	0.025
4 640	0.00440	14 500	0.060
6 950	0.00660	14 600	0.080
9 290	0.00880	14 800	0.100
11 600	0.0110	14 600	0.120
12 600	0.0150	13 600	Fracture

Plot the stress-strain diagram and determine the following mechanical properties: (a) proportional limit; (b) modulus of elasticity; (c) yield point; (d) yield strength at 0.2% offset; (e) ultimate strength; and (f) rupture strength.

Solution 204

$$\text{Area} = \frac{1}{4} \pi (0.505)^2 = 0.0638\pi \text{ in}^2; \quad \text{Length, } L = 2.0 \text{ in.}$$

$$\text{Strain} = \text{Elongation}/\text{Length}; \quad \text{Stress} = \text{Load}/\text{Area}$$



PL (0.0055, 57914.24)
 EL (0.0075, 62906.85)
 YP (0.01, 69896.49)
 US (0.05, 73890.58)
 RS (Failure, 67899.45)

From stress-strain diagram:

(a) Proportional Limit = 57,914.24 psi

(b) Modulus of Elasticity:

$$E = \frac{57914.24}{0.0055} = 10,529,861.82 \text{ psi}$$

$$E = 10,529.86 \text{ ksi}$$

(c) Yield Point = 69,896.49 psi

(d) Yield Strength at 0.2% Offset:

Strain of Elastic Limit

$$= \epsilon \text{ at PL} + 0.002$$

$$= 0.0055 + 0.002$$

$$= 0.0075 \text{ in/in}$$

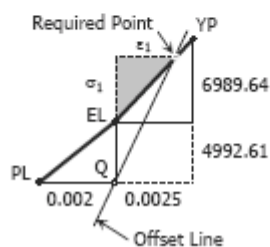
The offset line will pass through Q (See figure):

Slope of 0.2% offset

$$= E = 10,529,861.82 \text{ psi}$$

Test for location

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$



$$10,529,861.82 = \frac{6989.64 + 4992.61}{\text{run}}$$

run = 0.00113793 < 0.0025, therefore,
the required point is just
before YP.

Slope of EL to YP

$$\frac{\sigma_1}{\varepsilon_1} = \frac{6989.64}{0.0025}$$

$$\frac{\sigma_1}{\varepsilon_1} = 2\,795\,856$$

$$\varepsilon_1 = \frac{\sigma_1}{2\,795\,856}$$

For required point

$$E = \frac{4992.61 + \sigma_1}{\varepsilon_1}$$

$$10\,529\,861.82 = \frac{4992.61 + \sigma_1}{\frac{\sigma_1}{2\,795\,856}}$$

$$3.7662\sigma_1 = 4992.61 + \sigma_1$$

$$\sigma_1 = 1804.84 \text{ psi}$$

Yield Strength at 0.2% Offset

$$= EL + \sigma_1$$

$$= 62906.85 + 1804.84$$

$$= 64,711.69 \text{ psi}$$

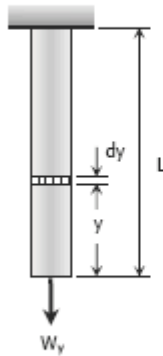
(e) Ultimate Strength = 73,890.58 psi

(f) Rupture Strength = 67,899.45 psi

Problem 205

A uniform bar of length L , cross-sectional area A , and unit mass ρ is suspended vertically from one end. Show that its total elongation is $\delta = \rho g L^2 / 2E$. If the total mass of the bar is M , show also that $\delta = MgL/2AE$.

Solution 205



$$\delta = \frac{PL}{AE}$$

From the figure:

$$\delta = d\delta$$

$$P = W_y = (\rho A y)g$$

$$L = dy$$

$$d\delta = \frac{(\rho A y)g dy}{AE}$$

$$\delta = \frac{\rho g}{E} \int_0^L y dy = \frac{\rho g}{E} \left[\frac{y^2}{2} \right]_0^L$$

$$\delta = \frac{\rho g}{2E} [L^2 - 0^2] = \rho g L^2 / 2E \quad \text{ok!}$$

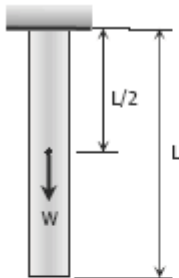
Given the total mass M :

$$\rho = M/V = M/AL$$

$$\delta = \rho g L^2 / 2E = (M/AL)(g L^2 / 2E)$$

$$\delta = MgL / 2AE \quad \text{ok!}$$

Another Solution:



$$\delta = \frac{PL}{AE}$$

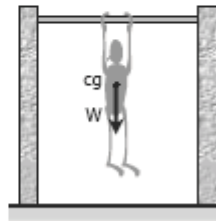
$$\text{Where: } P = W = (\rho AL)g$$

$$L = L/2$$

$$\delta = \frac{[(\rho AL)g](L/2)}{AE}$$

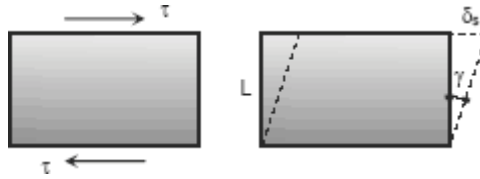
$$\delta = \frac{\rho g L^2}{2E} \quad \text{ok!}$$

For you to feel the situation, position yourself in pull-up exercise with your hands on the bar and your body hang freely above the ground. Notice that your arms suffer all your weight and your lower body feels no stress (center of weight is approximately just below the chest). If your body is the bar, the elongation will occur at the upper half of it.



Shearing Deformation

Shearing forces cause shearing deformation. An element subject to shear does not change in length but undergoes a change in shape.



The change in angle at the corner of an original rectangular element is called the **shear strain** and is expressed as

$$\gamma = \frac{\delta_s}{L}$$

The ratio of the shear stress τ and the shear strain γ is called the **modulus of elasticity** in shear or modulus of rigidity and is denoted as G , in MPa.

$$G = \frac{\tau}{\gamma}$$

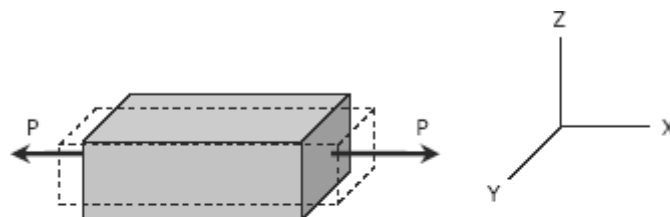
The relationship between the shearing deformation and the applied shearing force is

$$\delta_s = \frac{VL}{A_s G} = \frac{\tau L}{G}$$

where V is the shearing force acting over an area A_s .

Poisson's Ratio

When a bar is subjected to a tensile loading there is an increase in length of the bar in the direction of the applied load, but there is also a decrease in a lateral dimension perpendicular to the load. The ratio of the sidewise deformation (or strain) to the longitudinal deformation (or strain) is called the Poisson's ratio and is denoted by ν . For most steel, it lies in the range of 0.25 to 0.3, and 0.20 for concrete.



$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x}$$

where ε_x is strain in the x-direction and ε_y and ε_z are the strains in the perpendicular direction. The negative sign indicates a decrease in the transverse dimension when ε_x is positive.

BIAXIAL DEFORMATION

If an element is subjected simultaneously by tensile stresses, σ_x and σ_y , in the x and y directions, the strain in the x-direction is σ_x / E and the strain in the y direction is σ_y / E . Simultaneously, the stress in the y direction will produce a lateral contraction on the x direction of the amount $-\nu \varepsilon_y$ or $-\nu \sigma_y / E$. The resulting strain in the x direction will be

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \quad \text{or} \quad \sigma_x = \frac{(\varepsilon_x + \nu \varepsilon_y)E}{1 - \nu^2}$$

and

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} \quad \text{or} \quad \sigma_y = \frac{(\varepsilon_y + \nu \varepsilon_x)E}{1 - \nu^2}$$

TRIAXIAL DEFORMATION

If an element is subjected simultaneously by three mutually perpendicular normal stresses σ_x , σ_y , and σ_z , which are accompanied by strains ε_x , ε_y , and ε_z , respectively,

$$\begin{aligned}\varepsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \varepsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \varepsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]\end{aligned}$$

Tensile stresses and elongation are taken as positive. Compressive stresses and contraction are taken as negative.

Relationship Between E, G, and ν

The relationship between modulus of elasticity E, shear modulus G and Poisson's ratio ν is:

$$G = \frac{E}{2(1 + \nu)}$$

Bulk Modulus of Elasticity or Modulus of Volume Expansion, K

The bulk modulus of elasticity K is a measure of a resistance of a material to change in volume without change in shape or form. It is given as

$$K = \frac{E}{3(1-2\nu)} = \frac{\sigma}{\Delta V/V}$$

where V is the volume and ΔV is change in volume. The ratio $\Delta V / V$ is called volumetric strain and can be expressed as

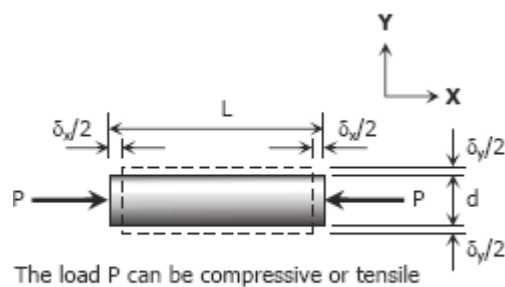
$$\frac{\Delta V}{V} = \frac{\sigma}{K} = \frac{3(1-2\nu)}{E}$$

Solved Problems in Shearing Deformation

Problem 222

A solid cylinder of diameter d carries an axial load P. Show that its change in diameter is $4P\nu / \pi E d$.

Solution 222



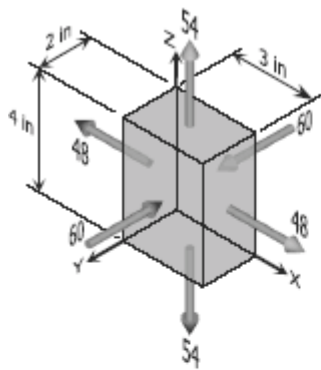
$$\begin{aligned} \nu &= -\frac{\epsilon_y}{\epsilon_x} \\ \epsilon_y &= -\nu \epsilon_x \\ \epsilon_y &= -\nu \frac{\sigma_x}{E} \\ \frac{\delta_y}{d} &= -\nu \frac{-P}{AE} \\ \delta_y &= \nu \frac{Pd}{\frac{1}{4}\pi d^2 E} \\ \delta_y &= \frac{4P\nu}{\pi E d} \quad \text{ok!} \end{aligned}$$

Problem 223

A rectangular steel block is 3 inches long in the x direction, 2 inches long in the y direction, and 4 inches long in the z direction. The block is subjected to a triaxial loading of three uniformly distributed forces as follows: 48 kips tension in the x direction, 60 kips compression in the y direction, and 54 kips tension in the z direction. If $\nu = 0.30$ and $E = 29 \times 10^6$ psi, determine the single uniformly distributed load in the x direction that would produce the same deformation in the y direction as the original loading.

Solution 223

For triaxial deformation (tensile triaxial stresses):
(compressive stresses are negative stresses)



$$\epsilon_y = \frac{1}{E} [\sigma_y - \nu(\sigma_x + \sigma_z)]$$

$$\sigma_x = \frac{P_x}{A_{yz}} = \frac{48}{4(2)} = 6.0 \text{ ksi (tension)}$$

$$\sigma_y = \frac{P_y}{A_{xz}} = \frac{60}{4(3)} = 5.0 \text{ ksi (compression)}$$

$$\sigma_z = \frac{P_z}{A_{xy}} = \frac{54}{2(3)} = 9.0 \text{ ksi (tension)}$$

$$\epsilon_y = \frac{1}{29 \times 10^6} [-5000 - 0.30(6000 + 9000)]$$

$$\epsilon_y = -3.276 \times 10^{-4}$$

ϵ_y is negative, thus tensile force is required in the x-direction to produce the same deformation in the y-direction as the original forces.

For equivalent single force in the x-direction:
(uniaxial stress)

$$\nu = -\frac{\epsilon_y}{\epsilon_x}$$

$$-\nu \epsilon_x = \epsilon_y$$

$$-\nu \frac{\sigma_x}{E} = \epsilon_y$$

$$-0.30 \left(\frac{\sigma_x}{29 \times 10^6} \right) = -3.276 \times 10^{-4}$$

$$\sigma_x = 31\,666.67 \text{ psi}$$

$$\sigma_x = \frac{P_x}{4(2)} = 31\,666.67$$

$$P_x = 253\,333.33 \text{ lb (tension)}$$

$$P_x = 253.33 \text{ kips (tension)}$$

Problem 224

For the block loaded triaxially as described in Prob. 223, find the uniformly distributed load that must be added in the x direction to produce no deformation in the z direction.

Solution 224

$$\epsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)]$$

$$\sigma_x = 6.0 \text{ ksi (tension)}$$

$$\sigma_y = 5.0 \text{ ksi (compression)}$$

$$\sigma_z = 9.0 \text{ ksi (tension)}$$

$$\epsilon_z = \frac{1}{29 \times 10^6} [9000 - 0.3(6000 - 5000)]$$

$$\epsilon_z = 2.07 \times 10^{-5}$$

ϵ_z is positive, thus positive stress is needed in the x-direction to eliminate deformation in z-direction.

The application of loads is still simultaneous:

(No deformation means zero strain)

$$\epsilon_z = \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] = 0$$

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\sigma_y = 5.0 \text{ ksi} \quad \rightarrow \text{(compression)}$$

$$\sigma_z = 9.0 \text{ ksi} \quad \rightarrow \text{(tension)}$$

$$9000 = 0.30(\sigma_x - 5000)$$

$$\sigma_x = 35\,000 \text{ psi}$$

$$\sigma_{added} + 6000 = 35\,000$$

$$\sigma_{added} = 29\,000 \text{ psi}$$

$$\frac{P_{added}}{2(4)} = 29\,000$$

$$P_{added} = 232\,000 \text{ lb}$$

$$P_{added} = 232 \text{ kips}$$

Problem 225

A welded steel cylindrical drum made of a 10-mm plate has an internal diameter of 1.20 m. Compute the change in diameter that would be caused by an internal pressure of 1.5 MPa. Assume that Poisson's ratio is 0.30 and $E = 200 \text{ GPa}$.

Solution 225 σ_y = longitudinal stress

$$\sigma_y = \frac{pD}{4t} = \frac{1.5(1200)}{4(10)}$$

$$\sigma_y = 45 \text{ MPa}$$

 σ_x = tangential stress

$$\sigma_x = \frac{pD}{2t} = \frac{1.5(1200)}{2(10)}$$

$$\sigma_x = 90 \text{ MPa}$$

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$

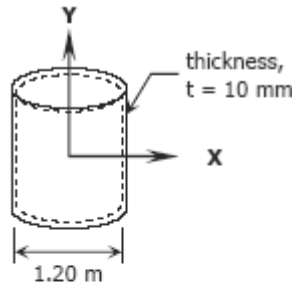
$$\epsilon_x = \frac{90}{200000} - 0.3 \left(\frac{45}{200000} \right)$$

$$\epsilon_x = 3.825 \times 10^{-4}$$

$$\epsilon_x = \frac{\Delta D}{D}$$

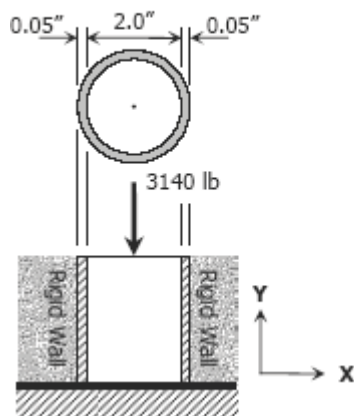
$$\Delta D = \epsilon_x D = (3.825 \times 10^{-4})(1200)$$

$$\Delta D = 0.459 \text{ mm}$$

**Problem 226**

A 2-in.-diameter steel tube with a wall thickness of 0.05 inch just fits in a rigid hole.

Find the tangential stress if an axial compressive load of 3140 lb is applied. Assume $\nu = 0.30$ and neglect the possibility of buckling.

Solution 226

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} = 0$$

$$\sigma_x = \nu \sigma_y$$

where σ_x = tangential stress

σ_y = longitudinal stress

$$\sigma_y = \frac{P_y}{A} = \frac{3140}{\pi(2)(0.05)}$$

$$\sigma_y = 31400/\pi \text{ psi}$$

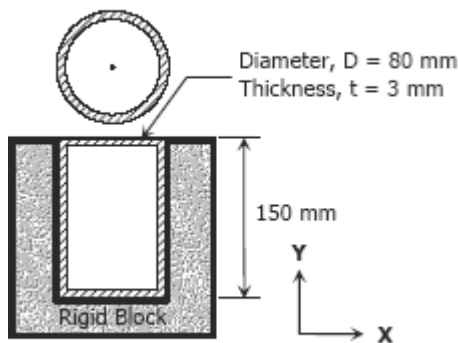
$$\sigma_x = 0.30(31400/\pi)$$

$$\sigma_x = 9430/\pi \text{ psi}$$

$$\sigma_x = 2298.5 \text{ psi}$$

Problem 227

A 150-mm-long bronze tube, closed at its ends, is 80 mm in diameter and has a wall thickness of 3 mm. It fits without clearance in an 80-mm hole in a rigid block. The tube is then subjected to an internal pressure of 4.00 MPa. Assuming $\nu = 1/3$ and $E = 83$ GPa, determine the tangential stress in the tube.

Solution 227

Longitudinal stress:

$$\sigma_y = \frac{pD}{4t} = \frac{4(80)}{4(3)}$$

$$\sigma_y = \frac{80}{3} \text{ MPa}$$

The strain in the x -direction is:

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} = 0$$

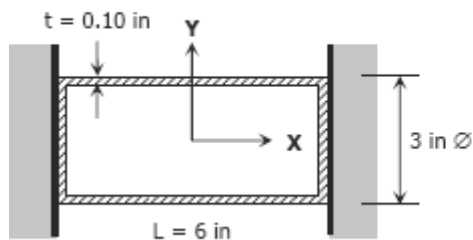
$$\sigma_x = \nu \sigma_y = \text{tangential stress}$$

$$\sigma_x = \frac{1}{3} \left(\frac{80}{3} \right)$$

$$\sigma_x = 8.89 \text{ MPa}$$

Problem 228

A 6-in.-long bronze tube, with closed ends, is 3 in. in diameter with a wall thickness of 0.10 in. With no internal pressure, the tube just fits between two rigid end walls. Calculate the longitudinal and tangential stresses for an internal pressure of 6000 psi. Assume $\nu = 1/3$ and $E = 12 \times 10^6$ psi.

Solution 228

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} = 0$$

$$\sigma_x = \nu \sigma_y = \sigma_l \rightarrow \text{longitudinal stress}$$

$$\sigma_t = \sigma_y \rightarrow \text{tangential stress}$$

$$\sigma_t = \frac{pD}{2t} = \frac{6000(3)}{2(0.10)}$$

$$\sigma_t = 90,000 \text{ psi}$$

$$\sigma_l = \nu \sigma_y = \frac{1}{3} (90,000)$$

$$\sigma_l = 30,000 \text{ psi}$$

Statically Indeterminate Members

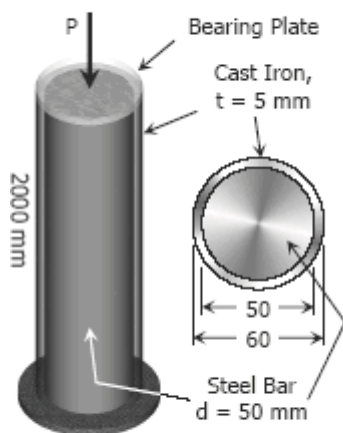
When the reactive forces or the internal resisting forces over a cross section exceed the number of independent equations of equilibrium, the structure is called **statically indeterminate**. These cases require the use of additional relations that depend on the elastic deformations in the members.

Solved Problems in Statically Indeterminate Members

Problem 233

A steel bar 50 mm in diameter and 2 m long is surrounded by a shell of a cast iron 5 mm thick. Compute the load that will compress the combined bar a total of 0.8 mm in the length of 2 m. For steel, $E = 200$ GPa, and for cast iron, $E = 100$ GPa.

Solution 233



$$\delta = \frac{PL}{AE}$$

$$\delta = \delta_{\text{cast iron}} = \delta_{\text{steel}} = 0.8 \text{ mm}$$

$$\delta_{\text{cast iron}} = \frac{P_{\text{cast iron}}(2000)}{\left[\frac{1}{4}\pi(60^2 - 50^2)\right](100\,000)} = 0.8$$

$$P_{\text{cast iron}} = 11\,000\pi \text{ N}$$

$$\delta_{\text{steel}} = \frac{P_{\text{steel}}(2000)}{\left[\frac{1}{4}\pi(50^2)\right](200\,000)} = 0.8$$

$$P_{\text{steel}} = 50\,000\pi \text{ N}$$

$$\sum F_V = 0$$

$$P = P_{\text{cast iron}} + P_{\text{steel}}$$

$$P = 11\,000\pi + 50\,000\pi$$

$$P = 61\,000\pi \text{ N}$$

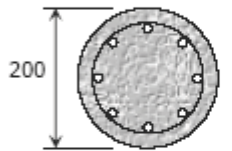
$$P = 191.64 \text{ kN}$$

Problem 234

A reinforced concrete column 200 mm in diameter is designed to carry an axial compressive load of 300 kN. Determine the required area of the reinforcing steel if the allowable stresses are 6 MPa and 120 MPa for the concrete and steel, respectively. Use $E_{co} = 14 \text{ GPa}$ and $E_{st} = 200 \text{ GPa}$.

Solution 234

$$\begin{aligned}\delta_{co} &= \delta_{st} = \delta \\ \left(\frac{PL}{AE} \right)_{co} &= \left(\frac{PL}{AE} \right)_{st} \\ \left(\frac{\sigma L}{E} \right)_{co} &= \left(\frac{\sigma L}{E} \right)_{st} \\ \frac{\sigma_{co} L}{14000} &= \frac{\sigma_{st} L}{200000} \\ 100\sigma_{co} &= 7\sigma_{st}\end{aligned}$$



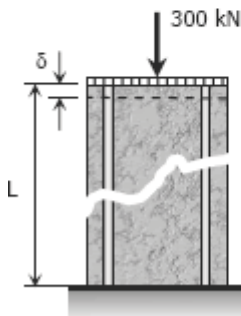
When $\sigma_{st} = 120 \text{ MPa}$

$$\begin{aligned}100\sigma_{co} &= 7(120) \\ \sigma_{co} &= 8.4 \text{ MPa} > 6 \text{ MPa (not ok!)}\end{aligned}$$

When $\sigma_{co} = 6 \text{ MPa}$

$$\begin{aligned}100(6) &= 7\sigma_{st} \\ \sigma_{st} &= 85.71 \text{ MPa} < 120 \text{ MPa (ok!)}\end{aligned}$$

Use $\sigma_{co} = 6 \text{ MPa}$ and $\sigma_{st} = 85.71 \text{ MPa}$

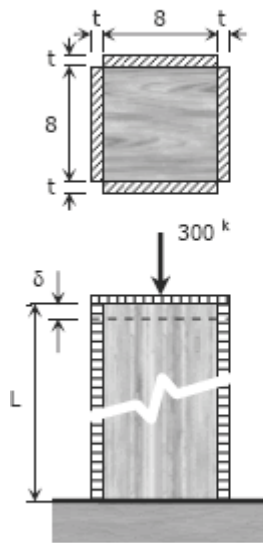


$$\begin{aligned}\sum F_V &= 0 \\ P_{st} + P_{co} &= 300 \\ \sigma_{st} A_{st} + \sigma_{co} A_{co} &= 300 \\ 85.71 A_{st} + 6 \left[\frac{1}{4} \pi (200)^2 - A_{st} \right] &= 300(1000) \\ 79.71 A_{st} + 60\,000\pi &= 300\,000 \\ A_{st} &= 1398.9 \text{ mm}^2\end{aligned}$$

Problem 235

A timber column, 8 in. $\times$ 8 in. in cross section, is reinforced on each side by a steel plate 8 in. wide and t in. thick. Determine the thickness t so that the column will support an axial load of 300 kips without exceeding a maximum timber stress of 1200 psi or a maximum steel stress of 20 ksi. The moduli of elasticity are $1.5 \times 10^6 \text{ psi}$ for timber, and $29 \times 10^6 \text{ psi}$ for steel.

Solution 235



$$\delta_{st} = \delta_t$$

$$\left(\frac{\sigma L}{E} \right)_{st} = \left(\frac{\sigma L}{E} \right)_{timber}$$

$$\frac{\sigma_{st} L}{29 \times 10^6} = \frac{\sigma_{timber} L}{1.5 \times 10^6}$$

$$1.5 \sigma_{st} = 29 \sigma_{timber}$$

When $\sigma_{timber} = 1200$ psi

$$1.5 \sigma_{st} = 29(1200)$$

$$\sigma_{st} = 23\,200 \text{ psi} = 23.2 \text{ ksi} > 20 \text{ ksi (not ok!)}$$

When $\sigma_{st} = 20$ ksi

$$1.5(20 \times 1000) = 29 \sigma_{timber}$$

$$\sigma_{timber} = 1034.48 \text{ psi} < 1200 \text{ psi (ok!)}$$

Use $\sigma_{st} = 20$ ksi and $\sigma_{timber} = 1.03$ ksi

$$\sum F_V = 0$$

$$F_{steel} + F_{timber} = 300$$

$$\sigma_{st} A_{st} + \sigma_{timber} A_{timber} = 300$$

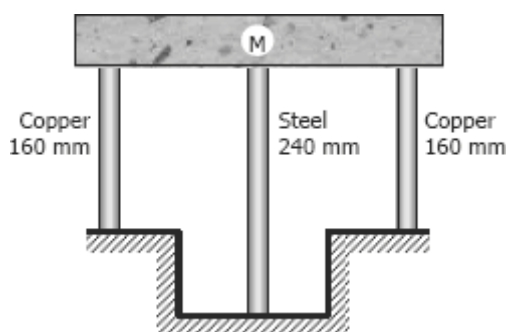
$$20[4(8t)] + 1.03(8^2) = 300$$

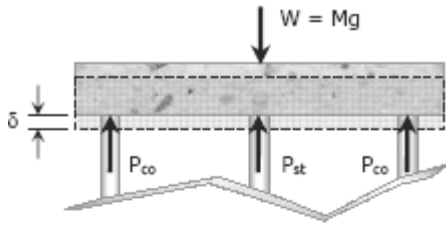
$$t = 0.365 \text{ in}$$

Problem 236

A rigid block of mass M is supported by three symmetrically spaced rods as shown in fig P-236. Each copper rod has an area of 900 mm^2 ; $E = 120 \text{ GPa}$; and the allowable stress is 70 MPa . The steel rod has an area of 1200 mm^2 ; $E = 200 \text{ GPa}$; and the allowable stress is 140 MPa . Determine the largest mass M which can be supported.

Figure P-236 and P-237



Solution 236

$$\delta_{co} = \delta_{st}$$

$$\left(\frac{\sigma L}{E}\right)_{co} = \left(\frac{\sigma L}{E}\right)_{st}$$

$$\frac{\sigma_{co}(160)}{120\,000} = \frac{\sigma_{st}(240)}{200\,000}$$

$$10\sigma_{co} = 9\sigma_{st}$$

When $\sigma_{st} = 140$ MPa

$$\sigma_{co} = \frac{9}{10}(140)$$

$$\sigma_{co} = 126 \text{ MPa} > 70 \text{ MPa (not ok!)}$$

When $\sigma_{co} = 70$ MPa

$$\sigma_{st} = \frac{10}{9}(70)$$

$$\sigma_{st} = 77.78 \text{ MPa} < 140 \text{ MPa (ok!)}$$

Use $\sigma_{co} = 70$ MPa and $\sigma_{st} = 77.78$ MPa

$$\sum F_V = 0$$

$$2P_{co} + P_{st} = W$$

$$2(\sigma_{co}A_{co}) + \sigma_{st}A_{st} = Mg$$

$$2[70(900)] + 77.78(1200) = M(9.81)$$

$$M = 22\,358.4 \text{ kg}$$

Problem 237

In Prob. 236, how should the lengths of the two identical copper rods be changed so that each material will be stressed to its allowable limit?

Solution 237

Use $\sigma_{co} = 70$ MPa and $\sigma_{st} = 140$ MPa

$$\delta_{co} = \delta_{st}$$

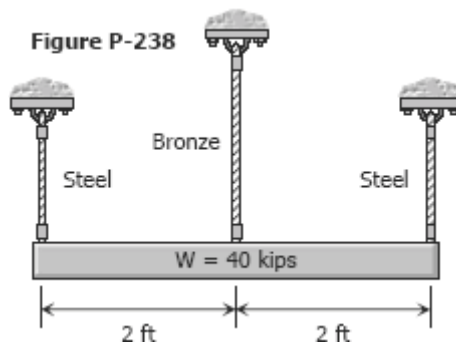
$$\left(\frac{\sigma L}{E}\right)_{co} = \left(\frac{\sigma L}{E}\right)_{st}$$

$$\frac{70L_{co}}{120\,000} = \frac{140(240)}{200\,000}$$

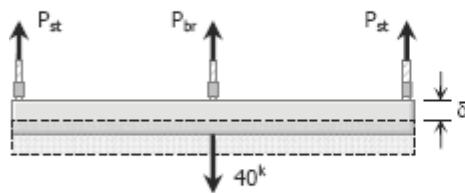
$$L_{co} = 288 \text{ mm}$$

Problem 238

The lower ends of the three bars in Fig. P-238 are at the same level before the uniform rigid block weighing 40 kips is attached. Each steel bar has a length of 3 ft, and area of 1.0 in.², and $E = 29 \times 10^6$ psi. For the bronze bar, the area is 1.5 in.² and $E = 12 \times 10^6$ psi. Determine (a) the length of the bronze bar so that the load on each steel bar is twice the load on the bronze bar, and (b) the length of the bronze that will make the steel stress twice the bronze stress.



Solution 238



(a) Condition: $P_{st} = 2P_{br}$

$$\sum F_V = 0$$

$$2P_{st} + P_{br} = 40$$

$$2(2P_{br}) + P_{br} = 40$$

$$P_{br} = 8 \text{ kips}$$

$$P_{st} = 2(8) = 16 \text{ kips}$$

$$\delta_{br} = \delta_{st}$$

$$\left(\frac{PL}{AE}\right)_{br} = \left(\frac{PL}{AE}\right)_{st}$$

$$\frac{8000L_{br}}{1.5(12 \times 10^6)} = \frac{16000(3 \times 12)}{1.0(29 \times 10^6)}$$

$$L_{br} = 44.69 \text{ in}$$

$$L_{br} = 3.72 \text{ ft}$$

(b) Condition: $\sigma_{st} = 2\sigma_{br}$

$$\sum F_V = 0$$

$$2P_{st} + P_{br} = 40$$

$$2(\sigma_{st}A_{st}) + \sigma_{br}A_{br} = 40$$

$$2[(2\sigma_{br})A_{st}] + \sigma_{br}A_{br} = 40$$

$$4\sigma_{br}(1.0) + \sigma_{br}(1.5) = 40$$

$$\sigma_{br} = 7.27 \text{ ksi}$$

$$\sigma_{st} = 2(7.27) = 14.54 \text{ ksi}$$

$$\delta_{br} = \delta_{st}$$

$$\left(\frac{\sigma L}{E}\right)_{br} = \left(\frac{\sigma L}{E}\right)_{st}$$

$$\frac{7.27(1000)L_{br}}{12 \times 10^6} = \frac{14.54(1000)(3 \times 12)}{29 \times 10^6}$$

$$L_{br} = 29.79 \text{ in}$$

$$L_{br} = 2.48 \text{ ft}$$

Problem 239

The rigid platform in Fig. P-239 has negligible mass and rests on two steel bars, each 250.00 mm long. The center bar is aluminum and 249.90 mm long. Compute the stress in the aluminum bar after the center load $P = 400 \text{ kN}$ has been applied. For each steel bar, the area is 1200 mm^2 and $E = 200 \text{ GPa}$. For the aluminum bar, the area is 2400 mm^2 and $E = 70 \text{ GPa}$.

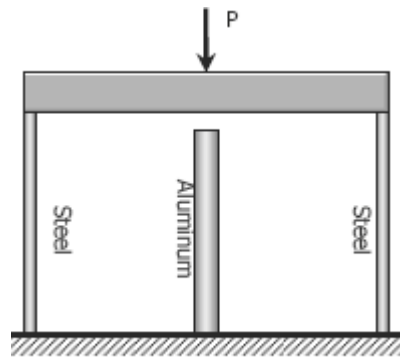
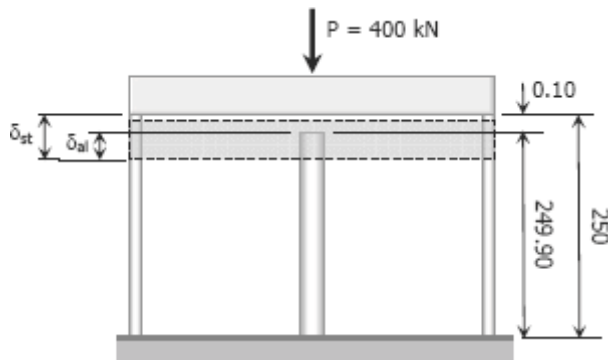


Figure P-239

Solution 239



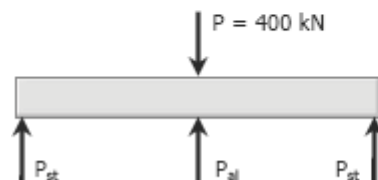
$$\delta_{st} = \delta_{al} + 0.10$$

$$\left(\frac{\sigma L}{E}\right)_{st} = \left(\frac{\sigma L}{E}\right)_{al} + 0.10$$

$$\frac{\sigma_{st}(250)}{200\,000} = \frac{\sigma_{al}(249.90)}{70\,000} + 0.10$$

$$0.00125\sigma_{st} = 0.00357\sigma_{al} + 0.10$$

$$\sigma_{st} = 2.856\sigma_{al} + 80$$



$$\sum F_V = 0$$

$$2P_{st} + P_{al} = 400\,000$$

$$2\sigma_{st}A_{st} + \sigma_{al}A_{al} = 400\,000$$

$$2(2.856\sigma_{al} + 80)1200 + \sigma_{al}(2400) = 400\,000$$

$$9254.4\sigma_{al} + 192\,000 = 400\,000$$

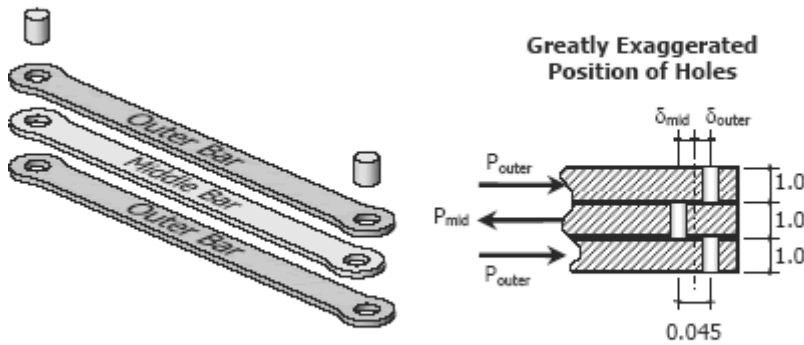
$$\sigma_{al} = 22.48 \text{ MPa}$$

Problem 240

Three steel eye-bars, each 4 in. by 1 in. in section, are to be assembled by driving rigid 7/8-in.-diameter drift pins through holes drilled in the ends of the bars. The center-line spacing between the holes is 30 ft in the two outer bars, but 0.045 in. shorter in the middle bar. Find the shearing stress developed in the drip pins. Neglect local deformation at the holes.

Solution 240

Middle bar is 0.045 inch shorter between holes than outer bars.



$$\sum F_H = 0$$

$$P_{mid} = 2P_{outer}$$

$$\delta_{outer} + \delta_{mid} = 0.045$$

$$\left(\frac{PL}{AE} \right)_{outer} + \left(\frac{PL}{AE} \right)_{mid} = 0.045$$

$$\frac{P_{outer}(30 \times 12)}{[1.0(4.0)]E} + \frac{P_{mid}(30 \times 12 - 0.045)}{[1.0(4.0)]E} = 0.045$$

$$360P_{outer} + 359.955P_{mid} = 0.18E$$

$$360P_{outer} + 359.955(2P_{outer}) = 0.18E$$

$$(\text{For steel: } E = 29 \times 10^6 \text{ psi})$$

$$1079.91P_{outer} = 0.18(29 \times 10^6)$$

$$P_{outer} = 4833.74 \text{ lb}$$

$$P_{mid} = 2(4833.74)$$

$$P_{mid} = 9667.48 \text{ lb}$$

Use shear force $V = P_{mid}$

Shearing stress of drip pins (double shear):

$$\tau = \frac{V}{A} = \frac{9667.48}{2\left[\frac{1}{4}\pi\left(\frac{7}{8}\right)^2\right]}$$

$$\tau = 8038.54 \text{ psi}$$

Problem 241

As shown in Fig. P-241, three steel wires, each 0.05 in.² in area, are used to lift a load $W = 1500 \text{ lb}$. Their unstressed lengths are 74.98 ft, 74.99 ft, and 75.00 ft. (a) What stress exists in the longest wire? (b) Determine the stress in the shortest wire if $W = 500 \text{ lb}$.

Solution 241

Let $L_1 = 74.98$ ft; $L_2 = 74.99$ ft; and $L_3 = 75.00$ ft

(a) Bring L_1 and L_2 into $L_3 = 75$ ft length:

(For steel: $E = 29 \times 10^6$ psi)

$$\delta = \frac{PL}{AE}$$

For L_1 :

$$(75 - 74.98)(12) = \frac{P_1(74.98 \times 12)}{0.05(29 \times 10^6)}$$

$$P_1 = 386.77 \text{ lb}$$

For L_2

$$(75 - 74.99)(12) = \frac{P_2(74.99 \times 12)}{0.05(29 \times 10^6)}$$

$$P_2 = 193.36 \text{ lb}$$

Let $P = P_3$ (Load carried by L_3)

$P + P_2$ (Total load carried by L_2)

$P + P_1$ (Total load carried by L_1)



Figure P-241

$$\sum F_V = 0$$

$$(P + P_1) + (P + P_2) + P = W$$

$$3P + 386.77 + 193.36 = 1500$$

$$P = 306.62 \text{ lb} = P_3$$

$$\sigma_3 = \frac{P_3}{A} = \frac{306.62}{0.05}$$

$$\sigma_3 = 6132.47 \text{ psi}$$

(b) From the above solution:

$$P_1 + P_2 = 580.13 \text{ lb} > 500 \text{ lb} \text{ (} L_3 \text{ carries no load)}$$

Bring L_1 into $L_2 = 74.99$ ft

$$\left[\delta = \frac{PL}{AE} \right] \quad (74.99 - 74.98)(12) = \frac{P_1(74.98 \times 12)}{0.05(29 \times 10^6)}$$

$$P_1 = 193.38 \text{ lb}$$

Let $P = P_2$ (Load carried by L_2)

$P + P_1$ (Total load carried by L_1)

$$\sum F_V = 0$$

$$(P + P_1) + P = 500$$

$$2P + 193.38 = 500$$

$$P = 153.31 \text{ lb}$$

$$P + P_1 = 153.31 + 193.38$$

$$P + P_1 = 346.69 \text{ lb}$$

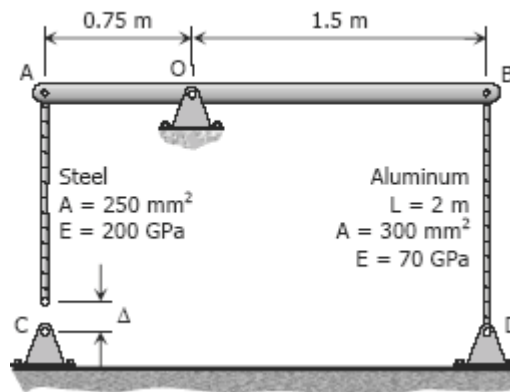
$$\sigma = \frac{P + P_1}{A} = \frac{346.69}{0.05}$$

$$\sigma = 6933.8 \text{ psi}$$

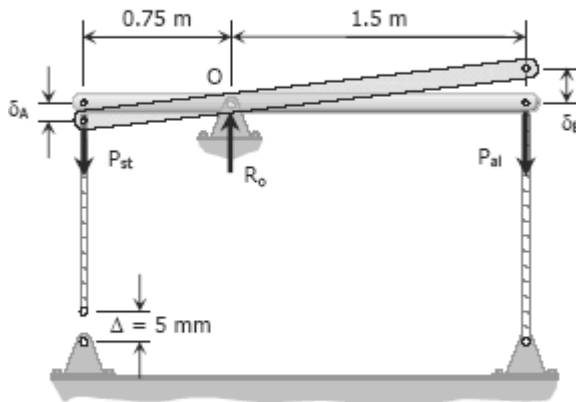
Problem 242

The assembly in Fig. P-242 consists of a light rigid bar AB, pinned at O, that is attached to the steel and aluminum rods. In the position shown, bar AB is horizontal and there is a gap, $\Delta = 5 \text{ mm}$, between the lower end of the steel rod and its pin support at C. Compute the stress in the aluminum rod when the lower end of the steel rod is attached to its support.

Figure P-242



Solution 242



$$\begin{aligned}\sum M_O &= 0 \\ 0.75P_{st} &= 1.5P_{al} \\ P_{st} &= 2P_{al} \\ \sigma_{st}A_{st} &= 2(\sigma_{al}A_{al}) \\ \sigma_{st} &= \frac{2(\sigma_{al}A_{al})}{A_{st}} \\ \sigma_{st} &= \frac{2[\sigma_{al}(300)]}{250} \\ \sigma_{st} &= 2.4\sigma_{al}\end{aligned}$$

$$\delta_{al} = \delta_B$$

By ratio and proportion:

$$\begin{aligned}\frac{\delta_A}{0.75} &= \frac{\delta_B}{1.5} \\ \delta_A &= 0.5\delta_B \\ \delta_A &= 0.5\delta_{al}\end{aligned}$$

$$\Delta = \delta_{st} + \delta_A$$

$$5 = \delta_{st} + 0.5\delta_{al}$$

$$5 = \frac{\sigma_{st}(2000 - 5)}{250(200000)} + 0.5 \left[\frac{\sigma_{al}(2000)}{300(70000)} \right]$$

$$5 = (3.99 \times 10^{-5}) \sigma_{st} + (4.76 \times 10^{-5}) \sigma_{al}$$

$$\sigma_{al} = 105000 - 0.8379\sigma_{st}$$

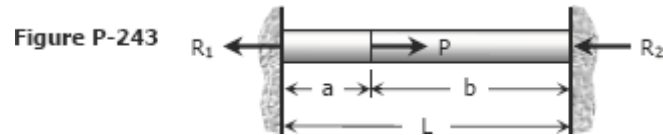
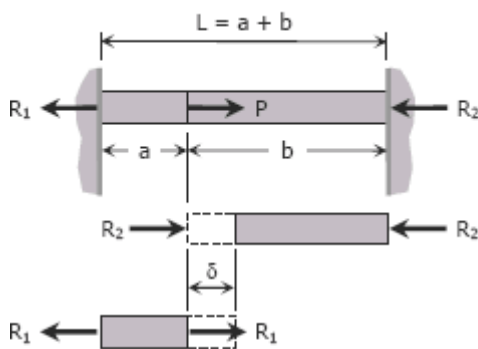
$$\sigma_{al} = 105000 - 0.8379(2.4\sigma_{al})$$

$$3.01096\sigma_{al} = 105000$$

$$\sigma_{al} = 34872.6 \text{ MPa}$$

Problem 243

A homogeneous rod of constant cross section is attached to unyielding supports. It carries an axial load P applied as shown in Fig. P-243. Prove that the reactions are given by $R_1 = Pb/L$ and $R_2 = Pa/L$.

**Solution 243**

$$\sum F_H = 0$$

$$R_1 + R_2 = P$$

$$R_2 = P - R_1$$

$$\delta_1 = \delta_2 = \delta$$

$$\left(\frac{PL}{AE} \right)_1 = \left(\frac{PL}{AE} \right)_2$$

$$\frac{R_1 a}{AE} = \frac{R_2 b}{AE}$$

$$R_1 a = R_2 b$$

$$R_1 a = (P - R_1) b$$

$$R_1 a = Pb - R_1 b$$

$$R_1(a + b) = Pb$$

$$R_1 L = Pb$$

$$R_1 = Pb/L \quad \text{ok!}$$

$$R_2 = P - Pb/L$$

$$R_2 = \frac{P(L - b)}{L}$$

$$R_2 = Pa/L \quad \text{ok!}$$

Problem 244

A homogeneous bar with a cross sectional area of 500 mm^2 is attached to rigid supports. It carries the axial loads $P_1 = 25 \text{ kN}$ and $P_2 = 50 \text{ kN}$, applied as shown in Fig. P-244. Determine the stress in segment BC. (Hint: Use the results of Prob. 243, and compute the reactions caused by P_1 and P_2 acting separately. Then use the principle of superposition to compute the reactions when both loads are applied.)

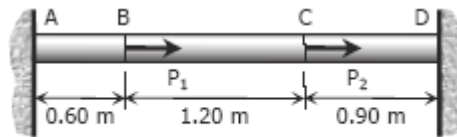
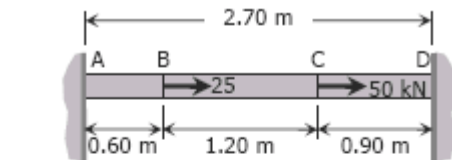


Figure P-244

Solution 244



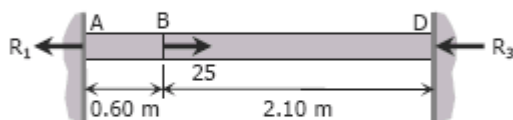
From the result of Prob. 243:

$$R_1 = 25(2.10)/2.70$$

$$R_1 = 19.44 \text{ kN}$$

$$R_2 = 50(0.90)/2.70$$

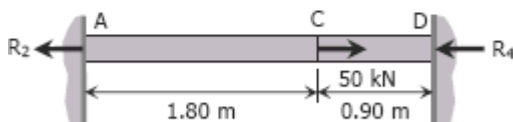
$$R_2 = 16.67 \text{ kN}$$



$$R_A = R_1 + R_2$$

$$R_A = 19.44 + 16.67$$

$$R_A = 36.11 \text{ kN}$$



For segment BC

$$P_{BC} + 25 = R_A$$

$$P_{BC} + 25 = 36.11$$

$$P_{BC} = 11.11 \text{ kN}$$



$$\sigma_{BC} = \frac{P_{BC}}{A} = \frac{11.11(1000)}{500}$$

$$\sigma_{BC} = 22.22 \text{ MPa}$$

Problem 245

The composite bar in Fig. P-245 is firmly attached to unyielding supports. Compute the stress in each material caused by the application of the axial load $P = 50$ kips.

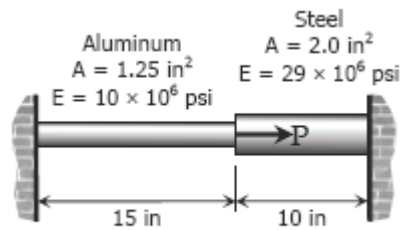


Figure P-245 and P-246

Solution 245

$$\begin{aligned}\sum F_H &= 0 \\ R_1 + R_2 &= 50\,000 \\ R_1 &= 50\,000 - R_2\end{aligned}$$

$$\begin{aligned}\delta_{al} &= \delta_{st} = \delta \\ \left(\frac{PL}{AE}\right)_{al} &= \left(\frac{PL}{AE}\right)_{st} \\ \frac{R_1(15)}{1.25(10 \times 10^6)} &= \frac{R_2(10)}{2.0(29 \times 10^6)} \\ R_2 &= 6.96R_1 \\ R_2 &= 6.96(50\,000 - R_2) \\ 7.96R_2 &= 348\,000 \\ R_2 &= 43\,718.59 \text{ lb}\end{aligned}$$

$$\sigma_{st} = \frac{R_2}{A_{st}} = \frac{43\,718.59}{2.0}$$

$$\sigma_{st} = 21\,859.30 \text{ psi}$$

$$R_1 = 50\,000 - 43\,718.59$$

$$R_1 = 6\,281.41 \text{ lb}$$

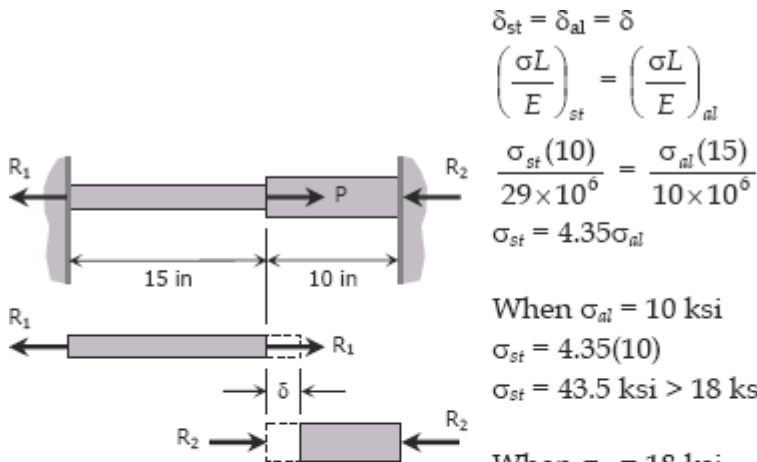
$$\sigma_{al} = \frac{R_1}{A_{al}} = \frac{6\,281.41}{1.25}$$

$$\sigma_{al} = 5\,025.12 \text{ psi}$$

Problem 246

Referring to the composite bar in Prob. 245, what maximum axial load P can be applied if the allowable stresses are 10 ksi for aluminum and 18 ksi for steel.

Solution 246



$$\delta_{st} = \delta_{al} = \delta$$

$$\left(\frac{\sigma L}{E} \right)_{st} = \left(\frac{\sigma L}{E} \right)_{al}$$

$$\frac{\sigma_{st}(10)}{29 \times 10^6} = \frac{\sigma_{al}(15)}{10 \times 10^6}$$

$$\sigma_{st} = 4.35 \sigma_{al}$$

When $\sigma_{al} = 10$ ksi

$$\sigma_{st} = 4.35(10)$$

$$\sigma_{st} = 43.5 \text{ ksi} > 18 \text{ ksi (not ok!)}$$

When $\sigma_{st} = 18$ ksi

$$18 = 4.35 \sigma_{al}$$

$$\sigma_{al} = 4.14 \text{ ksi} < 10 \text{ ksi (ok!)}$$

Use $\sigma_{al} = 4.14$ ksi and $\sigma_{st} = 18$ ksi

$$\sum F_H = 0$$

$$P = R_1 + R_2$$

$$P = \sigma_{al} A_{al} + \sigma_{st} A_{st}$$

$$P = 4.14(1.25) + 18(2.0)$$

$$P = 41.17 \text{ kips}$$

Problem 247

The composite bar in Fig. P-247 is stress-free before the axial loads P_1 and P_2 are applied. Assuming that the walls are rigid, calculate the stress in each material if $P_1 = 150$ kN and $P_2 = 90$ kN.

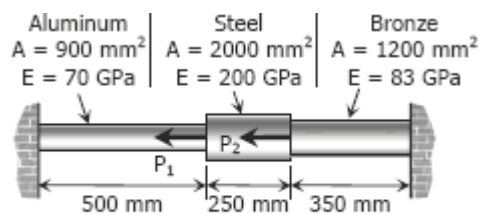
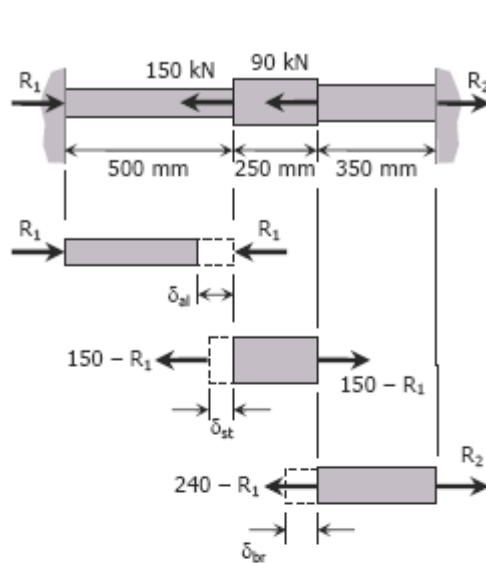


Figure P-247 and P-248

Solution 247

From the FBD of each material shown:



δ_{al} is shortening

δ_{st} and δ_{br} are lengthening

$$R_2 = 240 - R_1$$

$$P_{al} = R_1$$

$$P_{st} = 150 - R_1$$

$$P_{br} = R_2 = 240 - R_1$$

$$\delta_{al} = \delta_{st} + \delta_{br}$$

$$\left(\frac{PL}{AE}\right)_{al} = \left(\frac{PL}{AE}\right)_{st} + \left(\frac{PL}{AE}\right)_{br}$$

$$\frac{R_1(500)}{900(70\,000)} = \frac{(150 - R_1)(250)}{2000(200\,000)} + \frac{(240 - R_1)(350)}{1200(83\,000)}$$

$$\frac{R_1}{126\,000} = \frac{150 - R_1}{1600\,000} + \frac{(240 - R_1)7}{1992\,000}$$

$$\frac{1}{63} R_1 = \frac{1}{800} (150 - R_1) + \frac{7}{996} (240 - R_1)$$

$$\left(\frac{1}{63} + \frac{1}{800} + \frac{7}{996}\right) R_1 = \frac{1}{800} (150) + \frac{7}{996} (240)$$

$$R_1 = 77.60 \text{ kN}$$

$$P_{al} = R_1 = 77.60 \text{ kN}$$

$$P_{st} = 150 - 77.60 = 72.40 \text{ kN}$$

$$P_{br} = 240 - 77.60 = 162.40 \text{ kN}$$

$$\sigma = P/A$$

$$\begin{aligned}\sigma_{al} &= 77.60(1000)/900 \\ &= \mathbf{86.22 \text{ MPa}}\end{aligned}$$

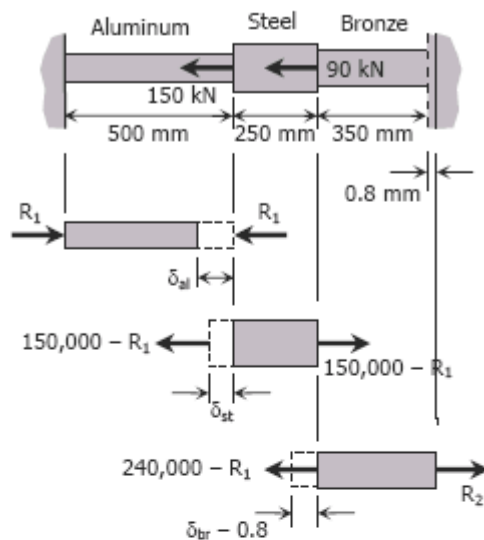
$$\begin{aligned}\sigma_{st} &= 72.40(1000)/2000 \\ &= \mathbf{36.20 \text{ MPa}}\end{aligned}$$

$$\begin{aligned}\sigma_{br} &= 162.40(1000)/1200 \\ &= \mathbf{135.33 \text{ MPa}}\end{aligned}$$

Problem 248

Solve Prob. 247 if the right wall yields 0.80 mm.

Solution 248



$$\delta_{al} = \delta_{st} + (\delta_{br} + 0.8)$$

$$\left(\frac{PL}{AE}\right)_{al} = \left(\frac{PL}{AE}\right)_{st} + \left(\frac{PL}{AE}\right)_{br} + 0.8$$

$$\frac{R_1(500)}{900(70000)} = \frac{(150000 - R_1)(250)}{2000(200000)} + \frac{(240000 - R_1)(350)}{1200(83000)} + 0.8$$

$$\frac{R_1}{126000} = \frac{150000 - R_1}{1600000} + \frac{7(240000 - R_1)}{1992000} + 0.8$$

$$\frac{1}{63} R_1 = \frac{1}{800} (150000 - R_1) + \frac{7}{996} (240000 - R_1) + 1600$$

$$\left(\frac{1}{63} + \frac{1}{800} + \frac{7}{996}\right) R_1 = \frac{1}{800} (150000) + \frac{7}{996} (240000) + 1600$$

$$R_1 = 143\,854\text{ N} = 143.854\text{ kN}$$

$$P_{al} = R_1 = 143.854\text{ kN}$$

$$P_{st} = 150 - R_1 = 150 - 143.854 = 6.146\text{ kN}$$

$$P_{br} = R_2 = 240 - R_1 = 240 - 143.854 = 96.146\text{ kN}$$

$$\sigma = P/A$$

$$\sigma_{al} = 143.854(1000)/900 = 159.84\text{ MPa}$$

$$\sigma_{st} = 6.146(1000)/2000 = 3.073\text{ MPa}$$

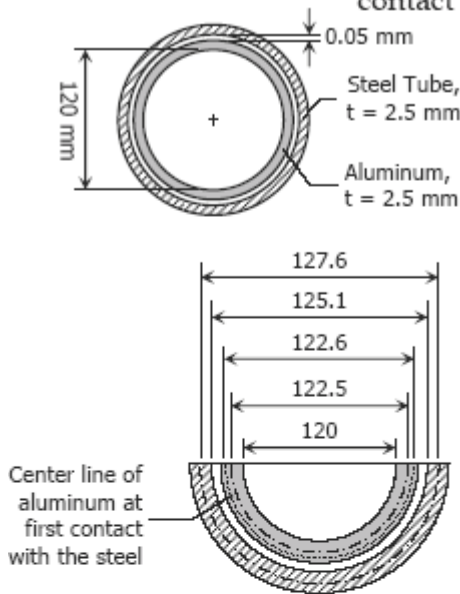
$$\sigma_{br} = 96.146(1000)/1200 = 80.122\text{ MPa}$$

Problem 249

There is a radial clearance of 0.05 mm when a steel tube is placed over an aluminum tube. The inside diameter of the aluminum tube is 120 mm, and the wall thickness of each tube is 2.5 mm. Compute the contact pressure and tangential stress in each tube when the aluminum tube is subjected to an internal pressure of 5.0 MPa.

Solution 249

Internal pressure of aluminum tube to cause contact with the steel:



$$\delta_{al} = \left(\frac{\sigma L}{E} \right)_{al}$$

$$\pi(122.6 - 122.5) = \frac{\sigma_1(122.5\pi)}{70000}$$

$$\sigma_1 = 57.143 \text{ MPa}$$

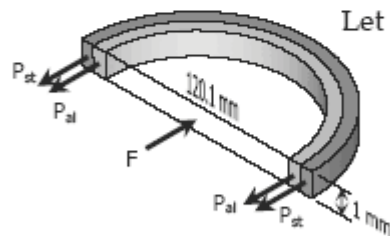
$$\frac{p_1 D}{2t} = 57.143$$

$$\frac{p_1(120)}{2(2.5)} = 57.143$$

$$p_1 = 2.381 \text{ MPa}$$

→ pressure that

causes aluminum to contact with the steel, further increase of pressure will expand both aluminum and steel tubes.



FBD for $p \geq 2.381 \text{ MPa}$

Let p_c = contact pressure between steel and aluminum tubes

$$2P_{st} + 2P_{al} = F$$

$$2P_{st} + 2P_{al} = 5.0(120.1)(1)$$

$$P_{st} + P_{al} = 300.25 \rightarrow \text{Equation (1)}$$

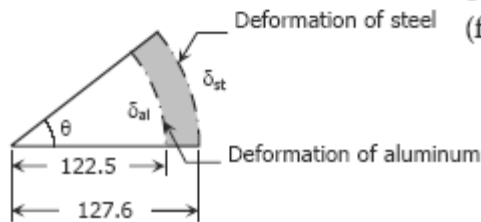
The relationship of deformations is (from the figure):

$$\delta_{st} = 127.6\theta$$

$$\theta = \delta_{st} / 127.6$$

$$\delta_{al} = 122.5\theta$$

$$\delta_{al} = 122.5(\delta_{st} / 127.6)$$



Geometric relation of deformations

$$\delta_{al} = 0.96\delta_{st}$$

$$\left(\frac{PL}{AE} \right)_{al} = 0.96 \left(\frac{PL}{AE} \right)_{st}$$

$$\frac{P_{al}(122.5\pi)}{2.5(70000)} = 0.96 \left[\frac{P_{st}(127.6\pi)}{2.5(200000)} \right]$$

$$P_{al} = 0.35P_{st} \rightarrow \text{Equation (2)}$$

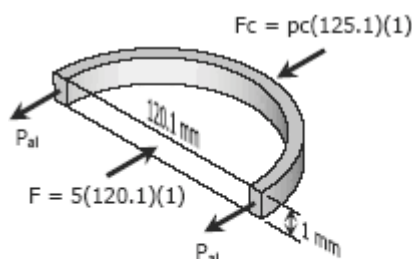
From Equation (1)

$$P_{st} + 0.35P_{st} = 300.25$$

$$P_{st} = 222.41 \text{ N}$$

$$P_{al} = 0.35(222.41)$$

$$P_{al} = 77.84 \text{ N}$$



Contact Force

$$F_c + 2P_{st} = F$$

$$p_c(125.1)(1) + 2(77.84) = 5(120.1)(1)$$

$$p_c = 3.56 \text{ MPa}$$

Problem 250

In the assembly of the bronze tube and steel bolt shown in Fig. P-250, the pitch of the bolt thread is $p = 1/32$ in.; the cross-sectional area of the bronze tube is 1.5 in.^2 and of steel bolt is $\frac{3}{4} \text{ in.}^2$. The nut is turned until there is a compressive stress of 4000 psi in the bronze tube. Find the stresses if the nut is given one additional turn. How many turns of the nut will reduce these stresses to zero? Use $E_{br} = 12 \times 10^6$ psi and $E_{st} = 29 \times 10^6$ psi.

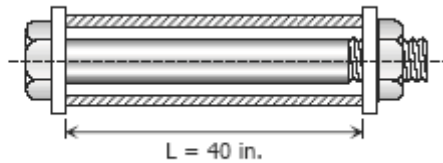


Figure P-250

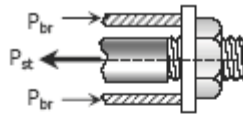
Solution 250

$$P_{st} = P_{br}$$

$$A_{st} \sigma_{st} = P_{br} \sigma_{br}$$

$$\frac{3}{4} \sigma_{st} = 1.5 \sigma_{br}$$

$$\sigma_{st} = 2 \sigma_{br}$$



For one turn of the nut:

$$\delta_{st} + \delta_{br} = \frac{1}{32}$$

$$\left(\frac{\sigma L}{E} \right)_{st} + \left(\frac{\sigma L}{E} \right)_{br} = \frac{1}{32}$$

$$\frac{\sigma_{st}(40)}{29 \times 10^6} + \frac{\sigma_{br}(40)}{12 \times 10^6} = \frac{1}{32}$$

$$\sigma_{st} + \frac{29}{12} \sigma_{br} = 22\,656.25$$

$$2\sigma_{br} + \frac{29}{12} \sigma_{br} = 22\,656.25$$

$$\sigma_{br} = 5129.72 \text{ psi}$$

$$\sigma_{st} = 2(5129.72) = 10\,259.43 \text{ psi}$$

Initial stresses:

$$\sigma_{br} = 4000 \text{ psi}$$

$$\sigma_{st} = 2(4000) = 8000 \text{ psi}$$

Final stresses:

$$\sigma_{br} = 4000 + 5129.72 = 9129.72 \text{ psi}$$

$$\sigma_{st} = 2(9129.72) = 18\,259.4 \text{ psi}$$

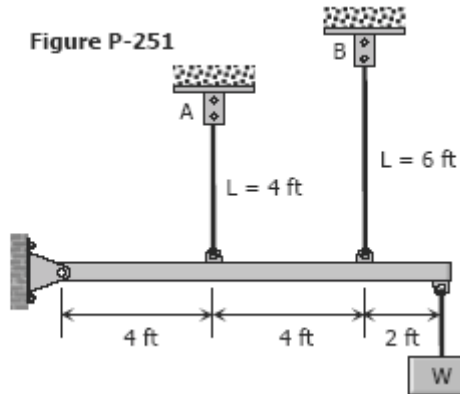
Required number of turns to reduce σ_{br} to zero:

$$n = \frac{9129.72}{5129.72} = 1.78 \text{ turns}$$

The nut must be turned back by 1.78 turns

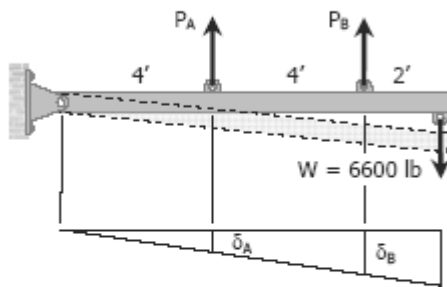
Problem 251

The two vertical rods attached to the light rigid bar in Fig. P-251 are identical except for length. Before the load W was attached, the bar was horizontal and the rods were stress-free. Determine the load in each rod if $W = 6600$ lb.



Solution 251

$$\begin{aligned}\sum M_{\text{pin support}} &= 0 \\ 4P_A + 8P_B &= 10(6600) \\ P_A + 2P_B &= 16500 \quad \rightarrow (1)\end{aligned}$$



$$\begin{aligned}\frac{\delta_A}{4} &= \frac{\delta_B}{8} \\ \delta_A &= 0.5\delta_B \\ \left(\frac{PL}{AE}\right)_A &= 0.5\left(\frac{PL}{AE}\right)_B \\ \frac{P_A(4)}{AE} &= \frac{0.5P_B(6)}{AE} \\ P_A &= 0.75P_B\end{aligned}$$

$$\begin{aligned}\text{From equation (1)} \\ 0.75P_B + 2P_B &= 16500 \\ P_B &= 6000 \text{ lb}\end{aligned}$$

$$\begin{aligned}P_A &= 0.75(6000) \\ P_A &= 4500 \text{ lb}\end{aligned}$$

Problem 252

The light rigid bar ABCD shown in Fig. P-252 is pinned at B and connected to two vertical rods. Assuming that the bar was initially horizontal and the rods stress-free, determine the stress in each rod after the load $P = 20$ kips is applied.

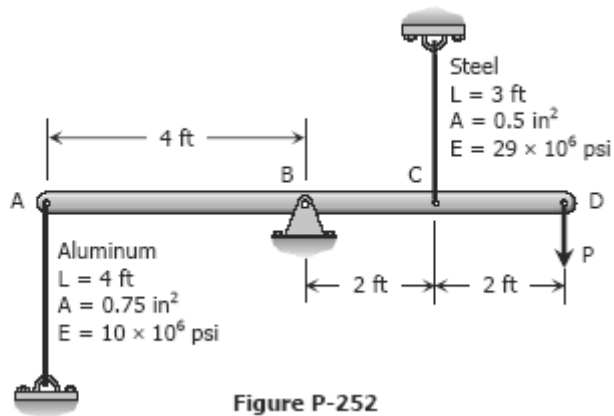
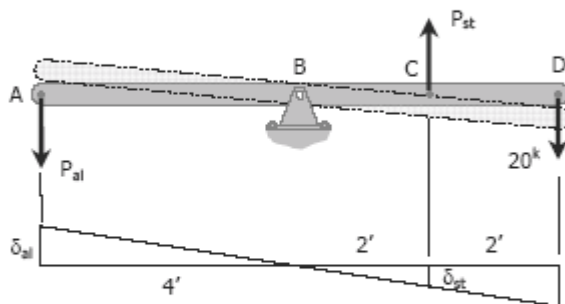


Figure P-252

Solution 252

$$\begin{aligned}\sum M_B &= 0 \\ 4P_{al} + 2P_{st} &= 4(20\,000) \\ 4(\sigma_{al} A_{al}) + 2\sigma_{st} A_{st} &= 80\,000 \\ 4[\sigma_{al} (0.75)] + 2[\sigma_{st} (0.5)] &= 80\,000 \\ 3\sigma_{al} + \sigma_{st} &= 80\,000 \quad \rightarrow (1) \\ \frac{\delta_{st}}{2} &= \frac{\delta_{al}}{4}\end{aligned}$$



$$\begin{aligned}\delta_{st} &= 0.5\delta_{al} \\ \left(\frac{\sigma L}{E}\right)_{st} &= 0.5\left(\frac{\sigma L}{E}\right)_{al} \\ \frac{\sigma_{st}(3)}{29 \times 10^6} &= 0.5\left[\frac{P_{al}(4)}{10 \times 10^6}\right] \\ \sigma_{st} &= \frac{29}{15} \sigma_{al}\end{aligned}$$

From equation (1)

$$\begin{aligned}3\sigma_{al} + \frac{29}{15} \sigma_{al} &= 80\,000 \\ \sigma_{al} &= 16\,216.22 \text{ psi} \\ \sigma_{al} &= 16.22 \text{ ksi}\end{aligned}$$

$$\begin{aligned}\sigma_{st} &= \frac{29}{15} (16.22) \\ \sigma_{st} &= 31.35 \text{ ksi}\end{aligned}$$

Problem 253

As shown in Fig. P-253, a rigid beam with negligible weight is pinned at one end and attached to two vertical rods. The beam was initially horizontal before the load $W = 50$ kips was applied. Find the vertical movement of W .

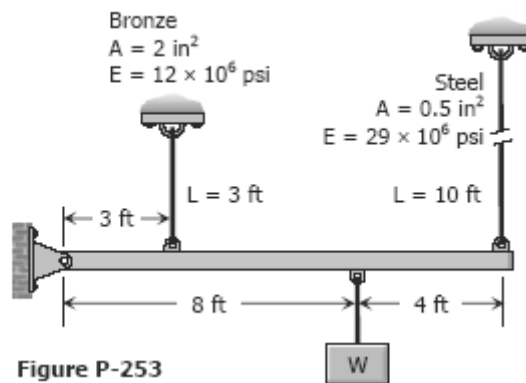


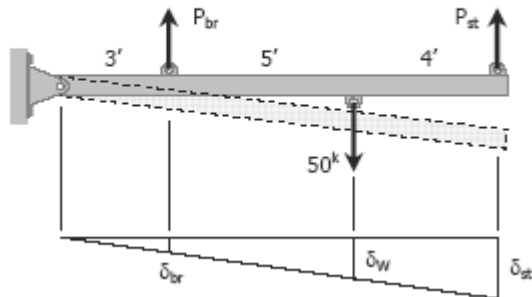
Figure P-253

Solution 253

$$\sum M_{pin \text{ support}} = 0$$

$$3P_{br} + 12P_{st} = 8(50\,000)$$

$$3P_{br} + 12P_{st} = 400\,000 \quad \rightarrow (1)$$



$$\frac{\delta_{st}}{12} = \frac{\delta_{br}}{3}; \delta_{st} = 4\delta_{br}$$

$$\left(\frac{PL}{AE}\right)_{st} = 4\left(\frac{PL}{AE}\right)_{br}$$

$$\frac{P_{st}(10)}{0.5(29 \times 10^6)} = 4\left[\frac{P_{br}(3)}{2(12 \times 10^6)}\right]$$

$$P_{st} = 0.725P_{br}$$

From equation (1)

$$3P_{br} + 12(0.725P_{br}) = 400\,000$$

$$P_{br} = 34\,188.03 \text{ lb}$$

$$\delta_{br} = \left(\frac{PL}{AE}\right)_{br} = \frac{34\,188.03(3 \times 12)}{2(12 \times 10^6)}$$

$$\delta_{br} = 0.0513 \text{ in}$$

$$\frac{\delta_W}{8} = \frac{\delta_{br}}{3}$$

$$\delta_W = \frac{8}{3} \delta_{br}$$

$$\delta_W = \frac{8}{3} (0.0513)$$

$$\delta_W = \mathbf{0.1368 \text{ in}}$$

Check by δ_{st} :

$$P_{st} = 0.725 P_{br} = 0.725(34\,188.03)$$

$$P_{st} = 24\,786.32 \text{ lb}$$

$$\delta_{st} = \left(\frac{PL}{AE} \right)_{st}$$

$$\delta_{st} = \frac{24\,786.32(10 \times 12)}{0.5(29 \times 10^6)}$$

$$\delta_{st} = 0.2051 \text{ in}$$

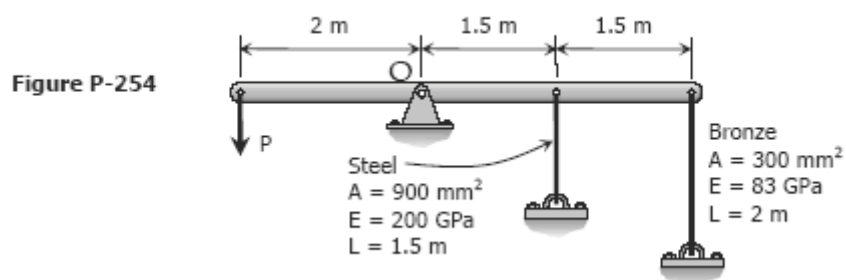
$$\frac{\delta_W}{8} = \frac{\delta_{st}}{12}$$

$$\delta_W = \frac{2}{3} \delta_{st}$$

$$\delta_W = \frac{2}{3} (0.2051) = \mathbf{0.1368 \text{ in}} \quad \text{ok!}$$

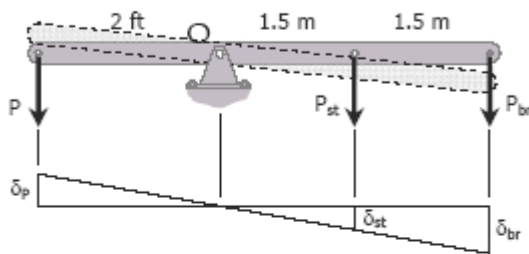
Problem 254

As shown in Fig. P-254, a rigid bar with negligible mass is pinned at O and attached to two vertical rods. Assuming that the rods were initially stress-free, what maximum load P can be applied without exceeding stresses of 150 MPa in the steel rod and 70 MPa in the bronze rod.



Solution 254

$$\begin{aligned}\sum M_O &= 0 \\ 2P &= 1.5P_{st} + 3P_{br} \\ 2P &= 1.5(\sigma_{st}A_{st}) + 3(\sigma_{br}A_{br}) \\ 2P &= 1.5[\sigma_{st}(900)] + 3[\sigma_{br}(300)] \\ 2P &= 1350\sigma_{st} + 900\sigma_{br} \\ P &= 675\sigma_{st} + 450\sigma_{br}\end{aligned}$$



$$\begin{aligned}\frac{\delta_{br}}{3} &= \frac{\delta_{st}}{1.5} \\ \delta_{br} &= 2\delta_{st} \\ \left(\frac{\sigma L}{E}\right)_{br} &= 2\left(\frac{\sigma L}{E}\right)_{st} \\ \frac{\sigma_{br}(2)}{83} &= 2\left[\frac{\sigma_{st}(1.5)}{200}\right] \\ \sigma_{br} &= 0.6225\sigma_{st}\end{aligned}$$

When $\sigma_{st} = 150 \text{ MPa}$
 $\sigma_{br} = 0.6225(150)$
 $\sigma_{br} = 93.375 \text{ MPa} > 70 \text{ MPa (not ok!)}$

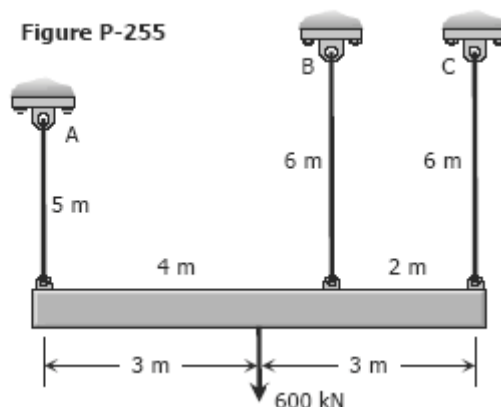
When $\sigma_{br} = 70 \text{ MPa}$
 $70 = 0.6225\sigma_{st}$
 $\sigma_{st} = 112.45 \text{ MPa} < 150 \text{ MPa (ok!)}$

Use $\sigma_{st} = 112.45 \text{ MPa}$ and $\sigma_{br} = 70 \text{ MPa}$

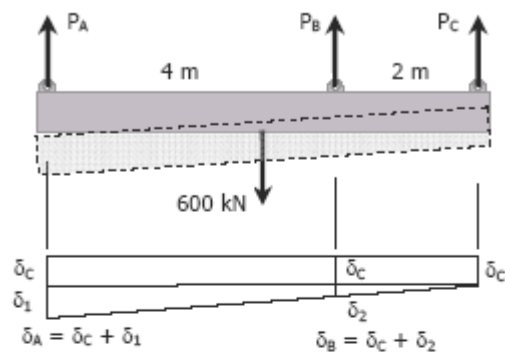
$$\begin{aligned}P &= 675\sigma_{st} + 450\sigma_{br} \\ P &= 675(112.45) + 450(70) \\ P &= 107\,403.75 \text{ N} \\ P &= \mathbf{107.4 \text{ kN}}\end{aligned}$$

Problem 255

Shown in Fig. P-255 is a section through a balcony. The total uniform load of 600 kN is supported by three rods of the same area and material. Compute the load in each rod. Assume the floor to be rigid, but note that it does not necessarily remain horizontal.



Solution 255



$$\delta_B = \delta_C + \delta_2$$

$$\delta_2 = \delta_B - \delta_C$$

$$\frac{\delta_1}{6} = \frac{\delta_2}{2}; \delta_1 = 3\delta_2$$

$$\delta_A = \delta_C + \delta_1 = \delta_C + 3\delta_2$$

$$\delta_A = \delta_C + 3(\delta_B - \delta_C)$$

$$\delta_A = 3\delta_B - 2\delta_C$$

$$\left(\frac{PL}{AE}\right)_A = 3\left(\frac{PL}{AE}\right)_B - 2\left(\frac{PL}{AE}\right)_C$$

$$\frac{P_A(5)}{AE} = \frac{3P_B(6)}{AE} - \frac{2P_C(6)}{AE}$$

$$P_A = 3.6P_B - 2.4P_C \quad \rightarrow (1)$$

$$[\sum F_V = 0] \quad P_A + P_B + P_C = 600$$

$$(3.6P_B - 2.4P_C) + P_B + P_C = 600$$

$$4.6P_B - 1.4P_C = 600 \quad \rightarrow (2)$$

$$[\sum M_A = 0] \quad 4P_B + 6P_C = 3(600)$$

$$P_B = 450 - 1.5P_C \quad \rightarrow (3)$$

Substitute $P_B = 450 - 1.5P_C$ to (2)

$$4.6(450 - 1.5P_C) - 1.4P_C = 600$$

$$8.3P_C = 1470$$

$$P_C = 177.11 \text{ kN}$$

From (3)

$$P_B = 450 - 1.5(177.11)$$

$$P_B = 184.34 \text{ kN}$$

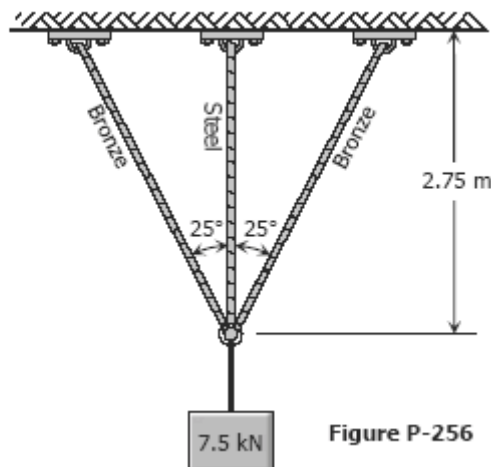
From (1)

$$P_A = 3.6(184.34) - 2.4(177.11)$$

$$P_A = 238.56 \text{ kN}$$

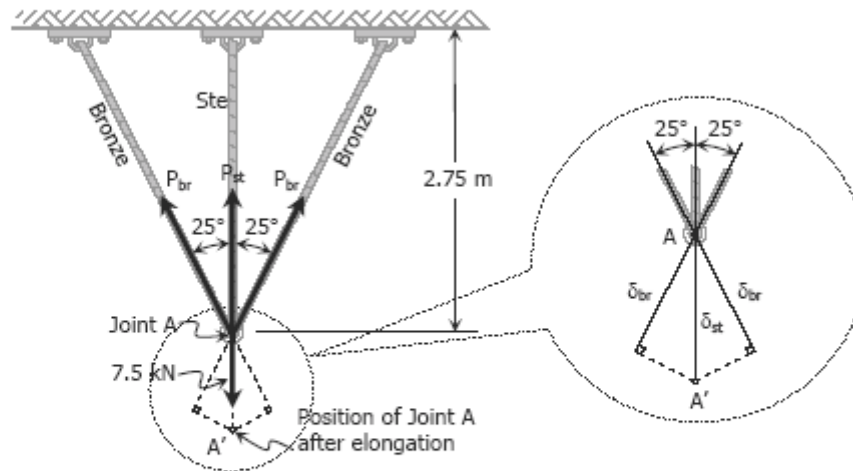
Problem 256

Three rods, each of area 250 mm^2 , jointly support a 7.5 kN load, as shown in Fig. P-256. Assuming that there was no slack or stress in the rods before the load was applied, find the stress in each rod. Use $E_{st} = 200 \text{ GPa}$ and $E_{br} = 83 \text{ GPa}$.



Solution 256

$$\cos 25^\circ = \frac{2.75}{L_{br}}; L_{br} = 3.03 \text{ m}$$



$$\sum F_V = 0$$

$$2P_{br} \cos 25^\circ + P_{st} = 7.5(1000)$$

$$P_{st} = 7500 - 1.8126P_{br}$$

$$\sigma_{st}A_{st} = 7500 - 1.8126\sigma_{br}A_{br}$$

$$\sigma_{st}(250) = 7500 - 1.8126[\sigma_{br}(250)]$$

$$\sigma_{st} = 30 - 1.8126 \sigma_{br} \rightarrow (1)$$

$$\cos 25^\circ = \frac{\delta_{br}}{\delta_{st}}$$

$$\delta_{br} = 0.9063 \delta_{st}$$

$$\left(\frac{\sigma L}{E}\right)_{br} = 0.9063 \left(\frac{\sigma L}{E}\right)_{st}$$

$$\frac{\sigma_{br}(3.03)}{83} = 0.9063 \left[\frac{\sigma_{st}(2.75)}{200} \right]$$

$$\sigma_{br} = 0.3414 \sigma_{st} \rightarrow (2)$$

From equation (1)

$$\sigma_{st} = 30 - 1.8126(0.3414\sigma_{st})$$

$$\sigma_{st} = 18.53 \text{ MPa}$$

From equation (2)

$$\sigma_{br} = 0.3414(18.53)$$

$$\sigma_{br} = 6.33 \text{ MPa}$$

Problem 257

Three bars AB, AC, and AD are pinned together as shown in Fig. P-257. Initially, the assembly is stressfree. Horizontal movement of the joint at A is prevented by a short horizontal strut AE. Calculate the stress in each bar and the force in the strut AE when the assembly is used to support the load $W = 10$ kips. For each steel bar, $A = 0.3 \text{ in.}^2$ and $E = 29 \times 10^6 \text{ psi}$. For the aluminum bar, $A = 0.6 \text{ in.}^2$ and $E = 10 \times 10^6 \text{ psi}$.

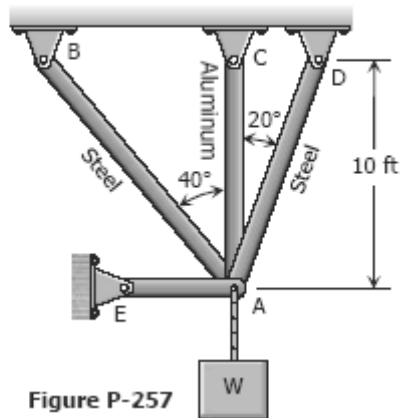
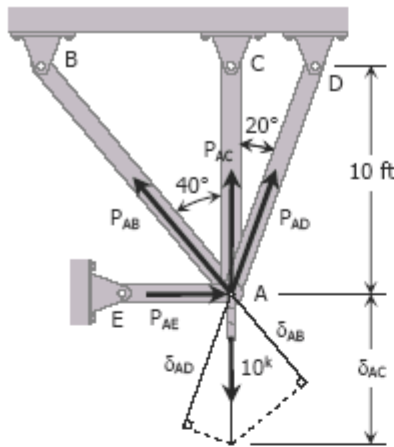


Figure P-257

Solution 257

$$\cos 40^\circ = 10 / L_{AB}; L_{AB} = 13.05 \text{ ft}$$

$$\cos 20^\circ = 10 / L_{AD}; L_{AD} = 10.64 \text{ ft}$$



$$\sum F_V = 0$$

$$P_{AB} \cos 40^\circ + P_{AC} + P_{AD} \cos 20^\circ = 10(1000)$$

$$0.7660P_{AB} + P_{AC} + 0.9397P_{AD} = 10\,000 \rightarrow (1)$$

$$\delta_{AB} = \cos 40^\circ \delta_{AC} = 0.7660 \delta_{AC}$$

$$\left(\frac{PL}{AE} \right)_{AB} = 0.7660 \left(\frac{PL}{AE} \right)_{AC}$$

$$\frac{P_{AB}(13.05)}{0.3(29 \times 10^6)} = 0.7660 \left[\frac{P_{AC}(10)}{0.6(10 \times 10^6)} \right]$$

$$P_{AB} = 0.8511P_{AC} \rightarrow (2)$$

$$\delta_{AD} = \cos 20^\circ \delta_{AC} = 0.9397 \delta_{AC}$$

$$\left(\frac{PL}{AE} \right)_{AD} = 0.9397 \left(\frac{PL}{AE} \right)_{AC}$$

$$\frac{P_{AD}(10.64)}{0.3(29 \times 10^6)} = 0.9397 \left[\frac{P_{AC}(10)}{0.6(10 \times 10^6)} \right]$$

$$P_{AD} = 1.2806P_{AC} \rightarrow (3)$$

Substitute P_{AB} of (2) and P_{AD} of (3) to (1)

$$0.7660(0.8511P_{AC}) + P_{AC} + 0.9397(1.2806P_{AC}) = 10\,000$$

$$2.8553P_{AC} = 10\,000$$

$$P_{AC} = 3\,502.23 \text{ lb}$$

$$P_{AB} = 0.8511(3\,502.23) \rightarrow \text{from (2)}$$

$$P_{AB} = 2\,980.75 \text{ lb}$$

$$P_{AD} = 1.2806(3\,502.23) \rightarrow \text{from (3)}$$

$$P_{AD} = 4\,484.96 \text{ lb}$$

Stresses:

$$\sigma = P/A$$

$$\sigma_{AB} = 2980.75/0.3 = 9\,935.83 \text{ psi}$$

$$\sigma_{AC} = 3502.23/0.6 = 5\,837.05 \text{ psi}$$

$$\sigma_{AD} = 4484.96/0.3 = 14\,949.87 \text{ psi}$$

$$\sum F_H = 0$$

$$P_{AE} + P_{AD} \sin 20^\circ = P_{AB} \sin 40^\circ$$

$$P_{AE} = 2\,980.75 \sin 40^\circ - 4\,484.96 \sin 20^\circ$$

$$P_{AE} = 382.04 \text{ lb}$$

Thermal Stress

Temperature changes cause the body to expand or contract. The amount δ_T , is given by

$$\delta_T = \alpha L (T_f - T_i) = \alpha L \Delta T$$

where α is the coefficient of thermal expansion in $\text{m/m}^\circ\text{C}$, L is the length in meter, and T_i and T_f are the initial and final temperatures, respectively in $^\circ\text{C}$.

For steel, $\alpha = 11.25 \times 10^{-6} / ^\circ\text{C}$.

If temperature deformation is permitted to occur freely, no load or stress will be induced in the structure. In some cases where temperature deformation is not permitted, an internal stress is created. The internal stress created is termed as thermal stress.

For a homogeneous rod mounted between unyielding supports as shown, the thermal stress is computed as:



deformation due to temperature changes;

$$\delta_T = \alpha L \Delta T$$

deformation due to equivalent axial stress;

$$\delta_P = \frac{PL}{AE} = \frac{\sigma L}{E}$$

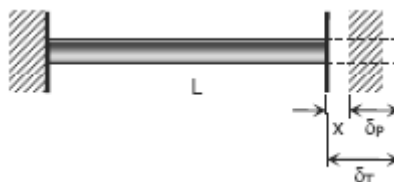
$$\delta_T = \delta_P$$

$$\alpha L \Delta T = \frac{\sigma L}{E}$$

$$\sigma = E \alpha \Delta T$$

where σ is the thermal stress in MPa and E is the modulus of elasticity of the rod in MPa.

If the wall yields a distance of x as shown, the following calculations will be made:



$$\delta_T = x + \delta_P$$

$$\alpha L \Delta T = x \frac{\sigma L}{E}$$

where σ represents the thermal stress.

Take note that as the temperature rises above the normal, the rod will be in compression, and if the temperature drops below the normal, the rod is in tension.

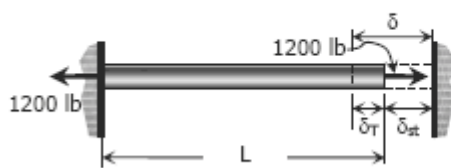
Solved Problems in Thermal Stress

Problem 261

A steel rod with a cross-sectional area of 0.25 in^2 is stretched between two fixed points. The tensile load at 70°F is 1200 lb. What will be the stress at 0°F ? At what temperature will the stress be zero? Assume $\alpha = 6.5 \times 10^{-6} \text{ in / (in}\cdot^\circ\text{F)}$ and $E = 29 \times 10^6 \text{ psi}$.

Solution 261

For the stress at 0°C :



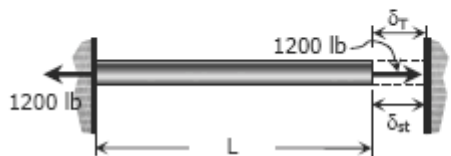
$$\delta = \delta_T + \delta_{st}$$

$$\frac{\sigma L}{E} = \alpha L (\Delta T) + \frac{P L}{AE}$$

$$\sigma = \alpha E (\Delta T) + \frac{P}{A}$$

$$\sigma = (6.5 \times 10^{-6})(29 \times 10^6)(70) + \frac{1200}{0.25}$$

$$\sigma = 17\,995 \text{ psi} = 18 \text{ ksi}$$



For the temperature that causes zero stress:

$$\delta_T = \delta_{st}$$

$$\alpha L (\Delta T) = \frac{P L}{AE}$$

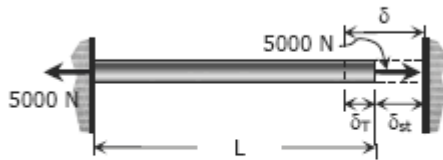
$$(6.5 \times 10^{-6})(T - 70) = \frac{1200}{0.25(29 \times 10^6)}$$

$$T = 95.46^\circ\text{C}$$

Problem 262

A steel rod is stretched between two rigid walls and carries a tensile load of 5000 N at 20°C. If the allowable stress is not to exceed 130 MPa at -20°C, what is the minimum diameter of the rod? Assume $\alpha = 11.7 \mu\text{m}/(\text{m}\cdot^\circ\text{C})$ and $E = 200 \text{ GPa}$.

Solution 262



$$\delta = \delta_T + \delta_{st}$$

$$\frac{\sigma L}{E} = \alpha L(\Delta T) + \frac{PL}{AE}$$

$$\sigma = \alpha E(\Delta T) + \frac{P}{A}$$

$$130 = (11.7 \times 10^{-6})(200\,000)(40) + \frac{5000}{A}$$

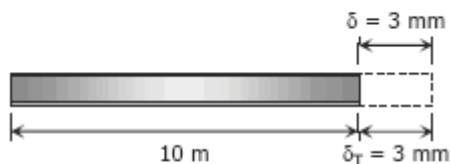
$$A = \frac{5000}{36.4} = 137.36 \text{ mm}^2$$

$$\frac{1}{4}\pi d^2 = 137.36; \quad d = 13.22 \text{ mm}$$

Problem 263

Steel railroad rails 10 m long are laid with a clearance of 3 mm at a temperature of 15°C. At what temperature will the rails just touch? What stress would be induced in the rails at that temperature if there were no initial clearance? Assume $\alpha = 11.7 \mu\text{m}/(\text{m}\cdot^\circ\text{C})$ and $E = 200 \text{ GPa}$.

Solution 263



Temperature at which $\delta_T = 3 \text{ mm}$:

$$\delta_T = \alpha L(\Delta T)$$

$$\delta_T = \alpha L(T_f - T_i)$$

$$3 = (11.7 \times 10^{-6})(10\,000)(T_f - 15)$$

$$T_f = 40.64^\circ\text{C}$$

Required stress:

$$\delta = \delta_T$$

$$\frac{\sigma L}{E} = \alpha L(\Delta T)$$

$$\sigma = \alpha E(T_f - T_i)$$

$$\sigma = (11.7 \times 10^{-6})(200\,000)(40.64 - 15)$$

$$\sigma = 60 \text{ MPa}$$

Problem 264

A steel rod 3 feet long with a cross-sectional area of 0.25 in.^2 is stretched between two fixed points. The tensile force is 1200 lb at 40°F. Using $E = 29 \times 10^6 \text{ psi}$ and $\alpha = 6.5 \times 10^{-6} \text{ in.}/(\text{in.}\cdot^\circ\text{F})$, calculate (a) the temperature at which the stress in the bar will be 10 ksi; and (b) the temperature at which the stress will be zero.

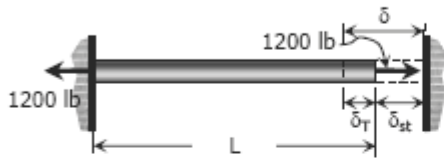
Solution 264

(a) Without temperature change:

$$\sigma = P/A = 1200/0.25 = 4800 \text{ psi}$$

$$\sigma = 4.8 \text{ ksi} < 10 \text{ ksi}$$

A drop of temperature is needed to increase the stress to 10 ksi. See accompanying figure.



$$\delta = \delta_T + \delta_{st}$$

$$\frac{\sigma L}{E} = \alpha L(\Delta T) + \frac{P L}{AE}$$

$$\sigma = \alpha E(\Delta T) + \frac{P}{A}$$

$$10\,000 = (6.5 \times 10^{-6})(29 \times 10^6)(\Delta T) + \frac{1200}{0.25}$$

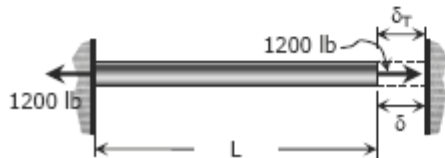
$$\Delta T = 27.59^\circ\text{F}$$

Required temperature:

(temperature must drop from 40°F)

$$T = 40 - 27.59 = 12.41^\circ\text{F}$$

(b) From the figure below:



$$\delta = \delta_T$$

$$\frac{P L}{AE} = \alpha L(\Delta T)$$

$$P = \alpha A E (T_f - T_i)$$

$$1200 = (6.5 \times 10^{-6})(0.25)(29 \times 10^6)(T_f - 40)$$

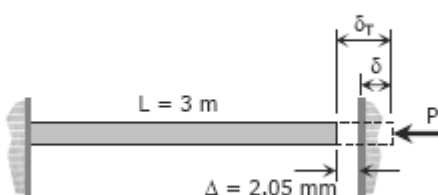
$$T_f = 65.46^\circ\text{F}$$

Problem 265

A bronze bar 3 m long with a cross sectional area of 320 mm^2 is placed between two rigid walls as shown in Fig. P-265. At a temperature of -20°C , the gap $\Delta = 25 \text{ mm}$. Find the temperature at which the compressive stress in the bar will be 35 MPa . Use $\alpha = 18.0 \times 10^{-6} \text{ m}/(\text{m}\cdot^\circ\text{C})$ and $E = 80 \text{ GPa}$.



Solution 265



$$\delta_T = \delta + \Delta$$

$$\alpha L(\Delta T) = \frac{\sigma L}{E} + 2.5$$

$$(18 \times 10^{-6})(3000)(\Delta T) = \frac{35(3000)}{80000} + 2.5$$

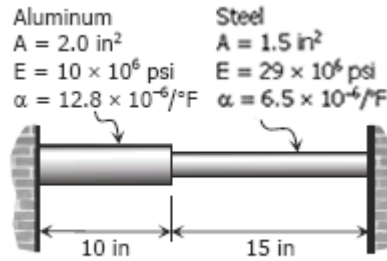
$$\Delta T = 70.6^\circ\text{C}$$

$$T = 70.6 - 20$$

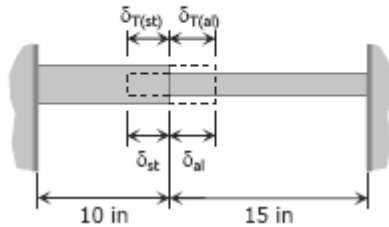
$$T = 50.6^\circ\text{C}$$

Problem 266

Calculate the increase in stress for each segment of the compound bar shown in Fig. P-266 if the temperature increases by 100°F . Assume that the supports are unyielding and that the bar is suitably braced against buckling.

**Figure P-266****Solution 266**

$$\delta_T = \alpha L \Delta T$$



$$\delta_{T(st)} = (6.5 \times 10^{-6})(15)(100)$$

$$\delta_{T(st)} = 0.00975$$

$$\delta_{T(al)} = (12.8 \times 10^{-6})(10)(100)$$

$$\delta_{T(al)} = 0.0128 \text{ in}$$

$$\delta_{st} + \delta_{al} = \delta_{T(st)} + \delta_{T(al)}$$

$$\left(\frac{PL}{AE} \right)_{st} + \left(\frac{PL}{AE} \right)_{al} = 0.00975 + 0.0128$$

$$\text{where } P = P_{st} = P_{al}$$

$$\frac{P(15)}{1.5(29 \times 10^6)} + \frac{P(10)}{2(10 \times 10^6)} = 0.02255$$

$$P = 26\,691.84 \text{ psi}$$

$$\sigma = \frac{P}{A}$$

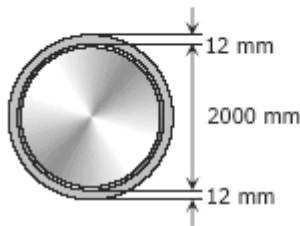
$$\sigma_{st} = \frac{26\,691.84}{1.5} = 17\,794.56 \text{ psi}$$

$$\sigma_{al} = \frac{26\,691.84}{2.0} = 13\,345.92 \text{ psi}$$

Problem 267

At a temperature of 80°C, a steel tire 12 mm thick and 90 mm wide that is to be shrunk onto a locomotive driving wheel 2 m in diameter just fits over the wheel, which is at a temperature of 25°C. Determine the contact pressure between the tire and wheel after the assembly cools to 25°C. Neglect the deformation of the wheel caused by the pressure of the tire. Assume $\alpha = 11.7 \mu\text{m}/(\text{m}\cdot^\circ\text{C})$ and $E = 200 \text{ GPa}$.

Solution 267



$$\delta = \delta_T$$

$$\frac{PL}{AE} = \alpha L \Delta T$$

$$P = \alpha \Delta T AE$$

$$P = (11.7 \times 10^{-6})(80 - 25)(90 \times 12)(200\,000)$$

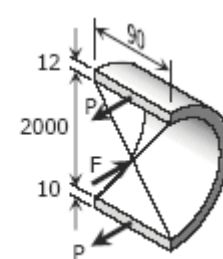
$$P = 138\,996 \text{ N}$$

$$F = 2P$$

$$pDL = 2P$$

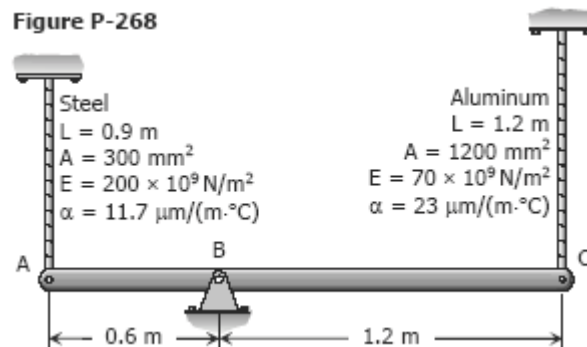
$$p(2000)(90) = 2(138\,996)$$

$$p = 1.5444 \text{ MPa}$$



Problem 268

The rigid bar ABC in Fig. P-268 is pinned at B and attached to the two vertical rods. Initially, the bar is horizontal and the vertical rods are stress-free. Determine the stress in the aluminum rod if the temperature of the steel rod is decreased by 40°C. Neglect the weight of bar ABC.



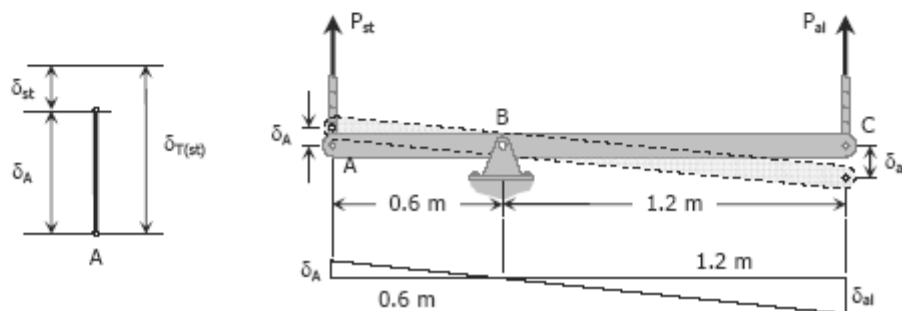
Solution 268

Contraction of steel rod, assuming complete freedom:

$$\begin{aligned}\delta_{T(st)} &= \alpha L \Delta T \\ &= (11.7 \times 10^{-6})(900)(40) \\ &= 0.4212 \text{ mm}\end{aligned}$$

The steel rod cannot freely contract because of the resistance of aluminum rod. The movement of A (referred to as δ_A), therefore, is less than 0.4212 mm. In terms of aluminum, this movement is (by ratio and proportion):

$$\begin{aligned}\frac{\delta_A}{0.6} &= \frac{\delta_{al}}{1.2} \\ \delta_A &= 0.5\delta_{al}\end{aligned}$$



$$\delta_{T(st)} - \delta_{st} = 0.5 \delta_{al}$$

$$0.4212 - \left(\frac{PL}{AE} \right)_{st} = 0.5 \left(\frac{PL}{AE} \right)_{al}$$

$$0.4212 - \frac{P_{st}(900)}{300(200\,000)} = 0.5 \left[\frac{P_{al}(1200)}{1200(70\,000)} \right]$$

$$28080 - P_{st} = 0.4762P_{al} \quad \rightarrow \text{Equation (1)}$$

$$\sum M_B = 0$$

$$0.6P_{st} = 1.2P_{al}$$

$$P_{st} = 2P_{al} \quad \rightarrow \text{Equation (2)}$$

Equations (1) and (2)

$$28080 - 2P_{al} = 0.4762P_{al}$$

$$P_{al} = 11340 \text{ N}$$

$$\sigma_{al} = \frac{P_{al}}{A_{al}} = \frac{11340}{1200}$$

$$\sigma_{al} = 9.45 \text{ MPa}$$

Problem 269

As shown in Fig. P-269, there is a gap between the aluminum bar and the rigid slab that is supported by two copper bars. At 10°C , $\Delta = 0.18 \text{ mm}$. Neglecting the mass of the slab, calculate the stress in each rod when the temperature in the assembly is increased to 95°C . For each copper bar, $A = 500 \text{ mm}^2$, $E = 120 \text{ GPa}$, and $\alpha = 16.8 \mu\text{m}/(\text{m}\cdot^\circ\text{C})$. For the aluminum bar, $A = 400 \text{ mm}^2$, $E = 70 \text{ GPa}$, and $\alpha = 23.1 \mu\text{m}/(\text{m}\cdot^\circ\text{C})$.

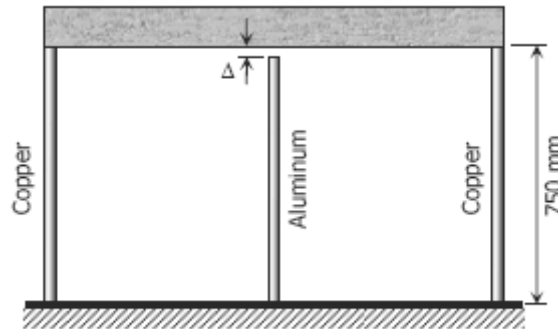


Figure P-269

Solution 269

Assuming complete freedom:

$$\delta_T = \alpha L \Delta T$$

$$\delta_{T(\text{co})} = (16.8 \times 10^{-6})(750)(95 - 10)$$

$$= 1.071 \text{ mm}$$

$$\delta_{T(\text{al})} = (23.1 \times 10^{-6})(750 - 0.18)(95 - 10)$$

$$= 1.472 \text{ mm}$$

From the figure:

$$\delta_{T(\text{al})} - \delta_{\text{al}} = \delta_{T(\text{co})} + \delta_{\text{co}}$$

$$1.472 - \left(\frac{PL}{AE} \right)_{\text{al}} = 1.071 + \left(\frac{PL}{AE} \right)_{\text{co}}$$

$$1.472 - \frac{2F(750 - 0.18)}{400(70000)} = 1.071 + \frac{F(750)}{500(120000)}$$

$$0.401 = (6.606 \times 10^{-5}) F$$

$$F = 6070.37 \text{ N}$$

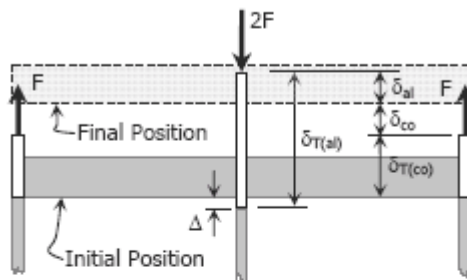
$$P_{\text{co}} = F = 6070.37 \text{ N}$$

$$P_{\text{al}} = 2F = 12140.74 \text{ N}$$

$$\sigma = P/A$$

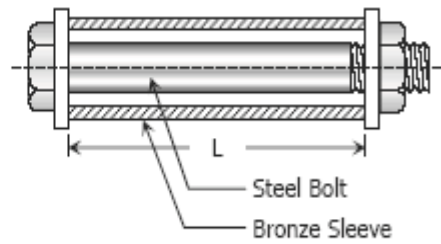
$$\sigma_{\text{co}} = \frac{6070.37}{500} = 12.14 \text{ MPa}$$

$$\sigma_{\text{al}} = \frac{12140.74}{400} = 30.35 \text{ MPa}$$

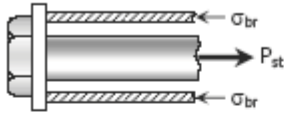


Problem 270

A bronze sleeve is slipped over a steel bolt and held in place by a nut that is turned to produce an initial stress of 2000 psi in the bronze. For the steel bolt, $A = 0.75 \text{ in}^2$, $E = 29 \times 10^6 \text{ psi}$, and $\alpha = 6.5 \times 10^{-6} \text{ in}/(\text{in}\cdot^\circ\text{F})$. For the bronze sleeve, $A = 1.5 \text{ in}^2$, $E = 12 \times 10^6 \text{ psi}$ and $\alpha = 10.5 \times 10^{-6} \text{ in}/(\text{in}\cdot^\circ\text{F})$. After a temperature rise of 100°F , find the final stress in each material.

Solution 270

Before temperature change:



$$\begin{aligned} P_{br} &= \sigma_{br} A_{br} \\ &= 2000(1.5) \\ &= 3000 \text{ lb compression} \end{aligned}$$

$$\begin{aligned} \Sigma F_H &= 0 \\ P_{st} &= P_{br} = 3000 \text{ lb tension} \\ \sigma_{st} &= P_{st} / A_{st} = 3000 / 0.75 \\ &= 4000 \text{ psi tensile stress} \end{aligned}$$

$$\delta = \frac{\sigma L}{E}$$

$$a = \delta_{br} = \frac{2000L}{12 \times 10^6} = 1.67 \times 10^{-4}L \text{ shortening}$$

$$b = \delta_{st} = \frac{4000L}{29 \times 10^6} = 1.38 \times 10^{-4}L \text{ lengthening}$$

With temperature rise of 100°F:

(Assuming complete freedom)

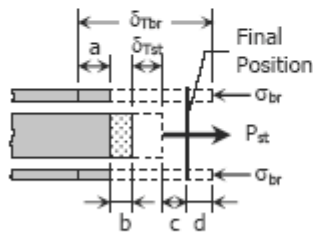
$$\begin{aligned} \delta_T &= \alpha L \Delta T \\ \delta_{Tbr} &= (10.5 \times 10^{-6})L (100) \\ &= 1.05 \times 10^{-3}L > a \end{aligned}$$

$$\begin{aligned} \delta_{Tst} &= (6.5 \times 10^{-6})L (100) \\ &= 6.5 \times 10^{-4}L \end{aligned}$$

$$\begin{aligned} \delta_{Tbr} - a &= 1.05 \times 10^{-3}L - 1.67 \times 10^{-4}L \\ &= 8.83 \times 10^{-4}L \end{aligned}$$

$$\begin{aligned} \delta_{Tst} + b &= 6.5 \times 10^{-4}L + 1.38 \times 10^{-4}L \\ &= 7.88 \times 10^{-4}L \end{aligned}$$

$$\delta_{Tbr} - a > \delta_{Tst} + b \text{ (see figure below)}$$



$$\delta_{Tbr} - a - d = b + \delta_{Tst} + c$$

$$\begin{aligned} 1.05 \times 10^{-3}L - 1.67 \times 10^{-4}L - \left(\frac{\sigma L}{E} \right)_{br} \\ = 1.38 \times 10^{-4}L + 6.5 \times 10^{-4}L + \left(\frac{PL}{AE} \right)_{st} \end{aligned}$$

$$\begin{aligned} 8.83 \times 10^{-4}L - \frac{\sigma_{br} L}{12 \times 10^6} \\ = 7.88 \times 10^{-4}L + \frac{P_{st} L}{0.75(29 \times 10^6)} \end{aligned}$$

$$9.5 \times 10^{-4} - \frac{P_{br}}{1.5(12 \times 10^6)} = \frac{P_{st}}{0.75(29 \times 10^6)}$$

$$P_{st} = 20\,662.5 - 1.2083P_{br} \rightarrow \text{Equation (1)}$$

$$\Sigma F_H = 0$$

$$P_{br} = P_{st} \rightarrow \text{Equation (2)}$$

Equations (1) and (2)

$$P_{st} = 20\,662.5 - 1.2083P_{st}$$

$$P_{st} = 9356.74 \text{ lb}$$

$$P_{br} = 9356.74 \text{ lb}$$

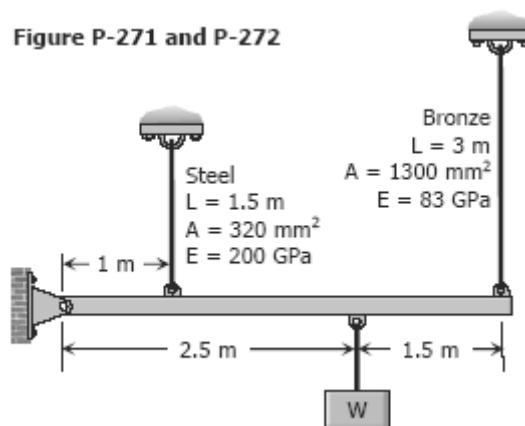
$$\sigma = P/A$$

$$\sigma_{br} = \frac{9356.74}{1.5} = 6237.83 \text{ psi compressive stress}$$

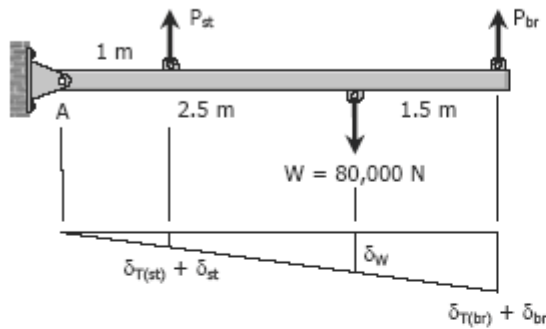
$$\sigma_{st} = \frac{9356.74}{0.74} = 12\,475.66 \text{ psi tensile stress}$$

Problem 271

A rigid bar of negligible weight is supported as shown in Fig. P-271. If $W = 80 \text{ kN}$, compute the temperature change that will cause the stress in the steel rod to be 55 MPa . Assume the coefficients of linear expansion are $11.7 \mu\text{m}/(\text{m}\cdot^\circ\text{C})$ for steel and $18.9 \mu\text{m}/(\text{m}\cdot^\circ\text{C})$ for bronze.



Solution 271



Stress in bronze when $\sigma_{st} = 55 \text{ MPa}$

$$\Sigma M_A = 0$$

$$4P_{br} + P_{st} = 2.5(80000)$$

$$4\sigma_{br}(1300) + 55(320) = 2.5(80000)$$

$$\sigma_{br} = 35.08 \text{ MPa}$$

By ratio and proportion:

$$\frac{\delta_{T(st)} + \delta_{st}}{1} = \frac{\delta_{T(br)} + \delta_{br}}{4}$$

$$\delta_{T(st)} + \delta_{st} = 0.25[\delta_{T(br)} + \delta_{br}]$$

$$(\alpha L \Delta T)_{st} + \left(\frac{\sigma L}{E}\right)_{st} = 0.25 \left[(\alpha L \Delta T)_{br} + \left(\frac{\sigma L}{E}\right)_{br} \right]$$

$$(11.7 \times 10^{-6})(1500) \Delta T + \frac{55(1500)}{2000}$$

$$= 0.25 \left[(18.9 \times 10^{-6})(3000) \Delta T + \frac{35.08(3000)}{83000} \right]$$

$$0.01755 \Delta T + 0.4125 = 0.014175 \Delta T + 0.317$$

$$\Delta T = -28.3^\circ \text{C}$$

A temperature drop of 28.3°C is needed to stress the steel to 55 MPa.

Problem 272

For the assembly in Fig. 271, find the stress in each rod if the temperature rises 30°C after a load $W = 120 \text{ kN}$ is applied.

Solution 272

$$\Sigma M_A = 0$$

$$4P_{br} + P_{st} = 2.5(80000)$$

$$4\sigma_{br}(1300) + \sigma_{st}(320) = 2.5(80000)$$

$$16.25\sigma_{br} + \sigma_{st} = 625$$

$$\sigma_{st} = 625 - 16.25\sigma_{br}$$

→ Equation (1)

$$\frac{\delta_{T(st)} + \delta_{st}}{1} = \frac{\delta_{T(br)} + \delta_{br}}{4}$$

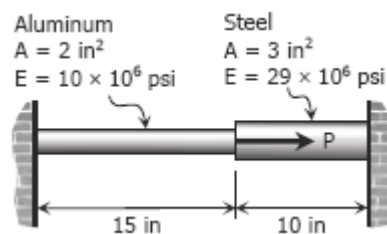
$$\delta_{T(st)} + \delta_{st} = 0.25[\delta_{T(br)} + \delta_{br}]$$

$$\begin{aligned}
 (\alpha L \Delta T)_{st} + \left(\frac{\sigma L}{E} \right)_{st} &= 0.25 \left[(\alpha L \Delta T)_{br} + \left(\frac{\sigma L}{E} \right)_{br} \right] \\
 (11.7 \times 10^{-6})(1500)(30) + \frac{\sigma_{st}(1500)}{200000} \\
 &= 0.25 \left[(18.9 \times 10^{-6})(3000)(30) + \frac{\sigma_{br}(3000)}{83000} \right] \\
 0.5265 + 0.0075\sigma_{st} &= 0.42525 + 0.00904\sigma_{br} \\
 0.0075\sigma_{st} - 0.00904\sigma_{br} &= -0.10125 \\
 0.0075(625 - 16.25\sigma_{br}) - 0.00904\sigma_{br} &= -0.10125 \\
 4.6875 - 0.121875\sigma_{br} - 0.00904\sigma_{br} &= -0.10125 \\
 4.78875 &= 0.130915\sigma_{br} \\
 \sigma_{br} &= 36.58 \text{ } ^\circ\text{C} \\
 \sigma_{st} &= 625 - 16.25(36.58) \\
 \sigma_{st} &= 30.58 \text{ } ^\circ\text{C}
 \end{aligned}$$

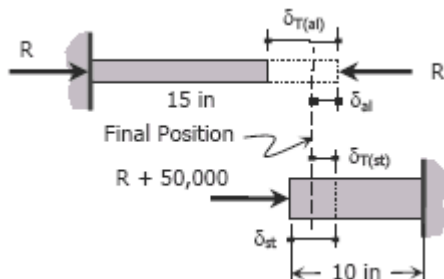
Problem 273

The composite bar shown in Fig. P-273 is firmly attached to unyielding supports. An axial force $P = 50$ kips is applied at 60°F . Compute the stress in each material at 120°F . Assume $\alpha = 6.5 \times 10^{-6} \text{ in}/(\text{in}\cdot^\circ\text{F})$ for steel and $12.8 \times 10^{-6} \text{ in}/(\text{in}\cdot^\circ\text{F})$ for aluminum.

Figure P-273 and P-274



Solution 273



$$\begin{aligned}
 \delta_{T(al)} &= (\alpha L \Delta T)_{al} \\
 \delta_{T(al)} &= (12.8 \times 10^{-6})(15)(120 - 60) \\
 \delta_{T(al)} &= 0.01152 \text{ inch} \\
 \delta_{T(st)} &= (\alpha L \Delta T)_{st} \\
 \delta_{T(st)} &= (6.5 \times 10^{-6})(10)(120 - 60) \\
 \delta_{T(st)} &= 0.0039 \text{ inch}
 \end{aligned}$$

$$\begin{aligned}\delta_{T(al)} - \delta_{al} &= \delta_{st} - \delta_{T(st)} \\ 0.011\,52 - \left(\frac{PL}{AE}\right)_{al} &= \left(\frac{PL}{AE}\right)_{st} - 0.0039 \\ 0.011\,52 - \frac{R(15)}{2(10 \times 10^6)} &= \frac{(R + 50\,000)(10)}{3(29 \times 10^6)} - 0.0039 \\ 100\,224 - 6.525R &= R + 50\,000 - 33\,930 \\ 84\,154 &= 7.525R \\ R &= 11\,183.25 \text{ lbs}\end{aligned}$$

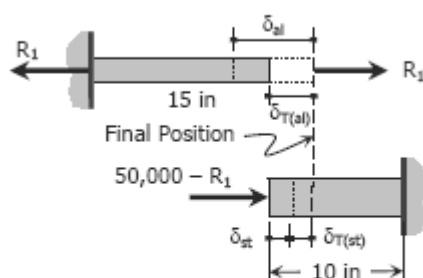
$$\begin{aligned}P_{al} &= R = 11\,183.25 \text{ lbs} \\ P_{st} &= R + 50\,000 = 61\,183.25 \text{ lbs}\end{aligned}$$

$$\begin{aligned}\sigma &= \frac{P}{A} \\ \sigma_{al} &= \frac{11\,183.25}{2} \\ &= 5\,591.62 \text{ psi} \\ \sigma_{st} &= \frac{61\,183.25}{3} \\ &= 20\,394.42 \text{ psi}\end{aligned}$$

Problem 274

At what temperature will the aluminum and steel segments in Prob. 273 have numerically equal stress?

Solution 274



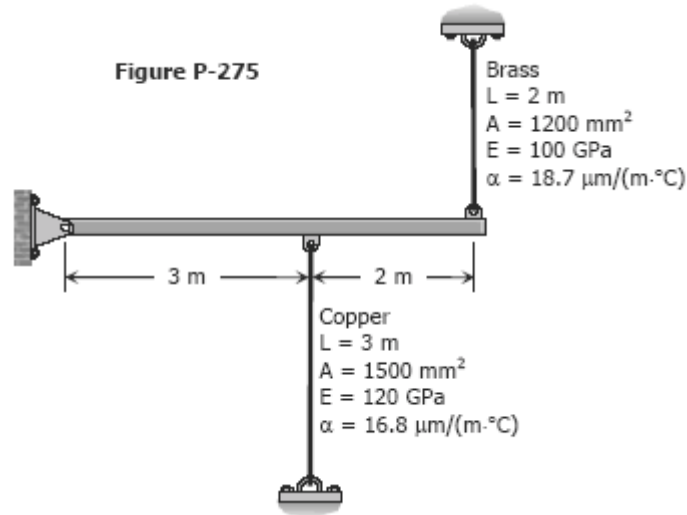
$$\begin{aligned}\sigma_{al} &= \sigma_{st} \\ \frac{R_1}{2} &= \frac{(50\,000 - R_1)}{3} \\ 3R_1 &= 100\,000 - 2R_1 \\ R_1 &= 20\,000 \text{ lbs} \\ \delta &= \frac{PL}{AE} \\ \delta_{al} &= \frac{20\,000(15)}{2(10 \times 10^6)} = 0.015 \text{ inch} \\ \delta_{st} &= \frac{(50\,000 - 20\,000)(10)}{3(29 \times 10^6)} = 0.003\,45 \text{ inch}\end{aligned}$$

$$\begin{aligned}\delta_{al} - \delta_{T(al)} &= \delta_{st} + \delta_{T(st)} \\ 0.015 - (12.8 \times 10^{-6})(15) \Delta T &= 0.003\,45 + (6.5 \times 10^{-6})(10) \Delta T \\ 0.011\,55 &= 0.000\,257 \Delta T \\ \Delta T &= 44.94^\circ\text{F}\end{aligned}$$

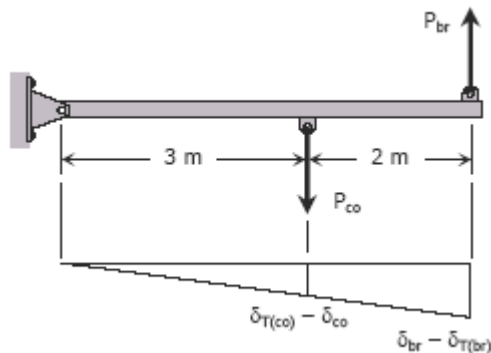
A drop of 44.94°F from the standard temperature will make the aluminum and steel segments equal in stress.

Problem 275

A rigid horizontal bar of negligible mass is connected to two rods as shown in Fig. P-275. If the system is initially stress-free. Calculate the temperature change that will cause a tensile stress of 90 MPa in the brass rod. Assume that both rods are subjected to the change in temperature.



Solution 275



$$\Sigma M_{\text{hinge support}} = 0$$

$$5P_{br} - 3P_{co} = 0$$

$$5\sigma_{br} A_{br} - 3\sigma_{co} A_{co} = 0$$

$$5(90)(1200) - 3\sigma_{co}(1500) = 0$$

$$\sigma_{co} = 120 \text{ MPa}$$

$$\delta = \sigma L / E$$

$$\delta_{br} = 90(2000) / 100\,000 = 1.8 \text{ mm}$$

$$\delta_{co} = 120(3000) / 120\,000 = 3 \text{ mm}$$

$$\frac{\delta_{T(co)} - \delta_{co}}{3} = \frac{\delta_{br} - \delta_{T(br)}}{5}$$

$$5\delta_{T(co)} - 5\delta_{co} = 3\delta_{br} - 3\delta_{T(br)}$$

$$5(16.8 \times 10^{-6})(3000) \Delta T - 5(3)$$

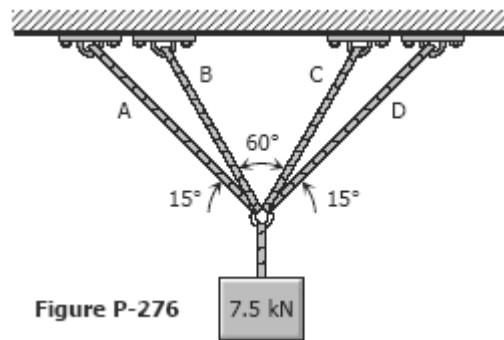
$$= 3(1.8) - 3(18.7 \times 10^{-6})(2000) \Delta T$$

$$0.3642 \Delta T = 20.4$$

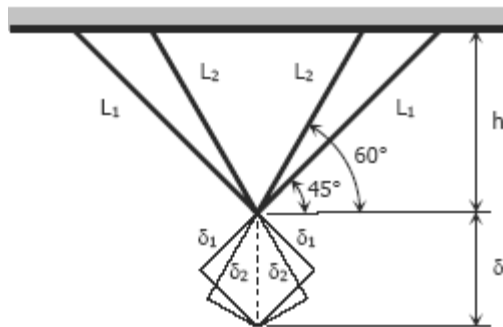
$$\Delta T = 56.01^\circ\text{C drop in temperature}$$

Problem 276

Four steel bars jointly support a mass of 15 Mg as shown in Fig. P-276. Each bar has a cross-sectional area of 600 mm². Find the load carried by each bar after a temperature rise of 50°C. Assume $\alpha = 11.7 \mu\text{m}/(\text{m}\cdot^\circ\text{C})$ and $E = 200 \text{ GPa}$.



Solution 276



$$h = L_1 \sin 45^\circ$$

$$h = L_2 \sin 60^\circ$$

$$h = h$$

$$L_1 \sin 45^\circ = L_2 \sin 60^\circ$$

$$L_1 = 1.2247 L_2$$

$$\delta_1 = \delta \sin 45^\circ$$

$$\delta_2 = \delta \sin 60^\circ$$

$$\frac{\delta_1}{\delta_2} = \frac{\delta \sin 45^\circ}{\delta \sin 60^\circ}$$

$$\delta_1 = 0.8165 \delta_2$$

$$\alpha L_1 \Delta T + \frac{P_1 L_1}{AE} = 0.8165 \left[\alpha L_2 \Delta T + \frac{P_2 L_2}{AE} \right]$$

$$(11.7 \times 10^{-6}) L_1 (50) + \frac{P_1 L_1}{600(200000)}$$

$$= 0.8165 \left[(11.7 \times 10^{-6}) L_2 (50) + \frac{P_2 L_2}{600(200000)} \right]$$

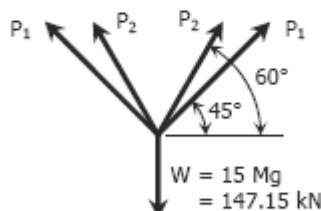
$$70,200 L_1 + P_1 L_1 = 0.8165 (70,200 L_2 + P_2 L_2)$$

$$(70,200 + P_1) L_1 = 0.8165 (70,200 + P_2) L_2$$

$$(70,200 + P_1) 1.2247 L_2 = 0.8165 (70,200 + P_2) L_2$$

$$1.5(70,200 + P_1) = 70,200 + P_2$$

$$P_2 = 1.5 P_1 + 35,100 \quad \rightarrow \text{Equation (1)}$$



$$\Sigma F_V = 0$$

$$2(P_1 \sin 45^\circ) + 2(P_2 \sin 60^\circ) = 147.15(1000)$$

$$P_1 \sin 45^\circ + P_2 \sin 60^\circ = 72,575$$

$$P_1 \sin 45^\circ + (1.5 P_1 + 35,100) \sin 60^\circ = 72,575$$

$$0.7071 P_1 + 1.299 P_1 + 30,397.49 = 72,575$$

$$2.0061 P_1 = 42,177.51$$

$$P_1 = 21,024.63 \text{ N}$$

$$P_2 = 1.5(21,024.63) + 35,100$$

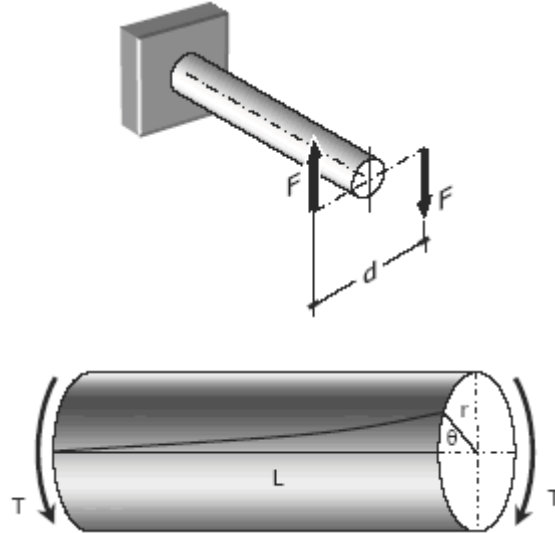
$$P_2 = 66,636.94 \text{ N}$$

$$P_A = P_D = P_1 = 21.02 \text{ kN}$$

$$P_B = P_C = P_2 = 66.64 \text{ kN}$$

Torsion

Consider a bar to be rigidly attached at one end and twisted at the other end by a torque or twisting moment T equivalent to $F \times d$, which is applied perpendicular to the axis of the bar, as shown in the figure. Such a bar is said to be in torsion.



TORSIONAL SHEARING STRESS, τ

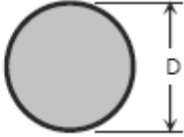
For a solid or hollow circular shaft subject to a twisting moment T , the torsional shearing stress τ at a distance ρ from the center of the shaft is

$$\tau = \frac{T\rho}{J} \text{ and } \tau_{\max} = \frac{Tr}{J}$$

where J is the polar moment of inertia of the section and r is the outer radius.

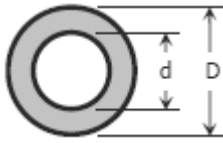
For solid cylindrical shaft:

$$J = \frac{\pi}{32} D^4$$

$$\tau_{\max} = \frac{16T}{\pi D^3}$$


For hollow cylindrical shaft:

$$J = \frac{\pi}{32} (D^4 - d^4)$$

$$\tau_{\max} = \frac{16TD}{\pi(D^4 - d^4)}$$


ANGLE OF TWIST

The angle θ through which the bar length L will twist is

$$\theta = \frac{TL}{JG} \text{ in radians}$$

where T is the torque in N·mm, L is the length of shaft in mm, G is shear modulus in MPa, J is the polar moment of inertia in mm⁴, D and d are diameter in mm, and r is the radius in mm.

POWER TRANSMITTED BY THE SHAFT

A shaft rotating with a constant angular velocity ω (in radians per second) is being acted by a twisting moment T . The power transmitted by the shaft is

$$P = T\omega = 2\pi T f$$

where T is the torque in N·m, f is the number of revolutions per second, and P is the power in watts.

Solved Problems in Torsion

Problem 304

A steel shaft 3 ft long that has a diameter of 4 in. is subjected to a torque of 15 kip·ft. Determine the maximum shearing stress and the angle of twist. Use $G = 12 \times 10^6$ psi.

Solution 304

$$\tau_{\max} = \frac{16T}{\pi D^3} = \frac{16(15)(1000)(12)}{\pi(4^3)}$$

$$\tau_{\max} = 14\,324 \text{ psi}$$

$$\tau_{\max} = 14.3 \text{ ksi}$$

$$\theta = \frac{TL}{JG} = \frac{15(3)(1000)(12^2)}{\frac{1}{32}\pi(4^4)(12 \times 10^6)}$$

$$\theta = 0.0215 \text{ rad}$$

$$\theta = 1.23^\circ$$

Problem 305

What is the minimum diameter of a solid steel shaft that will not twist through more than 3° in a 6-m length when subjected to a torque of 12 kN·m? What maximum shearing stress is developed? Use $G = 83 \text{ GPa}$.

Solution 305

$$\theta = \frac{TL}{JG}$$

$$3^\circ \left(\frac{\pi}{180^\circ} \right) = \frac{12(6)(1000^3)}{\frac{1}{32} \pi d^4 (83000)}$$

$$d = 113.98 \text{ mm}$$

$$\tau_{\max} = \frac{16T}{\pi d^3} = \frac{16(12)(1000^2)}{\pi (113.98^3)}$$

$$\tau_{\max} = 41.27 \text{ MPa}$$

Problem 306

A steel marine propeller shaft 14 in. in diameter and 18 ft long is used to transmit 5000 hp at 189 rpm. If $G = 12 \times 10^6 \text{ psi}$, determine the maximum shearing stress.

Solution 306

$$T = \frac{P}{2\pi f} = \frac{5000(396000)}{2\pi(189)}$$

$$T = 1\,667\,337.5 \text{ lb}\cdot\text{in}$$

$$\tau_{\max} = \frac{16T}{\pi d^3} = \frac{16(1\,667\,337.5)}{\pi (14^3)}$$

$$\tau_{\max} = 3094.6 \text{ psi}$$

Problem 307

A solid steel shaft 5 m long is stressed at 80 MPa when twisted through 4° . Using $G = 83 \text{ GPa}$, compute the shaft diameter. What power can be transmitted by the shaft at 20 Hz?

Solution 307

$$\theta = \frac{TL}{JG}$$

$$4^\circ \left(\frac{\pi}{180^\circ} \right) = \frac{T(5)(1000)}{\frac{1}{32} \pi d^4 (83\,000)}$$

$$T = 0.1138d^4$$

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

$$80 = \frac{16(0.1138d^4)}{\pi d^3}$$

$$d = 138 \text{ mm}$$

$$T = \frac{P}{2\pi f}$$

$$0.1138d^4 = \frac{P}{2\pi(20)}$$

$$P = 14.3d^4 = 14.3(138^4)$$

$$P = 5\,186\,237\,285 \text{ N}\cdot\text{mm}/\text{sec}$$

$$P = 5\,186\,237.28 \text{ W}$$

$$P = 5.19 \text{ MW}$$

Problem 308

A 2-in-diameter steel shaft rotates at 240 rpm. If the shearing stress is limited to 12 ksi, determine the maximum horsepower that can be transmitted.

Solution 308

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

$$12(1000) = \frac{16T}{\pi(2^3)}$$

$$T = 18\,849.56 \text{ lb}\cdot\text{in}$$

$$T = \frac{P}{2\pi f}$$

$$18\,849.56 = \frac{P(396\,000)}{2\pi(240)}$$

$$P = 71.78 \text{ hp}$$

Problem 309

A steel propeller shaft is to transmit 4.5 MW at 3 Hz without exceeding a shearing stress of 50 MPa or twisting through more than 1° in a length of 26 diameters. Compute the proper diameter if $G = 83 \text{ GPa}$.

Solution 309

$$T = \frac{P}{2\pi f} = \frac{4.5(1000000)}{2\pi(3)}$$

$$T = 238\,732.41 \text{ N}\cdot\text{m}$$

Based on maximum allowable shearing stress:

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

$$50 = \frac{16(238732.41)(1000)}{\pi d^3}$$

$$d = 289.71 \text{ mm}$$

Based on maximum allowable angle of twist:

$$\theta = \frac{TL}{JG}$$

$$1^\circ \left(\frac{\pi}{180^\circ} \right) = \frac{238732.41(26d)(1000)}{\frac{1}{32} \pi d^4 (83\,000)}$$

$$d = 352.08 \text{ mm}$$

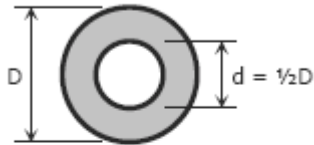
Use the bigger diameter, $d = 352 \text{ mm}$

Problem 310

Show that the hollow circular shaft whose inner diameter is half the outer diameter has a torsional strength equal to 15/16 of that of a solid shaft of the same outside diameter.

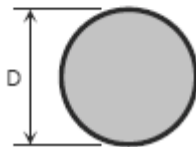
Solution 310

Hollow circular shaft:



$$\begin{aligned}\tau_{\max-\text{hollow}} &= \frac{16TD}{\pi(D^4 - d^4)} \\ &= \frac{16TD}{\pi[D^4 - (\frac{1}{2}D)^4]} \\ &= \frac{16TD}{\pi(\frac{15}{16}D^4)} \\ &= \frac{16^2 T}{15\pi D^3}\end{aligned}$$

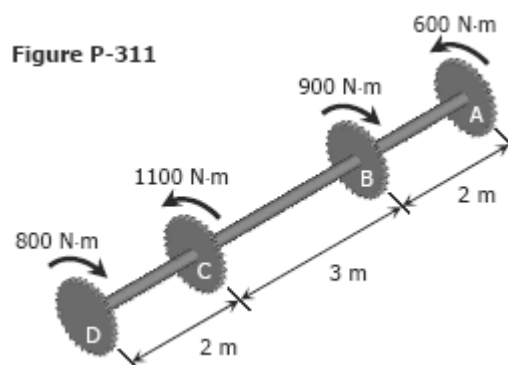
Solid circular shaft:



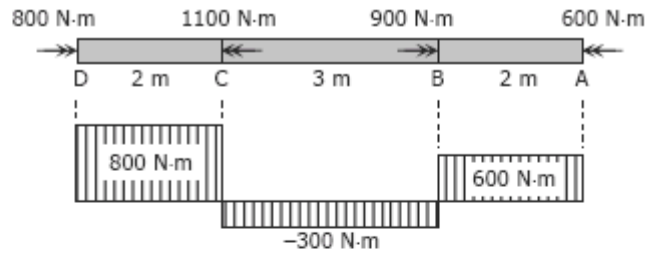
$$\begin{aligned}\tau_{\max-\text{solid}} &= \frac{16T}{\pi D^3} \\ &= \frac{15}{16} \left[\frac{16^2 T}{15\pi D^3} \right] \\ &= \frac{15}{16} \times \tau_{\max-\text{hollow}} \quad \text{ok!}\end{aligned}$$

Problem 311

An aluminum shaft with a constant diameter of 50 mm is loaded by torques applied to gears attached to it as shown in Fig. P-311. Using $G = 28 \text{ GPa}$, determine the relative angle of twist of gear D relative to gear A.



Solution 311



$$\theta = \frac{TL}{JG}$$

Rotation of D relative to A:

$$\theta_{D/A} = \frac{1}{JG} \sum TL$$

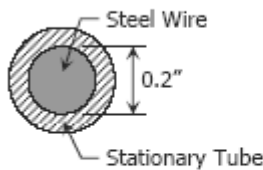
$$\theta_{D/A} = \frac{1}{\frac{1}{32} \pi (50^4) (28000)} [800(2) - 300(3) + 600(2)] (100^2)$$

$$\theta_{D/A} = 0.1106 \text{ rad}$$

$$\theta_{D/A} = 6.34^\circ$$

Problem 312

A flexible shaft consists of a 0.20-in-diameter steel wire encased in a stationary tube that fits closely enough to impose a frictional torque of 0.50 lb·in/in. Determine the maximum length of the shaft if the shearing stress is not to exceed 20 ksi. What will be the angular deformation of one end relative to the other end? $G = 12 \times 10^6$ psi.



$$\tau_{\max} = \frac{16T}{\pi d^3}$$

$$20(1000) = \frac{16T}{\pi(0.20)^3}$$

$$T = 10\pi \text{ lb}\cdot\text{in}$$

$$L = \frac{T}{0.50 \text{ lb}\cdot\text{in}/\text{in}} = \frac{10\pi \text{ lb}\cdot\text{in}}{0.50 \text{ lb}\cdot\text{in}/\text{in}}$$

$$L = 20\pi \text{ in} = 62.83 \text{ in}$$

$$\theta = \frac{TL}{JG}$$

If $\theta = d\theta$, $T = 0.5L$ and $L = dL$

$$\int d\theta = \frac{1}{JG} \int_0^{20\pi} (0.5L) dL$$

$$\theta = \frac{1}{JG} \left[\frac{0.5L^2}{2} \right]_0^{20\pi} = \frac{1}{JG} [0.25(20\pi)^2 - 0.25(0)^2]$$

$$\theta = \frac{100\pi^2}{\frac{1}{32} \pi (0.20^4) (12 \times 10^6)}$$

$$\theta = 0.5234 \text{ rad} = 30^\circ$$

Problem 313

Determine the maximum torque that can be applied to a hollow circular steel shaft of 100-mm outside diameter and an 80-mm inside diameter without exceeding a shearing stress of 60 MPa or a twist of 0.5 deg/m. Use $G = 83 \text{ GPa}$.

Solution 313

Based on maximum allowable shearing stress:

$$\tau_{\max} = \frac{16TD}{\pi(D^4 - d^4)}$$

$$60 = \frac{16T(100)}{\pi(100^4 - 80^4)}$$

$$T = 6\,955\,486.14 \text{ N}\cdot\text{mm}$$

$$T = 6\,955.5 \text{ N}\cdot\text{m}$$

Based on maximum allowable angle of twist:

$$\theta = \frac{TL}{JG}$$

$$0.5^\circ \left(\frac{\pi}{180^\circ} \right) = \frac{T(1000)}{\frac{1}{32} \pi (100^4 - 80^4) (83\,000)}$$

$$T = 4\,198\,282.97 \text{ N}\cdot\text{mm}$$

$$T = 4\,198.28 \text{ N}\cdot\text{m}$$

Use the smaller torque, $T = 4\,198.28 \text{ N}\cdot\text{m}$

Problem 314

The steel shaft shown in Fig. P-314 rotates at 4 Hz with 35 kW taken off at A, 20 kW removed at B, and 55 kW applied at C. Using $G = 83 \text{ GPa}$, find the maximum shearing stress and the angle of rotation of gear A relative to gear C.

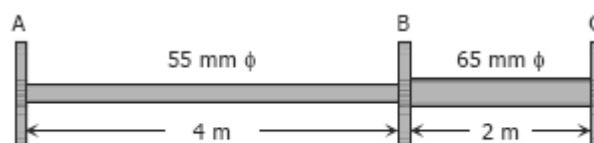


Figure P-314

Solution 314

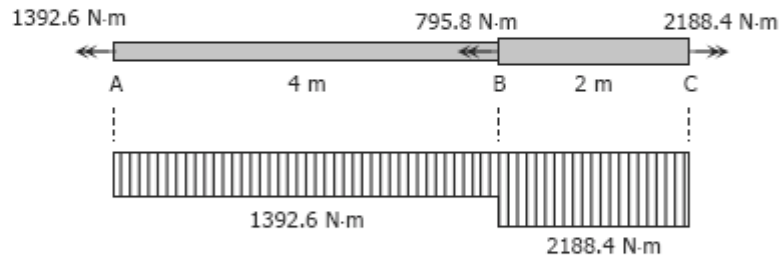
$$T = \frac{P}{2\pi f}$$

$$T_A = \frac{-35(1000)}{2\pi(4)} = -1392.6 \text{ N}\cdot\text{m}$$

$$T_B = \frac{-20(1000)}{2\pi(4)} = -795.8 \text{ N}\cdot\text{m}$$

$$T_C = \frac{55(1000)}{2\pi(4)} = 2188.4 \text{ N}\cdot\text{m}$$

Relative to C:



$$\tau_{\max} = \frac{16T}{\pi d^3}$$

$$\tau_{AB} = \frac{16(1392.6)(1000)}{\pi(55^3)} = 42.63 \text{ MPa}$$

$$\tau_{BC} = \frac{16(2188.4)(1000)}{\pi(65^3)} = 40.58 \text{ MPa}$$

$$\therefore \tau_{\max} = \tau_{AB} = 42.63 \text{ MPa}$$

$$\theta = \frac{TL}{JG}$$

$$\theta_{A/C} = \frac{1}{G} \sum \frac{TL}{J}$$

$$\theta_{A/C} = \frac{1}{83000} \left[\frac{1392.6(4)}{\frac{1}{32} \pi (55^4)} + \frac{2188.4(2)}{\frac{1}{32} \pi (65^4)} \right] (1000^2)$$

$$\theta_{A/C} = 0.104796585 \text{ rad}$$

$$\theta_{A/C} = 6.004^\circ$$

Problem 315

A 5-m steel shaft rotating at 2 Hz has 70 kW applied at a gear that is 2 m from the left end where 20 kW are removed. At the right end, 30 kW are removed and another 20 kW leaves the shaft at 1.5 m from the right end. (a) Find the uniform shaft diameter so that the shearing stress will not exceed 60 MPa. (b) If a uniform shaft diameter of 100 mm is specified, determine the angle by which one end of the shaft lags behind the other end. Use $G = 83 \text{ GPa}$.

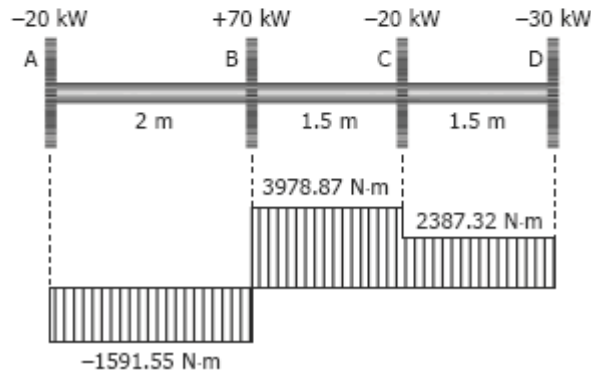
Solution 315

$$T = \frac{P}{2\pi f}$$

$$T_A = T_C = \frac{-20(1000)}{2\pi(2)} = -1591.55 \text{ N}\cdot\text{m}$$

$$T_B = \frac{70(1000)}{2\pi(2)} = 5570.42 \text{ N}\cdot\text{m}$$

$$T_D = \frac{-30(1000)}{2\pi(2)} = -2387.32 \text{ N}\cdot\text{m}$$



Part (a)

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

$$\text{For AB: } 60 = \frac{16(1591.55)(1000)}{\pi d^3}$$

$$d = 51.3 \text{ mm}$$

$$\text{For BC: } 60 = \frac{16(3978.87)(1000)}{\pi d^3}$$

$$d = 69.6 \text{ mm}$$

$$\text{For CD: } 60 = \frac{16(2387.32)(1000)}{\pi d^3}$$

$$d = 58.7 \text{ mm}$$

Use $d = 69.6 \text{ mm}$

Part (b)

$$\theta = \frac{TL}{JG}$$

$$\theta_{D/A} = \frac{1}{JG} \sum TL$$

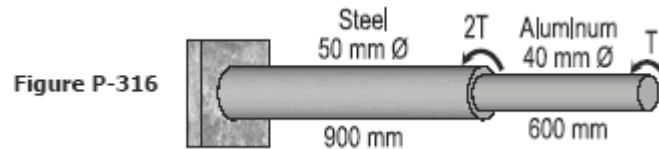
$$\theta_{D/A} = \frac{1}{\frac{1}{32}\pi(100^4)(83000)} [-1591.55(2) + 3978.87(1.5) + 2387.32(1.5)] (1000^2)$$

$$\theta_{D/A} = 0.007813 \text{ rad}$$

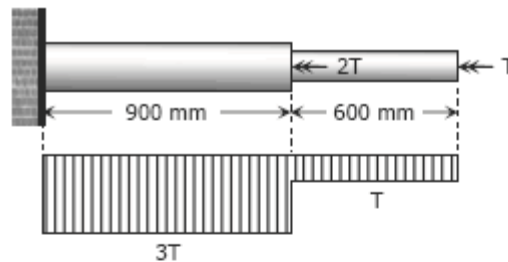
$$\theta_{D/A} = 0.448^\circ$$

Problem 316

A compound shaft consisting of a steel segment and an aluminum segment is acted upon by two torques as shown in Fig. P-316. Determine the maximum permissible value of T subject to the following conditions: $\tau_{st} = 83 \text{ MPa}$, $\tau_{al} = 55 \text{ MPa}$, and the angle of rotation of the free end is limited to 6° . For steel, $G = 83 \text{ GPa}$ and for aluminum, $G = 28 \text{ GPa}$.



Solution 316



Based on maximum shearing stress $\tau_{\max} = 16T / \pi d^3$:

$$\tau_{st} = \frac{16(3T)}{\pi(50^3)} = 83$$

$$T = 679\,042.16 \text{ N}\cdot\text{mm}$$

$$T = 679.04 \text{ N}\cdot\text{m}$$

$$\tau_{al} = \frac{16T}{\pi(40^3)} = 55$$

$$T = 691\,150.38 \text{ N}\cdot\text{mm}$$

$$T = 691.15 \text{ N}\cdot\text{m}$$

Based on maximum angle of twist:

$$\theta = \left(\frac{TL}{JG} \right)_{st} + \left(\frac{TL}{JG} \right)_{al}$$

$$6^\circ \left(\frac{\pi}{180^\circ} \right) = \frac{3T(900)}{\frac{1}{32} \pi (50^4) (83\,000)} + \frac{T(600)}{\frac{1}{32} \pi (40^4) (28\,000)}$$

$$T = 757\,316.32 \text{ N}\cdot\text{mm}$$

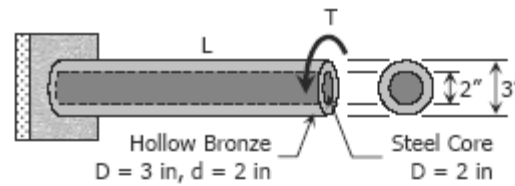
$$T = 757.32 \text{ N}\cdot\text{m}$$

Use $T = 679.04 \text{ N}\cdot\text{m}$

Problem 317

A hollow bronze shaft of 3 in. outer diameter and 2 in. inner diameter is slipped over a solid steel shaft 2 in. in diameter and of the same length as the hollow shaft. The two shafts are then fastened rigidly together at their ends. For bronze, $G = 6 \times 10^6 \text{ psi}$, and for steel, $G = 12 \times 10^6 \text{ psi}$. What torque can be applied to the composite shaft without exceeding a shearing stress of 8000 psi in the bronze or 12 ksi in the steel?

Solution 317



$$\theta_{st} = \theta_{br}$$

$$\left(\frac{TL}{JG} \right)_{st} = \left(\frac{TL}{JG} \right)_{br}$$

$$\frac{T_{st}L}{\frac{1}{32}\pi(2^4)(12 \times 10^6)} = \frac{T_{br}L}{\frac{1}{32}\pi(3^4 - 2^4)(6 \times 10^6)}$$

$$\frac{T_{st}}{192 \times 10^6} = \frac{T_{br}}{390 \times 10^6} \quad \rightarrow \text{Equation (1)}$$

Applied Torque = Resisting Torque

$$T = T_{st} + T_{br} \quad \rightarrow \text{Equation (2)}$$

Equation (1) with T_{st} in terms of T_{br} and Equation (2)

$$T = \frac{192 \times 10^6}{390 \times 10^6} T_{br} + T_{br}$$

$$T_{br} = 0.6701T$$

Equation (1) with T_{br} in terms of T_{st} and Equation (2)

$$T = T_{st} + \frac{390 \times 10^6}{192 \times 10^6} T_{st}$$

$$T_{st} = 0.3299T$$

Based on hollow bronze ($T_{br} = 0.6701T$)

$$\tau_{\max} = \left[\frac{16TD}{\pi(D^4 - d^4)} \right]_{br}$$

$$8000 = \frac{16(0.6701T)(3)}{\pi(3^4 - 2^4)}$$

$$T = 50\,789.32 \text{ lb}\cdot\text{in}$$

$$T = 4232.44 \text{ lb}\cdot\text{ft}$$

Based on steel core ($T_{st} = 0.3299T$):

$$\tau_{\max} = \left[\frac{16T}{\pi D^3} \right]_{st}$$

$$12\,000 = \frac{16(0.3299T)}{\pi(2^3)}$$

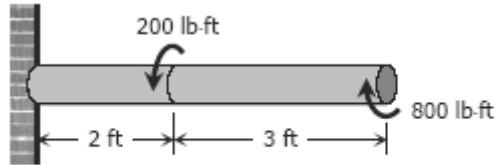
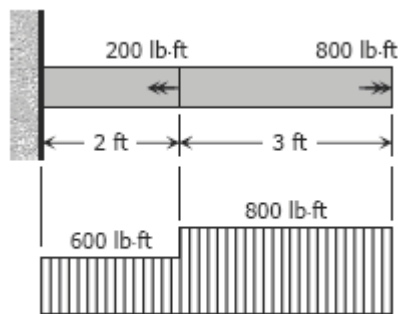
$$T = 57\,137.18 \text{ lb}\cdot\text{in}$$

$$T = 4761.43 \text{ lb}\cdot\text{ft}$$

Use $T = 4232.44 \text{ lb}\cdot\text{ft}$

Problem 318

A solid aluminum shaft 2 in. in diameter is subjected to two torques as shown in Fig. P-318. Determine the maximum shearing stress in each segment and the angle of rotation of the free end. Use $G = 4 \times 10^6$ psi.

**Figure P-318****Solution 318**

$$\tau_{\max} = \frac{16T}{\pi D^3}$$

For 2-ft segment:

$$\tau_{\max 2} = \frac{16(600)(12)}{\pi(2^3)} = 4583.66 \text{ psi}$$

For 3-ft segment:

$$\tau_{\max 3} = \frac{16(800)(12)}{\pi(2^3)} = 6111.55 \text{ psi}$$

$$\theta = \frac{TL}{JG}$$

$$\theta = \frac{1}{JG} \sum TL$$

$$\theta = \frac{1}{\frac{1}{32} \pi (2^4) (4 \times 10^6)} [600(2) + 800(3)] (12^2)$$

$$\theta = 0.0825 \text{ rad}$$

$$\theta = 4.73^\circ$$

Problem 319

The compound shaft shown in Fig. P-319 is attached to rigid supports. For the bronze segment AB, the diameter is 75 mm, $\tau \leq 60$ MPa, and $G = 35$ GPa. For the steel segment BC, the diameter is 50 mm, $\tau \leq 80$ MPa, and $G = 83$ GPa. If $a = 2$ m and $b = 1.5$ m, compute the maximum torque T that can be applied.

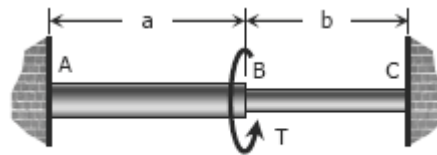
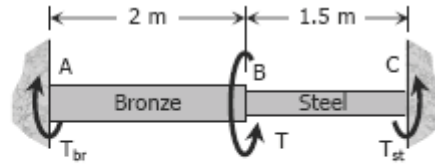


Figure P-319 and P-320

Solution 319



$$\Sigma M = 0$$

$$T = T_{br} + T_{st} \quad \rightarrow \text{Equation (1)}$$

$$\theta_{br} = \theta_{st}$$

$$\left(\frac{TL}{JG} \right)_{br} = \left(\frac{TL}{JG} \right)_{st}$$

$$\frac{T_{br}(2)(1000)}{\frac{1}{32}\pi(75^4)(35000)} = \frac{T_{st}(1.5)(1000)}{\frac{1}{32}\pi(50^4)(83000)}$$

$$\left. \begin{aligned} T_{br} &= 1.6011 T_{st} \\ T_{st} &= 0.6246 T_{br} \end{aligned} \right\} \text{Equations (2)}$$

$$\tau_{\max} = \frac{16T}{\pi D^3}$$

Based on $\tau_{br} \leq 60 \text{ MPa}$

$$60 = \frac{16T_{br}}{\pi(75^3)}$$

$$T_{br} = 4\,970\,097.75 \text{ N}\cdot\text{mm}$$

$$T_{br} = 4.970 \text{ kN}\cdot\text{m} \rightarrow \text{Maximum allowable torque for bronze}$$

$$T_{st} = 0.6246(4.970) \quad \rightarrow \text{From one of Equations (2)}$$

$$T_{st} = 3.104 \text{ kN}\cdot\text{m}$$

Based on $\tau_{st} \leq 80 \text{ MPa}$

$$80 = \frac{16T_{st}}{\pi(50^3)}$$

$$T_{st} = 1\,963\,495.41 \text{ N}\cdot\text{mm}$$

$$T_{st} = 1.963 \text{ kN}\cdot\text{m} \rightarrow \text{maximum allowable torque for steel}$$

$$T_{br} = 1.6011(1.963) \quad \rightarrow \text{From Equations (2)}$$

$$T_{br} = 3.142 \text{ kN}\cdot\text{m}$$

Use $T_{br} = 3.142 \text{ kN}\cdot\text{m}$ and $T_{st} = 1.963 \text{ kN}\cdot\text{m}$

$$T = 3.142 + 1.963 \quad \rightarrow \text{From Equation (1)}$$

$$T = 5.105 \text{ kN}\cdot\text{m}$$

Problem 320

In Prob. 319, determine the ratio of lengths b/a so that each material will be stressed to its permissible limit. What torque T is required?

Solution 320

From Solution 319:

$$\text{Maximum } T_{br} = 4.970 \text{ kN}\cdot\text{m}$$

$$\text{Maximum } T_{st} = 1.963 \text{ kN}\cdot\text{m}$$

$$\theta_{br} = \theta_{st}$$

$$\left(\frac{TL}{JG} \right)_{br} = \left(\frac{TL}{JG} \right)_{st}$$

$$\frac{4.973a (1000^2)}{\frac{1}{32} \pi (75^4) (35000)} = \frac{1.963b (1000^2)}{\frac{1}{32} \pi (50^4) (83000)}$$

$$b/a = 1.187$$

$$T = \max T_{br} + \max T_{st}$$

$$T = 4.970 + 1.963$$

$$T = 6.933 \text{ kN}\cdot\text{m}$$

Problem 321

A torque T is applied, as shown in Fig. P-321, to a solid shaft with built-in ends. Prove that the resisting torques at the walls are $T_1 = Tb/L$ and $T_2 = Ta/L$. How would these values be changed if the shaft were hollow?

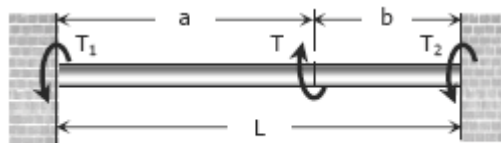


Figure P-321

Solution 321

$$\Sigma M = 0$$

$$T = T_1 + T_2 \quad \rightarrow \text{Equation (1)}$$

$$\theta_1 = \theta_2$$

$$\left(\frac{TL}{JG} \right)_1 = \left(\frac{TL}{JG} \right)_2$$

$$\frac{T_1 a}{JG} = \frac{T_2 b}{JG}$$

$$\left. \begin{aligned} T_1 &= \frac{b}{a} T_2 \\ T_2 &= \frac{a}{b} T_1 \end{aligned} \right\} \text{Equations (2)}$$

Equations (1) and (2) with T_2 in terms of T_1 :

$$T = T_1 + \frac{a}{b} T_1$$

$$T = \frac{T_1 b + T_1 a}{b}$$

$$T = \frac{(b+a)T_1}{b}$$

$$T = \frac{LT_1}{b}$$

$$T_1 = Tb/L$$

Equations (1) and (2) with T_1 in terms of T_2 :

$$T = \frac{b}{a} T_2 + T_2$$

$$T = \frac{T_2 b + T_2 a}{a}$$

$$T = \frac{(b+a)T_2}{a}$$

$$T = \frac{LT_2}{a}$$

$$T_2 = Ta/L$$

If the shaft were hollow, Equation (1) would be the same and the equality $\theta_1 = \theta_2$, by direct investigation, would yield the same result in Equations (2). Therefore, the values of T_1 and T_2 are the same (no change) if the shaft were hollow.

Problem 322

A solid steel shaft is loaded as shown in Fig. P-322. Using $G = 83 \text{ GPa}$, determine the required diameter of the shaft if the shearing stress is limited to 60 MPa and the angle of rotation at the free end is not to exceed 4 deg .

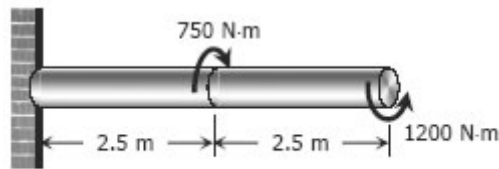
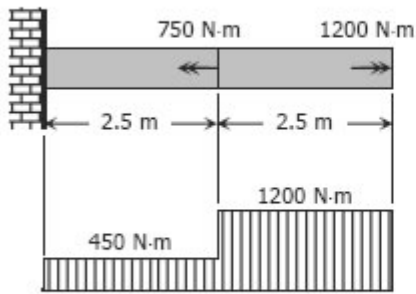


Figure P-322

Solution 322

Based on maximum allowable shear:

$$\tau_{\max} = \frac{16T}{\pi D^3}$$



For the 1st segment:

$$60 = \frac{450(2.5)(1000^2)}{\pi D^3}$$

$$D = 181.39 \text{ mm}$$

For the 2nd segment:

$$60 = \frac{1200(2.5)(1000^2)}{\pi D^3}$$

$$D = 251.54 \text{ mm}$$

Based on maximum angle of twist:

$$\theta = \frac{TL}{JG}$$

$$\theta = \frac{1}{JG} \sum TL$$

$$4^\circ \left(\frac{\pi}{180^\circ} \right) = \frac{1}{\frac{1}{32} \pi D^4 (83000)} [450(2.5) + 1200(2.5)] (1000^2)$$

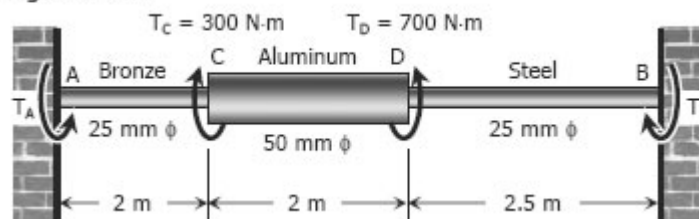
$$D = 51.89 \text{ mm}$$

Use $D = 251.54 \text{ mm}$

Problem 323

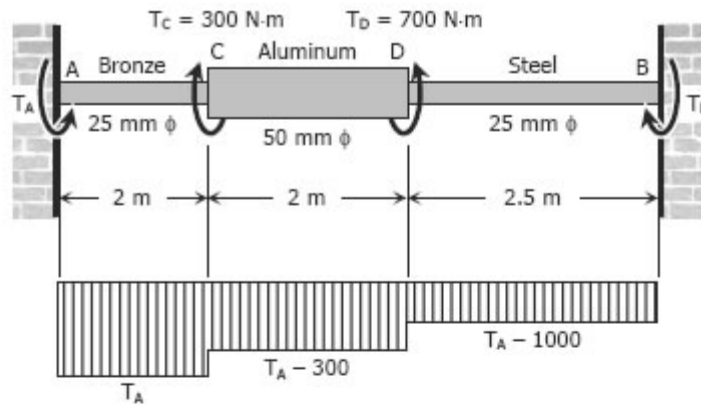
A shaft composed of segments AC, CD, and DB is fastened to rigid supports and loaded as shown in Fig. P-323. For bronze, $G = 35 \text{ GPa}$; aluminum, $G = 28 \text{ GPa}$, and for steel, $G = 83 \text{ GPa}$. Determine the maximum shearing stress developed in each segment.

Figure P-323



Solution 323

Stress developed in each segment with respect to T_A :



The rotation of B relative to A is zero.

$$\theta_{A/B} = 0$$

$$\left(\sum \frac{TL}{JG} \right)_{A/B} = 0$$

$$\frac{T_A(2)(1000^2)}{\frac{1}{32}\pi(25^4)(35000)} + \frac{(T_A - 300)(2)(1000^2)}{\frac{1}{32}\pi(50^4)(28000)} + \frac{(T_A - 1000)(2.5)(1000^2)}{\frac{1}{32}\pi(25^4)(83000)} = 0$$

$$\frac{2T_A}{(25^4)(35)} + \frac{2(T_A - 300)}{(50^4)(28)} + \frac{2.5(T_A - 1000)}{(25^4)(83)} = 0$$

$$\frac{16T_A}{35} + \frac{T_A - 300}{28} + \frac{20(T_A - 1000)}{83} = 0$$

$$\frac{16}{35}T_A + \frac{1}{28}T_A - \frac{75}{7} + \frac{20}{83}T_A - \frac{20000}{83} = 0$$

$$\frac{8527}{11620}T_A = 251.678$$

$$T_A = 342.97 \text{ N}\cdot\text{m}$$

$$\Sigma M = 0$$

$$T_A + T_B = 300 + 700$$

$$342.97 + T_B = 1000$$

$$T_B = 657.03 \text{ N}\cdot\text{m}$$

$$T_{br} = 342.97 \text{ N}\cdot\text{m}$$

$$T_{al} = 342.97 - 300 = 42.97 \text{ N}\cdot\text{m}$$

$$T_{st} = 342.97 - 1000 = -657.03 \text{ N}\cdot\text{m} = -T_B \text{ (ok!)}$$

$$\tau_{\max} = \frac{16T}{\pi D^3}$$

$$\tau_{br} = \frac{16(342.97)(1000)}{\pi(25^3)} = 111.79 \text{ MPa}$$

$$\tau_{al} = \frac{16(42.97)(1000)}{\pi(50^3)} = 1.75 \text{ MPa}$$

$$\tau_{st} = \frac{16(657.03)(1000)}{\pi(25^3)} = 214.16 \text{ MPa}$$

Problem 324

The compound shaft shown in Fig. P-324 is attached to rigid supports. For the bronze segment AB, the maximum shearing stress is limited to 8000 psi and for the steel segment BC, it is limited to 12 ksi. Determine the diameters of each segment so that each material will be simultaneously stressed to its permissible limit when a torque $T = 12 \text{ kip}\cdot\text{ft}$ is applied. For bronze, $G = 6 \times 10^6 \text{ psi}$ and for steel, $G = 12 \times 10^6$ psi.

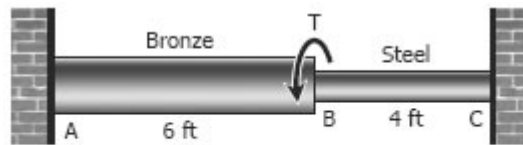


Figure P-324

Solution 324

$$\tau_{\max} = \frac{16T}{\pi D^3}$$

For bronze:

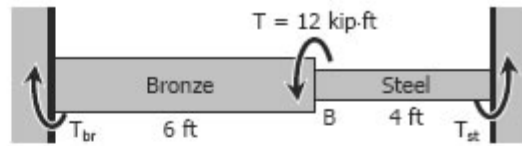
$$8000 = \frac{16T_{br}}{\pi D_{br}^3}$$

$$T_{br} = 500\pi D_{br}^3 \text{ lb}\cdot\text{in}$$

For steel:

$$12\,000 = \frac{16T_{st}}{\pi D_{st}^3}$$

$$T_{st} = 750\pi D_{st}^3 \text{ lb-in}$$



$$\Sigma M = 0$$

$$T_{br} + T_{st} = T$$

$$T_{br} + T_{st} = 12(1000)(12)$$

$$T_{br} + T_{st} = 144\,000 \text{ lb-in}$$

$$500\pi D_{br}^3 + 750\pi D_{st}^3 = 144\,000$$

$$D_{br}^3 = 288/\pi - 1.5 D_{st}^3 \rightarrow \text{equation (1)}$$

$$\theta_{br} = \theta_{st}$$

$$\left(\frac{TL}{JG}\right)_{br} = \left(\frac{TL}{JG}\right)_{st}$$

$$\frac{T_{br}(6)}{\frac{1}{32}\pi D_{br}^4(6 \times 10^6)} = \frac{T_{st}(4)}{\frac{1}{32}\pi D_{st}^4(12 \times 10^6)}$$

$$\frac{T_{br}}{D_{br}^4} = \frac{T_{st}}{3D_{st}^4}$$

$$\frac{500\pi D_{br}^3}{D_{br}^4} = \frac{750\pi D_{st}^3}{3D_{st}^4}$$

$$D_{st} = 0.5D_{br}$$

From Equation (1)

$$D_{br}^3 = 288/\pi - 1.5(0.5D_{br})^3$$

$$1.1875 D_{br}^3 = 288/\pi$$

$$D_{br} = 4.26 \text{ in.}$$

$$D_{st} = 0.5(4.26) = 2.13 \text{ in.}$$

Problem 325

The two steel shaft shown in Fig. P-325, each with one end built into a rigid support have flanges rigidly attached to their free ends. The shafts are to be bolted together at their flanges. However, initially there is a 6° mismatch in the location of the bolt holes as shown in the figure. Determine the maximum shearing stress in each shaft after the shafts are bolted together. Use $G = 12 \times 10^6$ psi and neglect deformations of the bolts and

flanges.

Solution 325

$$\theta_{\text{of } 6.5' \text{ shaft}} + \theta_{\text{of } 3.25' \text{ shaft}} = 6^\circ$$

$$\left(\frac{TL}{JG} \right)_{\text{of } 6.5' \text{ shaft}} + \left(\frac{TL}{JG} \right)_{\text{of } 3.25' \text{ shaft}} = 6^\circ \left(\frac{\pi}{180^\circ} \right)$$

$$\frac{T(6.5)(12^2)}{\frac{1}{32}\pi(2^4)(12 \times 10^6)} + \frac{T(3.25)(12^2)}{\frac{1}{32}\pi(1.5^4)(12 \times 10^6)} = \frac{\pi}{30}$$

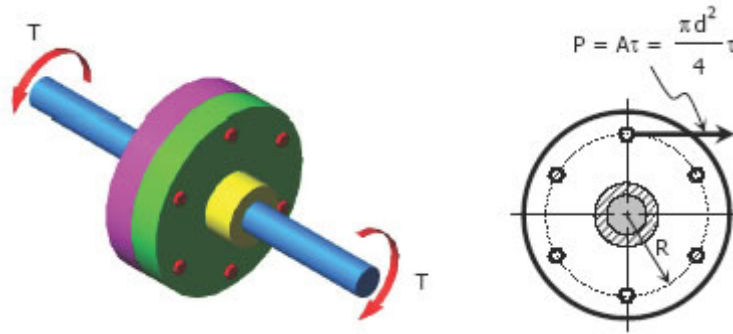
$$T = 817.32 \text{ lb-ft}$$

$$\tau_{\max} = \frac{16T}{\pi D^3}$$

$$\tau_{\text{of } 6.5' \text{ shaft}} = \frac{16(817.32)(12)}{\pi(2^3)} = 6243.86 \text{ psi}$$

$$\tau_{\text{of } 3.25' \text{ shaft}} = \frac{16(817.32)(12)}{\pi(1.5^3)} = 14\,800.27 \text{ psi}$$

Flanged Bolt Couplings

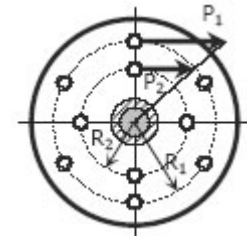


In shaft connection called flanged bolt couplings (see figure above), the torque is transmitted by the shearing force P created in the bolts that is assumed to be uniformly distributed. For any number of bolts n , the torque capacity of the coupling is

$$T = PRn = \frac{\pi d^2}{4} \tau Rn$$

If a coupling has two concentric rows of bolts, the torque capacity is

$$T = P_1 R_1 n_1 + P_2 R_2 n_2$$



where the subscript 1 refers to bolts on the outer circle and subscript 2 refers to bolts on the inner circle. See figure.

For rigid flanges, the shear deformations in the bolts are proportional to their radial distances from the shaft axis. The shearing strains are related by

$$\frac{\gamma_1}{R_1} = \frac{\gamma_2}{R_2}$$

Using Hooke's law for shear, $G = \tau / \gamma$, we have

$$\frac{\tau_1}{G_1 R_1} = \frac{\tau_2}{G_2 R_2} \quad \text{or} \quad \frac{P_1 / A_1}{G_1 R_1} = \frac{P_2 / A_2}{G_2 R_2}$$

If the bolts on the two circles have the same area, $A_1 = A_2$, and if the bolts are made of the same material, $G_1 = G_2$, the relation between P_1 and P_2 reduces to

$$\frac{P_1}{R_1} = \frac{P_2}{R_2}$$

Solved Problems in Flanged Bolt Couplings

Problem 326

A flanged bolt coupling consists of ten 20-mm-diameter bolts spaced evenly around a bolt circle 400 mm in diameter. Determine the torque capacity of the coupling if the allowable shearing stress in the bolts is 40 MPa.

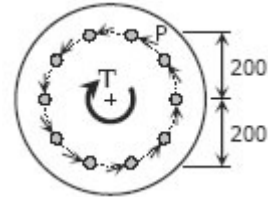
Solution 326

$$T = PRn = A\tau Rn = \frac{1}{4} \pi d^2 \tau Rn$$

$$T = \frac{1}{4} \pi (20^2)(40)(200)(10)$$

$$T = 8\,000\,000\pi \text{ N}\cdot\text{mm}$$

$$T = 8\pi \text{ kN}\cdot\text{m} = 25.13 \text{ kN}\cdot\text{m}$$



Problem 327

A flanged bolt coupling consists of ten steel $\frac{1}{2}$ -in.-diameter bolts spaced evenly around a bolt circle 14 in. in diameter. Determine the torque capacity of the coupling if the allowable shearing stress in the bolts is 6000 psi.

Solution 327

$$T = PRn = A\tau Rn = \frac{1}{4} \pi d^2 \tau Rn$$

$$T = \frac{1}{4} \pi (\frac{1}{2})^2 (6000)(7)(10)$$

$$T = 26250\pi \text{ lb}\cdot\text{in}$$

$$T = 2187.5\pi \text{ lb}\cdot\text{ft} = 6872.23 \text{ lb}\cdot\text{ft}$$

Problem 328

A flanged bolt coupling consists of eight 10-mm-diameter steel bolts on a bolt circle 400 mm in diameter, and six 10-mm-diameter steel bolts on a concentric bolt circle 300 mm in diameter, as shown in Fig. 3-7. What torque can be applied without exceeding a shearing stress of 60 MPa in the bolts?

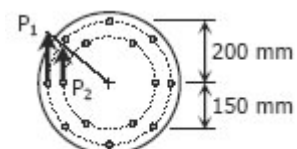


Figure 3-7

Solution 328

For one bolt in the outer circle:

$$P_1 = A\tau = \frac{\pi(10^2)}{4} (60)$$

$$P_1 = 1500\pi \text{ N}$$

For one bolt in the inner circle:

$$\frac{P_1}{R_1} = \frac{P_2}{R_2}$$

$$\frac{1500\pi}{200} = \frac{P_2}{150}$$

$$P_2 = 1125\pi \text{ N}$$

$$T = P_1 R_1 n_1 + P_2 R_2 n_2$$

$$T = 1500\pi(200)(8) + 1125\pi(150)(6)$$

$$T = 3\,412\,500\pi \text{ N}\cdot\text{mm}$$

$$T = 3.4125\pi \text{ kN}\cdot\text{m} = \mathbf{10.72 \text{ kN}\cdot\text{m}}$$

Problem 329

A torque of 700 lb-ft is to be carried by a flanged bolt coupling that consists of eight $\frac{1}{2}$ -in.-diameter steel bolts on a circle of diameter 12 in. and six $\frac{1}{2}$ -in.-diameter steel bolts on a circle of diameter 9 in. Determine the shearing stress in the bolts.

Solution 329

$$\frac{P_1}{R_1} = \frac{P_2}{R_2}$$

$$\frac{A\tau_1}{6} = \frac{A\tau_2}{4.5}$$

$$\tau_2 = 0.75\tau_1$$

$$T = P_1 R_1 n_1 + P_2 R_2 n_2$$

$$700(12) = \frac{1}{4} \pi (\frac{1}{2})^2 \tau_1 (6)(8) + \frac{1}{4} \pi (\frac{1}{2})^2 \tau_2 (4.5)(6)$$

$$8400 = 3\pi\tau_1 + 1.6875\pi(0.75\tau_1)$$

$$8400 = 13.4\tau_1$$

$$\tau_1 = \mathbf{626.87 \text{ psi}} \rightarrow \text{bolts in the outer circle}$$

$$\tau_2 = 0.75(626.87) = \mathbf{470.15 \text{ psi}} \rightarrow \text{bolts in the inner circle}$$

Problem 330

Determine the number of 10-mm-diameter steel bolts that must be used on the 400-mm bolt circle of the coupling described in Prob. 328 to increase the torque capacity to 14 kN·m

Solution 330

$$\begin{aligned}
 T &= P_1 R_1 n_1 + P_2 R_2 n_2 \\
 14(1000^2) &= 1500\pi(200)n_1 + 1125\pi(150)(6) \\
 n_1 &= 11.48 \text{ say } 12 \text{ bolts}
 \end{aligned}$$

Problem 331

A flanged bolt coupling consists of six $\frac{1}{2}$ -in. steel bolts evenly spaced around a bolt circle 12 in. in diameter, and four $\frac{3}{4}$ -in. aluminum bolts on a concentric bolt circle 8 in. in diameter. What torque can be applied without exceeding 9000 psi in the steel or 6000 psi in the aluminum? Assume $G_{st} = 12 \times 10^6$ psi and $G_{al} = 4 \times 10^6$ psi.

Solution 331

$$\begin{aligned}
 T &= (PRn)_{st} + (PRn)_{al} \\
 T &= (A\tau Rn)_{st} + (A\tau Rn)_{al} \\
 T &= \frac{1}{4}\pi(\frac{1}{2})^2\tau_{st}(6)(6) + \frac{1}{4}\pi(\frac{3}{4})^2\tau_{al}(4)(4) \\
 T &= 2.25\pi\tau_{st} + 2.25\pi\tau_{al} \\
 T &= 2.25\pi(\tau_{st} + \tau_{al}) \quad \rightarrow \text{Equation (1)}
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{\tau}{GR}\right)_{st} &= \left(\frac{\tau}{GR}\right)_{al} \\
 \frac{\tau_{st}}{(12 \times 10^6)(6)} &= \frac{\tau_{al}}{(4 \times 10^6)(4)} \\
 \tau_{st} &= \frac{9}{2}\tau_{al} \quad \rightarrow \text{Equation (2a)} \\
 \tau_{al} &= \frac{2}{9}\tau_{st} \quad \rightarrow \text{Equation (2b)}
 \end{aligned}$$

Equations (1) and (2a)

$$\begin{aligned}
 T &= 2.25\pi\left(\frac{9}{2}\tau_{al} + \tau_{al}\right) = 12.375\pi\tau_{al} \\
 T &= 12.375\pi(6000) = 74\,250\pi \text{ lb}\cdot\text{in} \\
 T &= 233.26 \text{ kip}\cdot\text{in}
 \end{aligned}$$

Equations (1) and (2b)

$$\begin{aligned}
 T &= 2.25\pi\left(\tau_{st} + \frac{2}{9}\tau_{st}\right) = 2.75\pi\tau_{st} \\
 T &= 2.25\pi(9000) = 24\,750\pi \text{ lb}\cdot\text{in} \\
 T &= 77.75 \text{ kip}\cdot\text{in}
 \end{aligned}$$

Use $T = 77.75 \text{ kip}\cdot\text{in}$

Problem 332

In a rivet group subjected to a twisting couple T , show that the torsion formula $\tau = T\rho/J$ can be used to find the shearing stress t at the center of any rivet. Let $J = \Sigma A\rho^2$, where A is the area of a rivet at the radial distance ρ from the centroid of the rivet group.

Solution 332

The shearing stress on each rivet is P/A

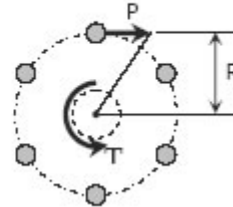
$$\tau = T\rho/J$$

Where: $T = PRn$
 $\rho = R$
 $J = \Sigma A\rho^2 = AR^2n$

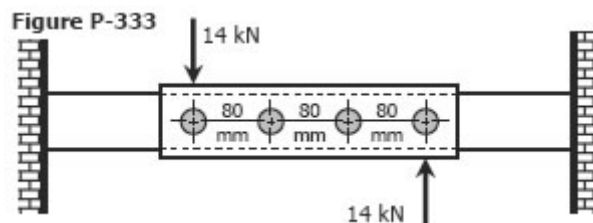
$$\tau = \frac{PRn(R)}{AR^2n}$$

$$\tau = \frac{P}{A} \quad (\text{ok!})$$

This shows that $\tau = T\rho/J$ can be used to find the shearing stress at the center of any rivet.

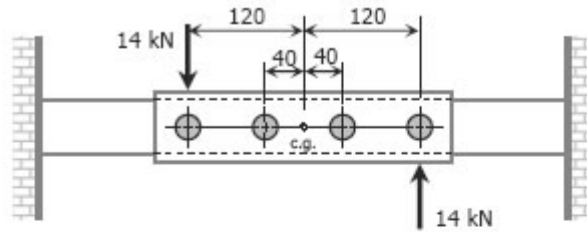
**Problem 333**

A plate is fastened to a fixed member by four 20-mm diameter rivets arranged as shown in Fig. P-333. Compute the maximum and minimum shearing stress developed.



Solution 333

$$\tau = \frac{T\rho}{J}$$



Where:

$$T = 14(1000)(120) = 1\,680\,000 \text{ N}\cdot\text{mm}$$

$$J = \sum A\rho^2 = \frac{1}{4}(\pi)(20)^2[2(40^2) + 2(120^2)]$$

$$= 3\,200\,000\pi \text{ mm}^4$$

Maximum shearing stress ($\rho = 120 \text{ mm}$):

$$\tau_{\max} = \frac{1\,680\,000(120)}{3\,200\,000\pi}$$

$$\tau_{\max} = 20.05 \text{ MPa}$$

Minimum shearing stress ($\rho = 40 \text{ mm}$):

$$\tau_{\min} = \frac{1\,680\,000(40)}{3\,200\,000\pi}$$

$$\tau_{\min} = 6.68 \text{ MPa}$$

Problem 334

Six 7/8-in.-diameter rivets fasten the plate in Fig. P-334 to the fixed member. Using the results of Prob. 332, determine the average shearing stress caused in each rivet by the 14 kip loads. What additional loads P can be applied before the shearing stress in any rivet exceeds 8000 psi?

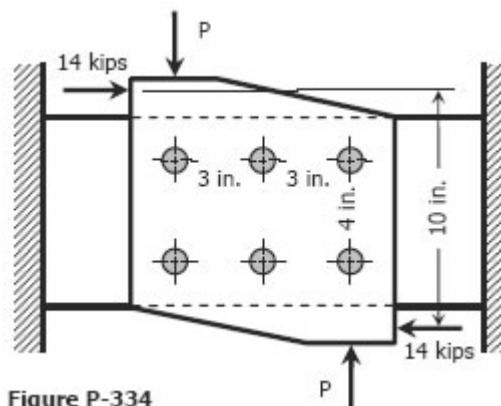


Figure P-334

Solution 334

Without the loads P :

$$\tau = \frac{T\rho}{J}$$

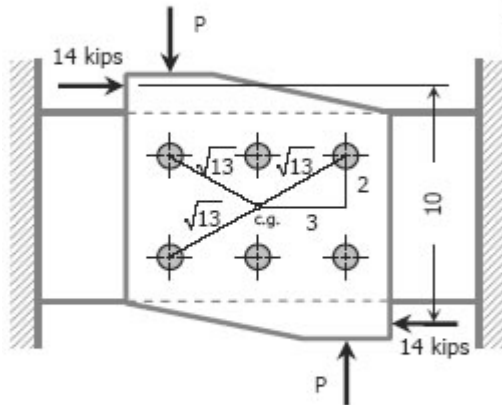
Where: $T = 14(10) = 140 \text{ kip}\cdot\text{in}$

$$\rho = \sqrt{13} \text{ in}$$

$$J = \sum A\rho^2 = \frac{1}{4}(\pi n)\left(\frac{7}{8}\right)^2 [4(\sqrt{13})^2 + 2(2)^2] = 36.08 \text{ in}^4$$

$$\tau_{\text{maximum}} = \frac{140(\sqrt{13})}{36.08} = 14.0 \text{ ksi}$$

$$\tau_{\text{minimum}} = \frac{140(2)}{36.08} = 7.76 \text{ ksi}$$



With the loads P , two cases will arise:

1st case ($P < 14 \text{ kips}$)

$$T = 10(14) - 6P = (140 - 6P) \text{ kip}\cdot\text{in}$$

$$\tau = \frac{T\rho}{J}$$

$$8000 = \frac{(140 - 6P)(1000)(\sqrt{13})}{36.08}$$

$$80.05 = 140 - 6P$$

$$P = 10.0 \text{ kips}$$

2nd case ($P > 14 \text{ kips}$)

$$T = 6P - 10(14) = (6P - 140) \text{ kip}\cdot\text{in}$$

$$\tau = \frac{T\rho}{J}$$

$$8000 = \frac{(6P - 140)(1000)(\sqrt{13})}{36.08}$$

$$80.05 = 6P - 140$$

$$P = 36.68 \text{ kips}$$

Problem 335

The plate shown in Fig. P-335 is fastened to the fixed member by five 10-mm-diameter rivets. Compute the value of the loads P so that the average shearing stress in any rivet does not exceed 70 MPa. (Hint: Use the results of Prob. 332.)

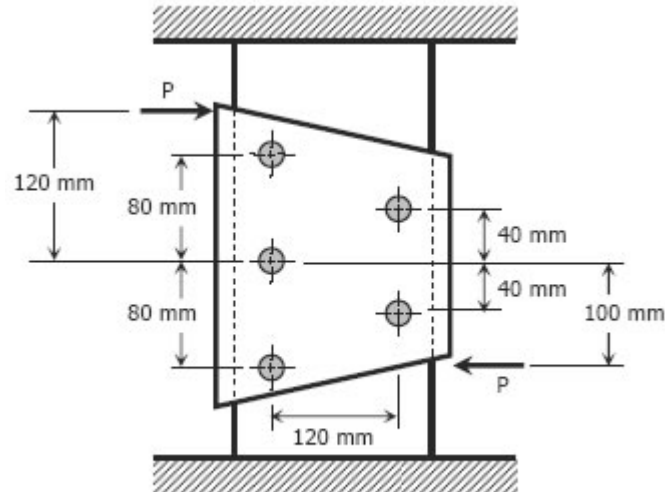


Figure P-335

Solution 335

Solving for location of centroid of rivets:

$$A X_G = \sum ax$$

$$\text{Where } A = \frac{1}{2} (80 + 160)(80) = 9600 \text{ mm}^2$$

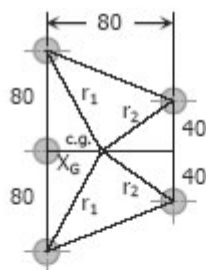
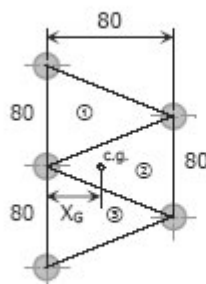
$$a_1 = a_2 = a_3 = \frac{1}{2} (80)(80) = 3200 \text{ mm}^2$$

$$x_1 = x_3 = \frac{1}{3} (80) = 80/3 \text{ mm}$$

$$x_2 = \frac{2}{3} (80) = 160/3 \text{ mm}$$

$$9600 X_G = 3200(80/3) + 3200(160/3) + 3200(80/3)$$

$$X_G = 320/9 \text{ mm}$$



$$r_1 = \sqrt{(320/9)^2 + 80^2} = 87.54 \text{ mm}$$

$$r_2 = \sqrt{(80 - 320/9)^2 + 40^2} = 59.79 \text{ mm}$$

$$J = \sum A \rho^2 = \frac{1}{4} \pi (10^2) (2r_1^2 + 2r_2^2 + X_G^2)$$

$$J = \frac{1}{4} \pi (10^2) [2(87.54)^2 + 2(59.79)^2 + (320/9)^2]$$

$$J = 1\,864\,565.79 \text{ mm}^4$$

$$T = (120 + 100)P = 220P$$

The critical rivets are at distance r_1 from centroid:

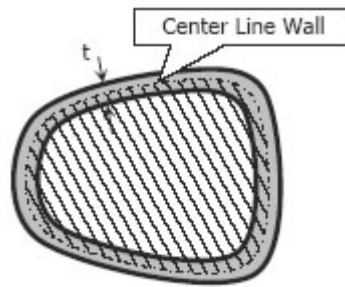
$$\tau = \frac{T \rho}{J}$$

$$70 = \frac{220P(87.54)}{1\,864\,565.79}$$

$$P = 6777.14 \text{ N}$$

Torsion of Thin-Walled Tubes

The torque applied to thin-walled tubes is expressed as



$$T = 2Aq$$

where T is the torque in $\text{N}\cdot\text{mm}$, A is the area enclosed by the centerline of the tube (as shown in the stripefilled portion) in mm^2 , and q is the shear flow in N/mm .

The average shearing stress across any thickness t is

$$\tau = \frac{q}{t} = \frac{T}{2At}$$

Thus, torque T can also be expressed as

$$T = 2A\tau t$$

Solved Problems in Torsion of Thin-Walled Tubes

Problem 337

A torque of $600 \text{ N}\cdot\text{m}$ is applied to the rectangular section shown in Fig. P-337.

Determine the wall thickness t so as not to exceed a shear stress of 80 MPa . What is the shear stress in the short sides? Neglect stress concentration at the corners.

Solution 337

$$T = 2A\tau t$$

$$\text{Where: } T = 600 \text{ N}\cdot\text{m} = 600\,000 \text{ N}\cdot\text{mm}$$

$$A = 30(80) = 2400 \text{ mm}^2$$

$$\tau = 80 \text{ MPa}$$

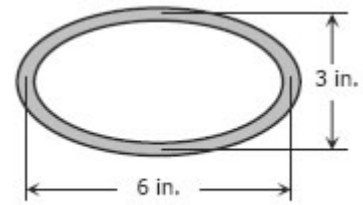
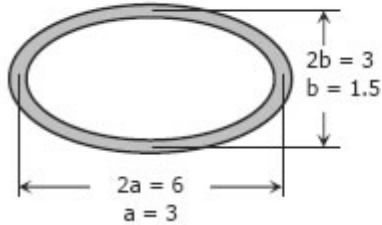
$$600\,000 = 2(2400)(t)(80)$$

$$t = 1.5625 \text{ mm}$$

At any convenient center O within the section, the farthest side is the shortest side, thus, it is induced with the maximum allowable shear stress of 80 MPa .

Problem 338

A tube 0.10 in. thick has an elliptical shape shown in Fig. P-338. What torque will cause a shearing stress of 8000 psi?

**Figure P-338****Solution 338**

$$T = 2At\tau$$

Where: $A = \pi ab = \pi(3)(1.5) = 4.5\pi \text{ in}^2$
 $t = 0.10 \text{ in}$
 $\tau = 8000 \text{ psi}$

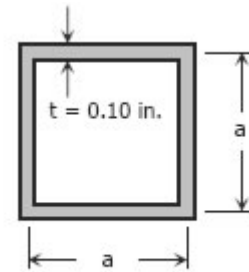
$$T = 2(4.5\pi)(0.10)(8000)$$

$$T = 22619.47 \text{ lb-in}$$

$$T = 22.62 \text{ kip-in}$$

Problem 339

A torque of 450 lb-ft is applied to the square section shown in Fig. P-339. Determine the smallest permissible dimension a if the shearing stress is limited to 6000 psi.

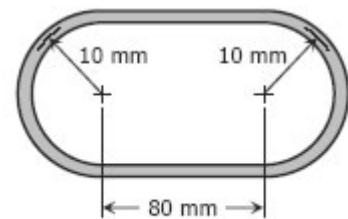
**Figure P-339****Solution 339**

$$T = 2At\tau$$

Where: $T = 450 \text{ lb-ft}$
 $T = 450(12) \text{ lb-in}$
 $A = a^2$
 $\tau = 6000 \text{ psi}$
 $450(12) = 2a^2(0.10)(6000)$
 $a = 2.12 \text{ in}$

Problem 340

A tube 2 mm thick has the shape shown in Fig. P-340. Find the shearing stress caused by a torque of 600 N·m.

**Figure P-340****Solution 340**

$$T = 2At\tau$$

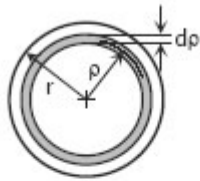
Where: $A = \pi(10^2) + 80(20) = 1914.16 \text{ mm}^2$
 $t = 2 \text{ mm}$
 $T = 600 \text{ N-m} = 600\,000 \text{ N-mm}$

$$600\,000 = 2(1914.16)(2)\tau$$

$$\tau = 78.36 \text{ MPa}$$

Problem 341

Derive the torsion formula $\tau = T\rho/J$ for a solid circular section by assuming the section is composed of a series of concentric thin circular tubes. Assume that the shearing stress at any point is proportional to its radial distance.

Solution 341

$$T = 2At\tau$$

$$\text{Where: } T = dT; A = \pi\rho^2; t = d\rho$$

$$\frac{\tau}{\rho} = \frac{\tau_{\max}}{r}; \tau = \frac{\tau_{\max}\rho}{r}$$

$$dT = 2\pi(\rho^2) d\rho \left(\frac{\tau_{\max}\rho}{r} \right)$$

$$T = \frac{2\pi\tau_{\max}}{r} \int_0^r \rho^3 d\rho$$

$$T = \frac{2\pi\tau_{\max}}{r} \left[\frac{\rho^4}{4} \right]_0^r$$

$$T = \frac{2\pi\tau_{\max}}{r} \left(\frac{r^4}{4} \right)$$

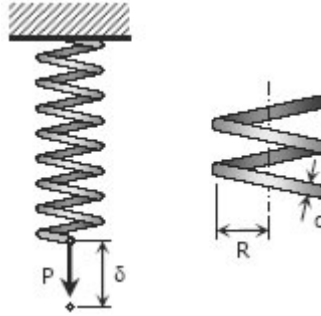
$$T = \frac{\tau_{\max}}{r} \left(\frac{\pi r^4}{2} \right)$$

$$T = \frac{\tau_{\max}}{r} J$$

$$\tau_{\max} = \frac{Tr}{J} \text{ and it follows that } \tau = \frac{T\rho}{J}$$

Helical Springs

When close-coiled helical spring, composed of a wire of round rod of diameter d wound into a helix of mean radius R with n number of turns, is subjected to an axial load P produces the following stresses and elongation:



The maximum shearing stress is the sum of the direct shearing stress $\tau_1 = P/A$ and the torsional shearing stress $\tau_2 = Tr/J$, with $T = PR$.

$$\tau = \tau_1 + \tau_2 = \frac{P}{\pi d^2/4} + \frac{16(PR)}{\pi d^3}$$

$$\tau = \frac{16PR}{\pi d^3} \left(1 + \frac{d}{4R} \right)$$

This formula neglects the curvature of the spring. This is used for light spring where the ratio $d/4R$ is small.

For heavy springs and considering the curvature of the spring, a more precise formula is given by: (A.M.Wahl Formula)

$$\tau = \frac{16PR}{\pi d^3} \left(\frac{4m-1}{4m-4} + \frac{0.615}{m} \right)$$

where m is called the spring index and $(4m - 1) / (4m - 4)$ is the Wahl Factor.

The elongation of the bar is

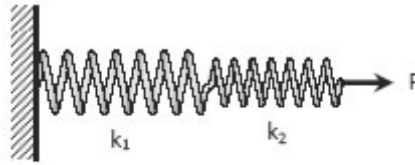
$$\delta = \frac{64PR^3n}{Gd^4}$$

Notice that the deformation δ is directly proportional to the applied load P . The ratio of P to δ is called the spring constant k and is equal to

$$k = \frac{P}{\delta} = \frac{Gd^4}{64R^3n} \text{ in N/mm}$$

SPRINGS IN SERIES

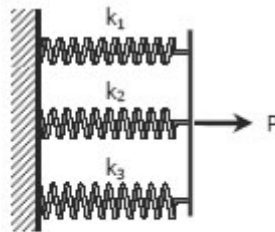
For two or more springs with spring laid in series, the resulting spring constant k is given by



$$1/k = 1/k_1 + 1/k_2 + \dots$$

where $k_1, k_2, \dots$ are the spring constants for different springs.

SPRINGS IN PARALLEL



$$k = k_1 + k_2 + \dots$$

Solved Problems in Helical Springs

Problem 343

Determine the maximum shearing stress and elongation in a helical steel spring composed of 20 turns of 20-mm-diameter wire on a mean radius of 90 mm when the spring is supporting a load of 1.5 kN. Use Eq. (3-10) and $G = 83 \text{ GPa}$.

Solution 343

$$\tau_{\max} = \frac{16PR}{\pi d^3} \left(\frac{4m-1}{4m-4} + \frac{0.615}{m} \right) \quad \rightarrow \text{Equation (3-10)}$$

Where $P = 1.5 \text{ kN} = 1500 \text{ N}; R = 90 \text{ mm}$
 $d = 20 \text{ mm}; n = 20 \text{ turns}$
 $m = 2R/d = 2(90)/20 = 9$

$$\tau_{\max} = \frac{16(1500)(90)}{\pi(20^3)} \left[\frac{4(9)-1}{4(9)-4} + \frac{0.615}{9} \right]$$

$$\tau_{\max} = 99.87 \text{ MPa}$$

$$\delta = \frac{64PR^3n}{Gd^4} = \frac{64(1500)(90^3)(20)}{83\,000(20^4)}$$

$$\delta = 105.4 \text{ mm}$$

Problem 344

Determine the maximum shearing stress and elongation in a bronze helical spring composed of 20 turns of 1.0-in.-diameter wire on a mean radius of 4 in. when the spring is supporting a load of 500 lb. Use Eq. (3-10) and $G = 6 \times 10^6$ psi.

Solution 344

$$\tau_{\max} = \frac{16PR}{\pi d^3} \left(\frac{4m-1}{4m-4} + \frac{0.615}{m} \right) \quad \rightarrow \text{Equation (3-10)}$$

Where $P = 500$ lb; $R = 4$ in
 $d = 1$ in; $n = 20$ turns
 $m = 2R/d = 2(4)/1 = 8$

$$\tau_{\max} = \frac{16(500)(4)}{\pi(1^3)} \left[\frac{4(8)-1}{4(8)-4} + \frac{0.615}{8} \right]$$

$$\tau_{\max} = 12\,060.3 \text{ psi} = 12.1 \text{ ksi}$$

$$\delta = \frac{64PR^3n}{Gd^4} = \frac{64(500)(4^3)(20)}{(6 \times 10^6)(1^4)}$$

$$\delta = 6.83 \text{ in}$$

Problem 345

A helical spring is fabricated by wrapping wire $\frac{3}{4}$ in. in diameter around a forming cylinder 8 in. in diameter. Compute the number of turns required to permit an elongation of 4 in. without exceeding a shearing stress of 18 ksi. Use Eq. (3-9) and $G = 12 \times 10^6$ psi.

Solution 345

$$\tau_{\max} = \frac{16PR}{\pi d^3} \left(1 + \frac{d}{4R} \right) \quad \rightarrow \text{Equation (3-9)}$$

$$18000 = \frac{16P(4)}{\pi(3/4)^3} \left[1 + \frac{3/4}{4(4)} \right]$$

$$P = 356.07 \text{ lb}$$

$$\delta = \frac{64PR^3n}{Gd^4}$$

$$4 = \frac{64(356.07)(4^3)n}{(12 \times 10^6)(3/4)^3}$$

$$n = 13.88 \text{ say } 14 \text{ turns}$$

Problem 346

Compute the maximum shearing stress developed in a phosphor bronze spring having mean diameter of 200 mm and consisting of 24 turns of 200-mm-diameter wire when the spring is stretched 100 mm. Use Eq. (3-10) and $G = 42$ GPa.

Solution 346

$$\delta = \frac{64PR^3}{Gd^4}$$

Where $\delta = 100 \text{ mm}; \quad R = 100 \text{ mm}$
 $d = 20 \text{ mm}; \quad n = 24 \text{ turns}$
 $G = 42\,000 \text{ MPa}$

$$100 = \frac{64P(100^3)24}{42\,000(20^4)}$$

$$P = 437.5 \text{ N}$$

$$\tau_{\max} = \frac{16PR}{\pi d^3} \left(\frac{4m-1}{4m-4} + \frac{0.615}{m} \right) \quad \rightarrow \text{Equation (3-10)}$$

Where $m = 2R/d$
 $= 2(100)/20 = 10$

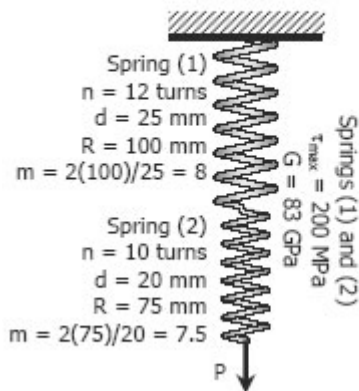
$$\tau_{\max} = \frac{16(437.5)(100)}{\pi(20^3)} \left[\frac{2(10)-1}{2(10)-4} + \frac{0.615}{10} \right]$$

$$\tau_{\max} = 34.79 \text{ MPa}$$

Problem 347

Two steel springs arranged in series as shown in Fig. P-347 supports a load P . The upper spring has 12 turns of 25-mm-diameter wire on a mean radius of 100 mm. The lower spring consists of 10 turns of 20-mm-diameter wire on a mean radius of 75 mm. If the maximum shearing stress in either spring must not exceed 200 MPa, compute the maximum value of P and the total elongation of the assembly. Use Eq. (3-10) and $G = 83 \text{ GPa}$. Compute the equivalent spring constant by dividing the load by the total elongation.

Solution 347



$$\tau_{\max} = \frac{16PR}{\pi d^3} \left(\frac{4m-1}{4m-4} + \frac{0.615}{m} \right) \rightarrow \text{Equation 3-10}$$

For Spring (1)

$$200 = \frac{16P(100)}{\pi(25^3)} \left[\frac{4(8)-1}{4(8)-4} + \frac{0.615}{8} \right]$$

$$P = 5182.29 \text{ N}$$

For Spring (2)

$$200 = \frac{16P(75)}{\pi(20^3)} \left[\frac{4(7.5)-1}{4(7.5)-4} + \frac{0.615}{7.5} \right]$$

$$P = 3498.28 \text{ N}$$

Use $P = 3498.28 \text{ N}$

Total elongation:

$$\delta = \delta_1 + \delta_2$$

$$\delta = \left(\frac{64PR^3n}{Gd^4} \right)_1 + \left(\frac{64PR^3n}{Gd^4} \right)_2$$

$$\delta = \frac{64(3498.28)(100^3)(12)}{83\,000(25^4)} + \frac{64(3498.28)(75^3)(10)}{83\,000(20^4)}$$

$$\delta = 153.99 \text{ mm}$$

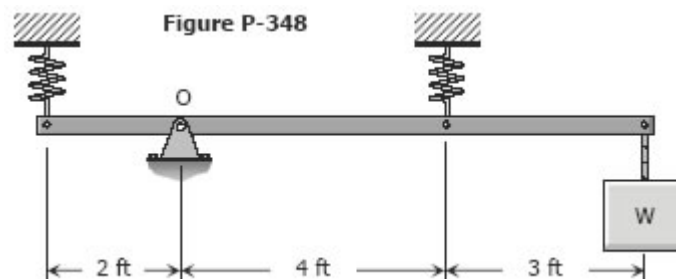
Equivalent spring constant, $k_{\text{equivalent}}$:

$$k_{\text{equivalent}} = \frac{P}{\delta} = \frac{3498.28}{153.99}$$

$$k_{\text{equivalent}} = 22.72 \text{ N/mm}$$

Problem 348

A rigid bar, pinned at O, is supported by two identical springs as shown in Fig. P-348. Each spring consists of 20 turns of $\frac{3}{4}$ -in-diameter wire having a mean diameter of 6 in. Determine the maximum load W that may be supported if the shearing stress in the springs is limited to 20 ksi. Use Eq. (3-9).



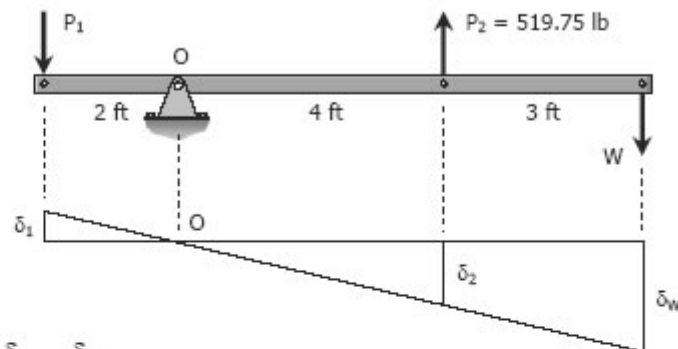
Solution 348

$$\tau_{\max} = \frac{16PR}{\pi d^3} \left(1 + \frac{d}{4R} \right) \quad \rightarrow \text{Equation (3-9)}$$

$$20\,000 = \frac{16P(3)}{\pi(3/4)^3} \left[1 + \frac{3/4}{4(3)} \right]$$

$$P = 519.75 \text{ lb}$$

For this problem, the critical spring is the one subjected to tension. Use $P_2 = 519.75 \text{ lb}$.



$$\frac{\delta_1}{2} = \frac{\delta_2}{4}$$

$$\delta_1 = \frac{1}{2} \delta_2$$

$$\frac{64P_1R^3n}{Gd^4} = \frac{1}{2} \left(\frac{64P_2R^3n}{Gd^4} \right)$$

$$P_1 = \frac{1}{2} P_2 = \frac{1}{2} (519.75)$$

$$P_1 = 259.875 \text{ lb}$$

$$\Sigma M_O = 0$$

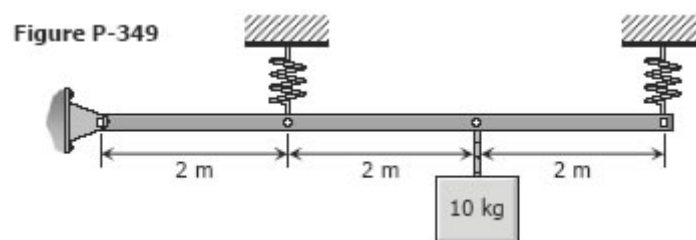
$$7W = 2P_1 + 4P_2$$

$$7W = 2(259.875) + 4(519.75)$$

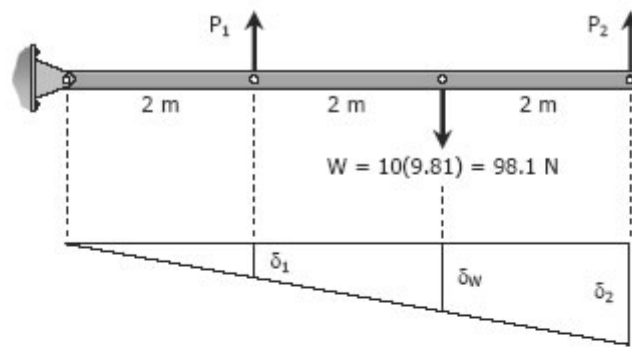
$$W = 371.25 \text{ lb}$$

Problem 349

A rigid bar, hinged at one end, is supported by two identical springs as shown in Fig. P-349. Each spring consists of 20 turns of 10-mm wire having a mean diameter of 150 mm. Compute the maximum shearing stress in the springs, using Eq. (3-9). Neglect the mass of the rigid bar.



Solution 349



$$\frac{\delta_1}{2} = \frac{\delta_2}{6}$$

$$\delta_1 = \frac{1}{3} \delta_2$$

$$\frac{64P_1R^3n}{Gd^4} = \frac{1}{3} \left(\frac{64P_2R^3n}{Gd^4} \right)$$

$$P_1 = \frac{1}{3} P_2$$

$$\sum M_{\text{at hinged support}} = 0$$

$$2P_1 + 6P_2 = 4(98.1)$$

$$2\left(\frac{1}{3} P_2\right) + 6P_2 = 4(98.1)$$

$$P_2 = 58.86 \text{ N}$$

$$P_1 = \frac{1}{3} (58.86) = 19.62 \text{ N}$$

$$\tau_{\max} = \frac{16PR}{\pi d^3} \left(1 + \frac{d}{4R} \right) \quad \rightarrow \text{Equation (3-9)}$$

For spring at left:

$$\tau_{\max 1} = \frac{16(19.62)(75)}{\pi(10^3)} \left[1 + \frac{10}{4(75)} \right]$$

$$\tau_{\max 1} = 7.744 \text{ MPa}$$

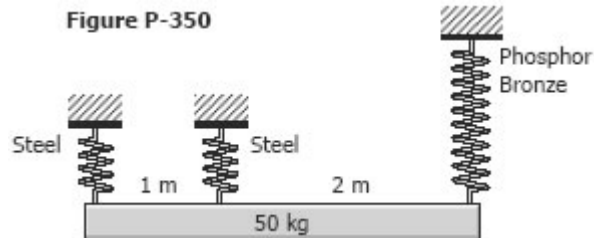
For spring at right:

$$\tau_{\max 2} = \frac{16(58.86)(75)}{\pi(10^3)} \left[1 + \frac{10}{4(75)} \right]$$

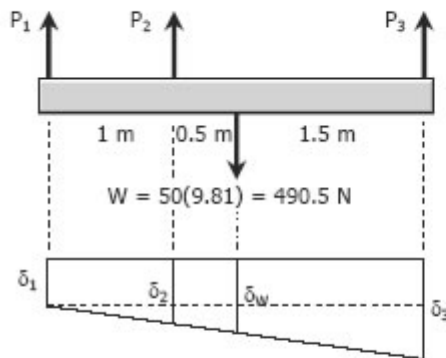
$$\tau_{\max 2} = 23.232 \text{ MPa}$$

Problem 350

As shown in Fig. P-350, a homogeneous 50-kg rigid block is suspended by the three springs whose lower ends were originally at the same level. Each steel spring has 24 turns of 10-mm-diameter on a mean diameter of 100 mm, and $G = 83$ GPa. The bronze spring has 48 turns of 20-mm-diameter wire on a mean diameter of 150 mm, and $G = 42$ GPa. Compute the maximum shearing stress in each spring using Eq. (3-9).



Solution 350



$$\sum F_V = 0$$

$$P_1 + P_2 + P_3 = 490.5 \quad \rightarrow \text{Equation (1)}$$

$$\sum M_1 = 0$$

$$P_2(1) + P_3(3) = 490.5(1.5)$$

$$P_2 + 3P_3 = 735.75 \quad \rightarrow \text{Equation (2)}$$

$$\frac{\delta_2 - \delta_1}{1} = \frac{\delta_3 - \delta_1}{3}$$

$$\delta_2 = \frac{1}{3} \delta_3 + \frac{2}{3} \delta_1$$

$$\frac{64P_2(50^3)(24)}{83000(10^4)} = \frac{1}{3} \left[\frac{64P_3(75^3)(48)}{42000(20^4)} \right] + \frac{2}{3} \left[\frac{64P_1(50^3)(24)}{83000(10^4)} \right]$$

$$\frac{3}{330} P_2 = \frac{9}{8960} P_3 + \frac{1}{415} P_1$$

$$\frac{3}{166} P_2 = \frac{9}{1792} P_3 + \frac{1}{83} P_1 \quad \rightarrow \text{Equation (3)}$$

From Equation (1)

$$P_1 = 490.5 - P_2 - P_3$$

Substitute P_1 to Equation (3)

$$\frac{3}{166} P_2 = \frac{9}{1792} P_3 + \frac{1}{83} (490.5 - P_2 - P_3)$$

$$\frac{3}{166} P_2 = \frac{9}{1792} P_3 + \frac{981}{166} - \frac{1}{83} P_2 - \frac{1}{83} P_3$$

$$\frac{5}{166} P_2 = \frac{981}{166} - \frac{1045}{148736} P_3 \quad \rightarrow \text{Equation (4)}$$

From Equation (2)

$$P_2 = 735.75 - 3P_3 = \frac{2943}{4} - 3P_3$$

Substitute P_2 to Equation (4)

$$\frac{5}{166} \left(\frac{2943}{4} - 3P_3 \right) = \frac{981}{166} - \frac{1045}{148736} P_3$$

$$\left(\frac{1045}{148736} - \frac{15}{166} \right) P_3 = \frac{981}{166} - \frac{14715}{664}$$

$$P_3 = 195.01 \text{ N}$$

$$P_2 = 735.75 - 3(195.01) = 150.72 \text{ N}$$

$$P_1 = 490.5 - 150.72 - 195.01 = 144.77 \text{ N}$$

$$\tau_{\max} = \frac{16PR}{\pi d^3} \left(1 + \frac{d}{4R} \right) \quad \rightarrow \text{Equation (3-9)}$$

For steel at left:

$$\tau_{\max 1} = \frac{16(144.77)(50)}{\pi(10^3)} \left[1 + \frac{10}{4(50)} \right] = 38.709 \text{ MPa}$$

For steel at right:

$$\tau_{\max 1} = \frac{16(150.72)(50)}{\pi(10^3)} \left[1 + \frac{10}{4(50)} \right] = 40.300 \text{ MPa}$$

For phosphor bronze:

$$\tau_{\max 3} = \frac{16(195.01)(75)}{\pi(20^3)} \left[1 + \frac{20}{4(75)} \right] = 9.932 \text{ MPa}$$

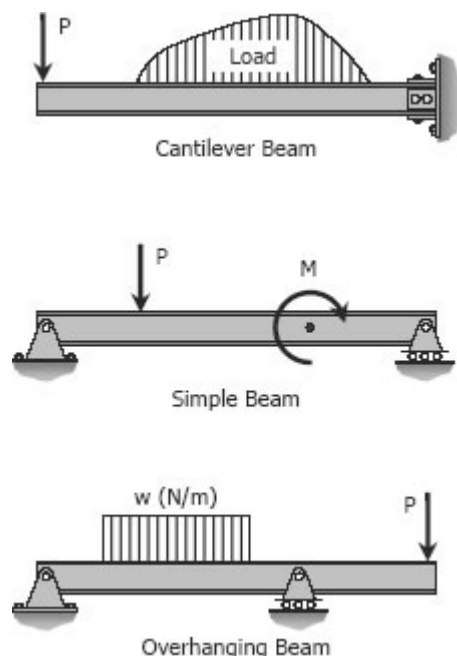
Shear & Moment in Beams

DEFINITION OF A BEAM

A beam is a bar subject to forces or couples that lie in a plane containing the longitudinal of the bar. According to determinacy, a beam may be determinate or indeterminate.

STATICALLY DETERMINATE BEAMS

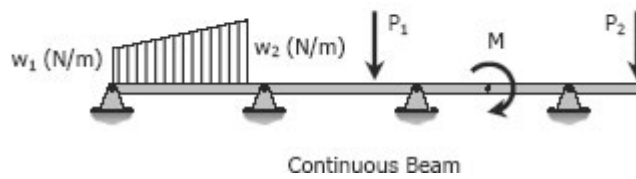
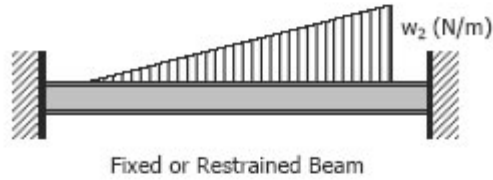
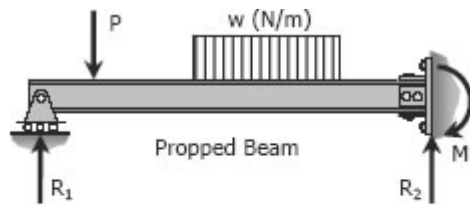
Statically determinate beams are those beams in which the reactions of the supports may be determined by the use of the equations of static equilibrium. The beams shown below are examples of statically determinate beams.



STATICALLY INDETERMINATE BEAMS

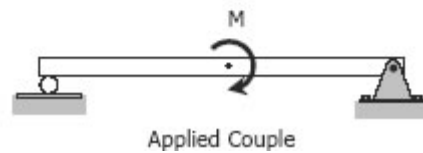
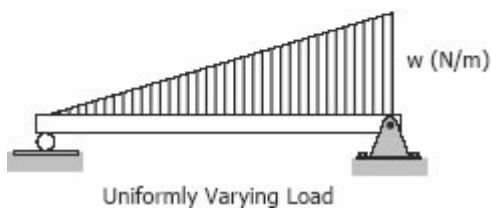
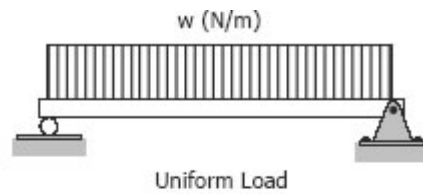
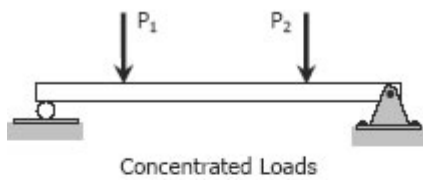
If the number of reactions exerted upon a beam exceeds the number of equations in static equilibrium, the beam is said to be statically indeterminate. In order to solve the reactions of the beam, the static equations must be supplemented by equations based upon the elastic deformations of the beam.

The degree of indeterminacy is taken as the difference between the number of reactions to the number of equations in static equilibrium that can be applied. In the case of the propped beam shown, there are three reactions R_1 , R_2 , and M and only two equations ($\sum M = 0$ and $\sum F_v = 0$) can be applied, thus the beam is indeterminate to the first degree ($3 - 2 = 1$).



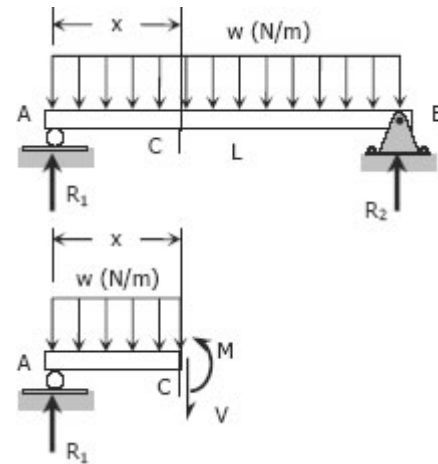
TYPES OF LOADING

Loads applied to the beam may consist of a concentrated load (load applied at a point), uniform load, uniformly varying load, or an applied couple or moment. These loads are shown in the following figures.



Shear and Moment Diagrams

Consider a simple beam shown of length L that carries a uniform load of w (N/m) throughout its length and is held in equilibrium by reactions R_1 and R_2 . Assume that the beam is cut at point C a distance of x from the left support and the portion of the beam to the right of C be removed. The portion removed must then be replaced by vertical shearing force V together with a couple M to hold the left portion of the bar in equilibrium under the action of R_1 and $w x$. The couple M is called the resisting moment or moment and the force V is called the resisting shear or shear. The sign of V and M are taken to be positive if they have the senses indicated above.



Solved Problems in Shear and Moment Diagrams

INSTRUCTION

Write shear and moment equations for the beams in the following problems. In each problem, let x be the distance measured from left end of the beam. Also, draw shear and moment diagrams, specifying values at all change of loading positions and at points of zero shear. Neglect the mass of the beam in each problem.

Problem 403

Beam loaded as shown in Fig. P-403.

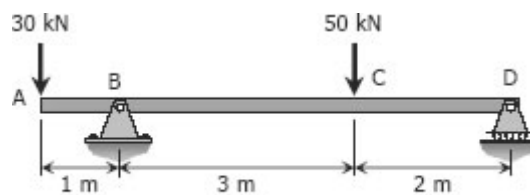


Figure P-403

Solution 403

From the load diagram:

$$\sum M_B = 0$$

$$5R_D + 1(30) = 3(50)$$

$$R_D = 24 \text{ kN}$$

$$\sum M_D = 0$$

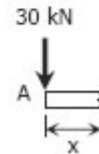
$$5R_B = 2(50) + 6(30)$$

$$R_B = 56 \text{ kN}$$

Segment AB:

$$V_{AB} = -30 \text{ kN}$$

$$M_{AB} = -30x \text{ kN}\cdot\text{m}$$



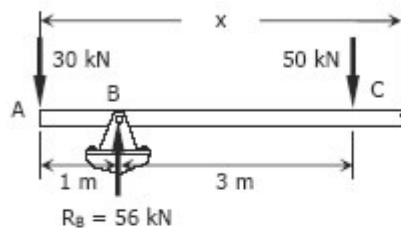
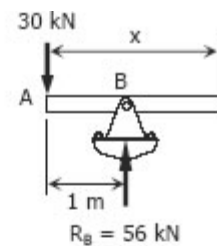
Segment BC:

$$V_{BC} = -30 + 56$$

$$= 26 \text{ kN}$$

$$M_{BC} = -30x + 56(x - 1)$$

$$= 26x - 56 \text{ kN}\cdot\text{m}$$



Segment CD:

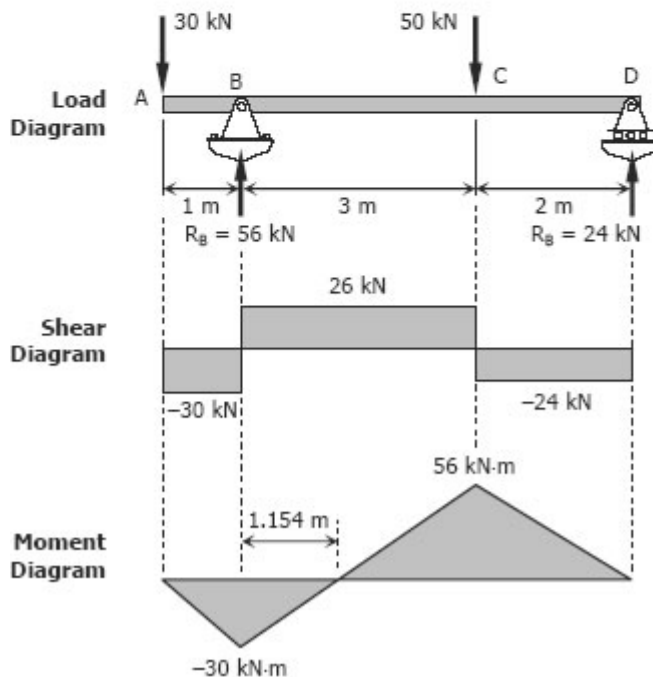
$$V_{CD} = -30 + 56 - 50$$

$$= -24 \text{ kN}$$

$$M_{CD} = -30x + 56(x - 1) - 50(x - 4)$$

$$= -30x + 56x - 56 - 50x + 200$$

$$= -24x + 144$$



To draw the Shear Diagram:

- (1) In segment AB, the shear is uniformly distributed over the segment at a magnitude of -30 kN .
- (2) In segment BC, the shear is uniformly distributed at a magnitude of 26 kN .
- (3) In segment CD, the shear is uniformly distributed at a magnitude of -24 kN .

To draw the Moment Diagram:

- (1) The equation $M_{AB} = -30x$ is linear, at $x = 0$, $M_{AB} = 0$ and at $x = 1 \text{ m}$, $M_{AB} = -30 \text{ kN}\cdot\text{m}$.
- (2) $M_{BC} = 26x - 56$ is also linear. At $x = 1 \text{ m}$, $M_{BC} = -30 \text{ kN}\cdot\text{m}$; at $x = 4 \text{ m}$, $M_{BC} = 48 \text{ kN}\cdot\text{m}$. When $M_{BC} = 0$, $x = 2.154 \text{ m}$, thus the moment is zero at 1.154 m from B.
- (3) $M_{CD} = -24x + 144$ is again linear. At $x = 4 \text{ m}$, $M_{CD} = 48 \text{ kN}\cdot\text{m}$; at $x = 6 \text{ m}$, $M_{CD} = 0$.

Problem 404

Beam loaded as shown in Fig. P-404.

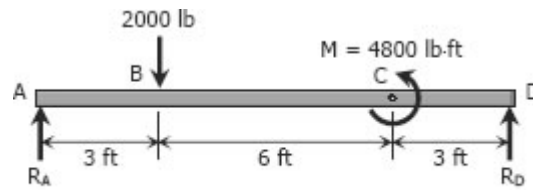


Figure P-404

Solution 404

$$\begin{aligned}\sum M_A &= 0 \\ 12R_D + 4800 &= 3(2000) \\ R_D &= 100 \text{ lb}\end{aligned}$$

$$\begin{aligned}\sum M_D &= 0 \\ 12R_A &= 9(2000) + 4800 \\ R_A &= 1900 \text{ lb}\end{aligned}$$

Segment AB:

$$V_{AB} = 1900 \text{ lb}$$

$$M_{AB} = 1900x \text{ lb-ft}$$

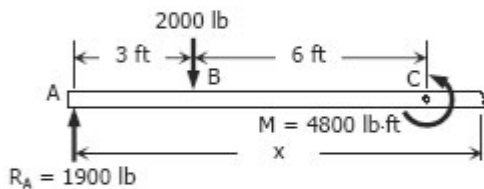
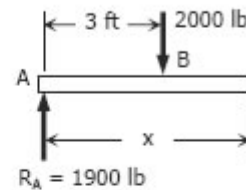


Segment BC:

$$V_{BC} = 1900 - 2000$$

$$= -100 \text{ lb}$$

$$\begin{aligned}M_{BC} &= 1900x - 2000(x - 3) \\ &= 1900x - 2000x + 6000 \\ &= -100x + 6000\end{aligned}$$

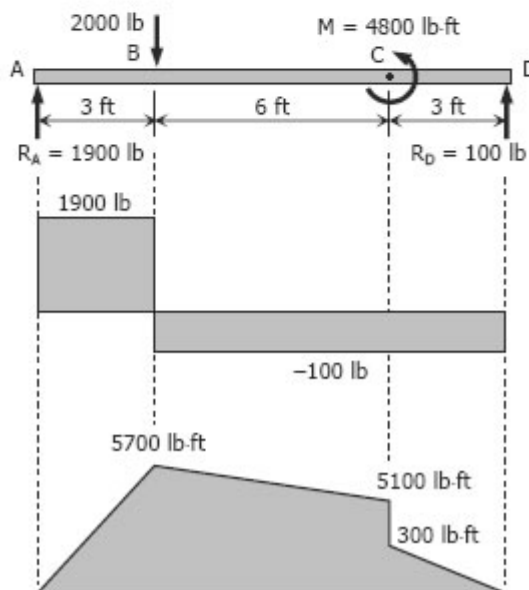


Segment CD:

$$V_{CD} = 1900 - 2000$$

$$= -100 \text{ lb}$$

$$\begin{aligned}M_{CD} &= 1900x - 2000(x - 3) - 4800 \\ &= 1900x - 2000x + 6000 - 4800 \\ &= -100x + 1200\end{aligned}$$



Load
Diagram

Shear
Diagram

Moment
Diagram

To draw the Shear Diagram:

- (1) At segment AB, the shear is uniformly distributed at 1900 lb.
- (2) A shear of -100 lb is uniformly distributed over segments BC and CD.

To draw the Moment Diagram:

- (1) $M_{AB} = 1900x$ is linear; at $x = 0$, $M_{AB} = 0$; at $x = 3$ ft, $M_{AB} = 5700$ lb-ft.
- (2) For segment BC, $M_{BC} = -100x + 6000$ is linear; at $x = 3$ ft, $M_{BC} = 5700$ lb-ft; at $x = 9$ ft, $M_{BC} = 5100$ lb-ft.
- (3) $M_{CD} = -100x + 1200$ is again linear; at $x = 9$ ft, $M_{CD} = 300$ lb-ft; at $x = 12$ ft, $M_{CD} = 0$.

Problem 405

Beam loaded as shown in Fig. P-405.

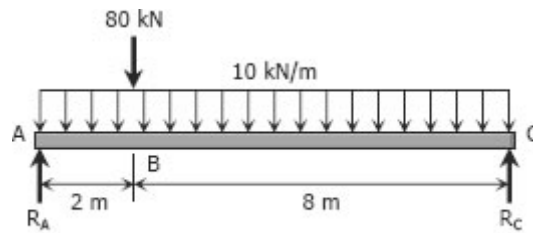
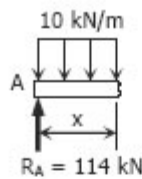


Figure P-405

Solution 405

$$\begin{aligned}\sum M_A &= 0 \\ 10R_C &= 2(80) + 5[10(10)] \\ R_C &= 66 \text{ kN}\end{aligned}$$

$$\begin{aligned}\sum M_C &= 0 \\ 10R_A &= 8(80) + 5[10(10)] \\ R_A &= 114 \text{ kN}\end{aligned}$$

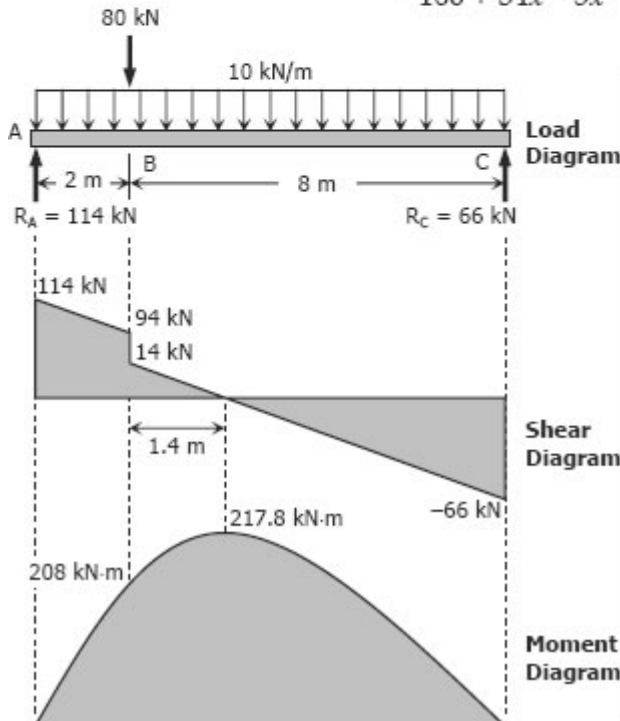
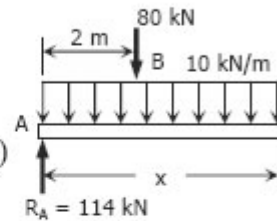


Segment AB:

$$\begin{aligned}V_{AB} &= 114 - 10x \text{ kN} \\ M_{AB} &= 114x - 10x(x/2) \\ &= 114x - 5x^2 \text{ kN}\cdot\text{m}\end{aligned}$$

Segment BC:

$$\begin{aligned}V_{BC} &= 114 - 80 - 10x \\ &= 34 - 10x \text{ kN} \\ M_{BC} &= 114x - 80(x - 2) - 10x(x/2) \\ &= 160 + 34x - 5x^2\end{aligned}$$



To draw the Shear Diagram:

- (1) For segment AB, $V_{AB} = 114 - 10x$ is linear; at $x = 0$, $V_{AB} = 114$ kN; at $x = 2$ m, $V_{AB} = 94$ kN.
- (2) $V_{BC} = 34 - 10x$ for segment BC is linear; at $x = 2$ m, $V_{BC} = 14$ kN; at $x = 10$ m, $V_{BC} = -66$ kN. When $V_{BC} = 0$, $x = 3.4$ m thus $V_{BC} = 0$ at 1.4 m from B.

To draw the Moment Diagram:

- (1) $M_{AB} = 114x - 5x^2$ is a second degree curve for segment AB; at $x = 0$, $M_{AB} = 0$; at $x = 2$ m, $M_{AB} = 208$ kN-m.
- (2) The moment diagram is also a second degree curve for segment BC given by $M_{BC} = 160 + 34x - 5x^2$; at $x = 2$ m, $M_{BC} = 208$ kN-m; at $x = 10$ m, $M_{BC} = 0$.
- (3) Note that the maximum moment occurs at point of zero shear. Thus, at $x = 3.4$ m, $M_{BC} = 217.8$ kN-m.

Problem 406

Beam loaded as shown in Fig. P-406.

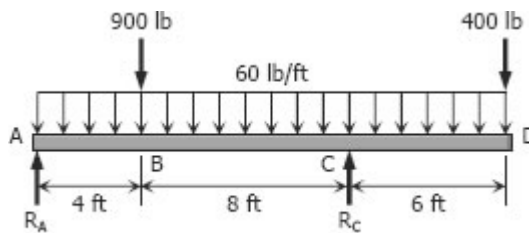


Figure P-406

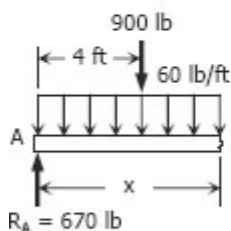
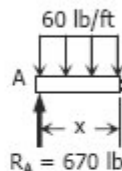
Solution 406

$$\begin{aligned}\sum M_A &= 0 \\ 12R_C &= 4(900) + 18(400) + 9[(60)(18)] \\ R_C &= 1710 \text{ lb}\end{aligned}$$

$$\begin{aligned}\sum M_C &= 0 \\ 12R_A + 6(400) &= 8(900) + 3[60(18)] \\ R_A &= 670 \text{ lb}\end{aligned}$$

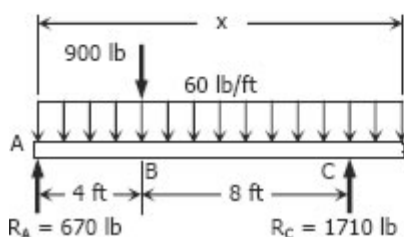
Segment AB:

$$\begin{aligned}V_{AB} &= 670 - 60x \text{ lb} \\ M_{AB} &= 670x - 60x(x/2) \\ &= 670x - 30x^2 \text{ lb-ft}\end{aligned}$$



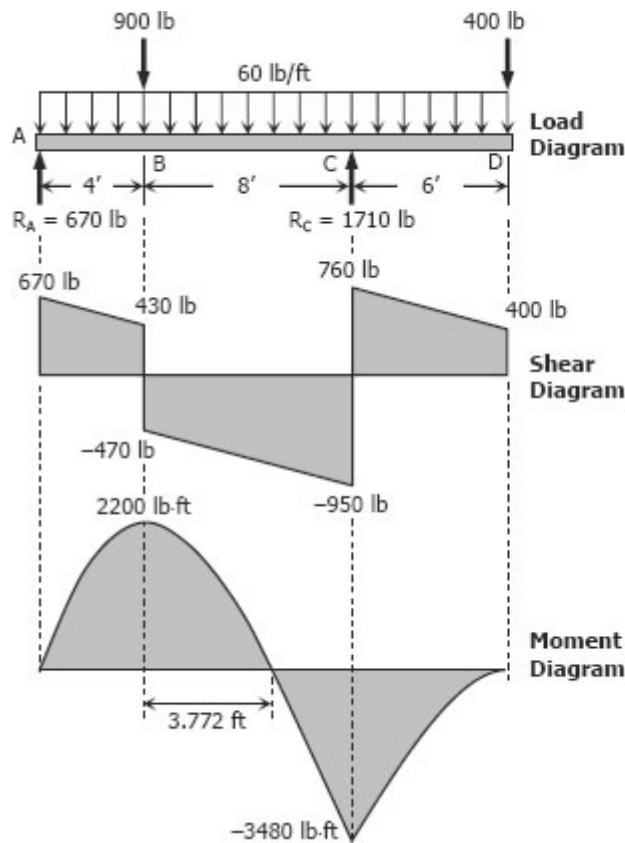
Segment BC:

$$\begin{aligned}V_{BC} &= 670 - 900 - 60x \\ &= -230 - 60x \text{ lb} \\ M_{BC} &= 670x - 900(x - 4) - 60x(x/2) \\ &= 3600 - 230x - 30x^2 \text{ lb-ft}\end{aligned}$$



Segment CD:

$$\begin{aligned}V_{CD} &= 670 + 1710 - 900 - 60x \\ &= 1480 - 60x \text{ lb} \\ M_{CD} &= 670x + 1710(x - 12) \\ &\quad - 900(x - 4) - 60x(x/2) \\ &= -16920 + 1480x - 30x^2 \text{ lb-ft}\end{aligned}$$



To draw the Shear Diagram:

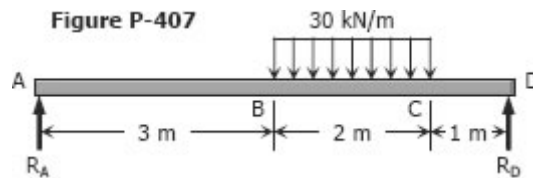
- (1) $V_{AB} = 670 - 60x$ for segment AB is linear; at $x = 0$, $V_{AB} = 670$ lb; at $x = 4$ ft, $V_{AB} = 430$ lb.
- (2) For segment BC, $V_{BC} = -230 - 60x$ is also linear; at $x = 4$ ft, $V_{BC} = -470$ lb, at $x = 12$ ft, $V_{BC} = -950$ lb.
- (3) $V_{CD} = 1480 - 60x$ for segment CD is again linear; at $x = 12$, $V_{CD} = 760$ lb; at $x = 18$ ft, $V_{CD} = 400$ lb.

To draw the Moment Diagram:

- (1) $M_{AB} = 670x - 30x^2$ for segment AB is a second degree curve; at $x = 0$, $M_{AB} = 0$; at $x = 4$ ft, $M_{AB} = 2200$ lb-ft.
- (2) For BC, $M_{BC} = 3600 - 230x - 30x^2$, is a second degree curve; at $x = 4$ ft, $M_{BC} = 2200$ lb-ft, at $x = 12$ ft, $M_{BC} = -3480$ lb-ft; When $M_{BC} = 0$, $3600 - 230x - 30x^2 = 0$, $x = -15.439$ ft and 7.772 ft. Take $x = 7.772$ ft, thus, the moment is zero at 3.772 ft from B.
- (3) For segment CD, $M_{CD} = -16920 + 1480x - 30x^2$ is a second degree curve; at $x = 12$ ft, $M_{CD} = -3480$ lb-ft; at $x = 18$ ft, $M_{CD} = 0$.

Problem 407

Beam loaded as shown in Fig. P-407.



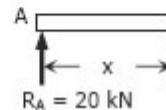
Solution 407

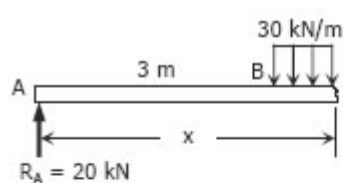
$$\begin{aligned}\sum M_A &= 0 \\ 6R_D &= 4[2(30)] \\ R_D &= 40 \text{ kN}\end{aligned}$$

$$\begin{aligned}\sum M_D &= 0 \\ 6R_A &= 2[2(30)] \\ R_A &= 20 \text{ kN}\end{aligned}$$

Segment AB:

$$\begin{aligned}V_{AB} &= 20 \text{ kN} \\ M_{AB} &= 20x \text{ kN}\cdot\text{m}\end{aligned}$$





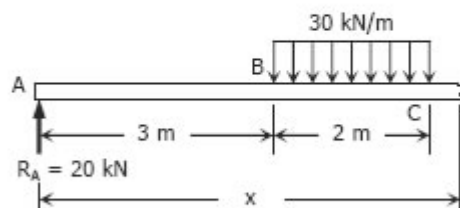
Segment BC:

$$V_{BC} = 20 - 30(x - 3)$$

$$= 110 - 30x \text{ kN}$$

$$M_{BC} = 20x - 30(x - 3)(x - 3)/2$$

$$= 20x - 15(x - 3)^2$$



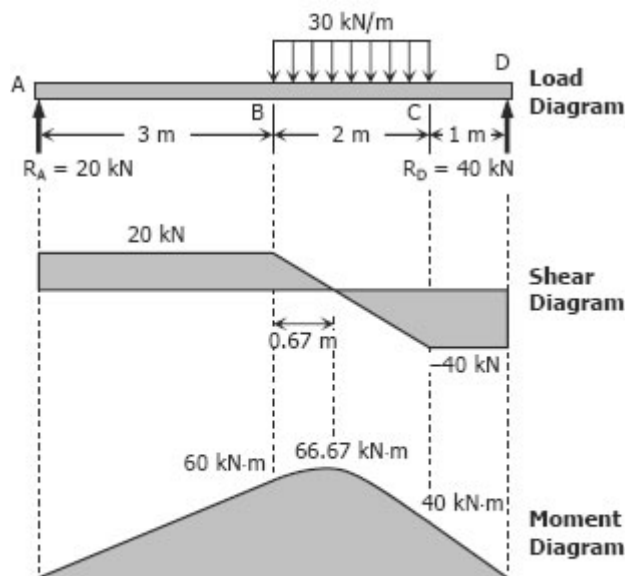
Segment CD:

$$V_{CD} = 20 - 30(2)$$

$$= -40 \text{ kN}$$

$$M_{CD} = 20x - 30(2)(x - 4)$$

$$= 20x - 60(x - 4)$$



To draw the Shear Diagram:

- (1) For segment AB, the shear is uniformly distributed at 20 kN.
- (2) $V_{BC} = 110 - 30x$ for segment BC; at $x = 3$ m, $V_{BC} = 20$ kN; at $x = 5$ m, $V_{BC} = -40$ kN. For $V_{BC} = 0$, $x = 3.67$ m or 0.67 m from B.
- (3) The shear for segment CD is uniformly distributed at -40 kN.

To draw the Moment Diagram:

- (1) For AB, $M_{AB} = 20x$; at $x = 0$, $M_{AB} = 0$; at $x = 3$ m, $M_{AB} = 60$ kN-m.
- (2) $M_{BC} = 20x - 15(x - 3)^2$ for segment BC is second degree curve; at $x = 3$ m, $M_{BC} = 60$ kN-m; at $x = 5$ m, $M_{BC} = 40$ kN-m. Note that maximum moment occurred at zero shear; at $x = 3.67$ m, $M_{BC} = 66.67$ kN-m.
- (3) $M_{CD} = 20x - 60(x - 4)$ for segment CD is linear; at $x = 5$ m, $M_{CD} = 40$ kN-m; at $x = 6$ m, $M_{CD} = 0$.

Problem 408

Beam loaded as shown in Fig. P-408.

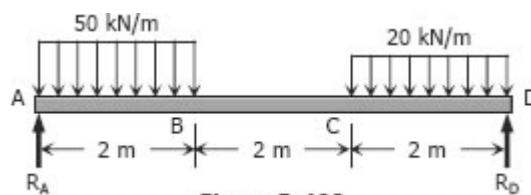
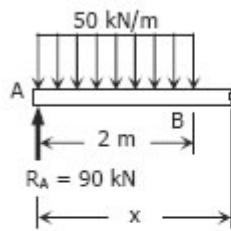


Figure P-408

Solution 408

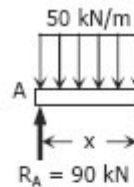
$$\begin{array}{l} \sum M_A = 0 \\ 6R_D = 1[2(50)] + 5[2(20)] \\ R_D = 50 \text{ kN} \end{array} \quad \left| \quad \begin{array}{l} \sum M_D = 0 \\ 6R_A = 5[2(50)] + 1[2(20)] \\ R_A = 90 \text{ kN} \end{array} \right.$$



Segment AB:

$$V_{AB} = 90 - 50x \text{ kN}$$

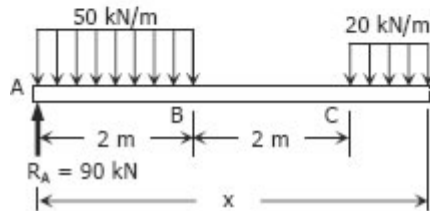
$$M_{AB} = 90x - 50x(x/2) \\ = 90x - 25x^2$$



Segment BC:

$$V_{BC} = 90 - 50(2) \\ = -10 \text{ kN}$$

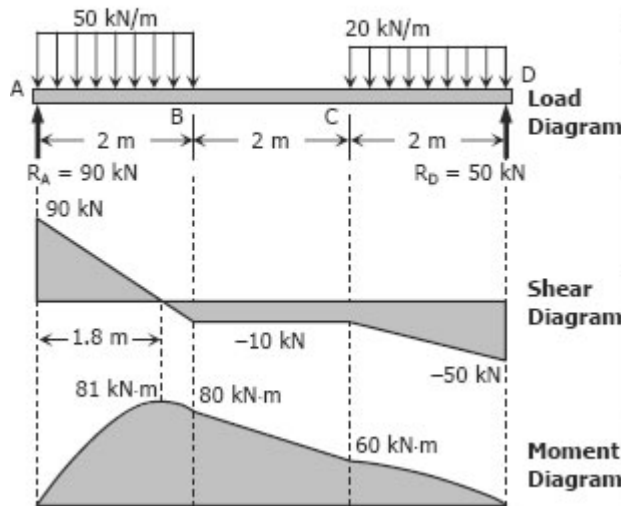
$$M_{BC} = 90x - 2(50)(x - 1) \\ = -10x + 100 \text{ kN}\cdot\text{m}$$



Segment CD:

$$V_{CD} = 90 - 2(50) - 20(x - 4) \\ = -20x + 70 \text{ kN}$$

$$M_{CD} = 90x - 2(50)(x - 1) \\ - 20(x - 4)(x - 4)/2 \\ = 90x - 100(x - 1) - 10(x - 4)^2 \\ = -10x^2 + 70x - 60 \text{ kN}\cdot\text{m}$$



To draw the Shear Diagram:

- (1) $V_{AB} = 90 - 50x$ is linear; at $x = 0$, $V_{BC} = 90 \text{ kN}$; at $x = 2 \text{ m}$, $V_{BC} = -10 \text{ kN}$. When $V_{AB} = 0$, $x = 1.8 \text{ m}$.
- (2) $V_{BC} = -10 \text{ kN}$ along segment BC.
- (3) $V_{CD} = -20x + 70$ is linear; at $x = 4 \text{ m}$, $V_{CD} = -10 \text{ kN}$; at $x = 6 \text{ m}$, $V_{CD} = -50 \text{ kN}$.

To draw the Moment Diagram:

- (1) $M_{AB} = 90x - 25x^2$ is second degree; at $x = 0$, $M_{AB} = 0$; at $x = 1.8 \text{ m}$, $M_{AB} = 81 \text{ kN}\cdot\text{m}$; at $x = 2 \text{ m}$, $M_{AB} = 80 \text{ kN}\cdot\text{m}$.
- (2) $M_{BC} = -10x + 100$ is linear; at $x = 2 \text{ m}$, $M_{BC} = 80 \text{ kN}\cdot\text{m}$; at $x = 4 \text{ m}$, $M_{BC} = 60 \text{ kN}\cdot\text{m}$.
- (3) $M_{CD} = -10x^2 + 70x - 60$; at $x = 4 \text{ m}$, $M_{CD} = 60 \text{ kN}\cdot\text{m}$; at $x = 6 \text{ m}$, $M_{CD} = 0$.

Problem 409

Cantilever beam loaded as shown in Fig. P-409.

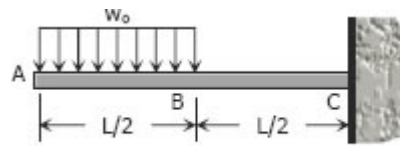


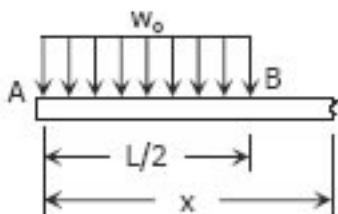
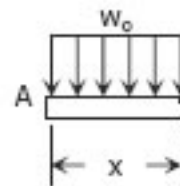
Figure P-409

Solution 409

Segment AB:

$$V_{AB} = -w_0x$$

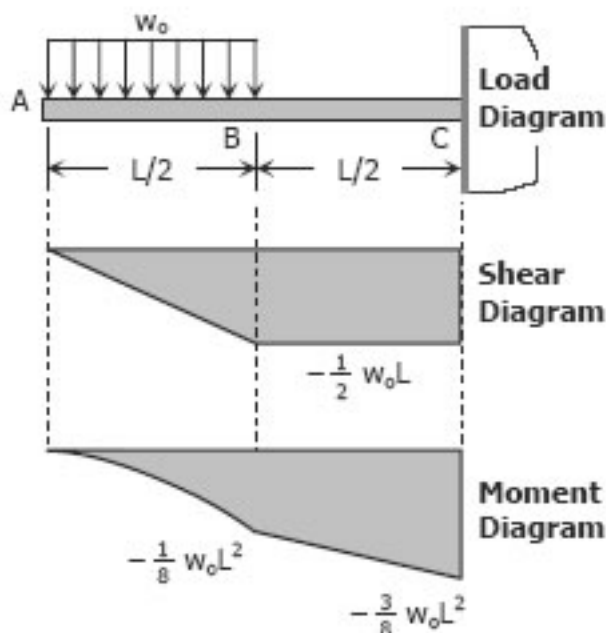
$$M_{AB} = -w_0x(x/2) \\ = -\frac{1}{2}w_0x^2$$



Segment BC:

$$V_{BC} = -w_0(L/2) \\ = -\frac{1}{2}w_0L$$

$$M_{BC} = -w_0(L/2)(x - L/4) \\ = -\frac{1}{2}w_0Lx + \frac{1}{8}w_0L^2$$



To draw the Shear Diagram:

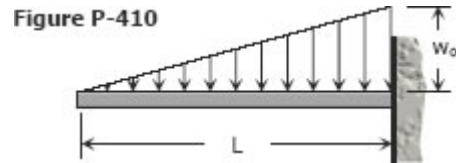
- (1) $V_{AB} = -w_0x$ for segment AB is linear; at $x = 0$, $V_{AB} = 0$; at $x = L/2$, $V_{AB} = -\frac{1}{2}w_0L$.
- (2) At BC, the shear is uniformly distributed by $-\frac{1}{2}w_0L$.

To draw the Moment Diagram:

- (1) $M_{AB} = -\frac{1}{2}w_0x^2$ is a second degree curve; at $x = 0$, $M_{AB} = 0$; at $x = L/2$, $M_{AB} = -\frac{1}{8}w_0L^2$.
- (2) $M_{BC} = -\frac{1}{2}w_0Lx + \frac{1}{8}w_0L^2$ is a second degree; at $x = L/2$, $M_{BC} = -\frac{1}{8}w_0L^2$; at $x = L$, $M_{BC} = -\frac{3}{8}w_0L^2$.

Problem 410

Cantilever beam carrying the uniformly varying load shown in Fig. P-410.



Solution 410

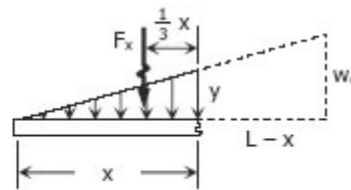
$$\frac{y}{x} = \frac{w_0}{L}$$

$$y = \frac{w_0}{L}x$$

$$F_x = \frac{1}{2}xy$$

$$= \frac{1}{2}x\left(\frac{w_0}{L}x\right)$$

$$= \frac{w_0}{2L}x^2$$



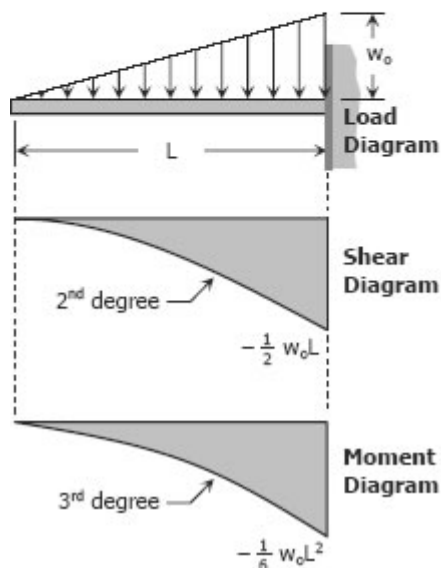
Shear equation:

$$V = -\frac{w_0}{2L}x^2$$

Moment equation:

$$M = -\frac{1}{3}xF_x = -\frac{1}{3}x\left(\frac{w_0}{2L}x^2\right)$$

$$= -\frac{w_0}{6L}x^3$$



To draw the Shear Diagram:

$$V = -\frac{w_0}{2L}x^2 \text{ is a second degree curve;}$$

$$\text{at } x = 0, V = 0; \text{ at } x = L, V = -\frac{1}{2}w_0L.$$

To draw the Moment Diagram:

$$M = -\frac{w_0}{6L}x^3 \text{ is a third degree curve; at}$$

$$x = 0, M = 0; \text{ at } x = L, M = -\frac{1}{6}w_0L^2.$$

Problem 411

Cantilever beam carrying a distributed load with intensity varying from w_0 at the free end to zero at the wall, as shown in Fig. P-411.

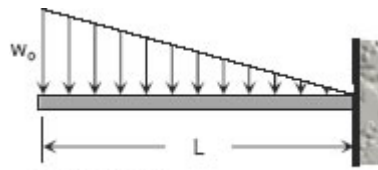
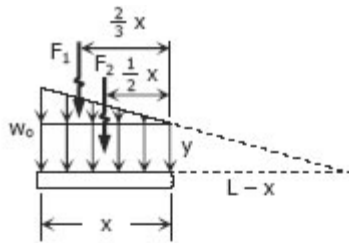


Figure P-411

Solution 411

$$\frac{y}{L-x} = \frac{w_0}{L}$$

$$y = \frac{w_0}{L}(L-x)$$



$$\begin{aligned} F_1 &= \frac{1}{2}x(w_0 - y) \\ &= \frac{1}{2}x\left[w_0 - \frac{w_0}{L}(L-x)\right] \\ &= \frac{1}{2}x\left[w_0 - w_0 + \frac{w_0}{L}x\right] \\ &= \frac{w_0}{2L}x^2 \end{aligned}$$

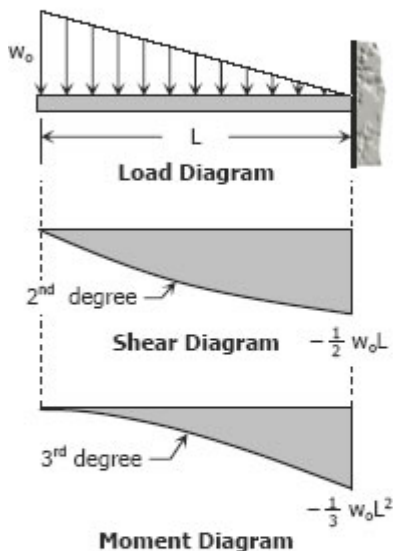
$$\begin{aligned} F_2 &= xy = x\left[\frac{w_0}{L}(L-x)\right] \\ &= \frac{w_0}{L}(Lx - x^2) \end{aligned}$$

Shear equation:

$$\begin{aligned} V &= -F_1 - F_2 = -\frac{w_0}{2L}x^2 - \frac{w_0}{L}(Lx - x^2) \\ &= -\frac{w_0}{2L}x^2 - w_0x + \frac{w_0}{L}x^2 \\ &= \frac{w_0}{2L}x^2 - w_0x \end{aligned}$$

Moment equation:

$$\begin{aligned} M &= -\frac{2}{3}xF_1 - \frac{1}{2}xF_2 \\ &= -\frac{1}{3}x\left(\frac{w_0}{2L}x^2\right) - \frac{1}{2}x\left[\frac{w_0}{L}(Lx - x^2)\right] \\ &= -\frac{w_0}{3L}x^3 - \frac{w_0}{2}x^2 + \frac{w_0}{2L}x^3 \\ &= -\frac{w_0}{2}x^2 + \frac{w_0}{6L}x^3 \end{aligned}$$



To draw the Shear Diagram:

$V = \frac{w_0}{2L}x^2 - w_0x$ is a concave upward second degree curve; at $x = 0$, $V = 0$; at $x = L$, $V = -\frac{1}{2}w_0L$.

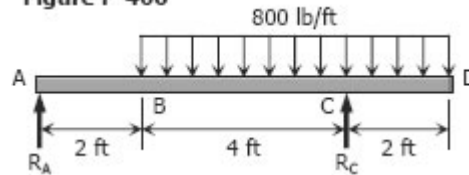
To draw the Moment diagram:

$M = -\frac{w_0}{2}x^2 + \frac{w_0}{6L}x^3$ is in third degree; at $x = 0$, $M = 0$; at $x = L$, $M = -\frac{1}{3}w_0L^2$.

Problem 412

Beam loaded as shown in Fig. P-412.

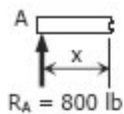
Figure P-406



Solution 412

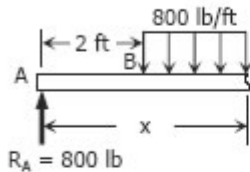
$$\begin{aligned}\sum M_A &= 0 \\ 6R_C &= 5[6(800)] \\ R_C &= 4000 \text{ lb}\end{aligned}$$

$$\begin{aligned}\sum M_C &= 0 \\ 6R_A &= 1[6(800)] \\ R_A &= 800 \text{ lb}\end{aligned}$$



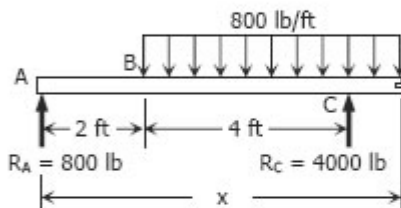
Segment AB:

$$\begin{aligned}V_{AB} &= 800 \text{ lb} \\ M_{AB} &= 800x\end{aligned}$$



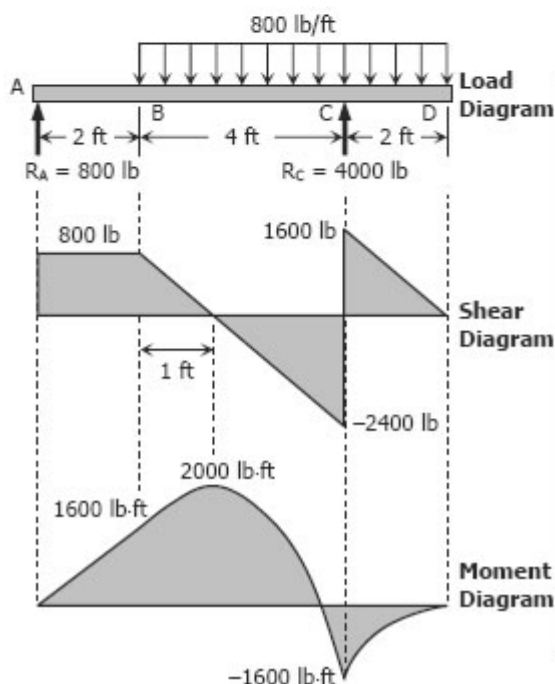
Segment BC:

$$\begin{aligned}V_{BC} &= 800 - 800(x - 2) \\ &= 2400 - 800x \\ M_{BC} &= 800x - 800(x - 2)(x - 2)/2 \\ &= 800x - 400(x - 2)^2\end{aligned}$$



Segment CD:

$$\begin{aligned}V_{CD} &= 800 + 4000 - 800(x - 2) \\ &= 4800 - 800x + 1600 \\ &= 6400 - 800x \\ M_{CD} &= 800x + 4000(x - 6) - 800(x - 2)(x - 2)/2 \\ &= 800x + 4000(x - 6) - 400(x - 2)^2\end{aligned}$$



To draw the Shear Diagram:

- (1) 800 lb of shear force is uniformly distributed along segment AB.
- (2) $V_{BC} = 2400 - 800x$ is linear; at $x = 2$ ft, $V_{BC} = 800$ lb; at $x = 6$ ft, $V_{BC} = -2400$ lb. When $V_{BC} = 0$, $2400 - 800x = 0$, thus $x = 3$ ft or $V_{BC} = 0$ at 1 ft from B.
- (3) $V_{CD} = 6400 - 800x$ is also linear; at $x = 6$ ft, $V_{CD} = 1600$ lb; at $x = 8$ ft, $V_{CD} = 0$.

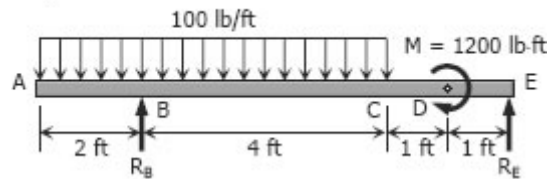
To draw the Moment Diagram:

- (1) $M_{AB} = 800x$ is linear; at $x = 0$, $M_{AB} = 0$; at $x = 2$ ft, $M_{AB} = 1600$ lb-ft.
- (2) $M_{BC} = 800x - 400(x - 2)^2$ is second degree curve; at $x = 2$ ft, $M_{BC} = 1600$ lb-ft; at $x = 6$ ft, $M_{BC} = -1600$ lb-ft; at $x = 3$ ft, $M_{BC} = 2000$ lb-ft.
- (3) $M_{CD} = 800x + 4000(x - 6) - 400(x - 2)^2$ is also a second degree curve; at $x = 6$ ft, $M_{CD} = -1600$ lb-ft; at $x = 8$ ft, $M_{CD} = 0$.

Problem 413

Beam loaded as shown in Fig. P-413.

Figure P-413



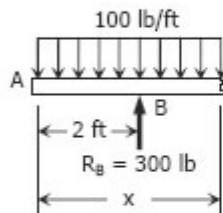
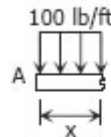
Solution 413

$$\begin{aligned}\sum M_B &= 0 \\ 6R_E &= 1200 + 1[6(100)] \\ R_E &= 300 \text{ lb}\end{aligned}$$

$$\begin{aligned}\sum M_E &= 0 \\ 6R_B + 1200 &= 5[6(100)] \\ R_B &= 300 \text{ lb}\end{aligned}$$

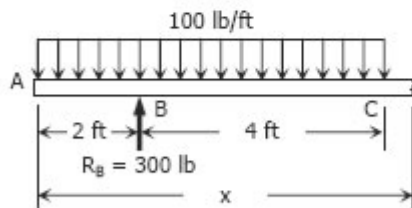
Segment AB:

$$\begin{aligned}V_{AB} &= -100x \text{ lb} \\ M_{AB} &= -100x(x/2) \\ &= -50x^2 \text{ lb}\cdot\text{ft}\end{aligned}$$



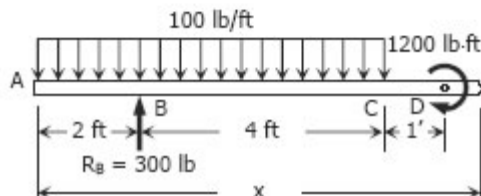
Segment BC:

$$\begin{aligned}V_{BC} &= -100x + 300 \text{ lb} \\ M_{BC} &= -100x(x/2) + 300(x - 2) \\ &= -50x^2 + 300x - 600 \text{ lb}\cdot\text{ft}\end{aligned}$$



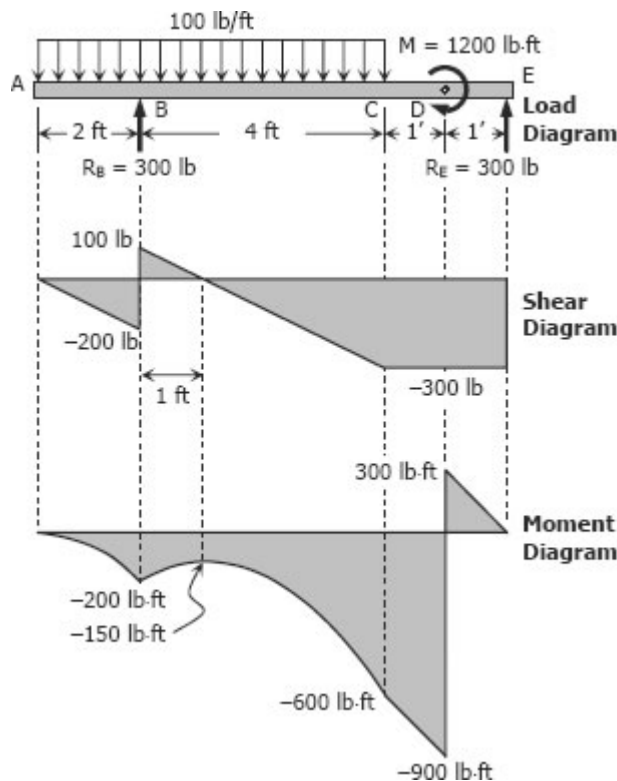
Segment CD:

$$\begin{aligned}V_{CD} &= -100(6) + 300 \\ &= -300 \text{ lb} \\ M_{CD} &= -100(6)(x - 3) + 300(x - 2) \\ &= -600x + 1800 + 300x - 600 \\ &= -300x + 1200 \text{ lb}\cdot\text{ft}\end{aligned}$$



Segment DE:

$$\begin{aligned}V_{DE} &= -100(6) + 300 \\ &= -300 \text{ lb} \\ M_{DE} &= -100(6)(x - 3) + 1200 + 300(x - 2) \\ &= -600x + 1800 + 1200 + 300x - 600 \\ &= -300x + 2400\end{aligned}$$



To draw the Shear Diagram:

- (1) $V_{AB} = -100x$ is linear; at $x = 0$, $V_{AB} = 0$; at $x = 2$ ft, $V_{AB} = -200$ lb.
- (2) $V_{BC} = 300 - 100x$ is also linear; at $x = 2$ ft, $V_{BC} = 100$ lb; at $x = 4$ ft, $V_{BC} = -300$ lb. When $V_{BC} = 0$, $x = 3$ ft, or $V_{BC} = 0$ at 1 ft from B.
- (3) The shear is uniformly distributed at -300 lb along segments CD and DE.

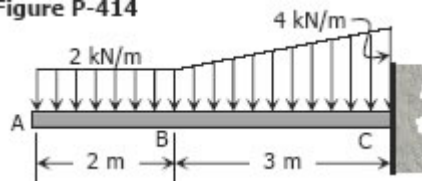
To draw the Moment Diagram:

- (1) $M_{AB} = -50x^2$ is a second degree curve; at $x = 0$, $M_{AB} = 0$; at $x = 2$ ft, $M_{AB} = -200$ lb-ft.
- (2) $M_{BC} = -50x^2 + 300x - 600$ is also second degree; at $x = 2$ ft, $M_{BC} = -200$ lb-ft; at $x = 6$ ft, $M_{BC} = -600$ lb-ft; at $x = 3$ ft, $M_{BC} = -150$ lb-ft.
- (3) $M_{CD} = -300x + 1200$ is linear; at $x = 6$ ft, $M_{CD} = -600$ lb-ft; at $x = 7$ ft, $M_{CD} = -900$ lb-ft.
- (4) $M_{DE} = -300x + 2400$ is again linear; at $x = 7$ ft, $M_{DE} = 300$ lb-ft; at $x = 8$ ft, $M_{DE} = 0$.

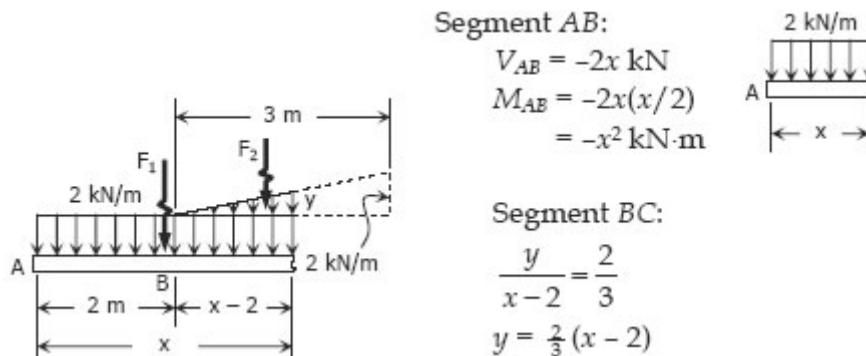
Problem 414

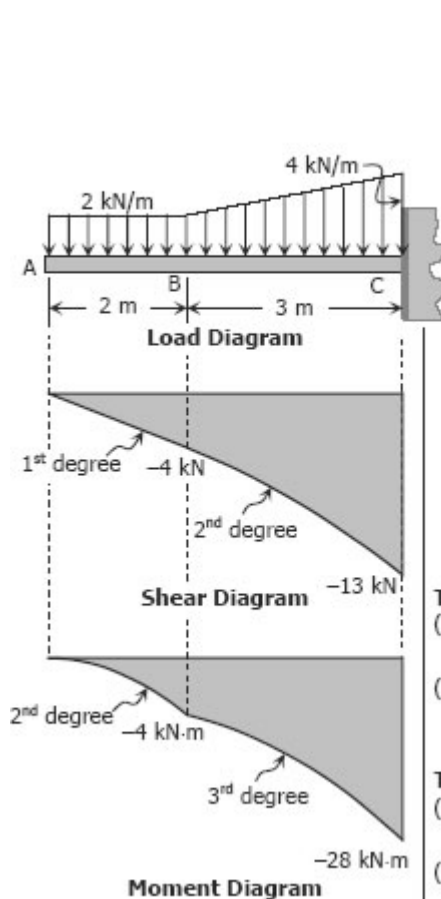
Cantilever beam carrying the load shown in Fig. P-414.

Figure P-414



Solution 414





$$F_1 = 2x$$

$$\begin{aligned} F_2 &= \frac{1}{2}(x-2)y \\ &= \frac{1}{2}(x-2)\left[\frac{2}{3}(x-2)\right] \\ &= \frac{1}{3}(x-2)^2 \end{aligned}$$

$$\begin{aligned} V_{BC} &= -F_1 - F_2 \\ &= -2x - \frac{1}{3}(x-2)^2 \end{aligned}$$

$$\begin{aligned} M_{BC} &= -(x/2)F_1 - \frac{1}{3}(x-2)F_2 \\ &= -(x/2)(2x) - \frac{1}{3}(x-2)\left[\frac{1}{3}(x-2)^2\right] \\ &= -x^2 - \frac{1}{9}(x-2)^3 \end{aligned}$$

To draw the Shear Diagram:

- (1) $V_{AB} = -2x$ is linear; at $x = 0$, $V_{AB} = 0$; at $x = 2$ m, $V_{AB} = -4$ kN.
- (2) $V_{BC} = -2x - \frac{1}{3}(x-2)^2$ is a second degree curve; at $x = 2$ m, $V_{BC} = -4$ kN; at $x = 5$ m, $V_{BC} = -13$ kN.

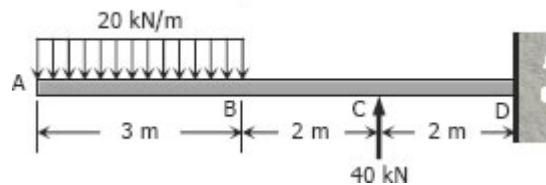
To draw the Moment Diagram:

- (1) $M_{AB} = -x^2$ is a second degree curve; at $x = 0$, $M_{AB} = 0$; at $x = 2$ m, $M_{AB} = -4$ kN-m.
- (2) $M_{BC} = -x^2 - \frac{1}{9}(x-2)^3$ is a third degree curve; at $x = 2$ m, $M_{BC} = -4$ kN-m; at $x = 5$ m, $M_{BC} = -28$ kN-m.

Problem 415

Cantilever beam loaded as shown in Fig. P-415.

Figure P-415

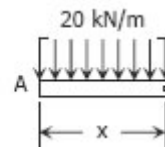


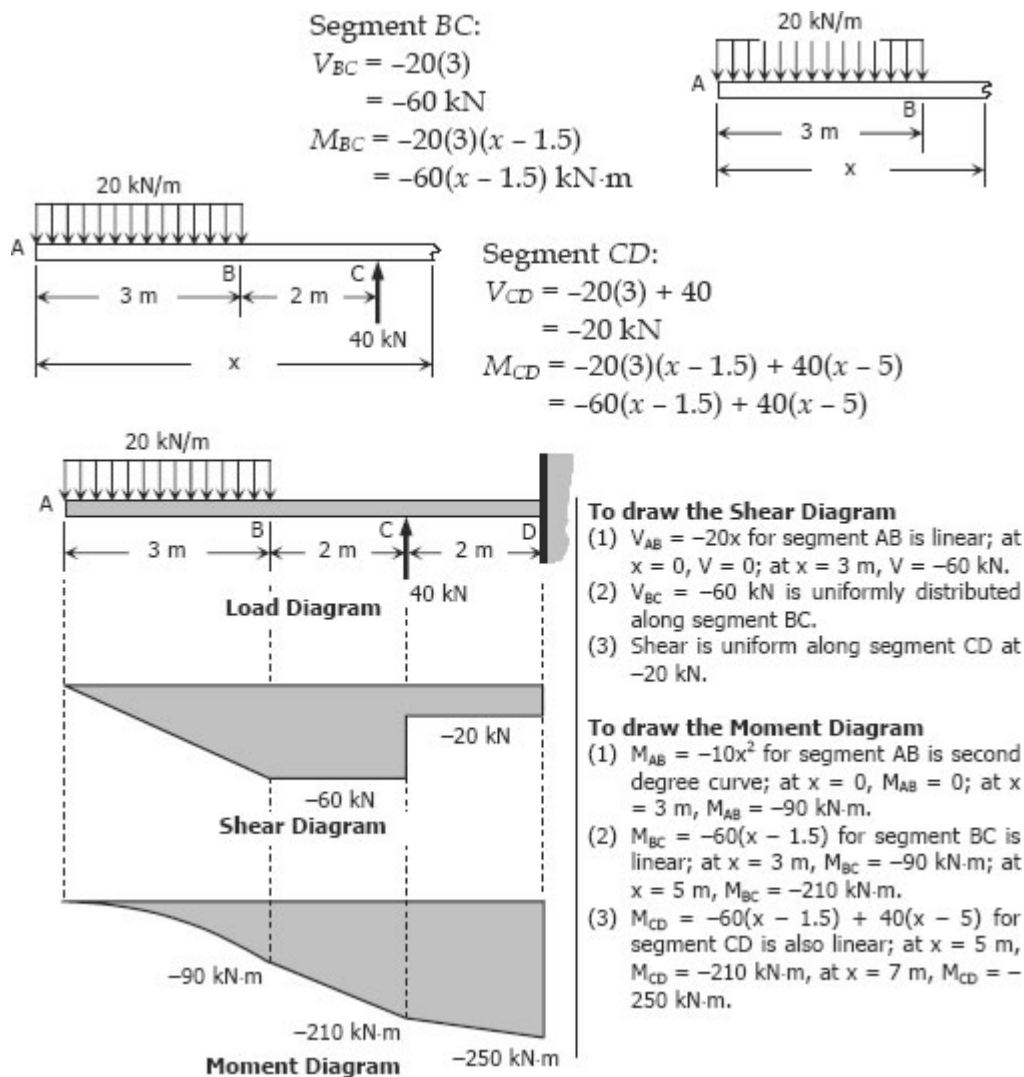
Solution 415

Segment AB:

$$V_{AB} = -20x \text{ kN}$$

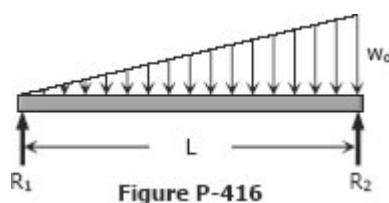
$$\begin{aligned} M_{AB} &= -20x(x/2) \\ &= -10x^2 \text{ kN-m} \end{aligned}$$





Problem 416

Beam carrying uniformly varying load shown in Fig. P-416.

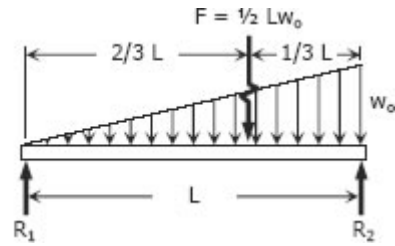


Solution 416

$$\sum M_{R2} = 0$$

$$LR_1 = \frac{1}{3} LF$$

$$R_1 = \frac{1}{3} \left(\frac{1}{2} Lw_0 \right) \\ = \frac{1}{6} Lw_0$$



$$\sum M_{R1} = 0$$

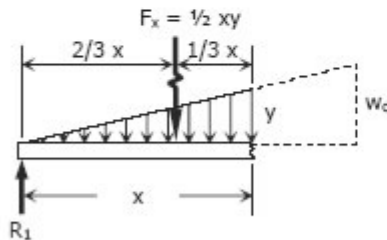
$$LR_2 = \frac{2}{3} LF$$

$$R_2 = \frac{2}{3} \left(\frac{1}{2} Lw_0 \right) \\ = \frac{1}{3} Lw_0$$

$$\frac{y}{x} = \frac{w_0}{L}$$

$$y = \frac{w_0}{L} x$$

$$F_x = \frac{1}{2} xy = \frac{1}{2} x \left(\frac{w_0}{L} x \right) \\ = \frac{w_0}{2L} x^2$$



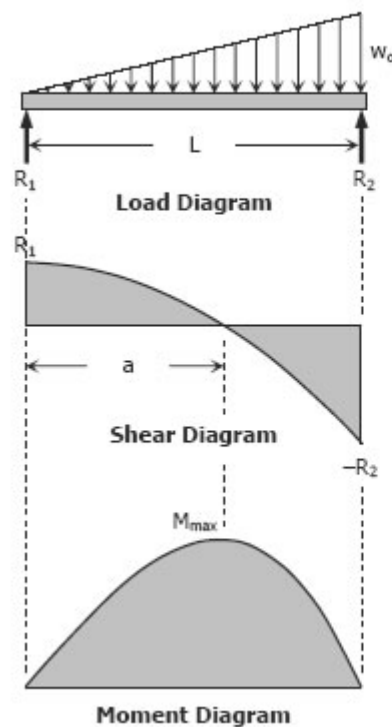
$$V = R_1 - F_x$$

$$= \frac{1}{6} Lw_0 - \frac{w_0}{2L} x^2$$

$$M = R_1 x - F_x \left(\frac{1}{3} x \right)$$

$$= \frac{1}{6} Lw_0 x - \frac{w_0}{2L} x^2 \left(\frac{1}{3} x \right)$$

$$= \frac{1}{6} Lw_0 x - \frac{w_0}{6L} x^3$$



To draw the Shear Diagram:

$V = 1/6 Lw_0 - w_0 x^2/2L$ is a second degree curve; at $x = 0$, $V = 1/6 Lw_0 = R_1$; at $x = L$, $V = -1/3 Lw_0 = -R_2$; If a is the location of zero shear from left end, $0 = 1/6 Lw_0 - w_0 x^2/2L$, $x = 0.5774L = a$; to check, use the squared property of parabola:

$$a^2/R_1 = L^2/(R_1 + R_2)$$

$$a^2/(1/6 Lw_0) = L^2/(1/6 Lw_0 + 1/3 Lw_0)$$

$$a^2 = (1/6 L^3 w_0)/(1/2 Lw_0) = 1/3 L^2$$

$$a = 0.5774L$$

$$a =$$

To draw the Moment Diagram

$M = 1/6 Lw_0 x - w_0 x^3/6L$ is a third degree curve; at $x = 0$, $M = 0$; at $x = L$, $M = 0$; at $x = a = 0.5774L$, $M = M_{max}$

$$M_{max} = 1/6 Lw_0(0.5774L) - w_0(0.5774L)^3/6L$$

$$M_{max} = 0.0962L^2 w_0 - 0.0321L^2 w_0$$

$$M_{max} = 0.0641L^2 w_0$$

Problem 417

Beam carrying the triangular loading shown in Fig. P- 417.

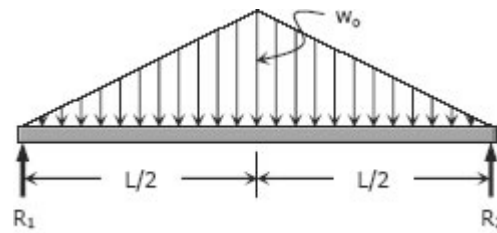


Figure P-417

Solution 417

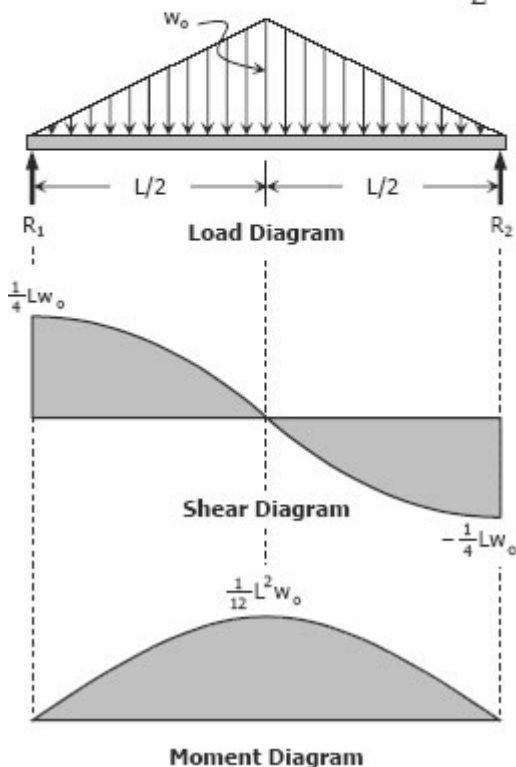
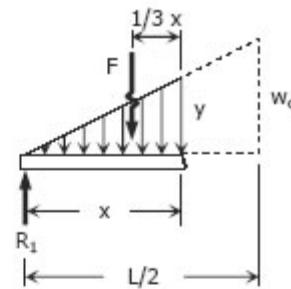
By symmetry:

$$R_1 = R_2 = \frac{1}{2} \left(\frac{1}{2} L w_0 \right) = \frac{1}{4} L w_0$$

$$\frac{y}{x} = \frac{w_0}{L/2}; \quad y = \frac{2w_0}{L} x$$

$$F = \frac{1}{2} xy = \frac{1}{2} x \left(\frac{2w_0}{L} x \right)$$

$$F = \frac{w_0}{L} x^2$$



$$V = R_1 - F$$

$$V = \frac{1}{4} L w_0 - \frac{w_0}{L} x^2$$

$$M = R_1 x - F \left(\frac{1}{3} x \right)$$

$$M = \frac{1}{4} L w_0 x - \left(\frac{w_0}{L} x^2 \right) \left(\frac{1}{3} x \right)$$

$$M = \frac{1}{4} L w_0 x - \frac{w_0}{3L} x^3$$

To draw the Shear Diagram:

$V = Lw_0/4 - w_0x^2/L$ is a second degree curve; at $x = 0$, $V = Lw_0/4$; at $x = L/2$, $V = 0$. The other half of the diagram can be drawn by the concept of symmetry.

To draw the Moment Diagram

$M = Lw_0x/4 - w_0x^3/3L$ is a third degree curve; at $x = 0$, $M = 0$; at $x = L/2$, $M = L^2w_0/12$. The other half of the diagram can be drawn by the concept of symmetry.

Problem 418

Cantilever beam loaded as shown in Fig. P-418.

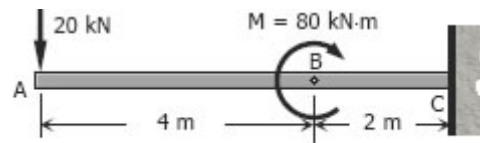
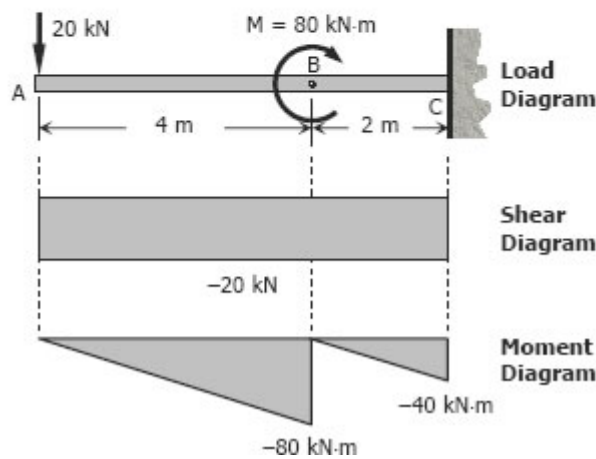
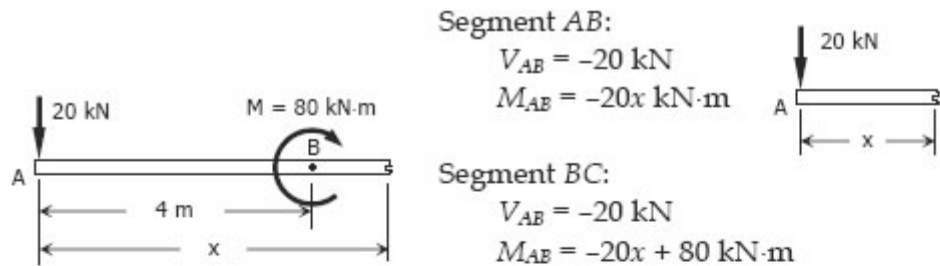


Figure P-418

Solution 418



To draw the Shear Diagram:

V_{AB} and V_{BC} are equal and constant at -20 kN .

To draw the Moment Diagram:

- (1) $M_{AB} = -20x$ is linear; when $x = 0$, $M_{AB} = 0$; when $x = 4 \text{ m}$, $M_{AB} = -80 \text{ kN}\cdot\text{m}$.
- (2) $M_{BC} = -20x + 80$ is also linear; when $x = 4 \text{ m}$, $M_{BC} = 0$; when $x = 6 \text{ m}$, $M_{BC} = -60 \text{ kN}\cdot\text{m}$

Problem 419

Beam loaded as shown in Fig. P-419.

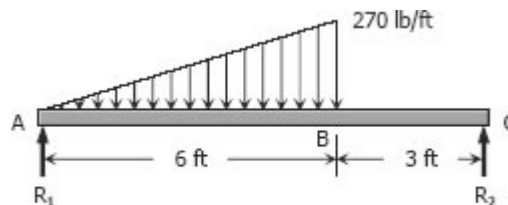
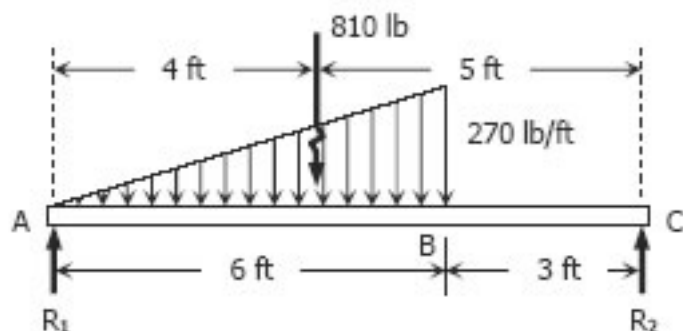


Figure P-419

Solution 419



$$[\sum M_C = 0] \quad 9R_1 = 5(810)$$

$$R_1 = 450 \text{ lb}$$

$$[\sum M_A = 0] \quad 9R_2 = 4(810)$$

$$R_2 = 360 \text{ lb}$$

Segment AB:

$$\frac{y}{x} = \frac{270}{6}$$

$$y = 45x$$

$$F = \frac{1}{2}xy = \frac{1}{2}x(45x)$$

$$F = 22.5x^2$$

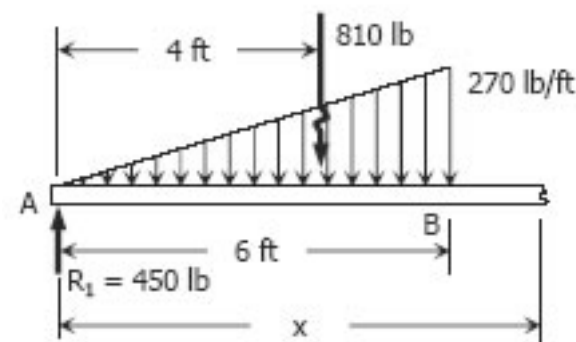
$$V_{AB} = R_1 - F$$

$$= 450 - 22.5x^2 \text{ lb}$$

$$M_{AB} = R_1x - F\left(\frac{1}{3}x\right)$$

$$= 450x - 22.5x^2\left(\frac{1}{3}x\right)$$

$$= 450x - 7.5x^3 \text{ lb}\cdot\text{ft}$$



Segment BC:

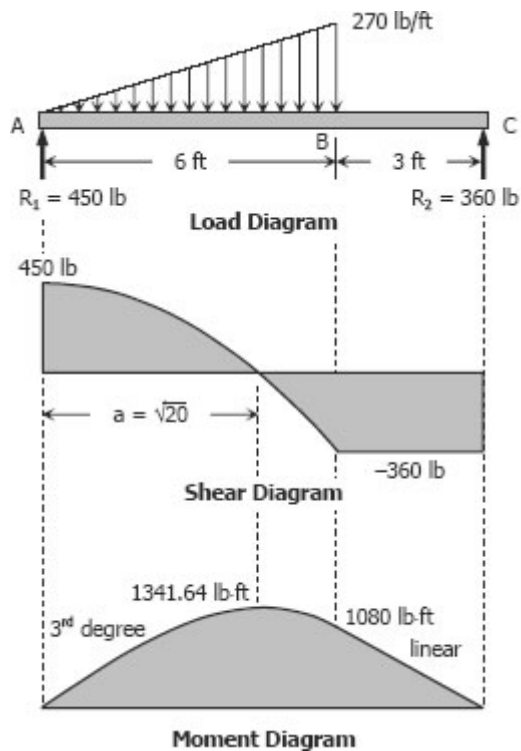
$$V_{BC} = 450 - 810$$

$$= -360 \text{ lb}$$

$$M_{BC} = 450x - 810(x - 4)$$

$$= 450x - 810x + 3240$$

$$= 3240 - 360x \text{ lb}\cdot\text{ft}$$



To draw the Shear Diagram:

- (1) $V_{AB} = 450 - 22.5x^2$ is a second degree curve; at $x = 0$, $V_{AB} = 450$ lb; at $x = 6$ ft, $V_{AB} = -360$ lb.
- (2) At $x = a$, $V_{AB} = 0$,
 $450 - 22.5x^2 = 0$
 $22.5x^2 = 450$
 $x^2 = 20$
 $x = \sqrt{20}$

To check, use the squared property of parabola.

$$\frac{a^2}{450} = \frac{6^2}{(450 + 360)}$$

$$a^2 = 20$$

$$a = \sqrt{20}$$

- (3) $V_{BC} = -360$ lb is constant.

To draw the Moment Diagram:

- (1) $M_{AB} = 450x - 7.5x^3$ for segment AB is third degree curve; at $x = 0$, $M_{AB} = 0$; at $x = \sqrt{20}$, $M_{AB} = 1341.64$ lb-ft; at $x = 6$ ft, $M_{AB} = 1080$ lb-ft.
- (2) $M_{BC} = 3240 - 360x$ for segment BC is linear; at $x = 6$ ft, $M_{BC} = 1080$ lb-ft; at $x = 9$ ft, $M_{BC} = 0$.

Problem 420

A total distributed load of 30 kips supported by a uniformly distributed reaction as shown in Fig. P-420.

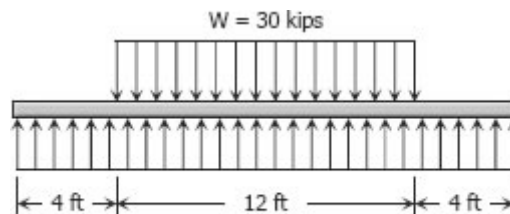
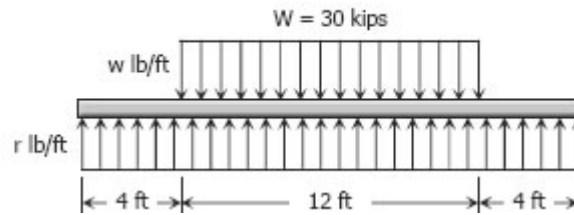


Figure P-420

Solution 420



$$w = 30(1000)/12$$

$$w = 2500 \text{ lb/ft}$$

$$\sum F_v = 0$$

$$R = W$$

$$20r = 30(1000)$$

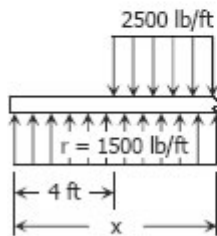
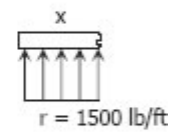
$$r = 1500 \text{ lb/ft}$$

First segment (from 0 to 4 ft from left):

$$V_1 = 1500x$$

$$M_1 = 1500x(x/2)$$

$$= 750x^2$$



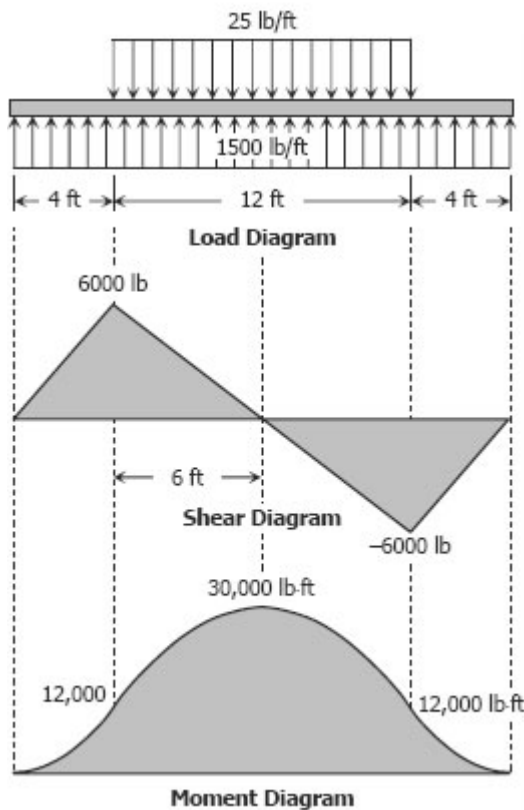
Second segment (from 4 ft to mid-span):

$$V_2 = 1500x - 2500(x - 4)$$

$$= 10000 - 1000x$$

$$M_2 = 1500x(x/2) - 2500(x - 4)(x - 4)/2$$

$$= 750x^2 - 1250(x - 4)^2$$



To draw the Shear Diagram:

- (1) For the first segment, $V_1 = 1500x$ is linear; at $x = 0$, $V_1 = 0$; at $x = 4$ ft, $V_1 = 6000$ lb.
- (2) For the second segment, $V_2 = 10000 - 1000x$ is also linear; at $x = 4$ ft, $V_1 = 6000$ lb; at mid-span, $x = 10$ ft, $V_1 = 0$.
- (3) For the next half of the beam, the shear diagram can be accomplished by the concept of symmetry.

To draw the Moment Diagram:

- (1) For the first segment, $M_1 = 750x^2$ is a second degree curve, an open upward parabola; at $x = 0$, $M_1 = 0$; at $x = 4$ ft, $M_1 = 12000$ lb-ft.
- (2) For the second segment, $M_2 = 750x^2 - 1250(x - 4)^2$ is a second degree curve, an downward parabola; at $x = 4$ ft, $M_2 = 12000$ lb-ft; at mid-span, $x = 10$ ft, $M_2 = 30000$ lb-ft.
- (2) The next half of the diagram, from $x = 10$ ft to $x = 20$ ft, can be drawn by using the concept of symmetry.

Problem 421

Write the shear and moment equations as functions of the angle θ for the built-in arch shown in Fig. P-421.

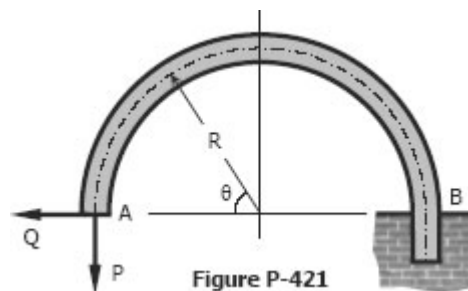


Figure P-421

Solution 421

For θ that is less than 90°

Components of Q and P :

$$Q_x = Q \sin \theta$$

$$Q_y = Q \cos \theta$$

$$P_x = P \sin (90^\circ - \theta)$$

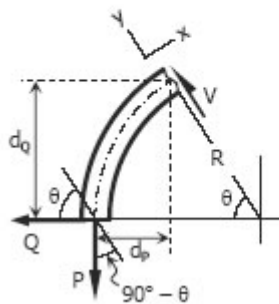
$$= P (\sin 90^\circ \cos \theta - \cancel{\cos 90^\circ} \sin \theta)$$

$$= P \cos \theta$$

$$P_y = P \cos (90^\circ - \theta)$$

$$= P (\cancel{\cos 90^\circ} \cos \theta + \sin 90^\circ \sin \theta)$$

$$= P \sin \theta$$



Shear:

$$V = \sum F_y$$

$$V = Q_y - P_y$$

$$V = Q \cos \theta - P \sin \theta$$

Moment arms:

$$d_Q = R \sin \theta$$

$$d_P = R - R \cos \theta$$

$$= R (1 - \cos \theta)$$

Moment:

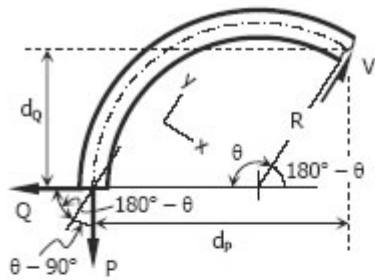
$$M = \sum M_{\text{counterclockwise}} - \sum M_{\text{clockwise}}$$

$$M = Q(d_Q) - P(d_P)$$

$$M = QR \sin \theta - PR(1 - \cos \theta)$$

For θ that is greater than 90°

Components of Q and P :



$$\begin{aligned} Q_x &= Q \sin (180^\circ - \theta) \\ &= Q (\sin 180^\circ \cos \theta - \cos 180^\circ \sin \theta) \\ &= Q \cos \theta \end{aligned}$$

$$\begin{aligned} Q_y &= Q \cos (180^\circ - \theta) \\ &= Q (\cos 180^\circ \cos \theta + \sin 180^\circ \sin \theta) \\ &= -Q \sin \theta \end{aligned}$$

$$\begin{aligned} P_x &= P \sin (\theta - 90^\circ) \\ &= P (\sin \theta \cos 90^\circ - \cos \theta \sin 90^\circ) \\ &= -P \cos \theta \end{aligned}$$

$$\begin{aligned} P_y &= P \cos (\theta - 90^\circ) \\ &= P (\cos \theta \cos 90^\circ + \sin \theta \sin 90^\circ) \\ &= P \sin \theta \end{aligned}$$

Shear:

$$\begin{aligned} V &= \sum F_y \\ V &= -Q_y - P_y \\ V &= -(-Q \sin \theta) - P \sin \theta \\ V &= Q \sin \theta - P \sin \theta \end{aligned}$$

Moment arms:

$$\begin{aligned} d_Q &= R \sin (180^\circ - \theta) \\ &= R (\sin 180^\circ \cos \theta - \cos 180^\circ \sin \theta) \\ &= R \sin \theta \end{aligned}$$

$$\begin{aligned} d_P &= R + R \cos (180^\circ - \theta) \\ &= R + R (\cos 180^\circ \cos \theta + \sin 180^\circ \sin \theta) \\ &= R - R \cos \theta \\ &= R(1 - \cos \theta) \end{aligned}$$

Moment:

$$\begin{aligned} M &= \sum M_{\text{counterclockwise}} - \sum M_{\text{clockwise}} \\ M &= Q(d_Q) - P(d_P) \\ M &= QR \sin \theta - PR(1 - \cos \theta) \end{aligned}$$

Problem 422

Write the shear and moment equations for the semicircular arch as shown in Fig. P-422 if (a) the load P is vertical as shown, and (b) the load is applied horizontally to the left at the top of the arch.

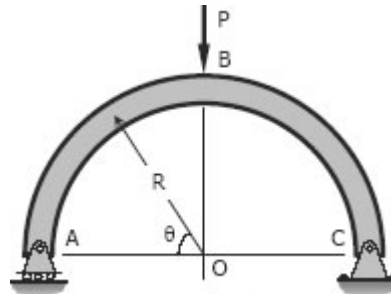
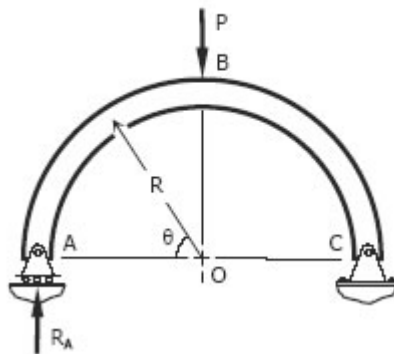


Figure P-422

Solution 422



$$\sum M_C = 0$$

$$2R(R_A) = RP$$

$$R_A = \frac{1}{2}P$$

For θ that is less than 90°

Shear:

$$V_{AB} = R_A \cos(90^\circ - \theta)$$

$$V_{AB} = \frac{1}{2}P (\cos 90^\circ \cos \theta + \sin 90^\circ \sin \theta)$$

$$V_{AB} = \frac{1}{2}P \sin \theta$$

Moment arm:

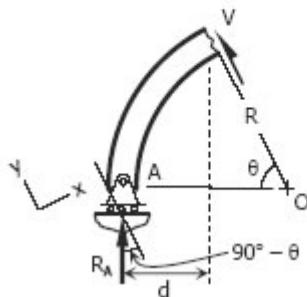
$$d = R - R \cos \theta$$

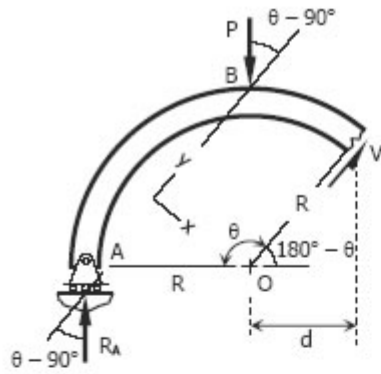
$$d = R(1 - \cos \theta)$$

Moment:

$$M_{AB} = R_s(d)$$

$$M_{AB} = \frac{1}{2}PR(1 - \cos \theta)$$





For θ that is greater than 90°

Components of P and R_A :

$$\begin{aligned} P_x &= P \sin (\theta - 90^\circ) \\ &= P (\sin \theta \cos 90^\circ - \cos \theta \sin 90^\circ) \\ &= -P \cos \theta \end{aligned}$$

$$\begin{aligned} P_y &= P \cos (\theta - 90^\circ) \\ &= P (\cos \theta \cos 90^\circ + \sin \theta \sin 90^\circ) \\ &= P \sin \theta \end{aligned}$$

$$\begin{aligned} R_{Ax} &= R_A \sin (\theta - 90^\circ) \\ &= \frac{1}{2} P (\sin \theta \cos 90^\circ - \cos \theta \sin 90^\circ) \\ &= -\frac{1}{2} P \cos \theta \end{aligned}$$

$$\begin{aligned} R_{Ay} &= R_A \cos (\theta - 90^\circ) \\ &= \frac{1}{2} P (\cos \theta \cos 90^\circ + \sin \theta \sin 90^\circ) \\ &= \frac{1}{2} P \sin \theta \end{aligned}$$

Shear:

$$V_{BC} = \Sigma F_y$$

$$V_{BC} = R_{Ay} - P_y$$

$$V_{BC} = \frac{1}{2} P \sin \theta - P \sin \theta$$

$$V_{BC} = -\frac{1}{2} P \sin \theta$$

Moment arm:

$$d = R \cos (180^\circ - \theta)$$

$$d = R (\cos 180^\circ \cos \theta + \sin 180^\circ \sin \theta)$$

$$d = -R \cos \theta$$

Moment:

$$M_{BC} = \Sigma M_{\text{counterclockwise}} - \Sigma M_{\text{clockwise}}$$

$$M_{BC} = R_A(R + d) - Pd$$

$$M_{BC} = \frac{1}{2} P(R - R \cos \theta) - P(-R \cos \theta)$$

$$M_{BC} = \frac{1}{2} PR - \frac{1}{2} PR \cos \theta + PR \cos \theta$$

$$M_{BC} = \frac{1}{2} PR + \frac{1}{2} PR \cos \theta$$

$$M_{BC} = \frac{1}{2} PR(1 + \cos \theta)$$

Relationship between Load, Shear, and Moment

The vertical shear at C in the figure shown in previous section is taken as

$$V_C = (\Sigma F_v)_L = R_1 - wx$$

where $R_1 = R_2 = wL/2$

$$V_C = \frac{wL}{2} - wx$$
$$M_C = (\Sigma M_C) = \frac{wL}{2}x - wx\left(\frac{x}{2}\right)$$
$$M_C = \frac{wLx}{2} - \frac{wx^2}{2}$$

If we differentiate M with respect to x:

$$\frac{dM}{dx} = \frac{wL}{2} \frac{dx}{dx} - \frac{w}{2} 2x \frac{dx}{dx}$$
$$\frac{dM}{dx} = \frac{wL}{2} - wx = \text{shear}$$

thus,

$$\frac{dM}{dx} = V$$

Thus, the rate of change of the bending moment with respect to x is equal to the shearing force, or **the slope of the moment diagram at the given point is the shear at that point.**

Differentiate V with respect to x gives

$$\frac{dV}{dx} = 0 - w = \text{load}$$
$$\frac{dV}{dx} = \text{Load}$$

Thus, the rate of change of the shearing force with respect to x is equal to the load or **the slope of the shear diagram at a given point equals the load at that point.**

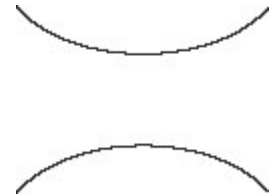
PROPERTIES OF SHEAR AND MOMENT DIAGRAMS

The following are some important properties of shear and moment diagrams:

1. The area of the shear diagram to the left or to the right of the section is equal to the moment at that section.
2. The slope of the moment diagram at a given point is the shear at that point.
3. The slope of the shear diagram at a given point equals the load at that point.

4. The maximum moment occurs at the point of zero shears.

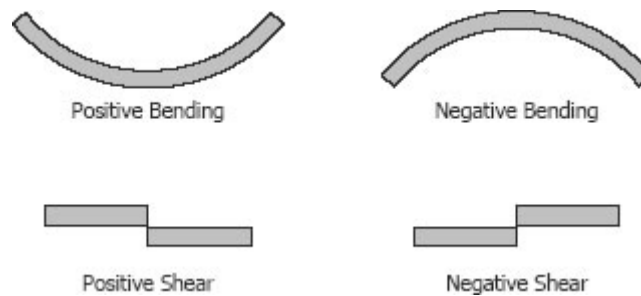
This is in reference to property number 2, that when the shear (also the slope of the moment diagram) is zero, the tangent drawn to the moment diagram is horizontal.



5. When the shear diagram is increasing, the moment diagram is concave upward.
6. When the shear diagram is decreasing, the moment diagram is concave downward.

SIGN CONVENTIONS

The customary sign conventions for shearing force and bending moment are represented by the figures below. A force that tends to bend the beam downward is said to produce a positive bending moment. A force that tends to shear the left portion of the beam upward with respect to the right portion is said to produce a positive shearing force.



An easier way of determining the sign of the bending moment at any section is that upward forces always cause positive bending moments regardless of whether they act to the left or to the right of the exploratory section.

Solved Problems in Relationship between Load, Shear, and Moment

INSTRUCTION

Without writing shear and moment equations, draw the shear and moment diagrams for the beams specified in the following problems. Give numerical values at all change of loading positions and at all points of zero shear. (Note to instructor: Problems 403 to 420 may also be assigned for solution by semi graphical method describes in this article.)

Problem 425

Beam loaded as shown in Fig. P-425.

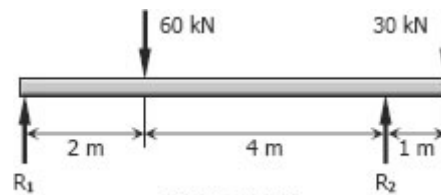
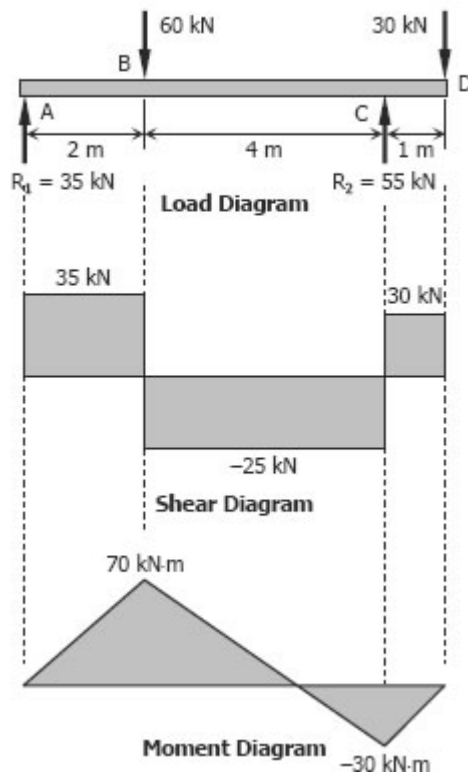


Figure P-425

Solution 425



$$\sum M_A = 0$$

$$6R_2 = 2(60) + 7(30)$$

$$R_2 = 55 \text{ kN}$$

$$\sum M_C = 0$$

$$6R_1 + 1(30) = 4(60)$$

$$R_1 = 35 \text{ kN}$$

To draw the Shear Diagram:

- (1) $V_A = R_1 = 35 \text{ kN}$
- (2) $V_B = V_A + \text{Area in load diagram} - 60 \text{ kN}$
 $V_B = 35 + 0 - 60 = -25 \text{ kN}$
- (3) $V_C = V_B + \text{area in load diagram} + R_2$
 $V_C = -25 + 0 + 55 = 30 \text{ kN}$
- (4) $V_D = V_C + \text{Area in load diagram} - 30 \text{ kN}$
 $V_D = 30 + 0 - 30 = 0$

To draw the Moment Diagram:

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 + 35(2) = 70 \text{ kN-m}$
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = 70 - 25(4) = -30 \text{ kN-m}$
- (4) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = -30 + 30(1) = 0$

Problem 426

Cantilever beam acted upon by a uniformly distributed load and a couple as shown in Fig. P-426.

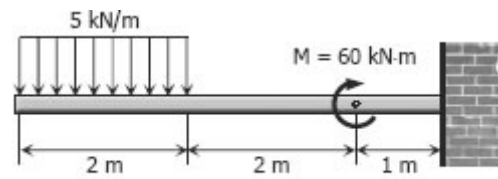
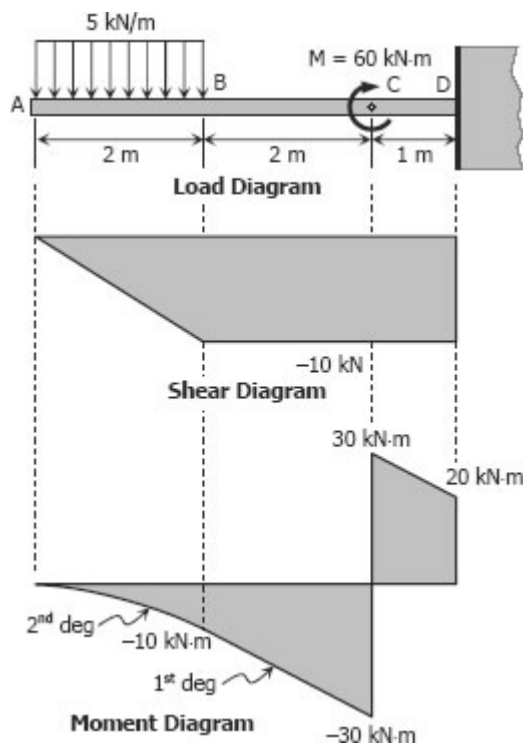


Figure P-426

Solution 426



To draw the Shear Diagram

- (1) $V_A = 0$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 0 - 5(2)$
 $V_B = -10 \text{ kN}$
- (3) $V_C = V_B + \text{Area in load diagram}$
 $V_C = -10 + 0$
 $V_C = -10 \text{ kN}$
- (4) $V_D = V_C + \text{Area in load diagram}$
 $V_D = -10 + 0$
 $V_D = -10 \text{ kN}$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 - \frac{1}{2}(2)(10)$
 $M_B = -10 \text{ kN-m}$
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = -10 - 10(2)$
 $M_C = -30 \text{ kN-m}$
 $M_{C2} = -30 + M = -30 + 60 = 30 \text{ kN-m}$
- (4) $M_D = M_{C2} + \text{Area in shear diagram}$
 $M_D = 30 - 10(1)$
 $M_D = 20 \text{ kN-m}$

Problem 427

Beam loaded as shown in Fig. P-427.

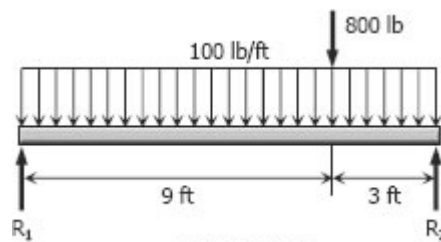
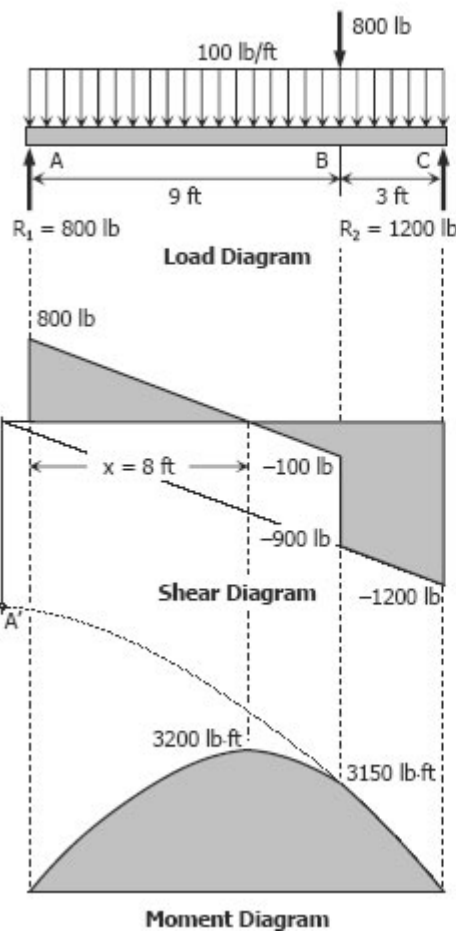


Figure P-427

Solution 427



$$\sum M_C = 0$$

$$12R_1 = 100(12)(6) + 800(3)$$

$$R_1 = 800 \text{ lb}$$

$$\sum M_A = 0$$

$$12R_2 = 100(12)(6) + 800(9)$$

$$R_2 = 1200 \text{ lb}$$

To draw the Shear Diagram

- (1) $V_A = R_1 = 800 \text{ lb}$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 800 - 100(9)$
 $V_B = -100 \text{ lb}$
 $V_{B2} = -100 - 800 = -900 \text{ lb}$
- (3) $V_C = V_{B2} + \text{Area in load diagram}$
 $V_C = -900 - 100(3)$
 $V_C = -1200 \text{ lb}$
- (4) Solving for x :
 $x / 800 = (9 - x) / 100$
 $100x = 7200 - 800x$
 $x = 8 \text{ ft}$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_x = M_A + \text{Area in shear diagram}$
 $M_x = 0 + \frac{1}{2}(8)(800) = 3200 \text{ lb-ft}$
- (3) $M_B = M_x + \text{Area in shear diagram}$
 $M_B = 3200 - \frac{1}{2}(1)(100) = 3150 \text{ lb-ft}$
- (4) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = 3150 - \frac{1}{2}(900 + 1200)(3) = 0$
- (5) The moment curve BC is downward parabola with vertex at A'. A' is the location of zero shear for segment BC.

Problem 428

Beam loaded as shown in Fig. P-428.

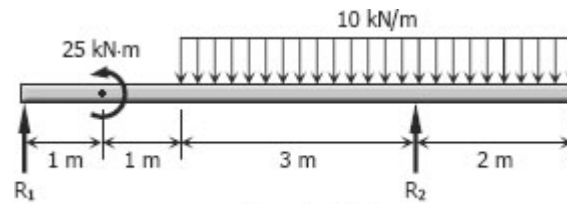
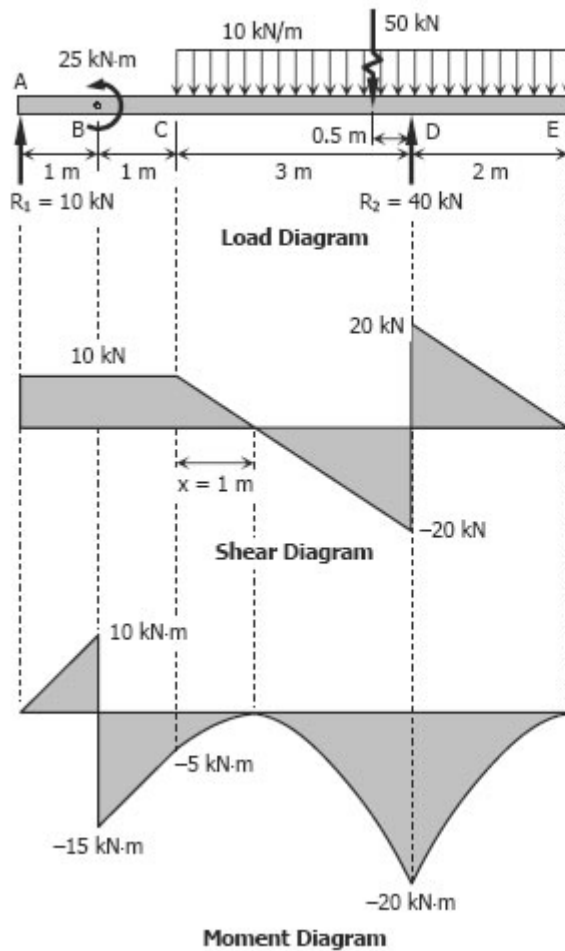


Figure P-428

Solution 428



$$\Sigma M_D = 0$$

$$5R_1 = 50(0.5) + 25$$

$$R_1 = 10 \text{ kN}$$

$$\Sigma M_A = 0$$

$$5R_2 + 25 = 50(4.5)$$

$$R_2 = 40 \text{ kN}$$

To draw the Shear Diagram

- (1) $V_A = R_1 = 10 \text{ kN}$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 10 + 0 = 10 \text{ kN}$
- (3) $V_C = V_B + \text{Area in load diagram}$
 $V_C = 10 + 0 = 10 \text{ kN}$
- (4) $V_D = V_C + \text{Area in load diagram}$
 $V_D = 10 - 10(3) = -20 \text{ kN}$
 $V_{D2} = -20 + R_2 = 20 \text{ kN}$
- (5) $V_E = V_{D2} + \text{Area in load diagram}$
 $V_E = 20 - 10(2) = 0$
- (6) Solving for x:
 $x / 10 = (3 - x) / 20$
 $20x = 30 - 10x$
 $x = 1 \text{ m}$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 + 1(10) = 10 \text{ kN-m}$
 $M_{B2} = 10 - 25 = -15 \text{ kN-m}$
- (3) $M_C = M_{B2} + \text{Area in shear diagram}$
 $M_C = -15 + 1(10) = -5 \text{ kN-m}$
- (4) $M_x = M_C + \text{Area in shear diagram}$
 $M_x = -5 + \frac{1}{2}(1)(10) = 0$
- (5) $M_D = M_x + \text{Area in shear diagram}$
 $M_D = 0 - \frac{1}{2}(2)(20) = -20 \text{ kN-m}$
- (6) $M_E = M_D + \text{Area in shear diagram}$
 $M_E = -20 + \frac{1}{2}(2)(20) = 0$

Problem 429

Beam loaded as shown in Fig. P-429.

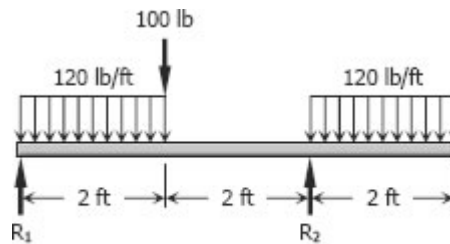
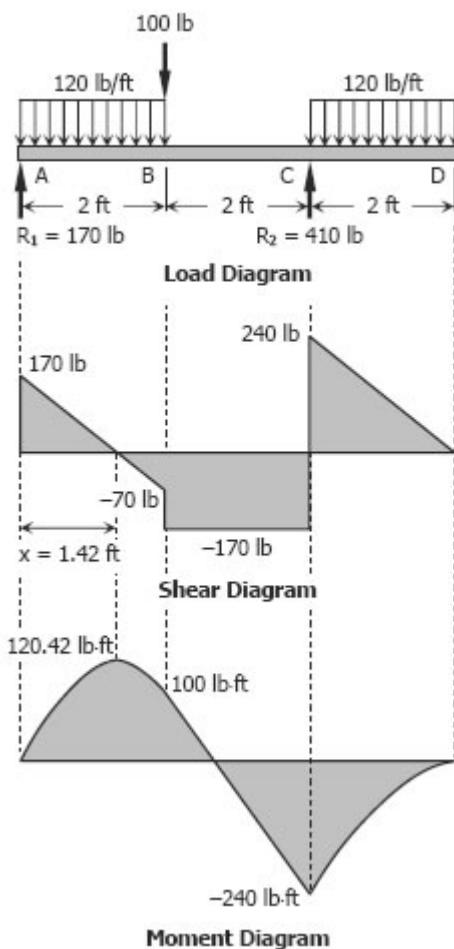


Figure P-429

Solution 429



$$\sum M_C = 0$$

$$4R_1 + 120(2)(1) = 100(2) + 120(2)(3)$$

$$R_1 = 170 \text{ lb}$$

$$\sum M_A = 0$$

$$4R_2 = 120(2)(1) + 100(2) + 120(2)(5)$$

$$R_2 = 410 \text{ lb}$$

To draw the Shear Diagram

- (1) $V_A = R_1 = 170 \text{ lb}$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 170 - 120(2) = -70 \text{ lb}$
 $V_{B2} = -70 - 100 = -170 \text{ lb}$
- (3) $V_C = V_{B2} + \text{Area in load diagram}$
 $V_C = -170 + 0 = -170 \text{ lb}$
 $V_{C2} = -170 + R_2$
 $V_{C2} = -170 + 410 = 240 \text{ lb}$
- (4) $V_D = V_{C2} + \text{Area in load diagram}$
 $V_D = 240 - 120(2) = 0$
- (5) Solving for x :

$$\frac{x}{170} = \frac{(2-x)}{70}$$

$$70x = 340 - 170x$$

$$x = 17/12 \text{ ft} = 1.42 \text{ ft}$$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_x = M_A + \text{Area in shear diagram}$
 $M_x = 0 + \frac{1}{2}(17/12)(170)$
 $M_x = 1445/12 = 120.42 \text{ lb-ft}$
- (3) $M_B = M_x + \text{Area in shear diagram}$
 $M_B = 1445/12 - \frac{1}{2}(2 - 17/12)(70)$
 $M_B = 100 \text{ lb-ft}$
- (4) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = 100 - 170(2) = -240 \text{ lb-ft}$
- (5) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = -240 + \frac{1}{2}(2)(240) = 0$

Problem 430

Beam loaded as shown in P-430.

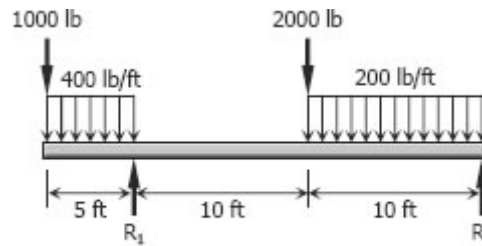
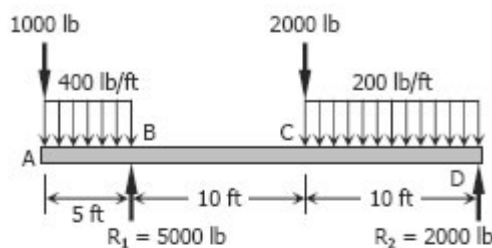
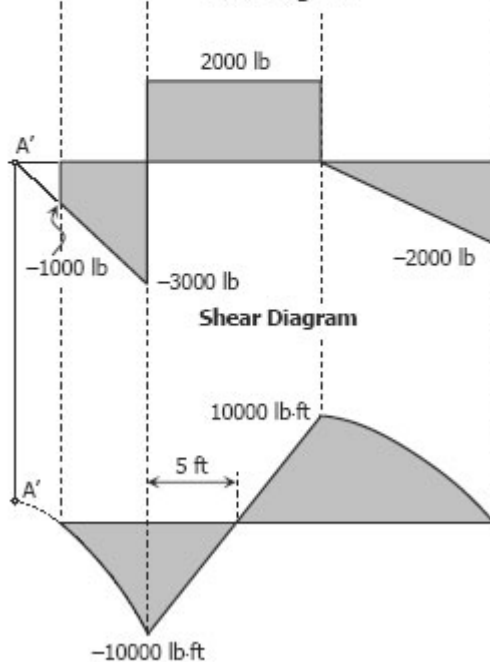


Figure P-430

Solution 430



Load Diagram



Moment Diagram

$$\sum M_D = 0$$

$$20R_1 = 1000(25) + 400(5)(22.5) + 2000(10) + 200(10)(15)$$

$$R_1 = 5000 \text{ lb}$$

$$\sum M_B = 0$$

$$20R_2 + 1000(5) + 400(5)(2.5) = 2000(10) + 200(10)(15)$$

$$R_2 = 2000 \text{ lb}$$

To draw the Shear Diagram

- (1) $V_A = -1000$ lb
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = -1000 - 400(5) = -3000$ lb
 $V_{B2} = -3000 + R_1 = 2000$ lb
- (3) $V_C = V_{B2} + \text{Area in load diagram}$
 $V_C = 2000 + 0 = 2000$ lb
 $V_{C2} = 2000 - 2000 = 0$
- (4) $V_D = V_{C2} + \text{Area in load diagram}$
 $V_D = 0 + 200(10) = 2000$ lb

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 - \frac{1}{2}(1000 + 3000)(5)$
 $M_B = -10000$ lb-ft
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = -10000 + 2000(10) = 10000$ lb-ft
- (4) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = 10000 - \frac{1}{2}(10)(2000) = 0$
- (5) For segment BC, the location of zero moment can be accomplished by symmetry and that is 5 ft from B.
- (6) The moment curve AB is a downward parabola with vertex at A'. A' is the location of zero shear for segment AB at point outside the beam.

Problem 431

Beam loaded as shown in Fig. P-431.

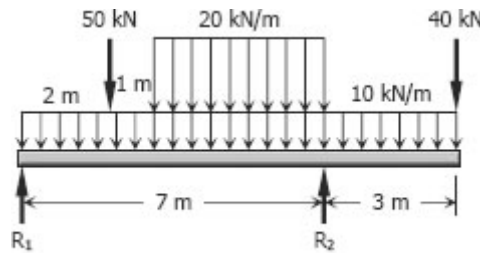
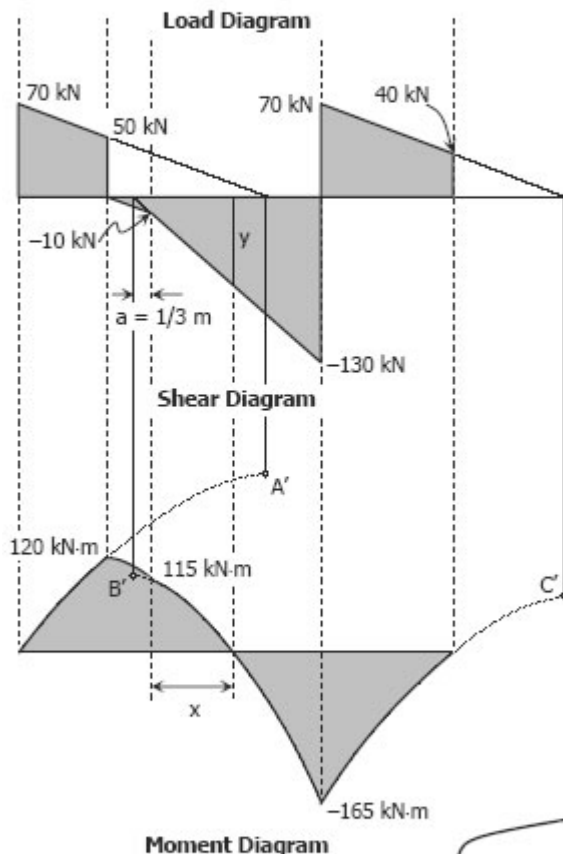
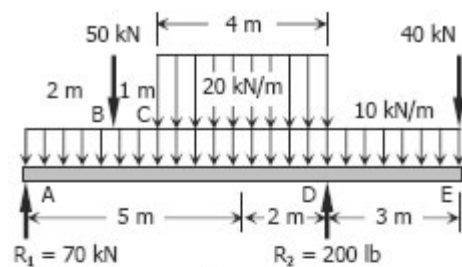


Figure P-431

Solution 431



$$\sum M_D = 0$$

$$7R_1 + 40(3) = 5(50) + 10(10)(2) + 20(4)(2)$$

$$R_1 = 70 \text{ kN}$$

$$\sum M_A = 0$$

$$7R_2 = 50(2) + 10(10)(5) + 20(4)(5) + 40(10)$$

$$R_2 = 200 \text{ lb}$$

To draw the Shear Diagram

- (1) $V_A = R_1 = 70 \text{ kN}$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 70 - 10(2) = 50 \text{ kN}$
 $V_{B2} = 50 - 50 = 0$
- (3) $V_C = V_{B2} + \text{Area in load diagram}$
 $V_C = 0 - 10(1) = -10 \text{ kN}$
- (4) $V_D = V_C + \text{Area in load diagram}$
 $V_D = -10 - 30(4) = -130 \text{ kN}$
 $V_{D2} = -130 + R_2$
 $V_{D2} = -130 + 200 = 70 \text{ kN}$
- (5) $V_E = V_{D2} + \text{Area in load diagram}$
 $V_E = 70 - 10(3) = 40 \text{ kN}$
 $V_{E2} = 40 - 40 = 0$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 + \frac{1}{2}(70 + 50)(2) = 120 \text{ kN-m}$
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = 120 - \frac{1}{2}(1)(10) = 115 \text{ kN-m}$
- (4) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = 115 - \frac{1}{2}(10 + 130)(4) = -165 \text{ kN-m}$
- (5) $M_E = M_D + \text{Area in shear diagram}$
 $M_E = -165 + \frac{1}{2}(70 + 40)(3) = 0$
- (6) Moment curves AB, CD and DE are downward parabolas with vertices at A', B' and C', respectively. A', B' and C' are corresponding zero shear points of segments AB, CD and DE.

- (7) Solving for point of zero moment:

$$\begin{aligned} a / 10 &= (a + 4) / 130 \\ 130a &= 10a + 40 \\ a &= 1/3 \text{ m} \end{aligned}$$

$$\begin{aligned} y / (x + a) &= 130 / (4 + a) \\ y &= 130(x + 1/3) / (4 + 1/3) \\ y &= 30x + 10 \end{aligned}$$

$$\begin{aligned} M_C &= 115 \text{ kN-m} \\ M_{\text{zero}} &= M_C + \text{Area in shear} \\ 0 &= 115 - \frac{1}{2}(10 + y)x \\ (10 + y)x &= 230 \\ (10 + 30x + 10)x &= 230 \\ 30x^2 + 20x - 230 &= 0 \\ 3x^2 + 2x - 23 &= 0 \\ x &= 2.46 \text{ m} \\ \text{zero moment is at 2.46 m from C} \end{aligned}$$

Another way to solve the location of zero moment is by the squared property of parabola (see Problem 434). This point is the appropriate location for construction joint of concrete structures.

Problem 432

Beam loaded as shown in Fig. P-432.

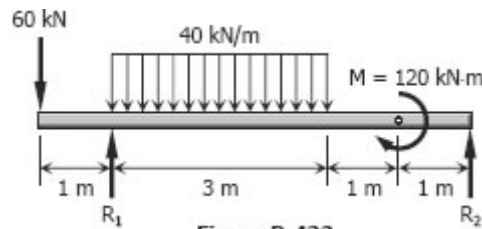
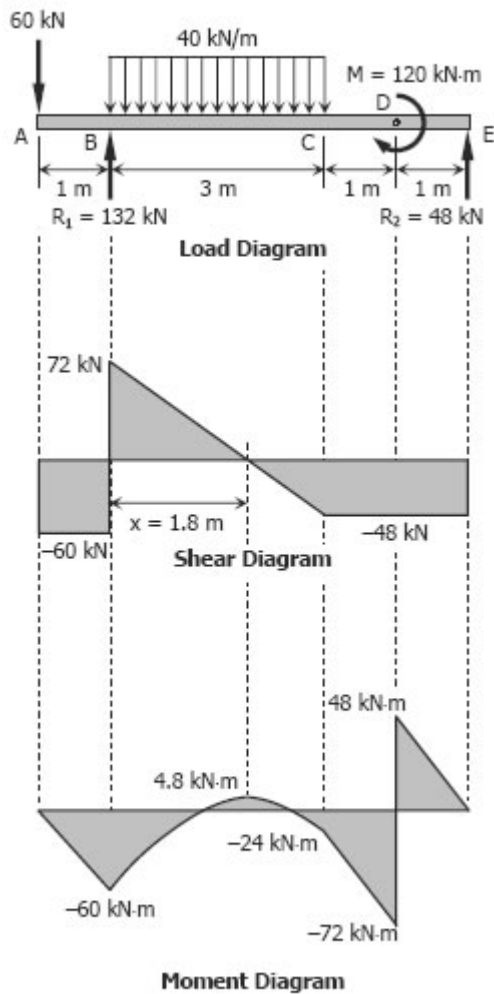


Figure P-432

Solution 432



$$\sum M_E = 0$$

$$5R_1 + 120 = 6(60) + 40(3)(3.5)$$

$$R_1 = 132 \text{ kN}$$

$$\sum M_B = 0$$

$$5R_2 + 60(1) = 40(3)(1.5) + 120$$

$$R_2 = 48 \text{ kN}$$

To draw the Shear Diagram

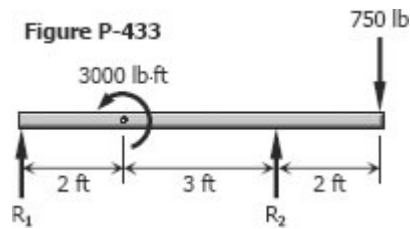
- (1) $V_A = -60 \text{ kN}$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = -60 + 0 = -60 \text{ kN}$
 $V_{B2} = V_B + R_1 = -60 + 132 = 72 \text{ kN}$
- (3) $V_C = V_{B2} + \text{Area in load diagram}$
 $V_C = 72 - 3(40) = -48 \text{ kN}$
- (4) $V_D = V_C + \text{Area in load diagram}$
 $V_D = -48 + 0 = -48 \text{ kN}$
- (5) $V_E = V_D + \text{Area in load diagram}$
 $V_E = -48 + 0 = -48 \text{ kN}$
 $V_{E2} = V_E + R_2 = -48 + 48 = 0$
- (6) Solving for x :
 $x / 72 = (3 - x) / 48$
 $48x = 216 - 72x$
 $x = 1.8 \text{ m}$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 - 60(1) = -60 \text{ kN-m}$
- (3) $M_x = M_B + \text{Area in shear diagram}$
 $M_x = -60 + \frac{1}{2}(1.8)(72) = 4.8 \text{ kN-m}$
- (4) $M_C = M_x + \text{Area in shear diagram}$
 $M_C = 4.8 - \frac{1}{2}(3 - 1.8)(48) = -24 \text{ kN-m}$
- (5) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = -24 - \frac{1}{2}(24 + 72)(1) = -72 \text{ kN-m}$
 $M_{D2} = -72 + 120 = 48 \text{ kN-m}$
- (6) $M_E = M_{D2} + \text{Area in shear diagram}$
 $M_E = 48 - 48(1) = 0$
- (7) The location of zero moment on segment BC can be determined using the squared property of parabola. See the solution of Problem 434.

Problem 433

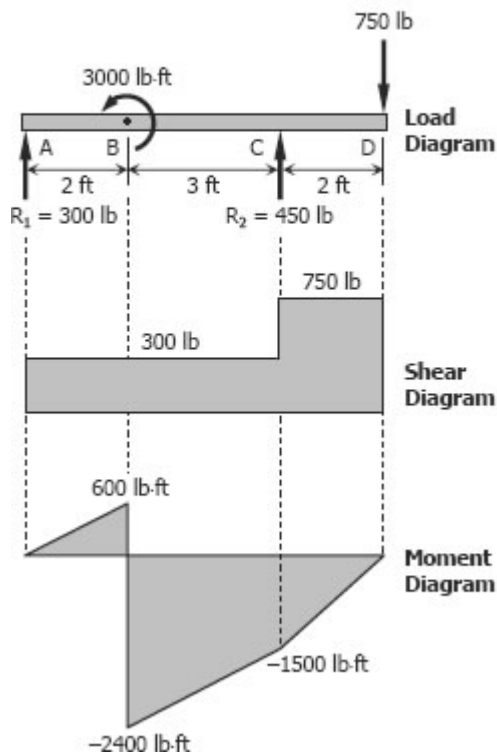
Overhang beam loaded by a force and a couple as shown in Fig. P-433.



Solution 433

$$\begin{aligned}\sum M_C &= 0 \\ 5R_1 + 2(750) &= 3000 \\ R_1 &= 300 \text{ lb}\end{aligned}$$

$$\begin{aligned}\sum M_A &= 0 \\ 5R_2 + 3000 &= 7(750) \\ R_2 &= 450 \text{ lb}\end{aligned}$$



To draw the Shear Diagram

- (1) $V_A = R_1 = 300 \text{ lb}$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 300 + 0 = 300 \text{ lb}$
- (3) $V_C = V_B + \text{Area in load diagram}$
 $V_C = 300 + 0 = 300 \text{ lb}$
 $V_{C2} = V_C + R_2 = 300 + 450 = 750 \text{ lb}$
- (5) $V_D = V_{C2} + \text{Area in load diagram}$
 $V_D = 750 + 0 = 750$
 $V_{D2} = V_D - 750 = 750 - 750 = 0$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = V_A + \text{Area in shear diagram}$
 $M_B = 0 + 300(2) = 600 \text{ lb-ft}$
 $M_{B2} = V_B - 3000$
 $M_{B2} = 600 - 3000 = -2400 \text{ lb-ft}$
- (3) $M_C = M_{B2} + \text{Area in shear diagram}$
 $M_C = -2400 + 300(3) = -1500 \text{ lb-ft}$
- (4) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = -1500 + 750(2) = 0$

Problem 434

Beam loaded as shown in Fig. P-434.

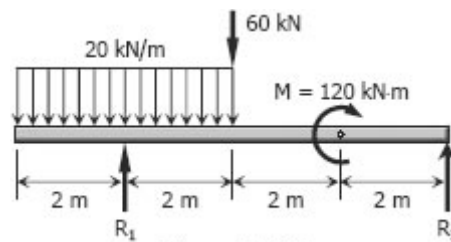
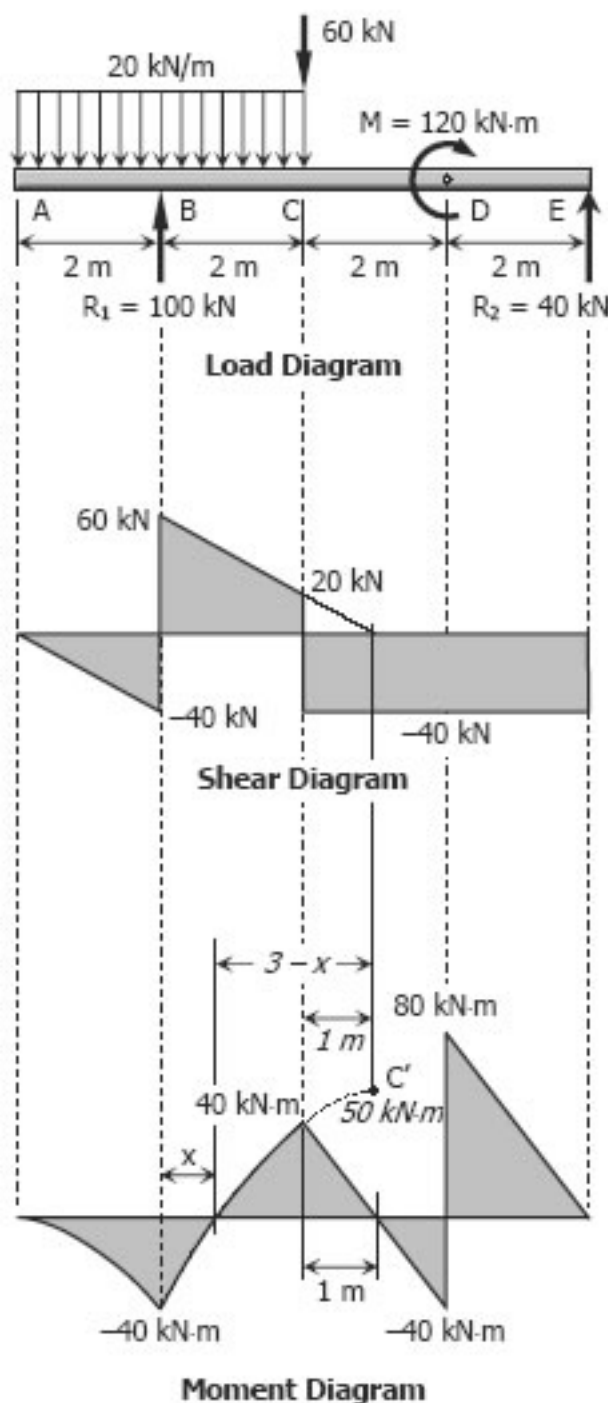


Figure P-434

Solution 434



$$\sum M_E = 0$$

$$6R_1 + 120 = 20(4)(6) + 60(4)$$

$$R_1 = 100 \text{ kN}$$

$$\sum M_B = 0$$

$$6R_2 = 20(4)(0) + 60(2) + 120$$

$$R_2 = 40 \text{ kN}$$

To draw the Shear Diagram

- (1) $V_A = 0$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 0 - 20(2) = -40 \text{ kN}$
 $V_{B2} = V_B + R_1 = -40 + 100 = 60 \text{ kN}$
- (3) $V_C = V_{B2} + \text{Area in load diagram}$
 $V_C = 60 - 20(2) = 20 \text{ kN}$
 $V_{C2} = V_C - 60 = 20 - 60 = -40 \text{ kN}$
- (4) $V_D = V_{C2} + \text{Area in load diagram}$
 $V_D = -40 + 0 = -40 \text{ kN}$
- (5) $V_E = V_D + \text{Area in load diagram}$
 $V_E = -40 + 0 = -40 \text{ kN}$
 $V_{E2} = V_E + R_2 = -40 + 40 = 0$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 - \frac{1}{2}(40)(2) = -40 \text{ kN-m}$
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = -40 + \frac{1}{2}(60 + 20)(2) = 40 \text{ kN-m}$
- (4) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = 40 - 40(2) = -40 \text{ kN-m}$
 $M_{D2} = M_D + M = -40 + 120 = 80 \text{ kN-m}$
- (5) $M_E = M_{D2} + \text{Area in shear diagram}$
 $M_E = 80 - 40(2) = 0$
- (6) Moment curve BC is a downward parabola with vertex at C'. C' is the location of zero shear for segment BC.
- (7) Location of zero moment at segment BC:
 By squared property of parabola:
 $(3 - x)^2 / 50 = 3^2 / (50 + 40)$
 $3 - x = 2.236$
 $x = 0.764 \text{ m from B}$

Problem 435

Beam loaded and supported as shown in Fig. P-435.

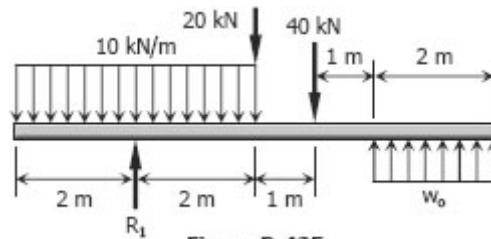
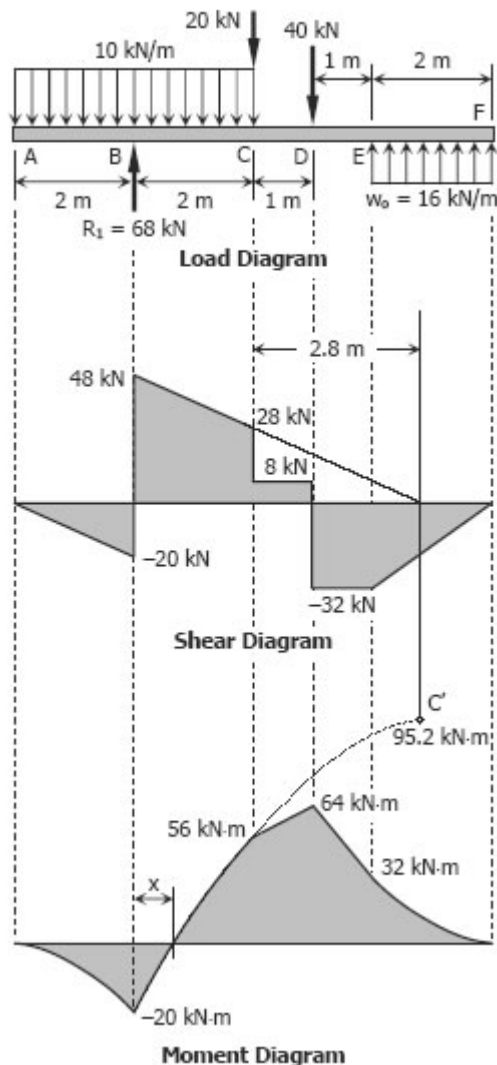


Figure P-435

Solution 435



$$\sum M_B = 0$$

$$2w_o(5) = 10(4)(0) + 20(2) + 40(3)$$

$$w_o = 16 \text{ kN/m}$$

$$\sum M_{\text{midpoint of EF}} = 0$$

$$5R_1 = 10(4)(5) + 20(3) + 40(2)$$

$$R_1 = 68 \text{ kN}$$

To draw the Shear Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in load diagram}$
 $M_B = 0 - 10(2) = -20 \text{ kN}$
 $M_{B2} + M_B + R_1 = -20 + 68 = 48 \text{ kN}$
- (3) $M_C = M_{B2} + \text{Area in load diagram}$
 $M_C = 48 - 10(2) = 28 \text{ kN}$
 $M_{C2} = M_C - 20 = 28 - 20 = 8 \text{ kN}$
- (4) $M_D = M_{C2} + \text{Area in load diagram}$
 $M_D = 8 + 0 = 8 \text{ kN}$
 $M_{D2} = M_D - 40 = 8 - 40 = -32 \text{ kN}$
- (5) $M_E = M_{D2} + \text{Area in load diagram}$
 $M_E = -32 + 0 = -32 \text{ kN}$
- (6) $M_F = M_E + \text{Area in load diagram}$
 $M_F = -32 + w_o(2)$
 $M_F = -32 + 16(2) = 0$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 - \frac{1}{2}(20)(2) = -20 \text{ kN-m}$
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = -20 + \frac{1}{2}(48 + 28)(2)$
 $M_C = 56 \text{ kN-m}$
- (4) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = 56 + 8(1) = 64 \text{ kN-m}$
- (5) $M_E = M_D + \text{Area in shear diagram}$
 $M_E = 64 - 32(1) = 32 \text{ kN-m}$
- (6) $M_F = M_E + \text{Area in shear diagram}$
 $M_F = 32 - \frac{1}{2}(32)(2) = 0$
- (7) The location and magnitude of moment at C' are determined from shear diagram. By squared property of parabola, $x = 0.44 \text{ m}$ from B.

Problem 436

A distributed load is supported by two distributed reactions as shown in Fig. P-436.

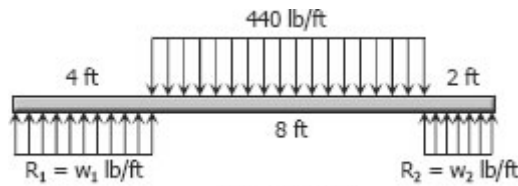
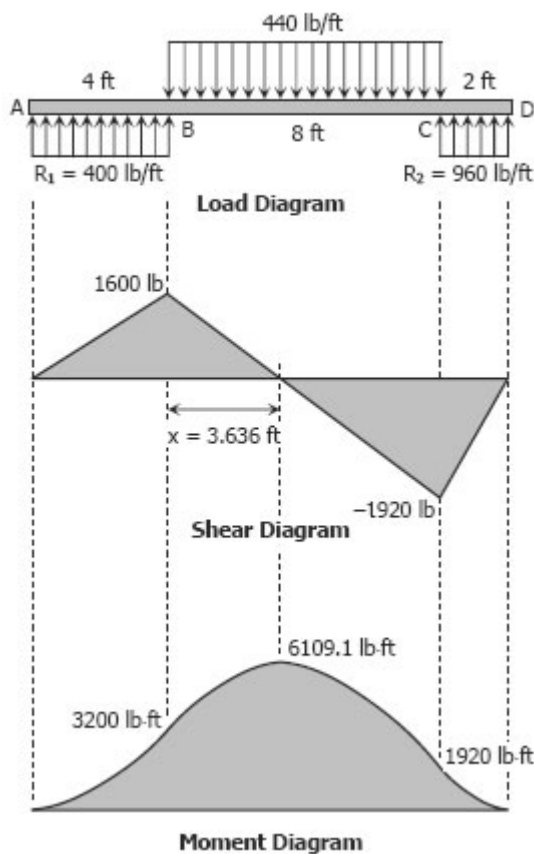


Figure P-436

Solution 436



$$\sum M_{\text{midpoint of CD}} = 0$$

$$4w_1(11) = 440(8)(5)$$

$$w_1 = 400 \text{ lb/ft}$$

$$\sum M_{\text{midpoint of AB}} = 0$$

$$2w_2(11) = 440(8)(6)$$

$$w_2 = 960 \text{ lb/ft}$$

To draw the Shear Diagram

- (1) $V_A = 0$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 0 + 400(4) = 1600 \text{ lb}$
- (3) $V_C = V_B + \text{Area in load diagram}$
 $V_C = 1600 - 440(8) = -1920 \text{ lb}$
- (4) $V_D = V_C + \text{Area in load diagram}$
 $V_D = -1920 + 960(2) = 0$
- (5) Location of zero shear:
 $x / 1600 = (8 - x) / 1920$
 $x = 40/11 \text{ ft} = 3.636 \text{ ft from B}$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 + \frac{1}{2}(1600)(4) = 3200 \text{ lb-ft}$
- (3) $M_x = M_B + \text{Area in shear diagram}$
 $M_x = 3200 + \frac{1}{2}(1600)(40/11)$
 $M_x = 6109.1 \text{ lb-ft}$
- (4) $M_C = M_x + \text{Area in shear diagram}$
 $M_C = 6109.1 - \frac{1}{2}(8 - 40/11)(1920)$
 $M_C = 1920 \text{ lb-ft}$
- (5) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = 1920 - \frac{1}{2}(1920)(2) = 0$

Problem 437

Cantilever beam loaded as shown in Fig. P-437

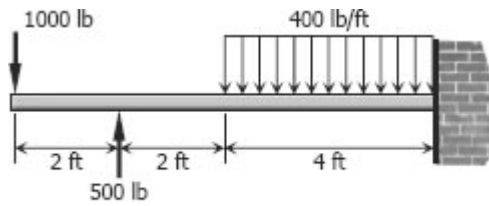
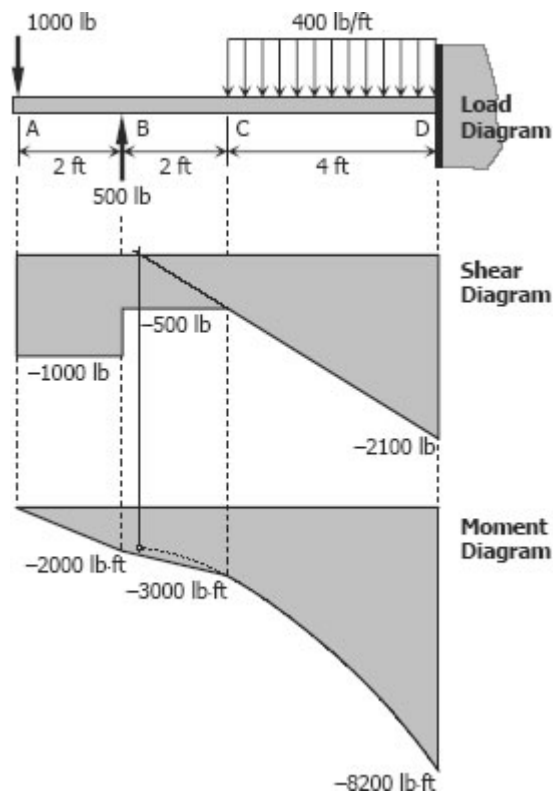


Figure P-437

Solution 437



To draw the Shear Diagram

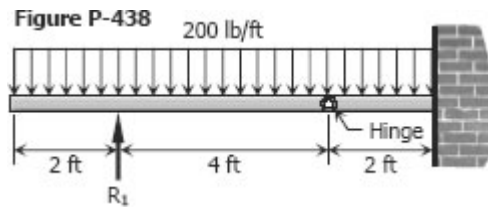
- (1) $V_A = -1000 \text{ lb}$
 $V_B = V_A + \text{Area in load diagram}$
 $V_B = -1000 + 0 = -1000 \text{ lb}$
 $V_{B2} = V_B + 500 = -1000 + 500$
 $V_{B2} = -500 \text{ lb}$
- (2) $V_C = V_{B2} + \text{Area in load diagram}$
 $V_C = -500 + 0 = -500 \text{ lb}$
- (3) $V_D = V_C + \text{Area in load diagram}$
 $V_D = -500 - 400(4) = -2100 \text{ lb}$

To draw the Moment Diagram

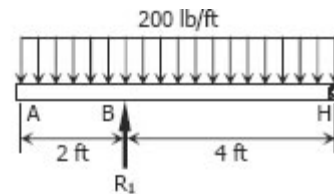
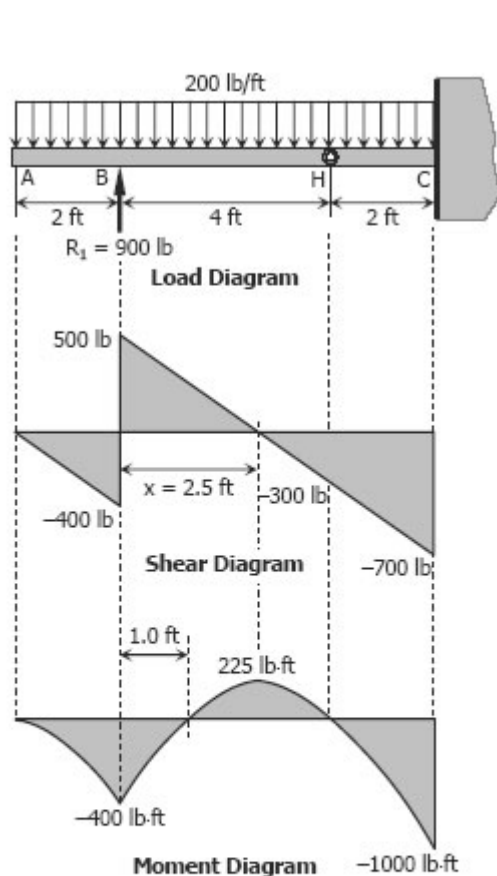
- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 - 1000(2) = -2000 \text{ lb-ft}$
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = -2000 - 500(2) = -3000 \text{ lb-ft}$
- (4) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = -3000 - \frac{1}{2}(500 + 2100)(4)$
 $M_D = -8200 \text{ lb-ft}$

Problem 438

The beam loaded as shown in Fig. P-438 consists of two segments joined by a frictionless hinge at which the bending moment is zero.



Solution 438



$$\begin{aligned}\sum M_H &= 0 \\ 4R_1 &= 200(6)(3) \\ R_1 &= 900 \text{ lb}\end{aligned}$$

To draw the Shear Diagram

- (1) $V_A = 0$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 0 - 200(2) = -400 \text{ lb}$
 $V_{B2} = V_B + R_1 = -400 + 900 = 500 \text{ lb}$
- (3) $V_H = V_{B2} + \text{Area in load diagram}$
 $V_H = 500 - 200(4) = -300 \text{ lb}$
- (4) $V_C = V_H + \text{Area in load diagram}$
 $V_C = -300 - 200(2) = -700 \text{ lb}$
- (5) Location of zero shear:
 $x / 500 = (4 - x) / 300$
 $300x = 2000 - 500x$
 $x = 2.5 \text{ ft}$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 - \frac{1}{2}(400)(2) = -400 \text{ lb-ft}$
- (3) $M_x = M_B + \text{Area in load diagram}$
 $M_x = -400 + \frac{1}{2}(500)(2.5)$
 $M_x = 225 \text{ lb-ft}$
- (4) $M_H = M_x + \text{Area in load diagram}$
 $M_H = 225 - \frac{1}{2}(300)(4 - 2.5) = 0 \text{ ok!}$
- (5) $M_C = M_H + \text{Area in load diagram}$
 $M_C = 0 - \frac{1}{2}(300 + 700)(2)$
 $M_C = -1000 \text{ lb-ft}$
- (6) The location of zero moment in segment BH can easily be found by symmetry.

Problem 439

A beam supported on three reactions as shown in Fig. P-439 consists of two segments joined by frictionless hinge at which the bending moment is zero.

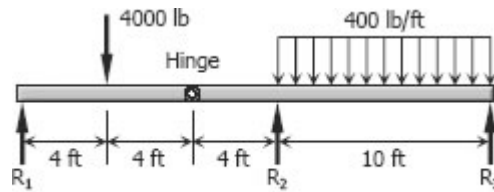
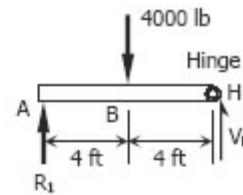
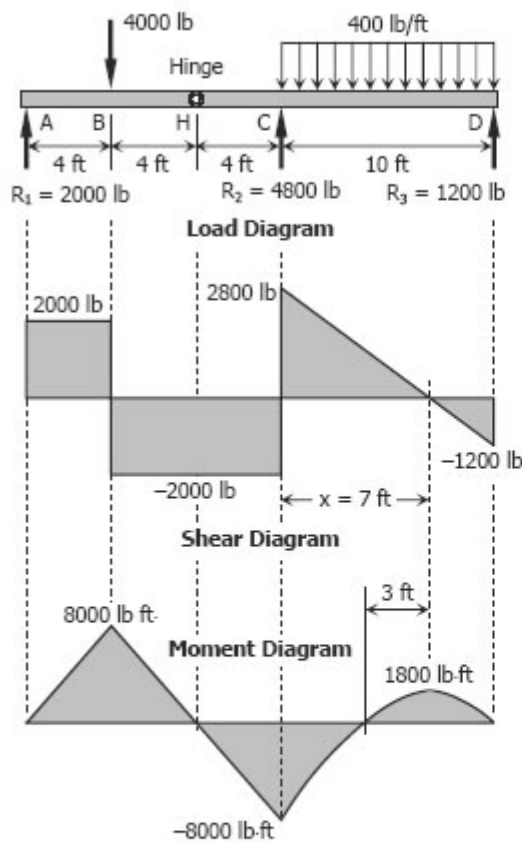
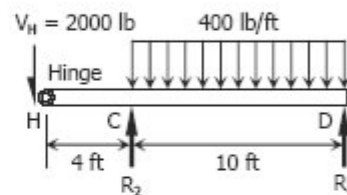


Figure P-439

Solution 439



$$\begin{array}{l|l} \sum M_H = 0 & \sum M_A = 0 \\ 8R_1 = 4000(4) & 8V_H = 4000(4) \\ R_1 = 2000 \text{ lb} & V_H = 2000 \text{ lb} \end{array}$$



$$\begin{array}{l} \sum M_D = 0 \\ 10R_2 = 2000(14) + 400(10)(5) \\ R_2 = 4800 \text{ lb} \end{array}$$

$$\begin{array}{l} \sum M_H = 0 \\ 14R_3 + 4(4800) = 400(10)(9) \\ R_3 = 1200 \text{ lb} \end{array}$$

To draw the Shear Diagram

- (1) $V_A = 0$
- (2) $V_B = 2000 \text{ lb}$
 $V_{B2} = 2000 - 4000 = -2000 \text{ lb}$
- (3) $V_H = -2000 \text{ lb}$
- (3) $V_C = -2000 \text{ lb}$
 $V_C = -2000 + 4800 = 2800 \text{ lb}$
- (4) $V_D = 2800 - 400(10) = -1200 \text{ lb}$

- (5) Location of zero shear:
 $x / 2800 = (10 - x) / 1200$
 $1200x = 28000 - 2800x$
 $x = 7 \text{ ft}$

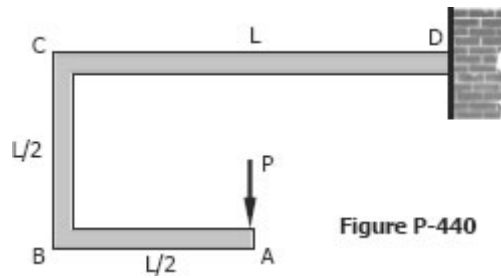
To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = 2000(4) = 8000 \text{ lb-ft}$

- (3) $M_H = 8000 - 4000(2) = 0$
- (4) $M_C = -400(2)$
 $M_C = -800 \text{ lb-ft}$
- (5) $M_x = -800 + \frac{1}{2}(2800)(7)$
 $M_x = 1800 \text{ lb-ft}$
- (6) $M_D = 1800 - \frac{1}{2}(1200)(3)$
 $M_D = 0$
- (7) Zero M is 4 ft from R_2

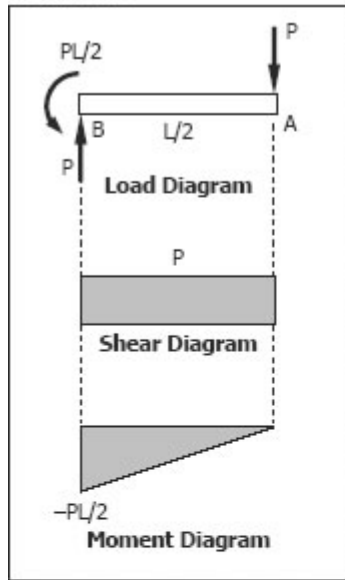
Problem 440

A frame ABCD, with rigid corners at B and C, supports the concentrated load as shown in Fig. P-440. (Draw shear and moment diagrams for each of the three parts of the frame.)

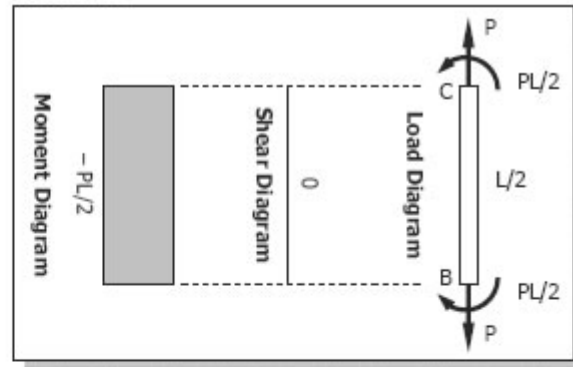


Solution 440

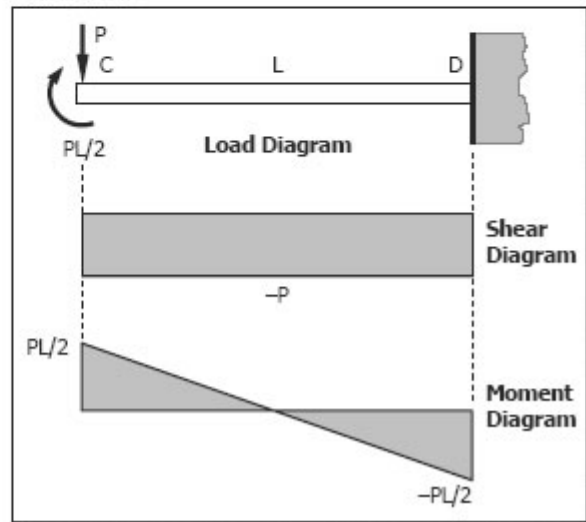
Member AB



Member BC

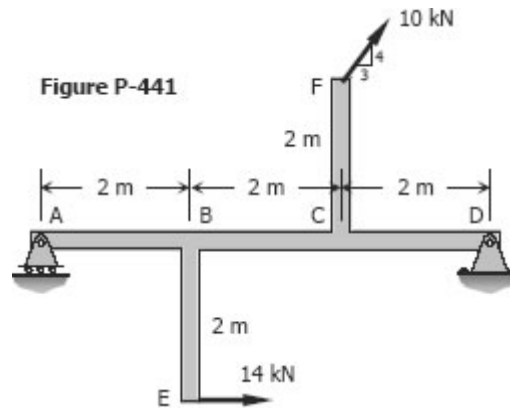


Member CD

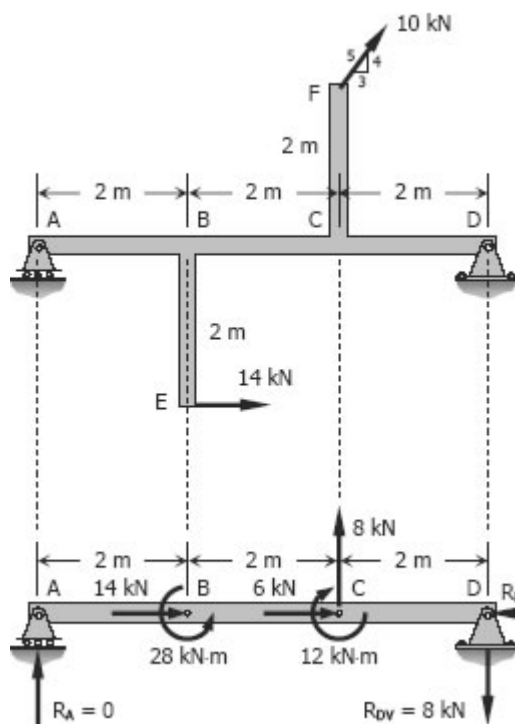


Problem 441

A beam ABCD is supported by a roller at A and a hinge at D. It is subjected to the loads shown in Fig. P-441, which act at the ends of the vertical members BE and CF. These vertical members are rigidly attached to the beam at B and C. (Draw shear and moment diagrams for the beam ABCD only.)



Solution 441



$$F_{BH} = 14 \text{ kN to the right}$$

$$M_B = 14(2)$$

$$= 28 \text{ kN}\cdot\text{m counterclockwise}$$

$$F_{CH} = 3/5 (10)$$

$$= 6 \text{ kN to the right}$$

$$F_{CV} = 4/5 (10)$$

$$= 8 \text{ kN upward}$$

$$M_C = F_{CH} (2) = 6(2)$$

$$= 12 \text{ kN}\cdot\text{m clockwise}$$

$$\Sigma M_D = 0$$

$$6R_A + 12 + 8(2) = 28$$

$$R_A = 0$$

$$\Sigma M_A = 0$$

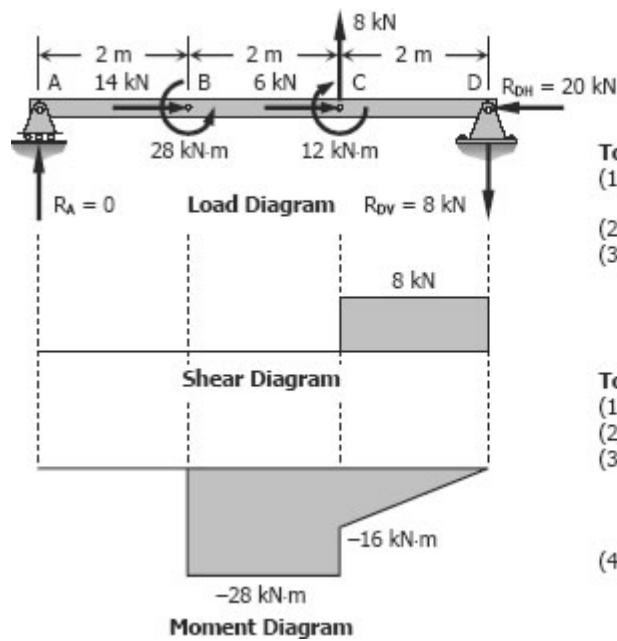
$$6R_{DV} + 12 = 28 + 8(4)$$

$$R_{DV} = 8 \text{ kN}$$

$$\Sigma F_H = 0$$

$$R_{DH} = 14 + 6$$

$$= 20 \text{ kN}$$



To draw the Shear Diagram

- (1) Shear in segments AB and BC is zero.
- (2) $V_C = 8$
- (3) $V_D = V_C + \text{Area in load diagram}$
 $V_D = 8 + 0 = 8$ kN
 $V_{D2} = V_D - R_{DV}$
 $V_{D2} = 8 - 8 = 0$

To draw the Moment Diagram

- (1) Moment in segment AB is zero
- (2) $M_B = -28$ kN-m
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = -28 + 0 = -28$ kN-m
 $M_{C2} = M_C + 12 = -28 + 12$
 $M_{C2} = -16$ kN-m
- (4) $M_D = M_{C2} + \text{Area in shear diagram}$
 $M_D = -16 + 8(2)$
 $M_D = 0$

Problem 442

Beam carrying the uniformly varying load shown in Fig. P-442.

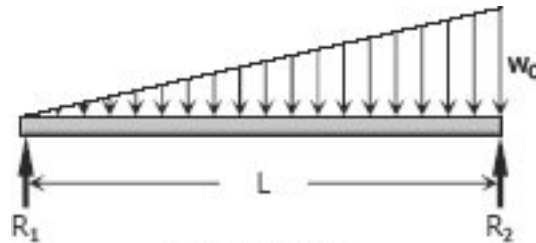
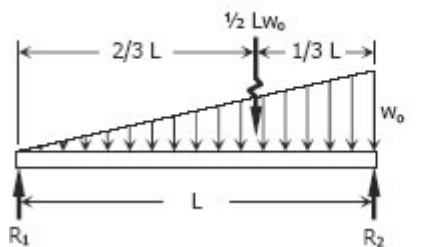


Figure P-442

Solution 442



$$\sum M_{R2} = 0$$

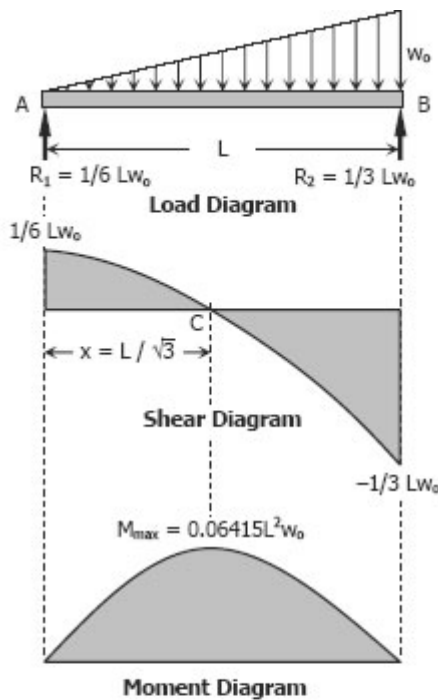
$$LR_1 = \frac{1}{3} L \left(\frac{1}{2} L w_0 \right)$$

$$R_1 = \frac{1}{6} L w_0$$

$$\sum M_{R1} = 0$$

$$LR_2 = \frac{2}{3} L \left(\frac{1}{2} L w_0 \right)$$

$$R_2 = \frac{1}{3} L w_0$$

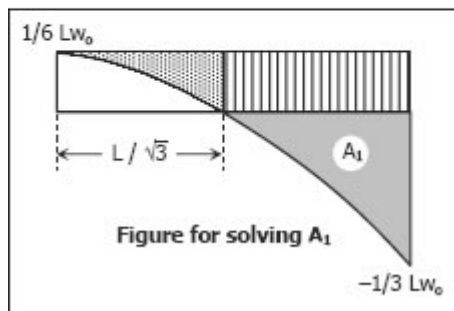


To draw the Shear Diagram

- (1) $V_A = R_1 = 1/6 L w_0$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 1/6 L w_0 - 1/2 L w_0$
 $V_B = -1/3 L w_0$
- (3) Location of zero shear C:
 By squared property of parabola:
 $x^2 / (1/6 L w_0) = L^2 / (1/6 L w_0 + 1/3 L w_0)$
 $6x^2 = 2L^2$
 $x = L / \sqrt{3}$
- (4) The shear in AB is a parabola with vertex at A, the starting point of uniformly varying load. The load in AB is 0 at A to downward w_0 or $-w_0$ at B, thus the slope of shear diagram is decreasing. For decreasing slope, the parabola is open downward.

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_C = M_A + \text{Area in shear diagram}$
 $M_C = 0 + 2/3 (L/\sqrt{3})(1/6 L w_0)$
 $M_C = 0.06415 L^2 w_0 = M_{\max}$
- (3) $M_B = M_C + \text{Area in shear diagram}$
 $M_B = M_C - A_1 \rightarrow \text{see figure for solving } A_1$
 For A_1 :
 $A_1 = 1/3 L(1/6 L w_0 + 1/3 L w_0)$
 $- 1/3 (L/\sqrt{3})(1/6 L w_0)$
 $- 1/6 L w_0 (L - L/\sqrt{3})$
 $A_1 = 0.16667 L^2 w_0 - 0.03208 L^2 w_0$
 $- 0.07044 L^2 w_0$
 $A_1 = 0.06415 L^2 w_0$
 $M_B = 0.06415 L^2 w_0 - 0.06415 L^2 w_0 = 0$
- (4) The shear diagram is second degree curve, thus the moment diagram is a third degree curve. The maximum moment (highest point) occurred at C, the location of zero shear. The value of shears in AC is positive then the moment in AC is increasing; at CB the shear is negative, then the moment in CB is decreasing.



Problem 443

Beam carrying the triangular loads shown in Fig. P-443.

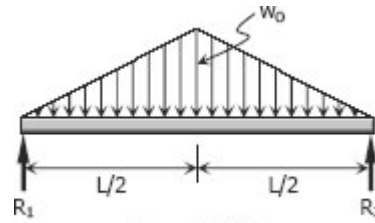


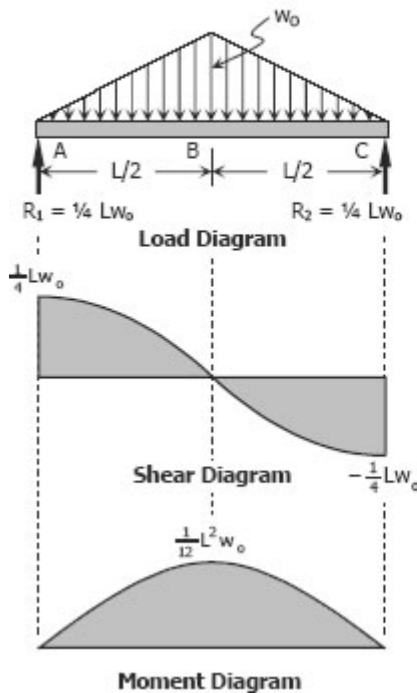
Figure P-443

Solution 443

By symmetry:

$$R_1 = R_2 = \frac{1}{2} \left(\frac{1}{2} L w_0 \right)$$

$$R_1 = R_2 = \frac{1}{4} L w_0$$



To draw the Shear Diagram

- (1) $V_A = R_1 = \frac{1}{4} L w_0$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = \frac{1}{4} L w_0 - \frac{1}{2} (L/2)(w_0) = 0$
- (3) $V_C = V_B + \text{Area in load diagram}$
 $V_C = 0 - \frac{1}{2} (L/2)(w_0) = -\frac{1}{4} L w_0$
- (4) Load in AB is linear, thus, V_{AB} is second degree or parabolic curve. The load is from 0 at A to w_0 (w_0 is downward or $-w_0$) at B, thus the slope of V_{AB} is decreasing.
- (5) V_{BC} is also parabolic since the load in BC is linear. The magnitude of load in BC is from $-w_0$ to 0 or increasing, thus the slope of V_{BC} is increasing.

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 + 2/3 (L/2)(1/4 L w_0) = 1/12 L w_0$
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = 1/12 L w_0 - 2/3 (L/2)(1/4 L w_0) = 0$
- (4) M_{AC} is third degree because the shear diagram in AC is second degree.
- (5) The shear from A to C is decreasing, thus the slope of moment diagram from A to C is decreasing.

Problem 444

Beam loaded as shown in Fig. P-444.

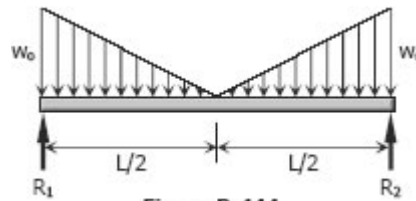


Figure P-444

Solution 444

Total load

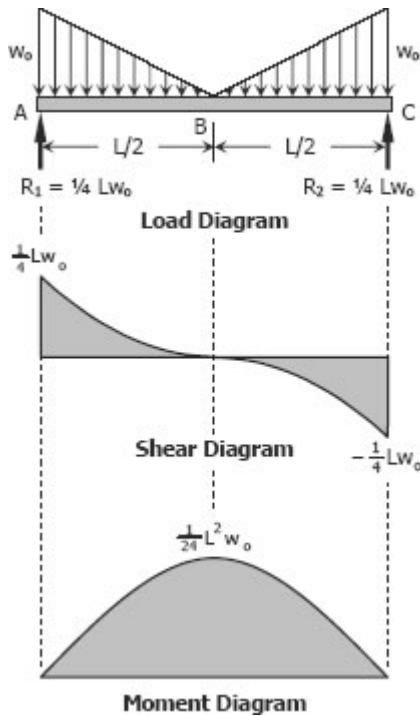
$$= 2\left[\frac{1}{2}(L/2)(w_0)\right]$$

$$= \frac{1}{2}Lw_0$$

By symmetry

$$R_1 = R_2 = \frac{1}{2} \times \text{total load}$$

$$R_1 = R_2 = \frac{1}{4}Lw_0$$



To draw the Shear Diagram

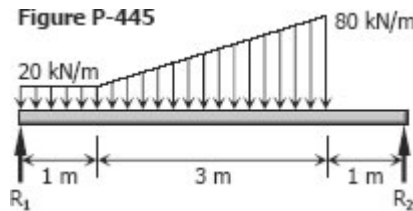
- (1) $V_A = R_1 = \frac{1}{4}Lw_0$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = \frac{1}{4}Lw_0 - \frac{1}{2}(L/2)(w_0) = 0$
- (3) $V_C = V_B + \text{Area in load diagram}$
 $V_C = 0 - \frac{1}{2}(L/2)(w_0) = -\frac{1}{4}Lw_0$
- (4) The shear diagram in AB is second degree curve. The shear in AB is from $-w_0$ (downward w_0) to zero or increasing, thus, the slope of shear at AB is increasing (upward parabola).
- (5) The shear diagram in BC is second degree curve. The shear in BC is from zero to $-w_0$ (downward w_0) or decreasing, thus, the slope of shear at BC is decreasing (downward parabola)

To draw the Moment Diagram

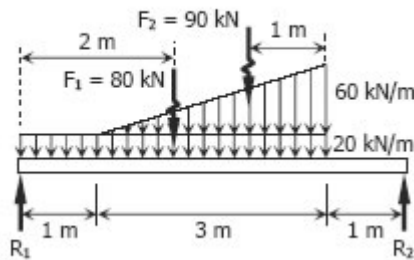
- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 + \frac{1}{3}(L/2)(\frac{1}{4}Lw_0) = \frac{1}{24}L^2w_0$
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = \frac{1}{24}L^2w_0 - \frac{1}{3}(L/2)(\frac{1}{4}Lw_0) = 0$
- (4) The shear diagram from A to C is decreasing, thus, the moment diagram is a concave downward third degree curve.

Problem 445

Beam carrying the loads shown in Fig. P-445.



Solution 445



$$\sum M_{R2} = 0$$

$$5R_1 = 80(3) + 90(2)$$

$$R_1 = 84 \text{ kN}$$

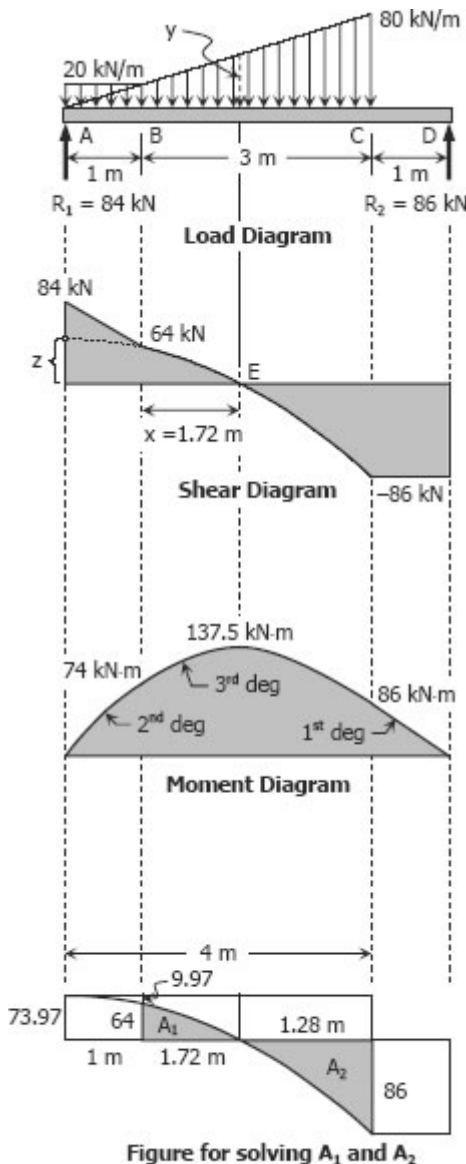
$$\sum M_{R1} = 0$$

$$5R_2 = 80(2) + 90(3)$$

$$R_2 = 86 \text{ kN}$$

Checking

$$R_1 + R_2 = F_1 + F_2 \text{ ok!}$$



To draw the Shear Diagram

- (1) $V_A = R_1 = 84 \text{ kN}$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 84 - 20(1) = 64 \text{ kN}$
- (3) $V_C = V_B + \text{Area in load diagram}$
 $V_C = 64 - \frac{1}{2}(20 + 80)(3) = -86 \text{ kN}$
- (4) $V_D = V_C + \text{Area in load diagram}$
 $V_D = -86 + 0 = -86 \text{ kN}$
 $V_{D2} = V_D + R_2 = -86 + 86 = 0$
- (5) Location of zero shear:
 From the load diagram:
 $y / (x + 1) = 80 / 4$
 $y = 20(x + 1)$
 $V_E = V_B + \text{Area in load diagram}$
 $0 = 64 - \frac{1}{2}(20 + y)x$
 $(20 + y)x = 128$
 $[20 + 20(x + 1)]x = 128$
 $20x^2 + 40x - 128 = 0$
 $5x^2 + 10x - 32 = 0$
 $x = 1.72 \text{ and } -3.72$
 use $x = 1.72 \text{ m}$ from B
- (5) By squared property of parabola:
 $z / (1 + x)^2 = (z + 86) / 4^2$
 $16z = 7.3984z + 636.2624$
 $8.6016z = 254.4224$
 $z = 73.97 \text{ kN}$

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 + \frac{1}{2}(84 + 64)(1) = 74 \text{ kN-m}$
- (3) $M_E = M_B + \text{Area in shear diagram}$
 $M_E = 74 + A_1 \rightarrow \text{see figure for } A_1 \text{ and } A_2$
 For A_1 :
 $A_1 = \frac{2}{3}(1 + 1.72)(73.97) - 64(1)$
 $A_1 = 63.5$
 $M_E = 74 + 63.5 = 137.5 \text{ kN-m}$
- (4) $M_C = M_E + \text{Area in shear diagram}$
 $M_C = M_E - A_2$
 For A_2 :
 $A_2 = \frac{1}{3}(4)(73.97 + 86)$
 $A_2 = 51.5$
 $M_C = 137.5 - 51.5 = 86 \text{ kN-m}$
- (5) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = 86 - 86(1) = 0$

Figure for solving A_1 and A_2

Problem 446

Beam loaded and supported as shown in Fig. P-446.

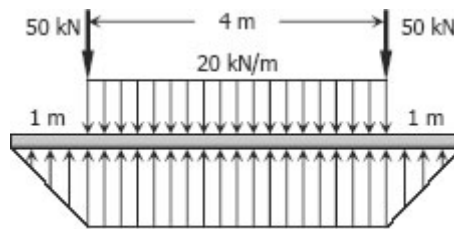
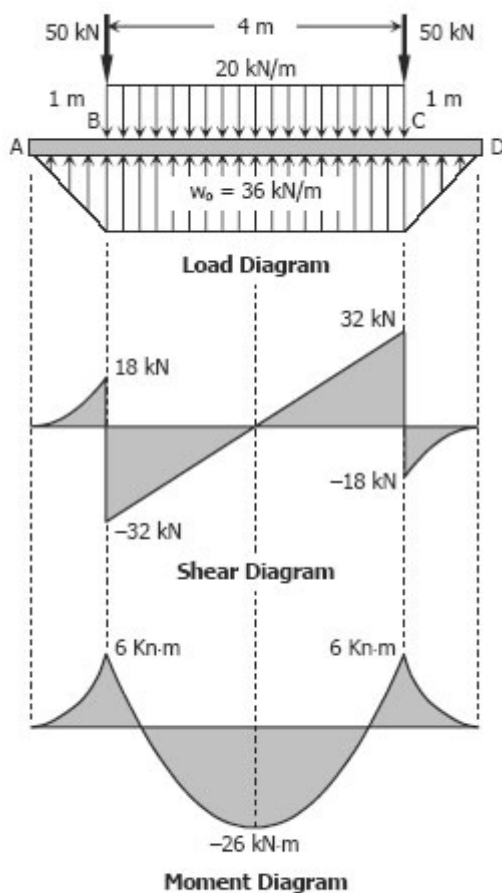


Figure P-446

Solution 446



$$\Sigma F_V = 0$$

$$4w_o + 2\left[\frac{1}{2}w_o(1)\right] = 20(4) + 2(50)$$

$$5w_o = 180$$

$$w_o = 36 \text{ kN/m}$$

To draw the Shear Diagram

- (1) $V_A = 0$
- (2) $V_B = V_A + \text{Area in load diagram}$
 $V_B = 0 + \frac{1}{2}(36)(1) = 18 \text{ kN}$
 $V_{B2} = V_B - 50 = 18 - 50$
 $V_{B2} = -32 \text{ kN}$
- (3) The net uniformly distributed load in segment BC is $36 - 20 = 16 \text{ kN/m}$ upward.
 $V_C = V_{B2} + \text{Area in load diagram}$
 $V_C = -32 + 16(4) = 32 \text{ kN}$
 $V_{C2} = V_C - 50 = 32 - 50$
 $V_{C2} = -18 \text{ kN}$
- (4) $V_D = V_{C2} + \text{Area in load diagram}$
 $V_D = -18 + \frac{1}{2}(36)(1) = 0$
- (5) The shape of shear at AB and CD are parabolic spandrel with vertex at A and D, respectively.
- (6) The location of zero shear is obviously at the midspan or 2 m from B.

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 + \frac{1}{3}(1)(18)$
 $M_B = 6 \text{ kN-m}$
- (3) $M_{\text{midspan}} = M_B + \text{Area in shear diagram}$
 $M_{\text{midspan}} = 6 - \frac{1}{2}(32)(2)$
 $M_{\text{midspan}} = -26 \text{ kN-m}$
- (4) $M_C = M_{\text{midspan}} + \text{Area in shear diagram}$
 $M_C = -26 + \frac{1}{2}(32)(2)$
 $M_C = 6 \text{ kN-m}$
- (5) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = 6 - \frac{1}{3}(1)(18) = 0$
- (6) The moment diagram at AB and CD are 3rd degree curve while at BC is 2nd degree curve.

Finding the Load & Moment Diagrams with Given Shear Diagram

INSTRUCTION

In the following problems, draw moment and load diagrams corresponding to the given shear diagrams. Specify values at all change of load positions and at all points of zero shear.

Problem 447

Shear diagram as shown in Fig. P-447.

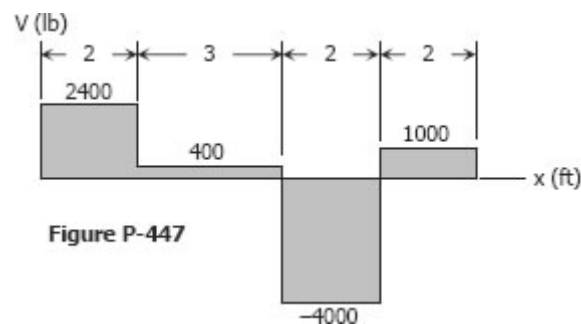
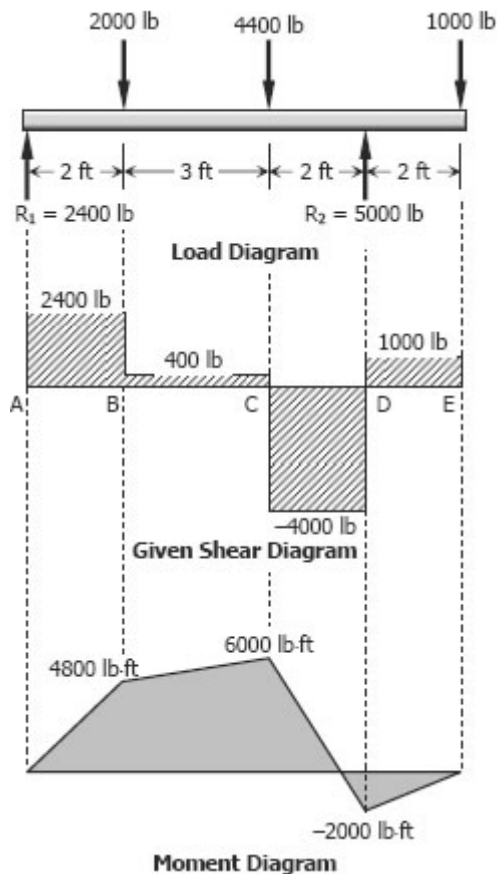


Figure P-447

Solution 447



To draw the Load Diagram

- (1) A 2400 lb upward force is acting at point A. No load in segment AB.
- (2) A point force of $2400 - 400 = 2000$ lb is acting downward at point B. No load in segment BC.
- (3) Another downward force of magnitude $400 + 4000 = 4400$ lb at point C. No load in segment CD.
- (4) Upward point force of $4000 + 1000 = 5000$ lb is acting at D. No load in segment DE.
- (5) A downward force of 1000 lb is concentrated at point E.

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 + 2400(2) = 4800 \text{ lb-ft}$
 M_{AB} is linear and upward
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = 4800 + 400(3) = 6000 \text{ lb-ft}$
 M_{BC} is linear and upward
- (4) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = 6000 - 4000(2) = -2000 \text{ lb-ft}$
 M_{CD} is linear and downward
- (5) $M_E = M_D + \text{Area in shear diagram}$
 $M_E = -2000 + 1000(2) = 0$
 M_{DE} is linear and upward

Problem 448

Shear diagram as shown in Fig. P-448.

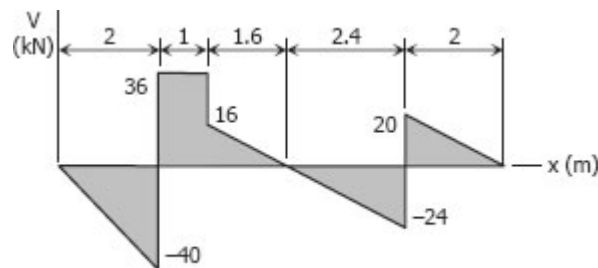
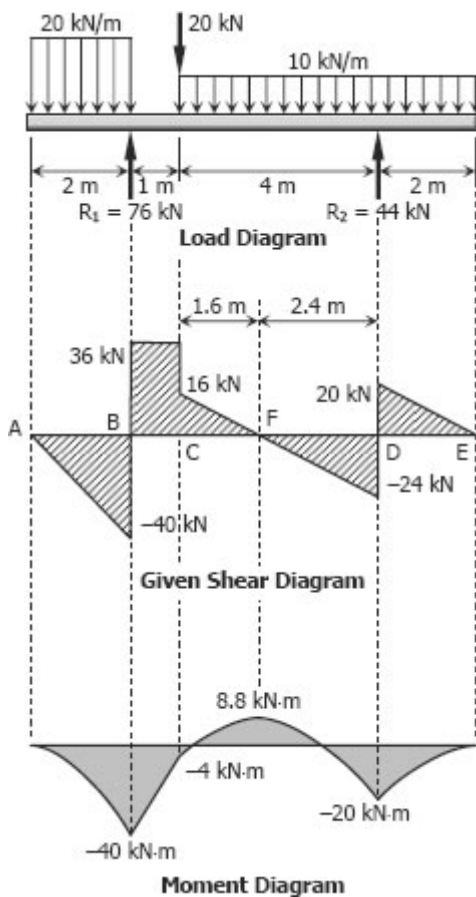


Figure P-448

Solution 448



To draw the Load Diagram

- (1) A uniformly distributed load in AB is acting downward at a magnitude of $40/2 = 20$ kN/m.
- (2) Upward concentrated force of $40 + 36 = 76$ kN acts at B. No load in segment BC.
- (3) A downward point force acts at C at a magnitude of $36 - 16 = 20$ kN.
- (4) Downward uniformly distributed load in CD has a magnitude of $(16 + 24)/4 = 10$ kN/m & causes zero shear at point F, 1.6 m from C.
- (5) Another upward concentrated force acts at D at a magnitude of $20 + 24 = 44$ kN.
- (6) The load in segment DE is uniform and downward at $20/2 = 10$ kN/m.

To draw the Moment Diagram

- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 - \frac{1}{2}(40)(2) = -40$ kN-m
 M_{AB} is downward parabola with vertex at A.
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = -40 + 36(1) = -4$ kN-m
 M_{BC} is linear and upward
- (4) $M_F = M_C + \text{Area in shear diagram}$
 $M_F = -4 + \frac{1}{2}(16)(1.6) = 8.8$ kN-m
- (5) $M_D = M_F + \text{Area in shear diagram}$
 $M_D = 8.8 - \frac{1}{2}(24)(2.4) = -20$ kN-m
 M_{CD} is downward parabola with vertex at F.
- (6) $M_E = M_D + \text{Area in shear diagram}$
 $M_E = -20 + \frac{1}{2}(20)(2) = 0$
 M_{DE} is downward parabola with vertex at E.

Problem 449

Shear diagram as shown in Fig. P-449.

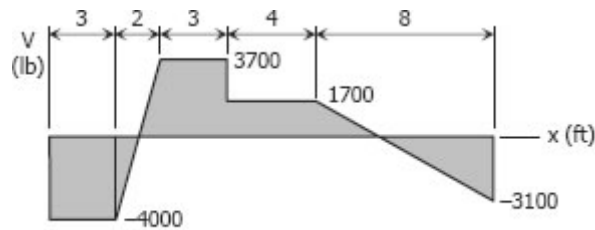
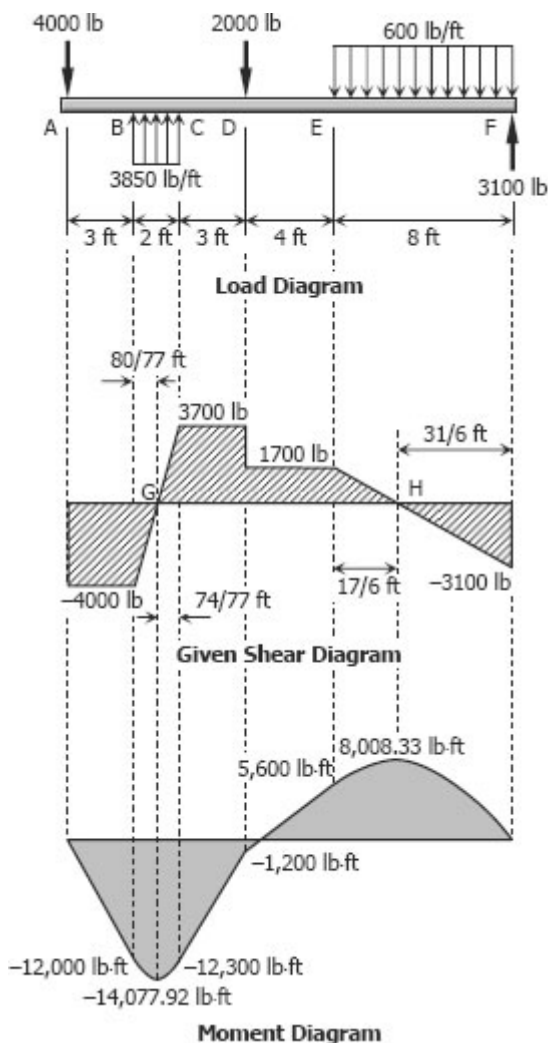


Figure P-449

Solution 449



To draw the Load Diagram

- (1) Downward 4000 lb force is concentrated at A and no load in segment AB.
- (2) The shear in BC is uniformly increasing, thus a uniform upward force is acting at a magnitude of $(3700 + 4000)/2 = 3850$ lb/ft. No load in segment CD.
- (3) Another point force acting downward with $3700 - 1700 = 1200$ lb at D and no load in segment DE.
- (4) The shear in EF is uniformly decreasing, thus a uniform downward force is acting with magnitude of $(1700 + 3100)/8 = 600$ lb/ft.
- (5) Upward force of 3100 lb is concentrated at end of span F.

To draw the Moment Diagram

- (1) The locations of zero shear (points G and H) can be easily determined by ratio and proportion of triangle.
- (2) $M_A = 0$
- (3) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 - 4000(3) = -12,000$ lb-ft
- (4) $M_G = M_B + \text{Area in shear diagram}$
 $M_G = -12,000 - \frac{1}{2}(80/77)(4000)$
 $M_G = -14,077.92$ lb-ft
- (5) $M_C = M_G + \text{Area in shear diagram}$
 $M_C = -14,077.92 + \frac{1}{2}(74/77)(3700)$
 $M_C = -12,300$ lb-ft
- (6) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = -12,300 + 3700(3) = -1,200$ lb-ft
- (7) $M_E = M_D + \text{Area in shear diagram}$
 $M_E = -1,200 + 1700(4) = 5600$ lb-ft
- (8) $M_H = M_E + \text{Area in shear diagram}$
 $M_H = 5600 + \frac{1}{2}(17/6)(1700)$
 $M_H = 8,008.33$ lb-ft
- (9) $M_F = M_H + \text{Area in shear diagram}$
 $M_F = 8,008.33 - \frac{1}{2}(31/6)(3100) = 0$

Problem 450

Shear diagram as shown in Fig. P-450.

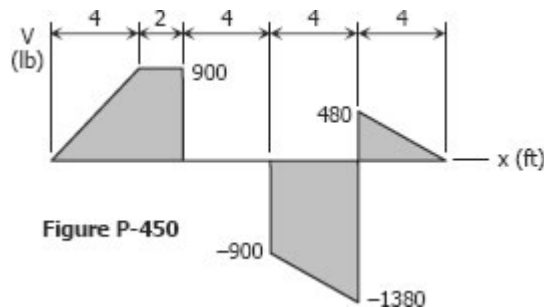
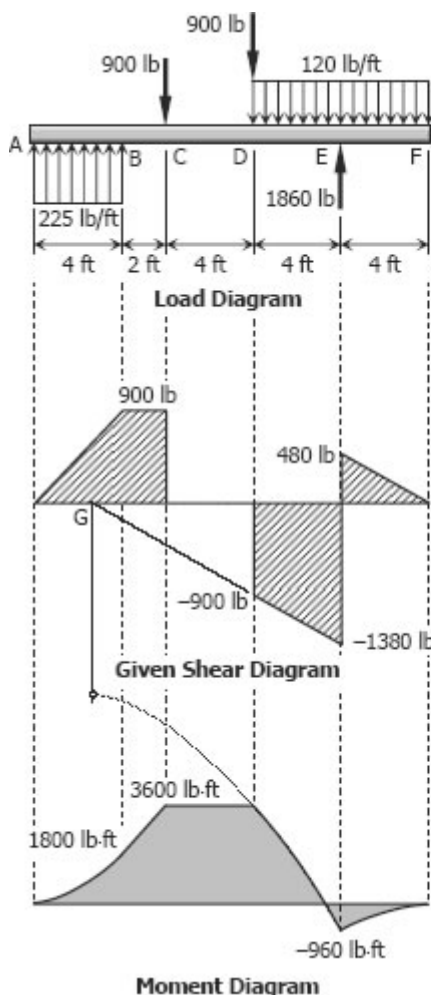


Figure P-450

Solution 450



To draw the Load Diagram

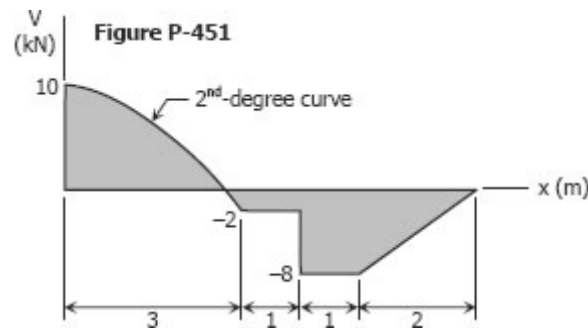
- (1) The shear diagram in AB is uniformly upward, thus the load is uniformly distributed upward at a magnitude of $900/4 = 225$ lb/ft. No load in segment BC.
- (2) A downward point force acts at point C with magnitude of 900 lb. No load in segment CD.
- (3) Another concentrated force is acting downward at D with a magnitude of 900 lb.
- (4) The load in DE is uniformly distributed downward at a magnitude of $(1380 - 900)/4 = 120$ lb/ft.
- (5) An upward load is concentrated at E with magnitude of $480 + 1380 = 1860$ lb.
- (6) $480/4 = 120$ lb/ft is distributed uniformly over the span EF.

To draw the Moment Diagram

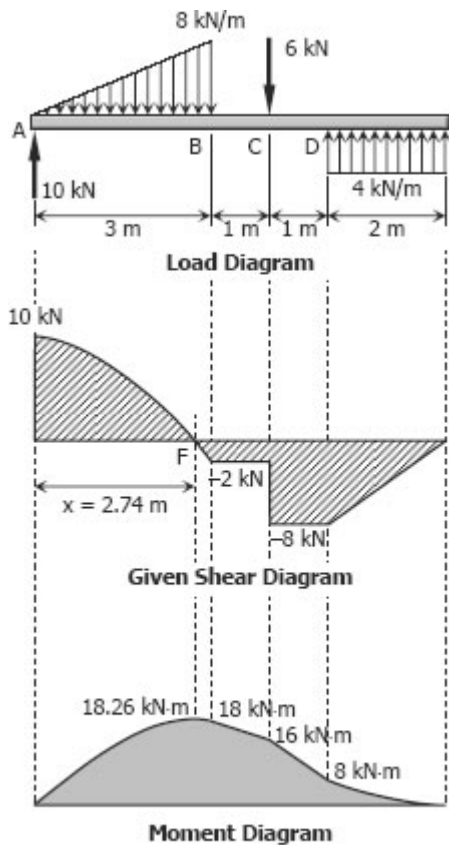
- (1) $M_A = 0$
- (2) $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 + \frac{1}{2}(4)(900) = 1800$ lb-ft
- (3) $M_C = M_B + \text{Area in shear diagram}$
 $M_C = 1800 + 900(2) = 3600$ lb-ft
- (4) $M_D = M_C + \text{Area in shear diagram}$
 $M_D = 3600 + 0 = 3600$ lb-ft
- (5) $M_E = M_D + \text{Area in shear diagram}$
 $M_E = 3600 - \frac{1}{2}(900 + 1380)(4)$
 $M_E = -960$ lb-ft
- (6) $M_F = M_E + \text{Area in shear diagram}$
 $M_F = -960 + \frac{1}{2}(480)(4) = 0$
- (7) The shape of moment diagram in AB is upward parabola with vertex at A, while linear in BC and horizontal in CD. For segment DE, the diagram is downward parabola with vertex at G. G is the point where the extended shear in DE intersects the line of zero shear.
- (8) The moment diagram in EF is a downward parabola with vertex at F.

Problem 451

Shear diagram as shown in Fig. P-451.



Solution 451



To draw the Load Diagram

- (1) Upward concentrated load at A is 10 kN.
- (2) The shear in AB is a 2nd-degree curve, thus the load in AB is uniformly varying. In this case, it is zero at A to $2(10 + 2)/3 = 8$ kN at B. No load in segment BC.
- (3) A downward point force is acting at C in a magnitude of $8 - 2 = 6$ kN.
- (4) The shear in DE is uniformly increasing, thus the load in DE is uniformly distributed and upward. This load is spread over DE at a magnitude of $8/2 = 4$ kN/m.

To draw the Moment Diagram

- (1) To find the location of zero shear, F:

$$x^2/10 = 3^2/(10 + 2)$$

$$x = 2.74 \text{ m}$$
- (2) $M_A = 0$
- (3) $M_F = M_A + \text{Area in shear diagram}$

$$M_F = 0 + 2/3 (2.74)(10) = 18.26 \text{ kN-m}$$
- (4) $M_B = M_F + \text{Area in shear diagram}$

$$M_B = 18.26 - [1/3 (10 + 2)(3) - 1/3 (2.74)(10) - 10(3 - 2.74)]$$

$$M_B = 18 \text{ kN-m}$$
- (5) $M_C = M_B + \text{Area in shear diagram}$

$$M_C = 18 - 2(1) = 16 \text{ kN-m}$$
- (6) $M_D = M_C + \text{Area in shear diagram}$

$$M_D = 16 - 8(1) = 8 \text{ kN-m}$$
- (7) $M_E = M_D + \text{Area in shear diagram}$

$$M_E = 8 - 1/2 (2)(8) = 0$$
- (8) The moment diagram in AB is a second degree curve, at BC and CD are linear and downward. For segment DE, the moment diagram is parabola open upward with vertex at E.

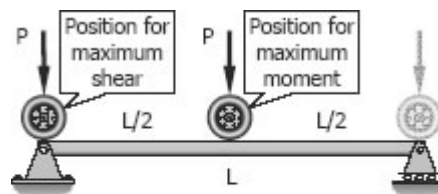
Moving Loads

From the previous section, we see that the maximum moment occurs at a point of zero shears. For beams loaded with concentrated loads, the point of zero shears usually occurs under a concentrated load and so the maximum moment.

Beams and girders such as in a bridge or an overhead crane are subject to moving concentrated loads, which are at fixed distance with each other. The problem here is to determine the moment under each load when each load is in a position to cause a maximum moment. The largest value of these moments governs the design of the beam.

SINGLE MOVING LOAD

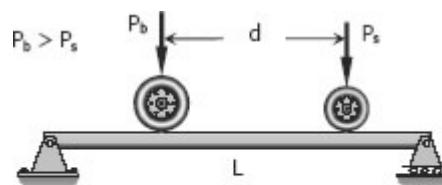
For a single moving load, the maximum moment occurs when the load is at the midspan and the maximum shear occurs when the load is very near the support (usually assumed to lie over the support).



$$M_{\max} = \frac{PL}{4} \text{ and } V_{\max} = P$$

TWO MOVING LOADS

For two moving loads, the maximum shear occurs at the reaction when the larger load is over that support. The maximum moment is given by



$$M_{\max} = \frac{(PL - P_s d)^2}{4PL}$$

where P_s is the smaller load, P_b is the bigger load, and P is the total load ($P = P_s + P_b$).

THREE OR MORE MOVING LOADS

In general, the bending moment under a particular load is a maximum when the center of the beam is midway between that load and the resultant of all the loads then on the span. With this rule, we compute the maximum moment under each load, and use the biggest of the moments for the design. Usually, the biggest of these moments occurs under the biggest load.

The maximum shear occurs at the reaction where the resultant load is nearest. Usually, it happens if the biggest load is over that support and as many a possible of the remaining loads are still on the span.

The maximum shear occurs at the reaction where the resultant load is nearest. Usually, it happens if the biggest load is over that support and as many a possible of the remaining loads are still on the span. In determining the largest moment and shear, it is sometimes necessary to check the condition when the bigger loads are on the span and the rest of the smaller loads are outside.

Solved Problems in Moving Loads

Problem 453

A truck with axle loads of 40 kN and 60 kN on a wheel base of 5 m rolls across a 10-m span. Compute the maximum bending moment and the maximum shearing force.

Solution 453

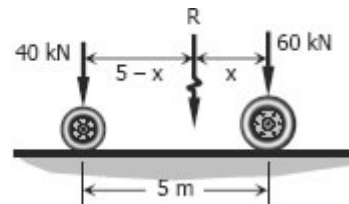
$$R = 40 + 60 = 100 \text{ kN}$$

$$xR = 40(5)$$

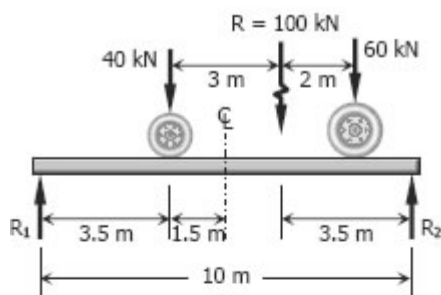
$$x = 200/R$$

$$x = 200/100$$

$$x = 2 \text{ m}$$



For maximum moment under 40 kN wheel:



$$\sum M_{R2} = 0$$

$$10R_1 = 3.5(100)$$

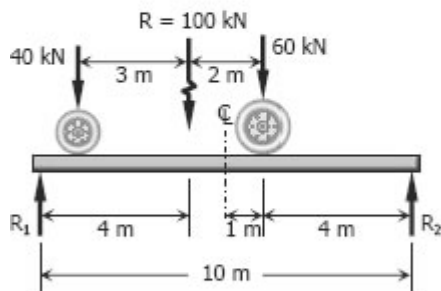
$$R_1 = 35 \text{ kN}$$

$$M_{\text{To the left of 40 kN}} = 3.5R_1$$

$$M_{\text{To the left of 40 kN}} = 3.5(35)$$

$$M_{\text{To the left of 40 kN}} = 122.5 \text{ kN}\cdot\text{m}$$

For maximum moment under 60 kN wheel:



$$\sum M_{R1} = 0$$

$$10R_2 = 4(100)$$

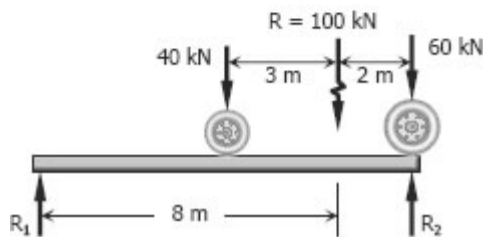
$$R_2 = 40 \text{ kN}$$

$$M_{\text{To the right of 60 kN}} = 4R_2$$

$$M_{\text{To the right of 60 kN}} = 4(40)$$

$$M_{\text{To the right of 60 kN}} = 160 \text{ kN}\cdot\text{m}$$

$$\text{Thus, } M_{\text{max}} = 160 \text{ kN}\cdot\text{m}$$



The maximum shear will occur when the 60 kN is over a support.

$$\sum M_{R1} = 0$$

$$10R_2 = 100(8)$$

$$R_2 = 80 \text{ kN}$$

$$\text{Thus, } V_{\text{max}} = 80 \text{ kN}$$

Problem 454

Repeat Prob. 453 using axle loads of 30 kN and 50 kN on a wheel base of 4 m crossing an 8-m span.

Solution 454

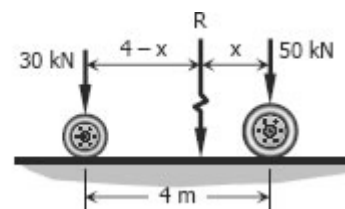
$$R = 30 + 50 = 80 \text{ kN}$$

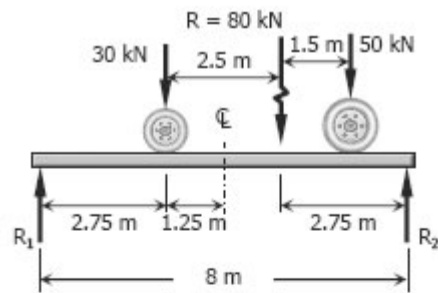
$$xR = 4(30)$$

$$x = 120/R$$

$$x = 120/80$$

$$x = 1.5 \text{ m}$$

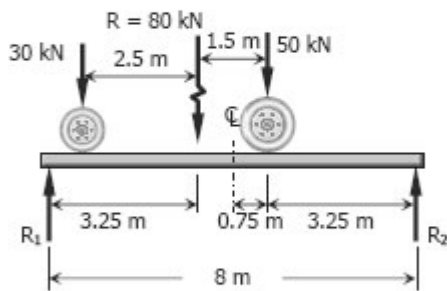




Maximum moment under 30 kN wheel:

$$\begin{aligned}\sum M_{R2} &= 0 \\ 8R_1 &= 2.75(80) \\ R_1 &= 27.5 \text{ kN}\end{aligned}$$

$$\begin{aligned}M_{\text{To the left of 30 kN}} &= 2.75R_1 \\ M_{\text{To the left of 30 kN}} &= 2.75(27.5) \\ M_{\text{To the left of 30 kN}} &= 75.625 \text{ kN}\cdot\text{m}\end{aligned}$$

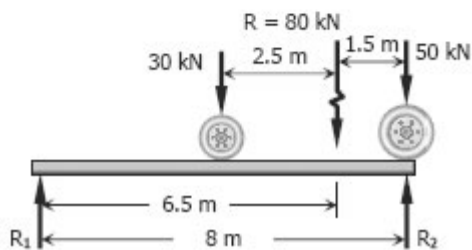


Maximum moment under 50 kN wheel:

$$\begin{aligned}\sum M_{R1} &= 0 \\ 8R_2 &= 3.25(80) \\ R_2 &= 32.5 \text{ kN}\end{aligned}$$

$$\begin{aligned}M_{\text{To the right of 50 kN}} &= 3.25R_2 \\ M_{\text{To the right of 50 kN}} &= 3.25(32.5) \\ M_{\text{To the right of 50 kN}} &= 105.625 \text{ kN}\cdot\text{m}\end{aligned}$$

Thus, $M_{\max} = 105.625 \text{ kN}\cdot\text{m}$



The maximum shear will occur when the 50 kN is over a support.

$$\begin{aligned}\sum M_{R1} &= 0 \\ 8R_2 &= 6.5(80) \\ R_2 &= 65 \text{ kN}\end{aligned}$$

Thus, $V_{\max} = 65 \text{ kN}$

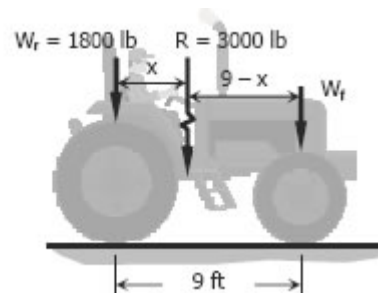
Problem 455

A tractor weighing 3000 lb, with a wheel base of 9 ft, carries 1800 lb of its load on the rear wheels. Compute the maximum moment and maximum shear when crossing a 14 ft-span.

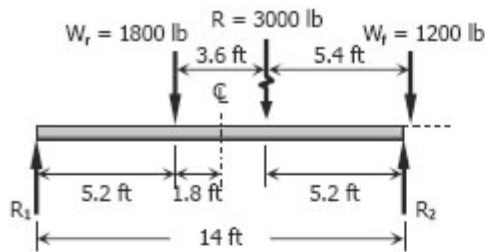
Solution 455

$$\begin{aligned}R &= W_r + W_f \\ 3000 &= 1800 + W_f \\ W_f &= 1200 \text{ lb}\end{aligned}$$

$$\begin{aligned}Rx &= 9W_f \\ 3000x &= 9(1200) \\ x &= 3.6 \text{ ft}\end{aligned}$$



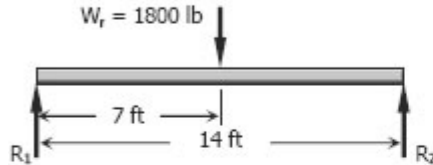
$$9 - x = 5.4 \text{ ft}$$



When the midspan is midway between W_r and R , the front wheel W_f will be outside the span (see figure). In this case, only the rear wheel $W_r = 1800 \text{ lb}$ is the load. The maximum moment for this condition is when the load is at the midspan.

$$R_1 = R_2 = \frac{1}{2} (1800)$$

$$R_1 = 900 \text{ lb}$$

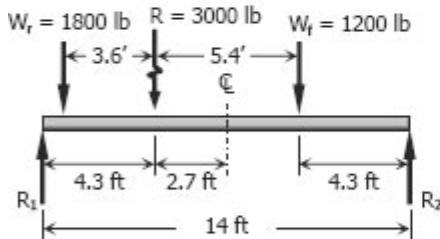


Maximum moment under W_r

$$M_{\text{To the left of rear wheel}} = 7R_1$$

$$M_{\text{To the left of rear wheel}} = 7(900)$$

$$M_{\text{To the left of rear wheel}} = 6300 \text{ lb-ft}$$



Maximum moment under W_f

$$\sum M_{R_1} = 0$$

$$14R_2 = 4.3R$$

$$14R_2 = 4.3(3000)$$

$$R_2 = 921.43 \text{ lb}$$

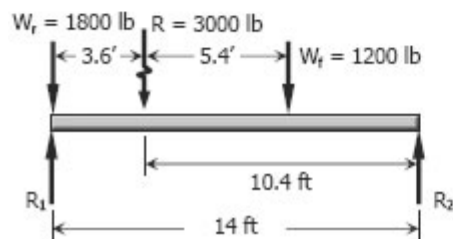
$$M_{\text{To the right of front wheel}} = 4.3R_2$$

$$M_{\text{To the right of front wheel}} = 4.3(921.43)$$

$$M_{\text{To the right of front wheel}} = 3962.1 \text{ lb-ft}$$

Thus, $M_{\max} = M_{\text{To the left of rear wheel}}$

$$M_{\max} = 6300 \text{ lb-ft}$$



The maximum shear will occur when the rear wheel (wheel of greater load) is directly over the support.

$$\sum M_{R_2} = 0$$

$$14R_1 = 10.4R$$

$$14R_1 = 10.4(3000)$$

$$R_1 = 2228.57 \text{ lb}$$

Thus, $V_{\max} = 2228.57 \text{ lb}$

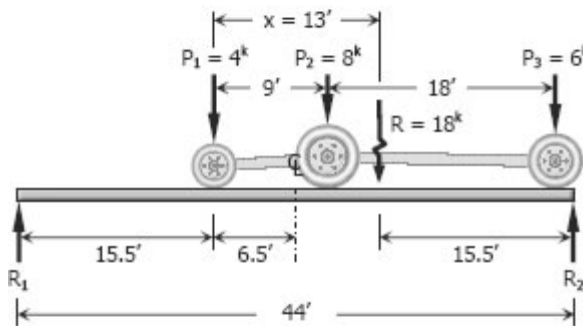
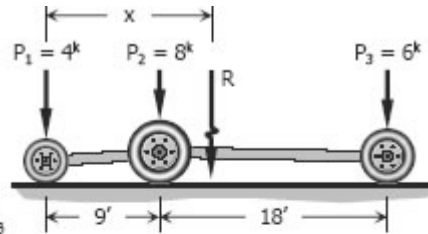
Problem 456

Three wheel loads roll as a unit across a 44-ft span. The loads are $P_1 = 4000$ lb and $P_2 = 8000$ lb separated by 9 ft, and $P_3 = 6000$ lb at 18 ft from P_2 . Determine the maximum moment and maximum shear in the simply supported span.

Solution 456

$$\begin{aligned} R &= P_1 + P_2 + P_3 \\ R &= 4^k + 8^k + 6^k \\ R &= 18 \text{ kips} \\ R &= 18,000 \text{ lbs} \end{aligned}$$

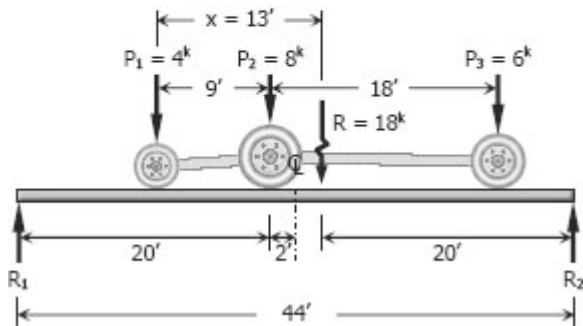
$$\begin{aligned} xR &= 9P_2 + (9 + 18)P_3 \\ x(18) &= 9(8) + (9 + 18)(6) \\ x &= 13 \text{ ft} \rightarrow \text{the resultant } R \text{ is } 13 \text{ ft from } P_1 \end{aligned}$$



Maximum moment under P_1

$$\begin{aligned} \sum M_{R2} &= 0 \\ 44R_1 &= 15.5R \\ 44R_1 &= 15.5(18) \\ R_1 &= 6.34091 \text{ kips} \\ R_1 &= 6,340.91 \text{ lbs} \end{aligned}$$

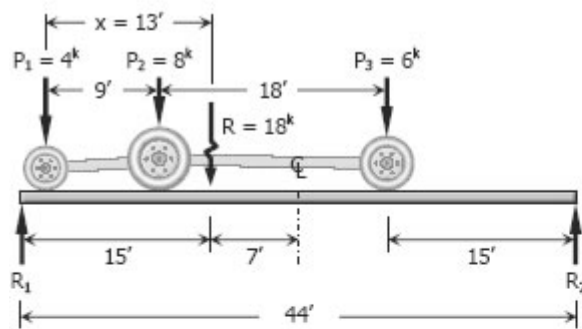
$$\begin{aligned} M_{\text{To the left of } P_1} &= 15.5R_1 \\ M_{\text{To the left of } P_1} &= 15.5(6340.91) \\ M_{\text{To the left of } P_1} &= 98,284.1 \text{ lb-ft} \end{aligned}$$



Maximum moment under P_2

$$\begin{aligned} \sum M_{R2} &= 0 \\ 44R_1 &= 20R \\ 44R_1 &= 20(18) \\ R_1 &= 8.18182 \text{ kips} \\ R_1 &= 8,181.82 \text{ lbs} \end{aligned}$$

$$\begin{aligned} M_{\text{To the left of } P_2} &= 20R_1 - 9P_1 \\ M_{\text{To the left of } P_2} &= 20(8,181.82) - 9(4000) \\ M_{\text{To the left of } P_2} &= 127,636.4 \text{ lb-ft} \end{aligned}$$



Maximum moment under P_3

$$\sum R_1 = 0$$

$$44R_2 = 15R$$

$$44R_2 = 15(18)$$

$$R_2 = 6.13636 \text{ kips}$$

$$R_2 = 6,136.36 \text{ lbs}$$

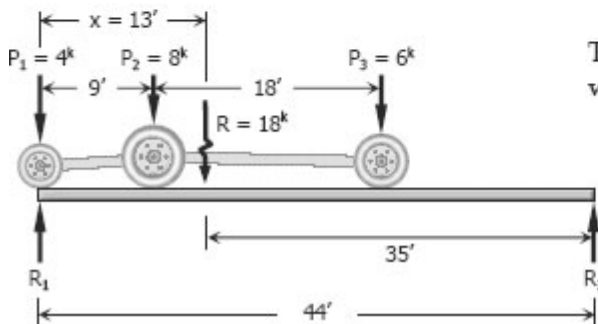
$$M_{\text{To the right of } P_3} = 15R_2$$

$$M_{\text{To the right of } P_3} = 15(6,136.36)$$

$$M_{\text{To the right of } P_3} = 92,045.4 \text{ lb}\cdot\text{ft}$$

$$\text{Thus, } M_{\max} = M_{\text{To the left of } P_2}$$

$$= 127,636.4 \text{ lb}\cdot\text{ft}$$



The maximum shear will occur when P_1 is over the support.

$$\sum M_{R2} = 0$$

$$44R_1 = 35R$$

$$44R_1 = 35(18)$$

$$R_1 = 14.3182 \text{ kips}$$

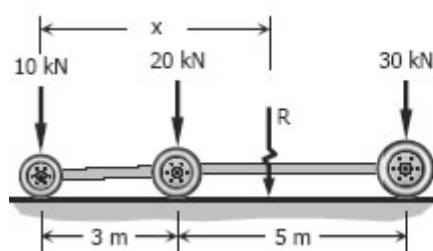
$$R_1 = 14,318.2 \text{ lbs}$$

$$\text{Thus, } V_{\max} = 14,318.2 \text{ lbs}$$

Problem 457

A truck and trailer combination crossing a 12-m span has axle loads of 10, 20, and 30 kN separated respectively by distances of 3 and 5 m. Compute the maximum moment and maximum shear developed in the span.

Solution 457



$$R = 10 + 20 + 30$$

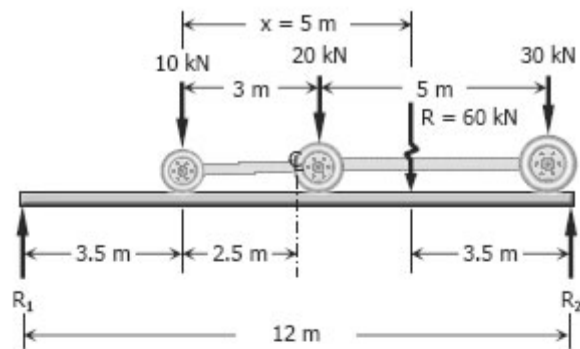
$$R = 60 \text{ kN}$$

$$xR = 3(20) + 8(30)$$

$$x(60) = 3(20) + 8(30)$$

$$x = 5 \text{ m}$$

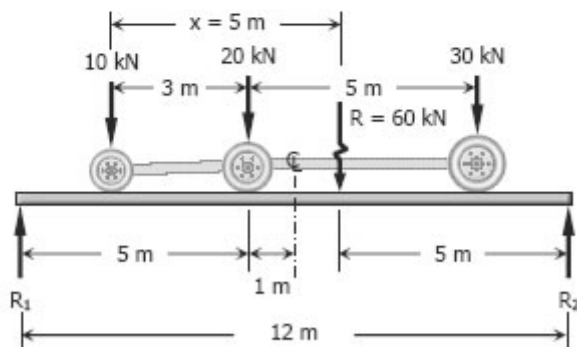
Maximum moment under 10 kN



$$\begin{aligned}\sum M_{R2} &= 0 \\ 12R_1 &= 3.5R \\ 12R_1 &= 3.5(60) \\ 12R_1 &= 210 \\ R_1 &= 12.7 \text{ kN}\end{aligned}$$

$$\begin{aligned}M_{\text{To the left of 10 kN}} &= 3.5R_1 \\ &= 3.5(12.7) \\ &= 61.25 \text{ kN}\cdot\text{m}\end{aligned}$$

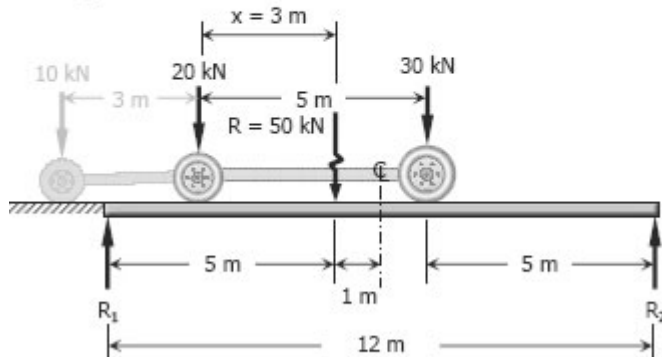
Maximum moment under 20 kN



$$\begin{aligned}\sum M_{R2} &= 0 \\ 12R_1 &= 5R \\ 12R_1 &= 5(60) \\ R_1 &= 25 \text{ kN}\end{aligned}$$

$$\begin{aligned}
 M_{\text{To the left of 20 kN}} &= 5R_1 - 3(10) \\
 &= 5(25) - 30 \\
 &= 95 \text{ kN}\cdot\text{m}
 \end{aligned}$$

When the centerline of the beam is midway between reaction $R = 60 \text{ kN}$ and 30 kN , the 10 kN comes off the span.



$$\begin{aligned}
 R &= 20 + 30 \\
 R &= 50 \text{ kN}
 \end{aligned}$$

$$\begin{aligned}
 xR &= 5(30) \\
 x(50) &= 150 \\
 x &= 3 \text{ m from 20 kN}
 \end{aligned}$$

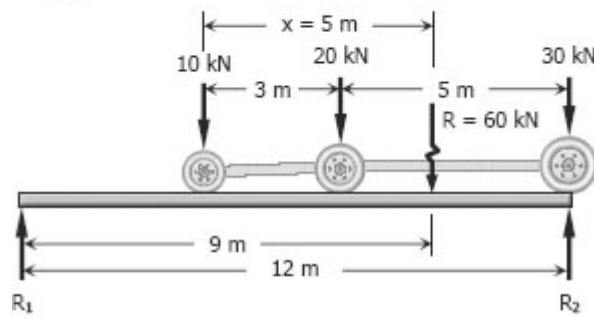
$$\begin{aligned}
 \sum M_{R1} &= 0 \\
 12R_2 &= 5R \\
 12R_2 &= 5(50) \\
 R_2 &= 20.83 \text{ kN}
 \end{aligned}$$

$$\begin{aligned}
 M_{\text{To the right of 30 kN}} &= 5R_2 \\
 &= 5(20.83) \\
 &= 104.17 \text{ kN}\cdot\text{m}
 \end{aligned}$$

Thus, the maximum moment will occur when only the 20 kN and 30 kN loads are on the span.

$$\begin{aligned}
 M_{\text{max}} &= M_{\text{To the right of 30 kN}} \\
 M_{\text{max}} &= \mathbf{104.17 \text{ kN}\cdot\text{m}}
 \end{aligned}$$

The maximum shear will occur when the three loads are on the span and the 30 kN load is directly over the support.



$$\Sigma M_{R1} = 0$$

$$12R_2 = 9R$$

$$12R_2 = 9(60)$$

$$R_2 = 45 \text{ kN}$$

$$\text{Thus, } V_{\max} = 45 \text{ kN}$$

Stresses in Beams

Forces and couples acting on the beam cause bending (flexural stresses) and shearing stresses on any cross section of the beam and deflection perpendicular to the longitudinal axis of the beam. If couples are applied to the ends of the beam and no forces act on it, the bending is said to be pure bending. If forces produce the bending, the bending is called ordinary bending.

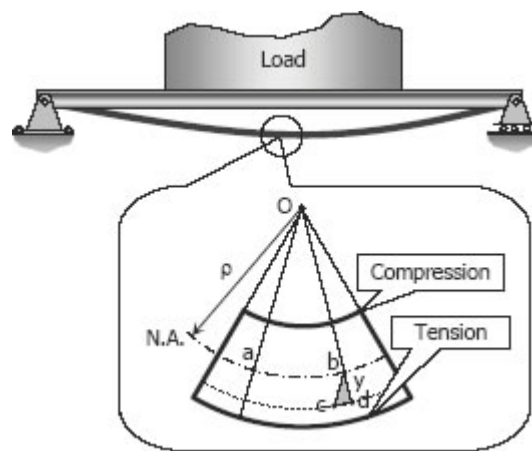
ASSUMPTIONS

In using the following formulas for flexural and shearing stresses, it is assumed that a plane section of the beam normal to its longitudinal axis prior to loading remains plane after the forces and couples have been applied, and that the beam is initially straight and of uniform cross section and that the moduli of elasticity in tension and compression are equal.

Flexure Formula

Stresses caused by the bending moment are known as flexural or bending stresses.

Consider a beam to be loaded as shown.



Consider a fiber at a distance y from the neutral axis, because of the beam's curvature, as the effect of bending moment, the fiber is stretched by an amount of cd . Since the curvature of the beam is very small, bcd and Oba are considered as similar triangles. The strain on this fiber is

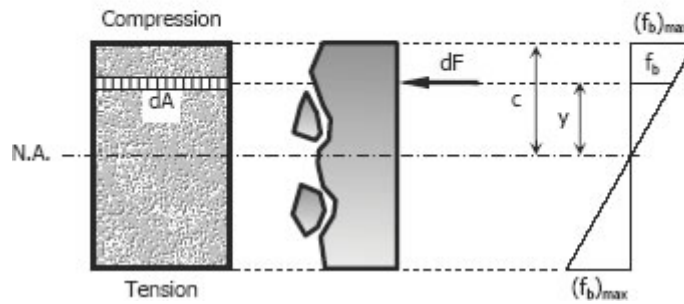
$$\epsilon = \frac{cd}{ab} = \frac{y}{\rho}$$

By Hooke's law, $\epsilon = \sigma / E$, then

$$\frac{\sigma}{E} = \frac{y}{\rho}; \quad \sigma = \frac{y}{\rho} E$$

which means that the stress is proportional to the distance y from the neutral axis.

For this chapter, the notation f_b will be used instead of σ , to denote flexural stresses.



Considering a differential area dA at a distance y from N.A., the force acting over the area is

$$dF = f_b dA = \frac{y}{\rho} E dA = \frac{E}{\rho} y dA$$

The resultant of all the elemental moment about N.A. must be equal to the bending moment on the section.

$$M = \int y dF = \int y \frac{E}{\rho} y dA$$

$$M = \frac{E}{\rho} \int y^2 dA$$

but $\int y^2 dA = I$, then

$$M = \frac{EI}{\rho} \text{ or } \rho = \frac{EI}{M}$$

substituting $\rho = Ey / f_b$

$$\frac{Ey}{f_b} = \frac{EI}{M}$$

then

$$f_b = \frac{My}{I}$$

and

$$(f_b)_{\max} = \frac{Mc}{I}$$

The bending stress due to beams curvature is

$$f_b = \frac{Mc}{I} = \frac{\frac{EI}{\rho}c}{I}$$

$$f_b = \frac{Ec}{\rho}$$

The beam curvature is:

$$k = 1 / \rho$$

where ρ is the radius of curvature of the beam in mm (in), M is the bending moment in N·mm (lb·in), f_b is the flexural stress in MPa (psi), I is the centroidal moment of inertia in mm⁴ (in⁴), and c is the distance from the neutral axis to the outermost fiber in mm (in).

SECTION MODULUS

In the formula

$$(f_b)_{\max} = \frac{Mc}{I} = \frac{M}{I/c},$$

the ratio I/c is called the section modulus and is usually denoted by S with units of mm³ (in³). The maximum bending stress may then be written as

$$(f_b)_{\max} = \frac{M}{S}$$

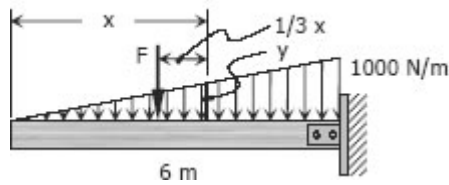
This form is convenient because the values of S are available in handbooks for a wide range of standard structural shapes.

Solved Problems in Flexure Formula

Problem 503

A cantilever beam, 50 mm wide by 150 mm high and 6 m long, carries a load that varies uniformly from zero at the free end to 1000 N/m at the wall. (a) Compute the magnitude and location of the maximum flexural stress. (b) Determine the type and magnitude of the stress in a fiber 20 mm from the top of the beam at a section 2 m from the free end.

Solution 503



$$M = F\left(\frac{1}{3}x\right)$$

$$\frac{y}{x} = \frac{1000}{6}$$

$$y = \frac{500}{3}x$$

$$F = \frac{1}{2}xy$$

$$F = \frac{1}{2}x\left(\frac{500}{3}x\right)$$

$$F = \frac{250}{3}x^2$$

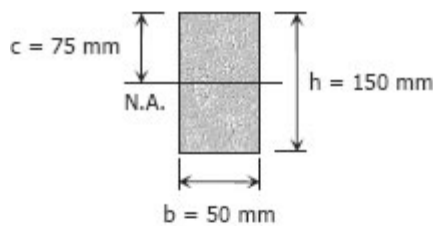
thus $M = \frac{250}{3}x^2 \left(\frac{1}{3}x\right)$

$$M = \frac{250}{9}x^3$$

- (a) The maximum moment occurs at the support (the wall) or at $x = 6$ m.

$$M = \frac{250}{9}x^3 = \frac{250}{9}(6^3)$$

$$= 6000 \text{ N}\cdot\text{m}$$

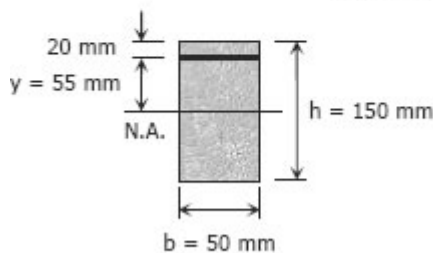


$$(f_b)_{\max} = \frac{Mc}{I} = \frac{Mc}{\frac{bh^3}{12}}$$

$$(f_b)_{\max} = \frac{6000(1000)(75)}{\frac{50(150)^3}{12}}$$

$$(f_b)_{\max} = 32 \text{ MPa}$$

- (b) At a section 2 m from the free end or at $x = 2$ m at fiber 20 mm from the top of the beam:



$$M = \frac{250}{9}x^3 = \frac{250}{9}(2)^3$$

$$M = \frac{2000}{9} \text{ N}\cdot\text{m}$$

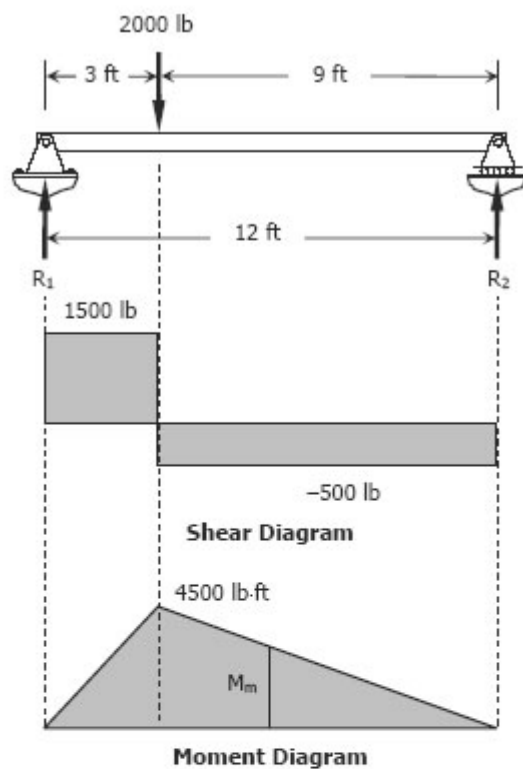
$$f_b = \frac{My}{I} = \frac{\left(\frac{2000}{9}\right)(1000)(55)}{\frac{50(150)^3}{12}}$$

$$f_b = 0.8691 \text{ MPa} = 869.1 \text{ kPa}$$

Problem 504

A simply supported beam, 2 in wide by 4 in high and 12 ft long is subjected to a concentrated load of 2000 lb at a point 3 ft from one of the supports. Determine the maximum fiber stress and the stress in a fiber located 0.5 in from the top of the beam at midspan.

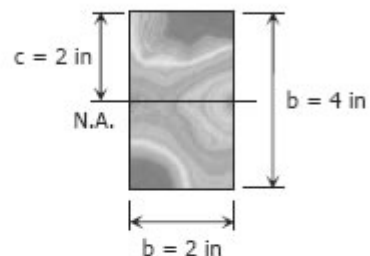
Solution 504



$$\begin{aligned}\Sigma M_{R2} &= 0 \\ 12R_1 &= 9(2000) \\ R_1 &= 1500 \text{ lb}\end{aligned}$$

$$\begin{aligned}\Sigma M_{R1} &= 0 \\ 12R_2 &= 3(2000) \\ R_2 &= 500 \text{ lb}\end{aligned}$$

Maximum fiber stress:



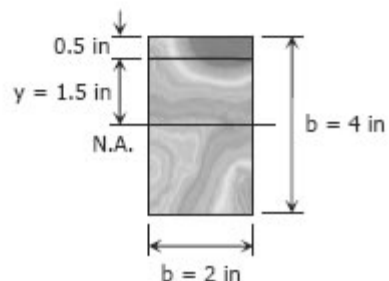
$$(f_b)_{\max} = \frac{Mc}{I} = \frac{4500(12)(2)}{\frac{2(4)^3}{12}}$$

$$(f_b)_{\max} = 10,125 \text{ psi}$$

Stress in a fiber located 0.5 in from the top of the beam at midspan:

$$\begin{aligned}\frac{M_m}{6} &= \frac{4500}{9} \\ M_m &= 3000 \text{ lb-ft}\end{aligned}$$

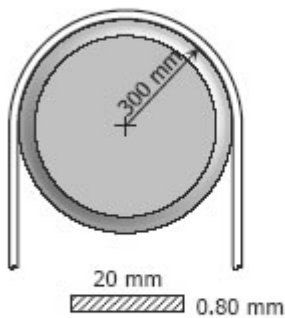
$$\begin{aligned}f_b &= \frac{My}{I} \\ f_b &= \frac{3000(12)(1.5)}{\frac{2(4^3)}{12}} \\ f_b &= 5,062.5 \text{ psi}\end{aligned}$$



Problem 505

A high strength steel band saw, 20 mm wide by 0.80 mm thick, runs over pulleys 600 mm in diameter. What maximum flexural stress is developed? What minimum diameter pulleys can be used without exceeding a flexural stress of 400 MPa? Assume $E = 200$ GPa.

Solution 505



Flexural stress developed:

$$M = \frac{EI}{\rho}$$

$$f_b = \frac{Mc}{I} = \frac{(EI/\rho)c}{I}$$

$$f_b = \frac{Ec}{\rho} = \frac{200000(0.80/2)}{300}$$

$$f_b = 266.67 \text{ MPa}$$

Minimum diameter of pulley:

$$f_b = \frac{Ec}{\rho}$$

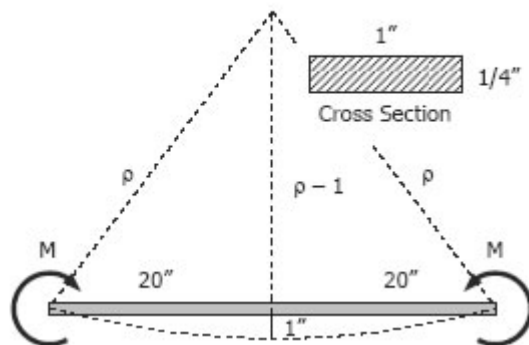
$$400 = \frac{200000(0.80/2)}{\rho}$$

$$\rho = 200 \text{ mm}$$

$$\text{diameter, } d = 400 \text{ mm}$$

Problem 506

A flat steel bar, 1 inch wide by $\frac{1}{4}$ inch thick and 40 inches long, is bent by couples applied at the ends so that the midpoint deflection is 1.0 inch. Compute the stress in the bar and the magnitude of the couples. Use $E = 29 \times 10^6$ psi.



$$(\rho - 1)^2 + 20^2 = \rho^2$$

$$\rho^2 - 2\rho + 1 + 400 = \rho^2$$

$$2\rho = 401$$

$$\rho = 200.5 \text{ in}$$

$$M = \frac{EI}{\rho}$$

$$f_b = \frac{Mc}{I} = \frac{(EI/\rho)c}{I}$$

$$f_b = \frac{Ec}{\rho} = \frac{(29 \times 10^6)(1/8)}{200.5}$$

$$f_b = 18\,079.8 \text{ psi}$$

$$f_b = 18.1 \text{ ksi}$$

$$M = \frac{EI}{\rho} = \frac{(29 \times 10^6) \frac{1(1/4)^3}{12}}{200.5}$$

$$M = 188.3 \text{ lb}\cdot\text{in}$$

Problem 507

In a laboratory test of a beam loaded by end couples, the fibers at layer AB in Fig. P-507 are found to increase $60 \times 10^{-3} \text{ mm}$ whereas those at CD decrease $100 \times 10^{-3} \text{ mm}$ in the 200-mm-gage length. Using $E = 70 \text{ GPa}$, determine the flexural stress in the top and bottom fibers.

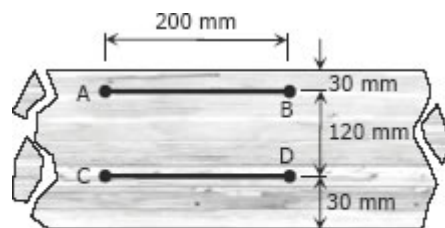
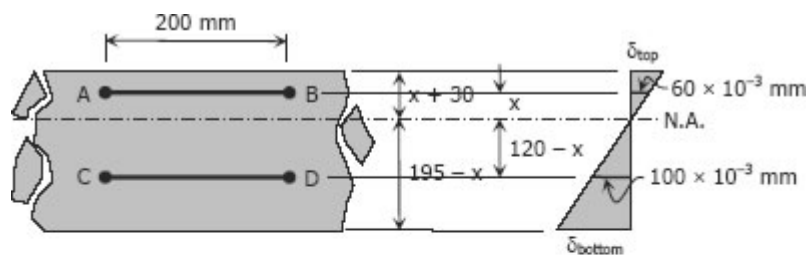


Figure P-507

Solution 507



$$\frac{x}{60 \times 10^{-3}} = \frac{120 - x}{100 \times 10^{-3}}$$

$$x = 0.6(120 - x)$$

$$x + 0.6x = 0.6(120)$$

$$1.6x = 72$$

$$x = 45 \text{ mm}$$

$$\frac{\delta_{top}}{x+30} = \frac{60 \times 10^{-3}}{x}$$

$$\delta_{top} = \frac{60 \times 10^{-3}}{45} (45+30)$$

$$\delta_{top} = 0.1 \text{ mm lengthening}$$

$$\frac{\delta_{bottom}}{195-x} = \frac{100 \times 10^{-3}}{120-x}$$

$$\delta_{bottom} = \frac{100 \times 10^{-3}}{120-45} (195-45)$$

$$\delta_{bottom} = 0.2 \text{ mm shortening}$$

From Hooke's Law

$$f_v = E\varepsilon$$

$$f_v = \frac{E\delta}{L}$$

$$(f_v)_{top} = \frac{70000(0.1)}{200}$$

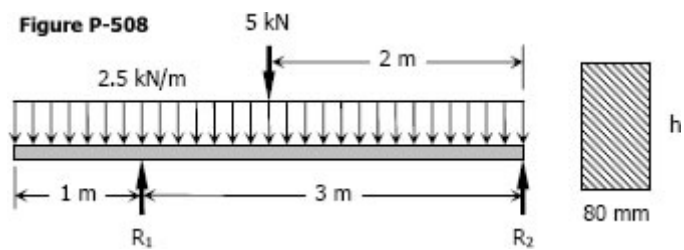
$$= 35 \text{ MPa tension}$$

$$(f_v)_{bottom} = \frac{70000(0.2)}{200}$$

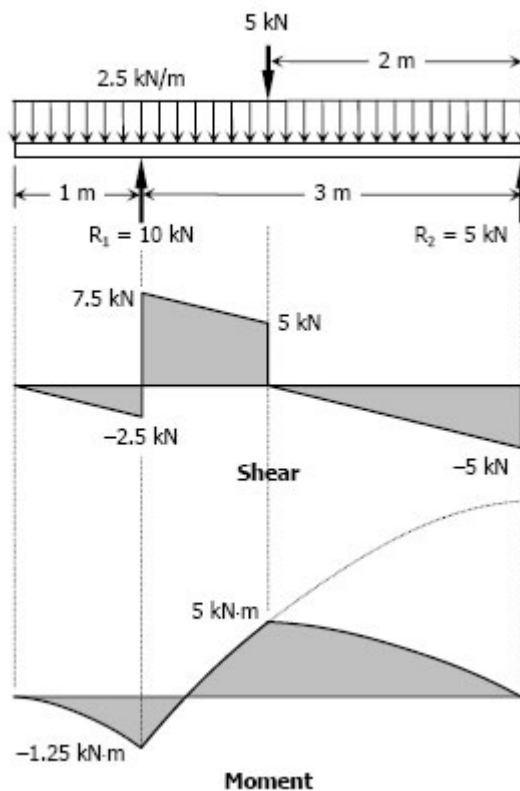
$$= 70 \text{ MPa compression}$$

Problem 508

Determine the minimum height h of the beam shown in Fig. P-508 if the flexural stress is not to exceed 20 MPa.



Solution 508



$$\begin{aligned}\Sigma M_{R2} &= 0 \\ 3R_1 &= 2(5) + 2(2.5)(4) \\ R_1 &= 10 \text{ kN}\end{aligned}$$

$$\begin{aligned}\Sigma M_{R1} &= 0 \\ 3R_2 &= 1(5) + 1(2.5)(4) \\ R_2 &= 5 \text{ kN}\end{aligned}$$

$$f_b = \frac{Mc}{I}$$

Where:

$$\begin{aligned}f_b &= 20 \text{ MPa} \\ M &= 5 \text{ kN}\cdot\text{m} \\ &= 5(1000)^2 \text{ N}\cdot\text{mm} \\ c &= \frac{1}{2}h \\ I &= \frac{bh^3}{12} = \frac{80h^3}{12} \\ &= \frac{20}{3}h^3\end{aligned}$$

Thus,

$$20 = \frac{5(1000)^2 (\frac{1}{2}h)}{\frac{20}{3}h^3}$$

$$h^2 = 18\,750$$

$$h = 137 \text{ mm}$$

Problem 509

A section used in aircraft is constructed of tubes connected by thin webs as shown in Fig. P-509. Each tube has a cross-sectional area of 0.20 in². If the average stress in the tubes is not to exceed 10 ksi, determine the total uniformly distributed load that can be supported in a simple span 12 ft long. Neglect the effect of the webs.

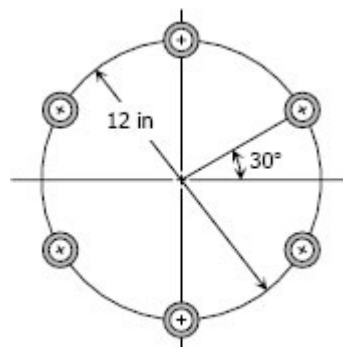
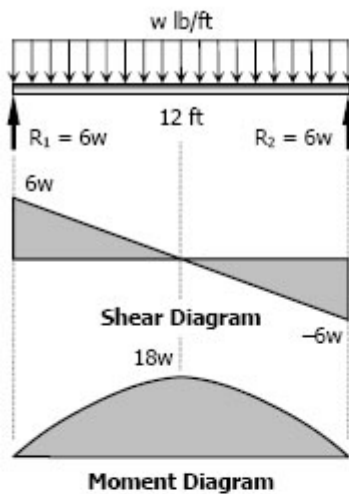


Figure P-509

Solution 509



$$R_1 = R_2 = \frac{1}{2} (12)(w)$$

$$R_1 = R_2 = 6w$$

$$f_b = 10 \text{ ksi} = 10,000 \text{ psi}$$

$$M = 18w \text{ lb-ft}$$

$$c = 6$$

Centroidal moment of inertia of one tube:

$$A = \pi r^2 = 0.20$$

$r = 0.2523 \text{ in} \rightarrow$ hollow portion of the tube was neglected

$$\bar{I}_x = \frac{\pi r^4}{4} = \frac{\pi (0.2523)^4}{4}$$

$$\bar{I}_x = 0.0032 \text{ in}^4$$

Moment of inertia at the center of the section:

$$d_1 = 6 \sin 30^\circ = 3 \text{ in}$$

$$I_1 = \bar{I}_x + A d_1^2$$

$$I_1 = 0.0032 + 0.2(3^2)$$

$$I_1 = 1.8 \text{ in}^4$$

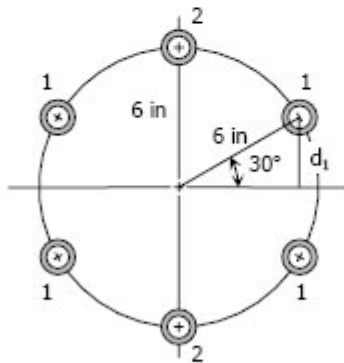
$$I_2 = \bar{I}_x + A d_2^2$$

$$I_2 = 0.0032 + 0.2(6^2)$$

$$I_2 = 7.2 \text{ in}^4$$

$$I = 4I_1 + 2I_2 = 4(1.8) + 2(7.2)$$

$$I = 21.6 \text{ in}^4$$



$$f_b = \frac{Mc}{I}$$

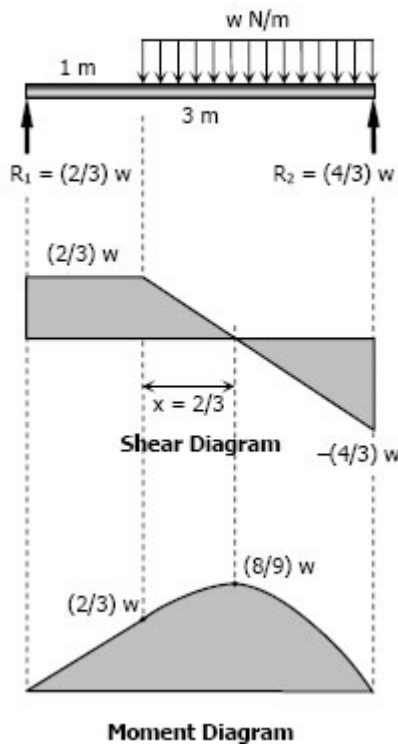
$$10,000 = \frac{18w(12)(6)}{21.6}$$

$$w = 166.7 \text{ lb/ft}$$

Problem 510

A 50-mm diameter bar is used as a simply supported beam 3 m long. Determine the largest uniformly distributed load that can be applied over the right two-thirds of the beam if the flexural stress is limited to 50 MPa.

Solution 510



$$\begin{aligned}\sum M_{R1} &= 0 \\ 3R_2 &= 2w(2) \\ R_2 &= \frac{4}{3}w\end{aligned}$$

$$\begin{aligned}\sum M_{R2} &= 0 \\ 3R_1 &= 2w(1) \\ R_1 &= \frac{2}{3}w\end{aligned}$$

$$(f_b)_{\max} = \frac{Mc}{I}$$

$$\text{where } (f_b)_{\max} = 50 \text{ MPa}$$

$$M = \frac{8}{9}w \text{ N}\cdot\text{m}$$

$$c = 25 \text{ mm}$$

$$I = \frac{\pi r^4}{4} = \frac{\pi(25)^4}{4}$$

$$I = 97656.25\pi \text{ mm}^4$$

$$50 = \frac{\frac{8}{9}w(1000)(25)}{97656.25\pi}$$

$$w = 690.29 \text{ N/m}$$

Problem 511

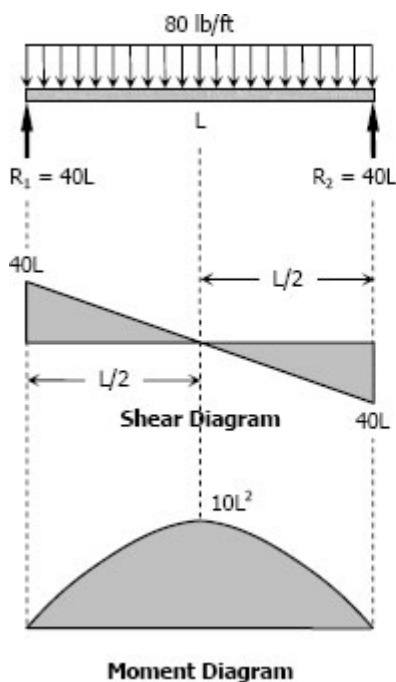
A simply supported rectangular beam, 2 in wide by 4 in deep, carries a uniformly distributed load of 80 lb/ft over its entire length. What is the maximum length of the beam if the flexural stress is limited to 3000 psi?

Solution 511

By symmetry:

$$R_1 = R_2 = \frac{1}{2}(80L)$$

$$R_1 = R_2 = 40L$$



$$(f_b)_{\max} = \frac{Mc}{I}$$

$$\text{where } (f_b)_{\max} = 3000 \text{ psi}$$

$$M = 10L^2 \text{ lb}\cdot\text{ft}$$

$$c = h/2 = 2 \text{ in}$$

$$\begin{aligned}I &= \frac{bh^3}{12} = \frac{2(4)^3}{12} \\ &= \frac{32}{3} \text{ in}^4\end{aligned}$$



$$h = 4 \text{ in}$$

$$b = 2 \text{ in}$$

$$3000 = \frac{10L^2(12)(2)}{32/3}$$

$$L = 11.55 \text{ ft}$$

Problem 512

The circular bar 1 inch in diameter shown in Fig. P-512 is bent into a semicircle with a mean radius of 2 ft. If $P = 400$ lb and $F = 200$ lb, compute the maximum flexural stress developed in section a-a. Neglect the deformation of the bar.

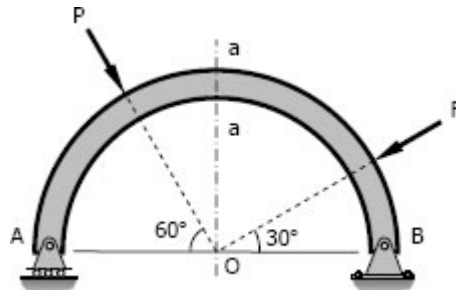
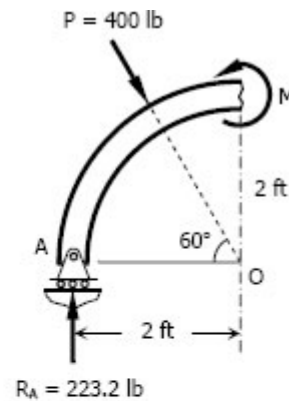
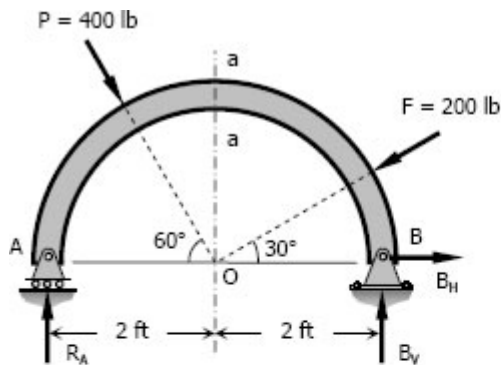


Figure P-512

Solution 512

$$\begin{aligned}\sum M_B &= 0 \\ 4R_A &= 2(400 \sin 60^\circ) + 2(200 \sin 30^\circ) \\ R_A &= 223.2 \text{ lb}\end{aligned}$$



$$\begin{aligned}M &= 2(223.2) - 2(400 \cos 60^\circ) \\ M &= 46.4 \text{ lb}\cdot\text{ft}\end{aligned}$$

$$\begin{aligned}(f_b)_{\max} &= \frac{Mc}{I} = \frac{Mr}{\pi r^4 / 4} \\ (f_b)_{\max} &= \frac{4M}{\pi r^3} = \frac{4(46.4)(12)}{\pi(0.5)^3} \\ (f_b)_{\max} &= 5671.52 \text{ psi}\end{aligned}$$

Problem 513

A rectangular steel beam, 2 in wide by 3 in deep, is loaded as shown in Fig. P-513. Determine the magnitude and the location of the maximum flexural stress.

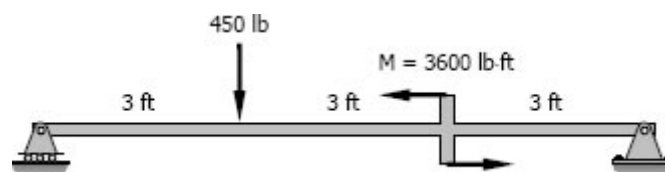
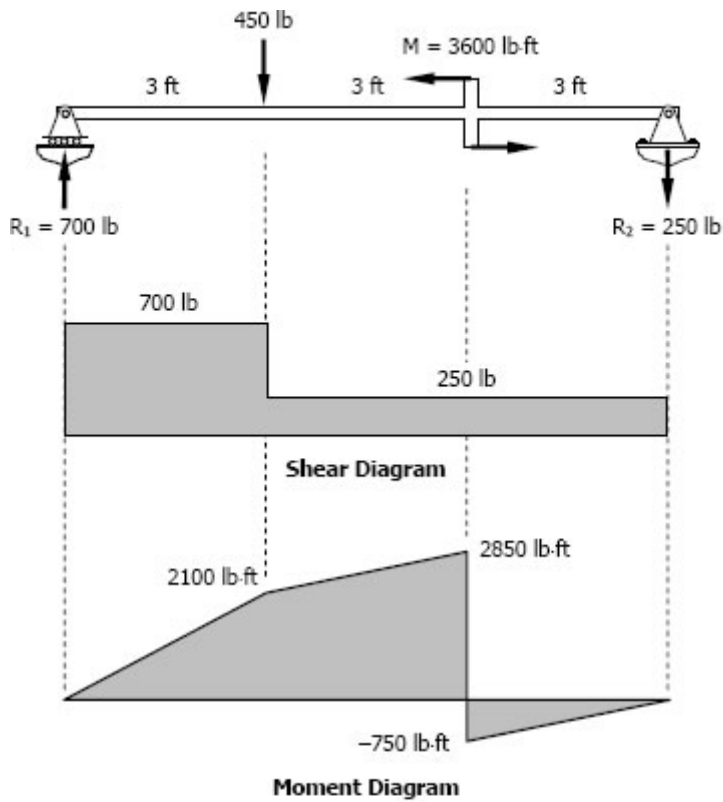


Figure P-513

Solution 513

$$\begin{aligned}\Sigma M_{R_2} &= 0 \\ 9R_1 &= 6(450) + 3600 \\ R_1 &= 700 \text{ lb}\end{aligned}$$

$$\begin{aligned}\Sigma M_{R_1} &= 0 \\ 9R_2 + 3(450) &= 3600 \\ R_2 &= 250 \text{ lb}\end{aligned}$$



$$(f_b)_{\max} = \frac{Mc}{I}$$

where $M = 2850 \text{ lb}\cdot\text{ft}$
 $c = h/2 = 3/2$
 $= 1.5 \text{ in}$

$$\begin{aligned}I &= \frac{bh^3}{12} = \frac{2(3^3)}{12} \\ &= 4.5 \text{ in}^4\end{aligned}$$

$$(f_b)_{\max} = \frac{2850(12)(1.5)}{4.5}$$

$$(f_b)_{\max} = 11400 \text{ psi @ 3 ft from right support}$$

Problem 514

The right-angled frame shown in Fig. P-514 carries a uniformly distributed loading equivalent to 200 N for each horizontal projected meter of the frame; that is, the total load is 1000 N. Compute the maximum flexural stress at section a-a if the cross-section is 50 mm square.

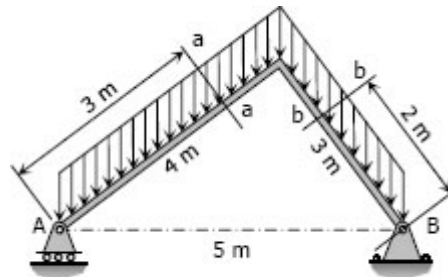
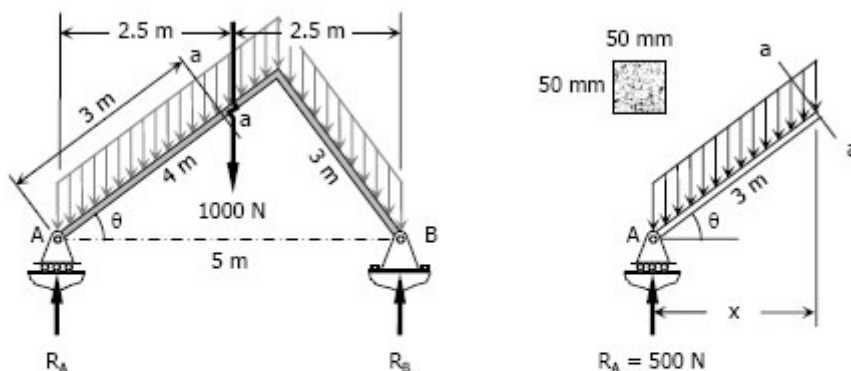


Figure P-514 and P-515

Solution 514

By symmetry
 $R_A = 500 \text{ N}$
 $R_B = 500 \text{ N}$



At section a-a:

$$\cos \theta = \frac{x}{3} = \frac{4}{5}$$

$$x = 2.4 \text{ m}$$

$$M = xR_A - 200x(x/2)$$

$$M = 2.4(500) - 200(2.4)(2.4/2)$$

$$M = 624 \text{ N}\cdot\text{m}$$

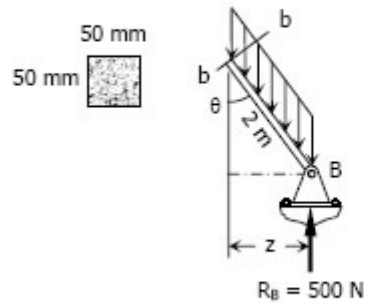
$$f_b = \frac{Mc}{I} = \frac{624(1000)(50/2)}{\frac{50(50^3)}{12}}$$

$$f_b = 29.952 \text{ MPa}$$

Problem 515

Repeat Prob. 524 to find the maximum flexural stress at section b-b.

Solution 515



At section b-b:

$$\sin \theta = \frac{z}{2} = \frac{3}{5}$$

$$z = 1.5 \text{ m}$$

$$M = zR_B - 200z(z/2)$$

$$M = 1.5(500) - 200(1.5)(1.5/2)$$

$$M = 525 \text{ N}\cdot\text{m}$$

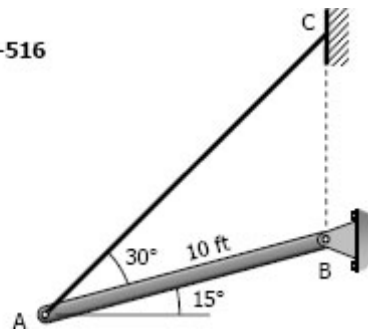
$$f_b = \frac{Mc}{I} = \frac{525(1000)(50/2)}{\frac{50(50)^3}{12}}$$

$$f_b = 25.2 \text{ MPa}$$

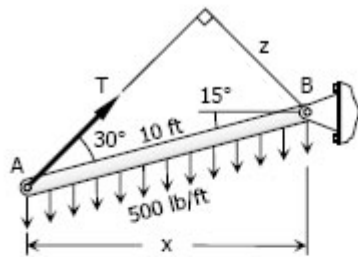
Problem 516

A timber beam AB, 6 in wide by 10 in deep and 10 ft long, is supported by a guy wire AC in the position shown in Fig. P-516. The beam carries a load, including its own weight, of 500 lb for each foot of its length. Compute the maximum flexural stress at the middle of the beam.

Figure P-516



Solution 516



$$x = 10 \cos 15^\circ$$

$$x = 9.66 \text{ ft}$$

$$z = 10 \sin 30^\circ$$

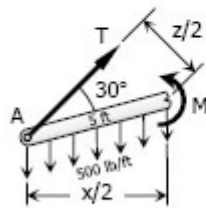
$$z = 5 \text{ ft}$$

$$\Sigma M_B = 0$$

$$zT = 500(10)(x/2)$$

$$5T = 500(10)(9.66/2)$$

$$T = 4829.63 \text{ lb}$$



At midspan:

$$M = T(z/2) - 500(5)(x/4)$$

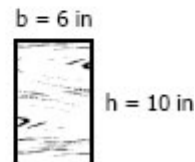
$$M = 4829.63(5/2) - 500(5)(9.66/4)$$

$$M = 6036.58 \text{ lb-ft}$$

$$f_b = \frac{Mc}{I} = \frac{M(h/2)}{\frac{bh^3}{12}}$$

$$f_b = \frac{6036.58(12)(10/2)}{\frac{6(10^3)}{12}}$$

$$f_b = 724.39 \text{ psi}$$



Problem 517

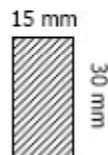
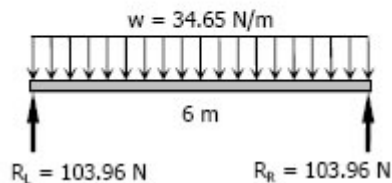
A rectangular steel bar, 15 mm wide by 30 mm high and 6 m long, is simply supported at its ends. If the density of steel is 7850 kg/m^3 , determine the maximum bending stress caused by the weight of the bar.

Solution 517

$$w = (7850 \text{ kg/m}^3)(0.015 \text{ m} \times 0.03 \text{ m})$$

$$= (3.5325 \text{ kg/m})(9.81 \text{ m/s}^2)$$

$$= 34.65 \text{ N/m}$$

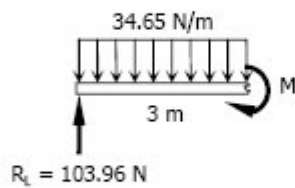


$$R_L = R_R = 6w/2$$

$$= 6(34.65)/2$$

$$= 103.96 \text{ N}$$

For simply supported beam subjected to uniformly distributed load, the maximum moment will occur at the midspan. At midspan:



$$M = 3(103.96) - 34.65(3)(3/2)$$

$$M = 155.955 \text{ N}\cdot\text{m}$$

$$(f_b)_{\max} = \frac{Mc}{I} = \frac{M(h/2)}{\frac{bh^3}{12}}$$

$$(f_b)_{\max} = \frac{155.955(1000)(30/2)}{\frac{15(30^3)}{12}}$$

$$(f_b)_{\max} = 69.31 \text{ MPa}$$

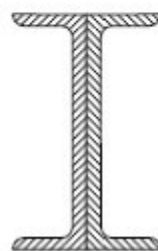
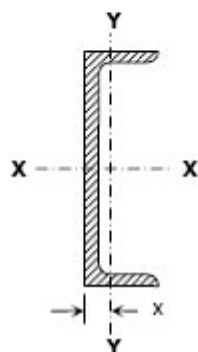
Problem 518

A cantilever beam 4 m long is composed of two C200 × 28 channels riveted back to back. What uniformly distributed load can be carried, in addition to the weight of the beam, without exceeding a flexural stress of 120 MPa if (a) the webs are vertical and (b) the webs are horizontal? Refer to Appendix B of text book for channel properties.

Solution 518

Relevant data from Appendix B, Table B-4 Properties of Channel Sections: SI Units, of text book.

Designation.....	C200 × 28
Area.....	3560 mm ²
Width	64 mm
S_{x-x}	$180 \times 10^3 \text{ mm}^3$
I_{y-y}	$0.825 \times 10^6 \text{ mm}^4$
x	14.4 mm



Webs are Vertical



Webs are horizontal

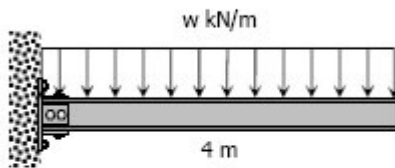
- a. Webs are vertical

$$(f_b)_{\max} = \frac{M}{S}$$

$$120 = \frac{M}{2(180 \times 10^3)}$$

$$M = 43,200,000 \text{ N}\cdot\text{mm}$$

$$M = 43.2 \text{ kN}\cdot\text{m}$$



From the figure:

$$M = 4w(2)$$

$$M = 8w$$

$$43.2 = 8w$$

$$w = 5.4 \text{ kN/m}$$

$$w = 550.46 \text{ kg/m}$$

w = dead load, DL + live load, LL

$$550.46 = 2(28) + LL$$

$$LL = 494.46 \text{ kg/m}$$

- b. Webs are horizontal

$$I_{\text{back}} = I_{Y-Y} + Ax^2$$

$$I_{\text{back}} = (0.825 \times 10^8) + 3560(14.4^2)$$

$$I_{\text{back}} = 1\,563\,201.6 \text{ mm}^4$$

$$I = 2I_{\text{back}} = 2(1\,563\,201.6)$$

$$I = 3\,126\,403.2 \text{ mm}^4$$

$$(f_b)_{\max} = \frac{Mc}{I}$$

$$120 = \frac{M(64)}{3\,126\,403.2}$$

$$M = 5\,862\,006 \text{ N}\cdot\text{mm}$$

$$M = 5.862 \text{ kN}\cdot\text{m}$$

From the figure:

$$M = 4w(2)$$

$$M = 8w$$

$$5.862 = 8w$$

$$w = 0.732\,75 \text{ kN/m}$$

$$w = 74.69 \text{ kg/m}$$

w = dead load, DL + live load, LL

$$74.69 = 2(28) + LL$$

$$LL = 18.69 \text{ kg/m}$$

Problem 519

A 30-ft beam, simply supported at 6 ft from either end carries a uniformly distributed load of intensity w_o over its entire length. The beam is made by welding two S18 × 70 (see appendix B of text book) sections along their flanges to form the section shown in Fig. P-519. Calculate the maximum value of w_o if the flexural stress is limited to 20 ksi. Be sure to include the weight of the beam.

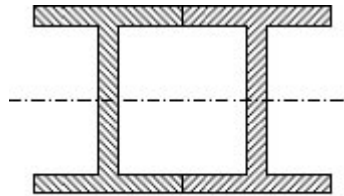
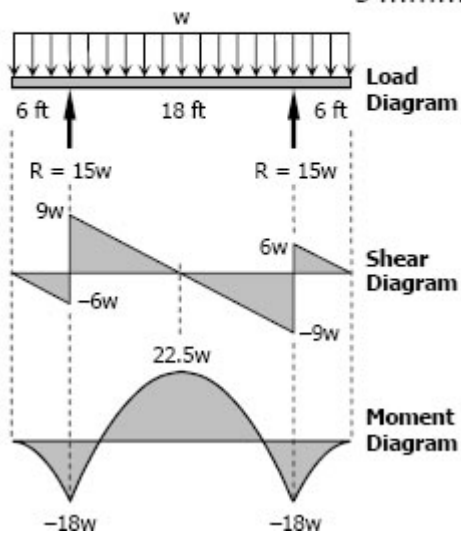


Figure P-519

Solution 519

Relevant data from Appendix B, Table B-8 Properties of I-Beam Sections (S-Shapes): US Customary Units, of text book.

Designation..... S18 × 70
 S 103 in³



$$(f_b)_{\max} = \frac{M}{S}$$

$$20 = \frac{M}{2(103)}$$

$$M = 4120 \text{ kip-in}$$

$$M = \frac{1030}{3} \text{ kip-ft}$$

From the moment diagram:

$$M = 22.5w$$

$$\frac{1030}{3} = 22.5w$$

$$w = 15.26 \text{ kip/ft}$$

w = dead load, DL + live load, w_o

$$15.26(1000) = 2(70) + w_o$$

$$w_o = 15\,120 \text{ lb/ft}$$

$$w_o = 15.12 \text{ kip/ft}$$

Problem 520

A beam with an S310 × 74 section (see Appendix B of textbook) is used as a simply supported beam 6 m long. Find the maximum uniformly distributed load that can be applied over the entire length of the beam, in addition to the weight of the beam, if the flexural stress is not to exceed 120 MPa.

Solution 520

Relevant data from Appendix B, Table B-4 Properties of I-Beam Sections (S-Shapes): SI Units, of text book.

Designation..... S310 × 74
 S $833 \times 10^3 \text{ mm}^3$

From the shear diagram:

$$M_{max} = \frac{1}{2} (3)(3w)$$

$$M_{max} = 4.5w \text{ N}\cdot\text{m}$$

$$(f_b)_{max} = \frac{M}{S}$$

$$120 = \frac{4.5w(1000)}{833 \times 10^3}$$

$$w = 22,213.33 \text{ N/m}$$

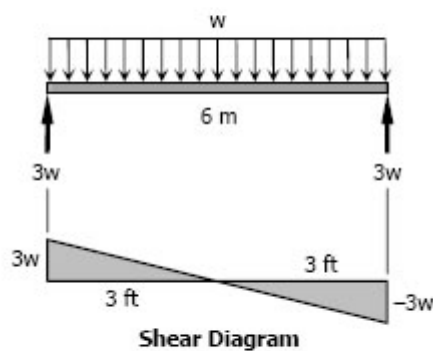
$$w = 2,264.36 \text{ kg/m}$$

$$w = DL + LL$$

$$2\,264.36 = 74 + LL$$

$$LL = 2,190.36 \text{ kg/m}$$

$$LL = 21.5 \text{ kN/m}$$



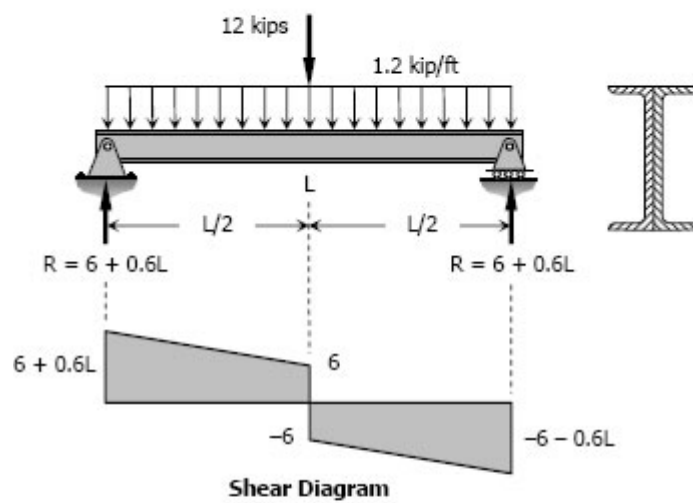
Problem 521

A beam made by bolting two C10 × 30 channels back to back, is simply supported at its ends. The beam supports a central concentrated load of 12 kips and a uniformly distributed load of 1200 lb/ft, including the weight of the beam. Compute the maximum length of the beam if the flexural stress is not to exceed 20 ksi.

Solution 521

Relevant data from Appendix B, Table B-9 Properties of Channel Sections: US Customary Units, of text book.

Designation..... C10 × 30
 S 20.7 in³



From the shear diagram:

$$M_{\max} = \frac{1}{2} [(6 + 0.6L) + 6](L/2)$$

$$M_{\max} = 3L + 0.15L^2$$

$$(f_b)_{\max} = \frac{M}{S}$$

$$20(1000) = \frac{(3L + 0.15L^2)(1000)(12)}{2(20.7)}$$

$$0.15L^2 + 3L - 69 = 0$$

$$L = 13.66 \text{ and } -33.66 \text{ (meaningless)}$$

$$\text{Use } L = 13.66 \text{ ft}$$

Problem 522

A box beam is composed of four planks, each 2 inches by 8 inches, securely spiked together to form the section shown in Fig. P-522. Show that $I_{NA} = 981.3 \text{ in}^4$. If $w_0 = 300 \text{ lb/ft}$, find P to cause a maximum flexural stress of 1400 psi.

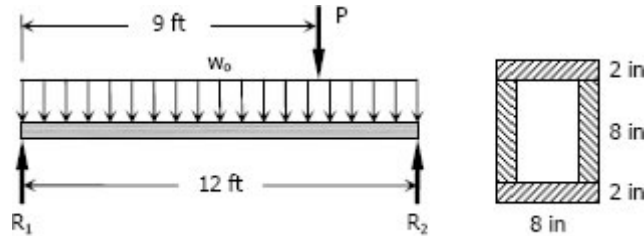
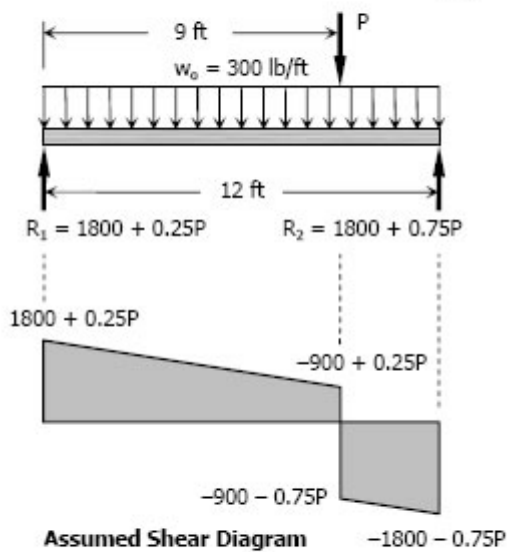
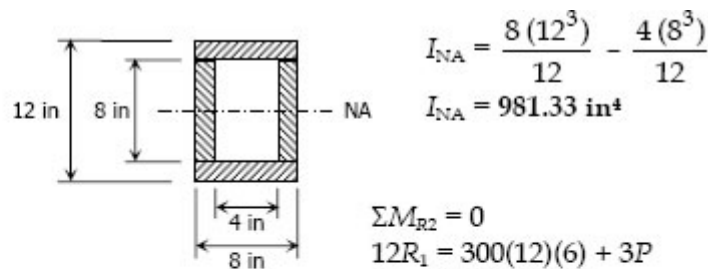


Figure P-522 and P-523

Solution 522



$$\Sigma M_{R2} = 0$$

$$12R_1 = 300(12)(6) + 3P$$

$$R_1 = 1800 + 0.25P$$

$$\Sigma M_{R1} = 0$$

$$12R_2 = 300(12)(6) + 9P$$

$$R_2 = 1800 + 0.75P$$

$$M = \frac{1}{2} [(1800 + 0.25P)(-900 + 0.25P)](9)$$

$$M = 4050 + 2.25P \text{ lb-ft}$$

$$(f_b)_{\max} = \frac{Mc}{I}$$

$$1400 = \frac{(4050 + 2.25P)(6)(12)}{981.33}$$

$$P = 6680.63 \text{ lb}$$

Check if the shear at P is positive as assumed

$$-900 + 0.25P = -900 + 0.25(6680.63)$$

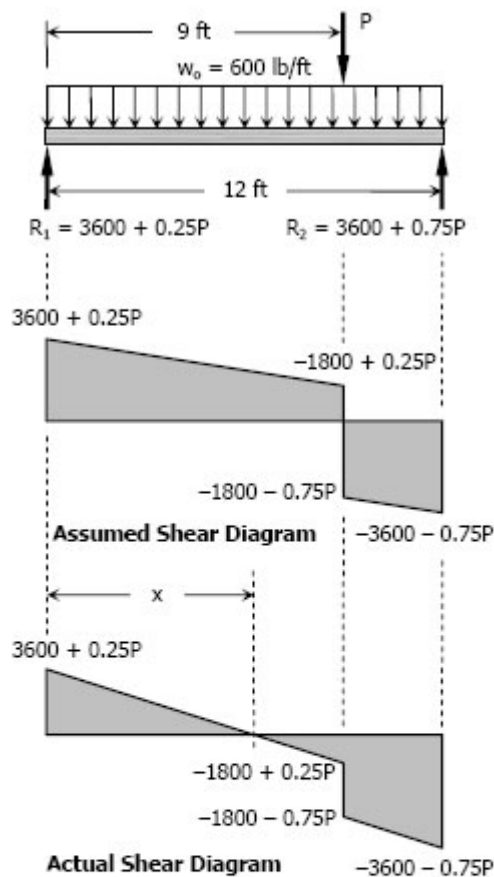
$$= 770.16 \text{ lb (ok!)}$$

Thus, $P = 6680.63 \text{ lb}$

Problem 523

Solve Prob. 522 if $w_0 = 600 \text{ lb/ft}$.

Solution 523



$$\begin{aligned}\Sigma M_{R2} &= 0 \\ 12R_1 &= 600(12)(6) + 3P \\ R_1 &= 3600 + 0.25P\end{aligned}$$

$$\begin{aligned}\Sigma M_{R1} &= 0 \\ 12R_2 &= 600(12)(6) + 9P \\ R_2 &= 3600 + 0.75P\end{aligned}$$

$$\begin{aligned}M &= \frac{1}{2} [(3600 + 0.25P) \\ &\quad + (-1800 + 0.25P)](9) \\ M &= 8100 + 2.25P \text{ lb-ft}\end{aligned}$$

$$\begin{aligned}(f_b)_{\max} &= \frac{Mc}{I} \\ 1400 &= \frac{(8100 + 2.25P)(6)(12)}{981.33} \\ P &= 4880.63 \text{ lb}\end{aligned}$$

Check if the shear at P is positive as assumed

$$\begin{aligned}-1800 + 0.25P &= -1800 + 0.25(4880.63) \\ &= -579.84 \text{ lb (not ok!)}\end{aligned}$$

From the actual shear diagram:

$$\begin{aligned}(3600 + 0.25P) - 600x &= 0 \\ x &= \frac{3600 + 0.25P}{600}\end{aligned}$$

$$\begin{aligned}M_{\max} &= \frac{1}{2} x (3600 + 0.25P) \\ M_{\max} &= \frac{1}{2} \left(\frac{3600 + 0.25P}{600} \right) (3600 + 0.25P) \\ M_{\max} &= \frac{(3600 + 0.25P)^2}{1200}\end{aligned}$$

$$\begin{aligned}(f_b)_{\max} &= \frac{Mc}{I} \\ 1400 &= \frac{\frac{(3600 + 0.25P)^2}{1200} (6)(12)}{981.33}\end{aligned}$$

$$\begin{aligned}22\,897\,700 &= (3600 + 0.25P)^2 \\ P &= 4740.62 \text{ lb}\end{aligned}$$

Problem 524

A beam with an S380 × 74 section carries a total uniformly distributed load of $3W$ and a concentrated load W , as shown in Fig. P-524. Determine W if the flexural stress is limited to 120 MPa.

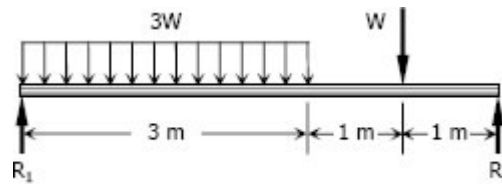
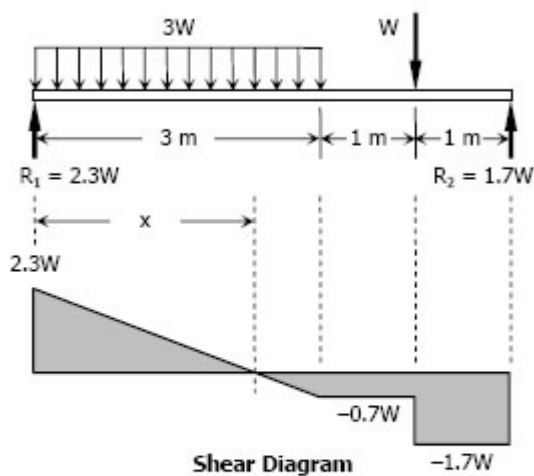


Figure P-524

Solution 524



$$\begin{aligned}\Sigma M_{R2} &= 0 \\ 5R_1 &= 3W(3.5) + W(1) \\ R_1 &= 2.3W\end{aligned}$$

$$\begin{aligned}\Sigma M_{R1} &= 0 \\ 5R_2 &= 3W(1.5) + W(4) \\ R_2 &= 1.7W\end{aligned}$$

$$\begin{aligned}2.3W - Wx &= 0 \\ x &= 2.3 \text{ m}\end{aligned}$$

$$\begin{aligned}M_{\max} &= \frac{1}{2} x (2.3W) \\ M_{\max} &= \frac{1}{2} (2.3)(2.3W) \\ M_{\max} &= 2.645W\end{aligned}$$

From Appendix B, Table B-3 Properties of I-Beam Sections (S-Shapes): SI Units, of text book.

Designation..... S380 × 74
S..... $1\,060 \times 10^3 \text{ mm}^3$

$$\begin{aligned}(f_b)_{\max} &= \frac{M}{S} \\ 120 &= \frac{2.645W(1000)}{1060 \times 10^3} \\ W &= 48\,090.74 \text{ N}\end{aligned}$$

Problem 525

A square timber beam used as a railroad tie is supported by a uniformly distributed load and carries two uniformly distributed loads each totaling 48 kN as shown in Fig. P-525. Determine the size of the section if the maximum stress is limited to 8 MPa.

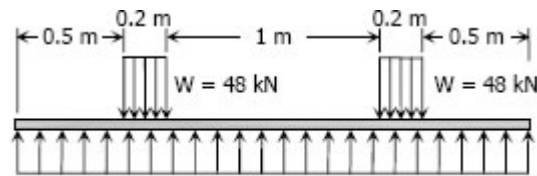
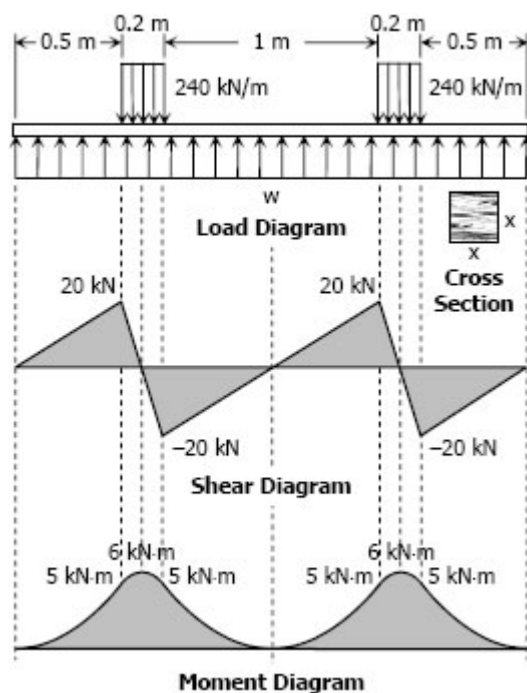


Figure P-524

Solution 525



$$\Sigma F_V = 0$$

$$2.4w = 240(0.2) + 240(0.2)$$

$$w = 40 \text{ kN/m}$$

$$(f_b)_{\max} = \frac{Mc}{I}$$

$$\text{Where: } f_b = 8 \text{ MPa}$$

$$M = 6 \text{ kN-m}$$

$$c = \frac{1}{2}x$$

$$I = \frac{bh^3}{12} = \frac{x(x^3)}{12}$$

$$= \frac{1}{12}x^4$$

$$8 = \frac{6(\frac{1}{2}x)(1000^2)}{\frac{1}{12}x^4}$$

$$x^3 = 4\,500\,000$$

$$x = 165.1 \text{ mm square}$$

Problem 526

A wood beam 6 in wide by 12 in deep is loaded as shown in Fig. P-526. If the maximum flexural stress is 1200 psi, find the maximum values of w_0 and P which can be applied simultaneously?

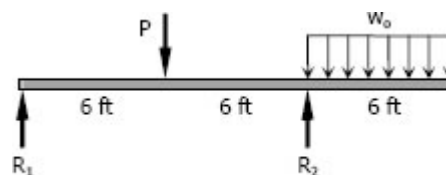
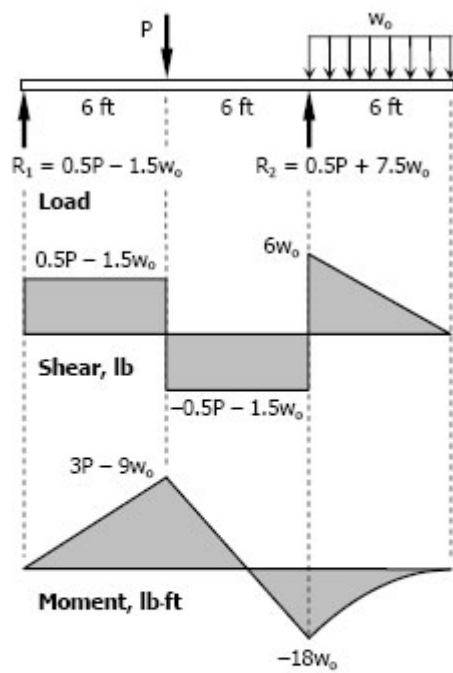


Figure P-526 and P-527

Solution 526



$$\begin{aligned}\Sigma M_{R2} &= 0 \\ 12R_1 + 3(6w_o) &= 6P \\ R_1 &= 0.5P - 1.5w_o\end{aligned}$$

$$\begin{aligned}\Sigma M_{R1} &= 0 \\ 12R_2 &= 6P + 15(6w_o) \\ R_2 &= 0.5P + 7.5w_o\end{aligned}$$

$$(f_b)_{\max} = \frac{Mc}{I}$$

Where: $f_b = 1200$ psi

$$c = \frac{1}{2}h = \frac{1}{2}(12) = 6 \text{ in}$$

$$\begin{aligned}I &= \frac{bh^3}{12} = \frac{6(12^3)}{12} \\ &= 864 \text{ in}^4\end{aligned}$$

For moment at R_2 :

$$1200 = \frac{18w_o(6)(12)}{864}$$

$$w_o = 800 \text{ lb/ft}$$

For moment under P :

$$1200 = \frac{(3P - 9w_o)(6)(12)}{864}$$

$$14\,400 = 3P - 9w_o$$

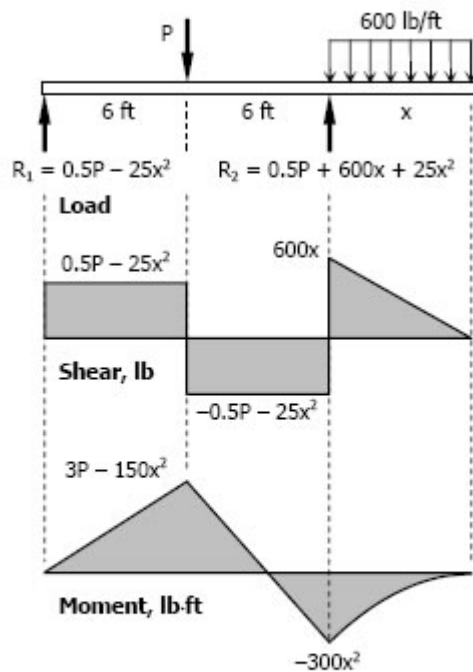
$$14\,400 = 3P - 9(800)$$

$$P = 7200 \text{ lb}$$

Problem 527

In Prob. 526, if the load on the overhang is 600 lb/ft and the overhang is x ft long, find the maximum values of P and x that can be used simultaneously.

Solution 527



$$\Sigma M_{R2} = 0$$

$$12R_1 + 600x(x/2) = 6P$$

$$R_1 = 0.5P - 25x^2$$

$$12R_2 = 6P + 600x(12 + \frac{1}{2}x)$$

$$R_2 = 0.5P + 600x + 25x^2$$

$$(f_b)_{\max} = \frac{Mc}{I}$$

Refer to Solution 526 for values of c and I .

For moment at R_2 :

$$1200 = \frac{(300x^2)(6)(12)}{864}$$

$$x^2 = 48$$

$$x = 6.93 \text{ ft}$$

For moment under P :

$$1200 = \frac{(3P - 150x^2)(6)(12)}{864}$$

$$14\,400 = 3P - 150x^2$$

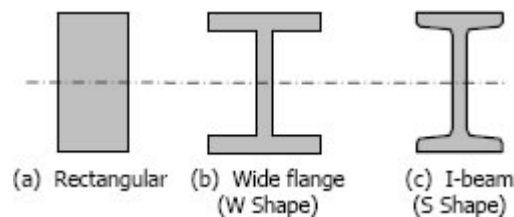
$$14\,400 = 3P - 150(6.93^2)$$

$$P = 7\,201.245 \text{ lb}$$

Economic Sections

From the flexure formula $f_b = My / I$, it can be seen that the bending stress at the neutral axis, where $y = 0$, is zero and increases linearly outwards. This means that for a rectangular or circular section a large portion of the cross section near the middle section is understressed.

For steel beams or composite beams, instead of adopting the rectangular shape, the area may be arranged so as to give more area on the outer fiber and maintaining the same overall depth, and saving a lot of weight.



When using a wide flange or I-beam section for long beams, the compression flanges tend to buckle horizontally sidewise. This buckling is a column effect, which may be prevented by providing lateral support such as a floor system so that the full allowable stresses may be used, otherwise the stress should be reduced. The reduction of stresses for these beams will be discussed in steel design. In selecting a structural section to be used as a beam, the resisting moment must be equal or greater than the applied bending moment. Note: $(f_b)_{\max} = M/S$.

$$S_{\text{required}} \geq S_{\text{live-load}} \text{ OR } S_{\text{required}} \geq \frac{M_{\text{live-load}}}{(f_b)_{\max}}$$

The equation above indicates that the required section modulus of the beam must be equal or greater than the ratio of bending moment to the maximum allowable stress. A check that includes the weight of the selected beam is necessary to complete the calculation. In checking, the beams resisting moment must be equal or greater than the sum of the live-load moment caused by the applied loads and the dead-load moment caused by dead weight of the beam.

$$M_{\text{resisting}} \geq M_{\text{live-load}} + M_{\text{dead-load}}$$

Dividing both sides of the above equation by $(f_b)_{\max}$, we obtain the checking equation

$$S_{\text{resisting}} \geq S_{\text{live-load}} + S_{\text{dead-load}}$$

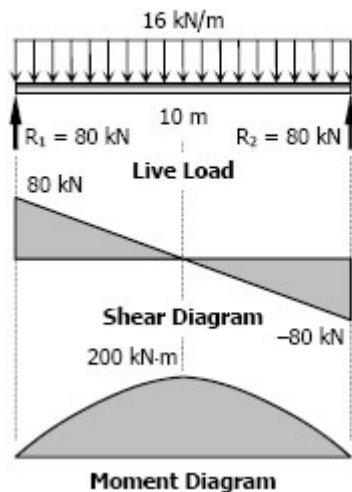
Assume that the beams in the following problems are properly braced against lateral deflection. Be sure to include the weight of the beam itself.

Solved Problems in Economic Sections

Problem 529

A 10-m beam simply supported at the ends carries a uniformly distributed load of 16 kN/m over its entire length. What is the lightest W shape beam that will not exceed a flexural stress of 120 MPa? What is the actual maximum stress in the beam selected?

Solution 529



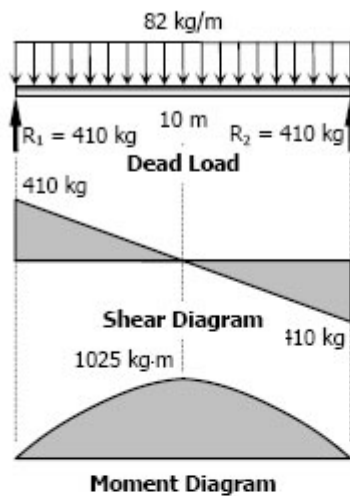
$$S_{\text{required}} \geq \frac{M_{\text{live-load}}}{(f_b)_{\text{max}}} \geq \frac{200(1000^2)}{120}$$

$$S_{\text{required}} \geq 1,666,666.67 \text{ mm}^3$$

$$S_{\text{required}} \geq 1\,666.67 \times 10^3 \text{ mm}^3$$

Starting at the bottom of Appendix B, Table B-2 Properties of Wide-Flange Sections (W Shapes): SI Units, of text book, the following are the first to exceed the S above:

Designation	Section Modulus
W250 × 149	$1\,840 \times 10^3 \text{ mm}^3$
W310 × 118	$1\,750 \times 10^3 \text{ mm}^3$
W360 × 101	$1\,690 \times 10^3 \text{ mm}^3$
W410 × 100	$1\,920 \times 10^3 \text{ mm}^3$



W460 × 89	$1\,770 \times 10^3 \text{ mm}^3$
W530 × 85	$1\,810 \times 10^3 \text{ mm}^3$
W610 × 82	$1\,870 \times 10^3 \text{ mm}^3$
W690 × 125	$3\,500 \times 10^3 \text{ mm}^3$

Use the lightest section W610 × 82

Checking:

$$S_{\text{resisting}} \geq S_{\text{live-load}} + S_{\text{dead-load}}$$

$$S_{\text{live-load}} = 1\,666.67 \times 10^3 \text{ mm}^3$$

$$S_{\text{dead-load}} = \frac{1025(9.81)(1000)}{120} = 83.79 \times 10^3 \text{ mm}^3$$

$$\begin{aligned} S_{\text{live-load}} + S_{\text{dead-load}} &= (1\,666.67 \times 10^3) + (83.79 \times 10^3) \\ &= 1\,750.46 \times 10^3 \text{ mm}^3 \end{aligned}$$

The resisting S of W610 × 82 is $1\,870 \times 10^3 \text{ mm}^3$, the S due to live-load and dead-load is only $1\,750.46 \times 10^3 \text{ mm}^3$, therefore, the chosen section is sufficient to resist the combined dead-load and live-load.

Actual bending moment due to dead and live loads:

$$\begin{aligned} M &= M_{\text{live-load}} + M_{\text{dead-load}} \\ M &= 200 + 1025(9.81/1000) \\ M &= 210.06 \text{ kN-m} \end{aligned}$$

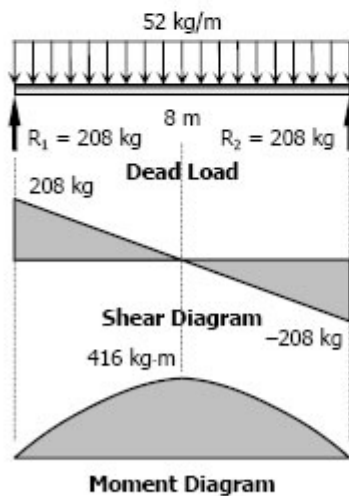
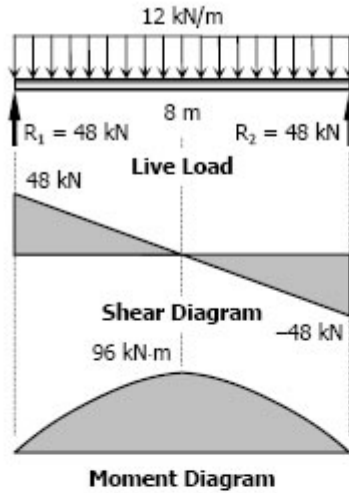
Actual stress:

$$\begin{aligned} (f_b)_{\text{max}} &= \frac{M}{S} \\ &= \frac{210.06(1000^2)}{1870 \times 10^3} \\ &= 112.33 \text{ MPa} \end{aligned}$$

Problem 530

Repeat Prob. 529 if the distributed load is 12 kN/m and the length of the beam is 8 m.

Solution 530



$$S_{\text{required}} \geq \frac{M_{\text{live-load}}}{(f_b)_{\text{max}}} \geq \frac{96(1000^2)}{120}$$

$$S_{\text{required}} \geq 800 \times 10^3 \text{ mm}^3$$

From Appendix B, Table B-2 Properties of Wide-Flange Sections (W Shapes): SI Units, of text book:

Designation	Section Modulus
W200 × 86	$853 \times 10^3 \text{ mm}^3$
W250 × 67	$806 \times 10^3 \text{ mm}^3$
W310 × 60	$849 \times 10^3 \text{ mm}^3$
W360 × 57	$897 \times 10^3 \text{ mm}^3$
W410 × 54	$924 \times 10^3 \text{ mm}^3$
W460 × 52	$943 \times 10^3 \text{ mm}^3$

Use the lightest section W460 × 60

Checking:

$$S_{\text{resisting}} \geq S_{\text{live-load}} + S_{\text{dead-load}}$$

$$S_{\text{live-load}} = 800 \times 10^3 \text{ mm}^3$$

$$S_{\text{dead-load}} = \frac{416(9.81)(1000)}{120} = 34 \times 10^3 \text{ mm}^3$$

$$S_{\text{live-load}} + S_{\text{dead-load}} = (800 \times 10^3) + (34 \times 10^3) = 834 \times 10^3 \text{ mm}^3$$

$$(943 \times 10^3 \text{ mm}^3) > (834 \times 10^3 \text{ mm}^3) \text{ (ok!)}$$

Actual bending moment:

$$M = M_{\text{live-load}} + M_{\text{dead-load}}$$

$$M = 96 + 416(9.81/1000) = 100.08 \text{ kN·m}$$

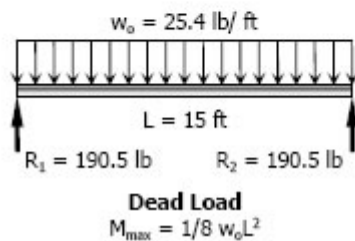
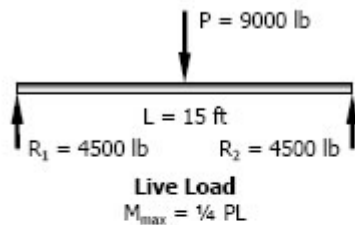
Actual stress:

$$(f_b)_{\text{max}} = \frac{M}{S} = \frac{100.08(1000^2)}{943 \times 10^3}$$

$$(f_b)_{\text{max}} = 106.13 \text{ MPa}$$

Problem 531

A 15-ft beam simply supported at the ends carries a concentrated load of 9000 lb at midspan. Select the lightest S section that can be employed using an allowable stress of 18 ksi. What is the actual maximum stress in the beam selected?

Solution 531

$$S_{\text{required}} \geq \frac{M_{\text{live-load}}}{(f_b)_{\max}} \geq \frac{\frac{1}{4}(9000)(15)(12)}{18000}$$

$$S_{\text{required}} \geq 22.5 \text{ in}^3$$

From Appendix B, Table B-8 Properties of I-Beam Sections (S Shapes): US Customary Units, of text book:

Use S10 × 25.4 with $S = 24.7 \text{ in}^3$

Checking:

$$S_{\text{resisting}} \geq S_{\text{live-load}} + S_{\text{dead-load}}$$

$$S_{\text{live-load}} = 22.5 \text{ in}^3$$

$$S_{\text{dead-load}} = \frac{\frac{1}{8}(25.4)(15^2)(12)}{18000}$$

$$= 0.48 \text{ in}^3$$

$$S_{\text{live-load}} + S_{\text{dead-load}} = 22.5 + 0.48$$

$$= 22.98 \text{ in}^3$$

$$(S_{\text{resisting}} = 24.7 \text{ in}^3) > 22.98 \text{ in}^3 \text{ (ok!)}$$

Actual bending moment:

$$M = M_{\text{live-load}} + M_{\text{dead-load}}$$

$$M = \frac{1}{4} PL + \frac{1}{8} w_0 L^2$$

$$M = \frac{1}{4} (9000)(15) + \frac{1}{8} (25.4)(15^2)$$

$$M = 34,464.38 \text{ lb-ft}$$

Actual stress:

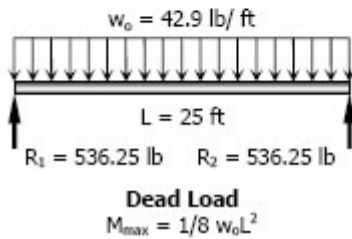
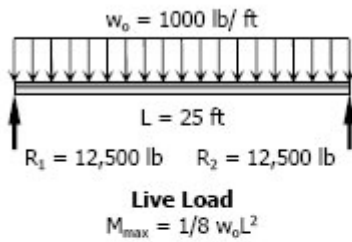
$$(f_b)_{\max} = \frac{M}{S} = \frac{34,464.38(12)}{24.7}$$

$$(f_b)_{\max} = 16,743.83 \text{ psi}$$

$$(f_b)_{\max} = 16.74 \text{ ksi}$$

Problem 532

A beam simply supported at the ends of a 25-ft span carries a uniformly distributed load of 1000 lb/ft over its entire length. Select the lightest S section that can be used if the allowable stress is 20 ksi. What is the actual maximum stress in the beam selected?

Solution 532

$$S_{\text{required}} \geq \frac{M}{(f_b)_{\max}} \geq \frac{\frac{1}{8}(1000)(25^2)(12)}{20000}$$

$$S_{\text{required}} \geq 46.875 \text{ in}^3$$

From Appendix B, Table B-8 Properties of I-Beam Sections (S Shapes): US Customary Units, of text book:

Use S15 × 42.9 with $S = 59.6 \text{ in}^3$

Checking:

$$S_{\text{resisting}} \geq S_{\text{live-load}} + S_{\text{dead-load}}$$

$$S_{\text{live-load}} = 46.875 \text{ in}^3$$

$$S_{\text{dead-load}} = \frac{\frac{1}{8}(42.9)(25^2)(12)}{20000}$$

$$= 2.011 \text{ in}^3$$

$$S_{\text{live-load}} + S_{\text{dead-load}} = 46.875 + 2.011$$

$$= 48.886 \text{ in}^3$$

$$(S_{\text{resisting}} = 59.6 \text{ in}^3) > 48.886 \text{ in}^3 \text{ (ok!)}$$

Actual bending moment:

$$M = M_{\text{live-load}} + M_{\text{dead-load}}$$

$$M = \left(\frac{1}{8} w_o L^2\right)_{\text{live-load}} + \left(\frac{1}{8} w_o L^2\right)_{\text{dead-load}}$$

$$M = \frac{1}{8} (1000)(25^2) + \frac{1}{8} (42.9)(25^2)$$

$$M = 81,476.56 \text{ lb-ft}$$

Actual stress:

$$(f_b)_{\max} = \frac{M}{S}$$

$$(f_b)_{\max} = \frac{81,476.56(12)}{59.6}$$

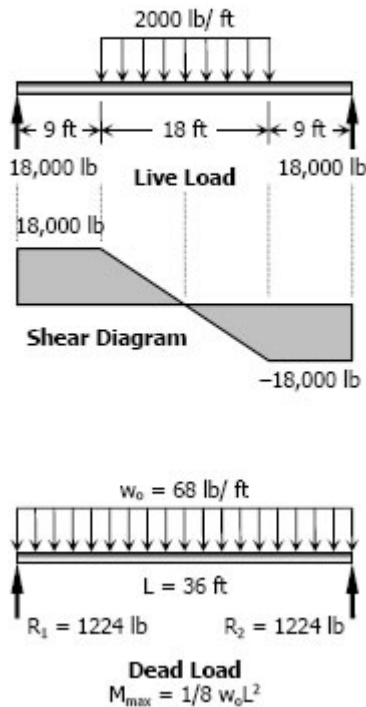
$$(f_b)_{\max} = 16,404.68 \text{ psi}$$

$$(f_b)_{\max} = 16.4 \text{ ksi}$$

Problem 533

A beam simply supported on a 36-ft span carries a uniformly distributed load of 2000 lb/ft over the middle 18 ft. Using an allowable stress of 20 ksi, determine the lightest suitable W shape beam. What is the actual maximum stress in the selected beam?

Solution 533



$$M_{\text{live-load}} = 18,000(9) + \frac{1}{2}(9)(18,000)$$

$$M_{\text{live-load}} = 243,000 \text{ lb-ft}$$

$$S_{\text{required}} \geq \frac{M_{\text{live-load}}}{(f_b)_{\text{max}}} \geq \frac{243,000(12)}{20,000}$$

$$S_{\text{required}} \geq 145.8 \text{ in}^3$$

From Appendix B, Table B-7 Properties of Wide-Flange Sections (W Shapes): US Customary Units, of text book:

Designation	Section Modulus
W12 × 120	163 in ³
W14 × 99	157 in ³
W16 × 89	155 in ³
W18 × 76	146 in ³
W21 × 73	151 in ³
W24 × 68	154 in ³

Use W24 × 68 with $S = 154 \text{ in}^3$

Checking:

$$S_{\text{resisting}} \geq S_{\text{live-load}} + S_{\text{dead-load}}$$

$$S_{\text{live-load}} = 145.8 \text{ in}^3$$

$$S_{\text{dead-load}} = \frac{\frac{1}{8}(68)(36^2)(12)}{20,000}$$

$$= 6.61 \text{ in}^3$$

$$S_{\text{live-load}} + S_{\text{dead-load}} = 145.8 + 6.61 = 152.41 \text{ in}^3$$

$$(S_{\text{resisting}} = 154 \text{ in}^3) > 152.41 \text{ in}^3 \text{ (ok!)}$$

Actual bending moment:

$$M = M_{\text{live-load}} + M_{\text{dead-load}}$$

$$M = 243,000 + \frac{1}{8}(68)(36^2)$$

$$M = 254,016 \text{ lb-ft}$$

Actual stress:

$$(f_b)_{\text{max}} = \frac{M}{S} = \frac{254,016(12)}{154}$$

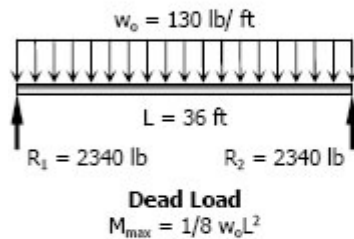
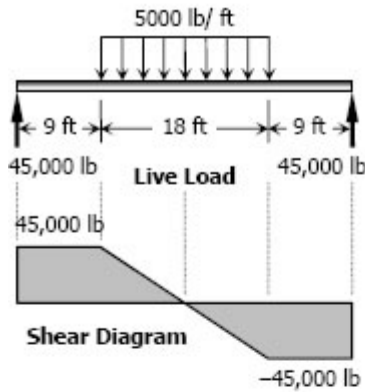
$$(f_b)_{\text{max}} = 19,793.45 \text{ psi}$$

$$(f_b)_{\text{max}} = 19.79 \text{ ksi}$$

Problem 534

Repeat Prob. 533 if the uniformly distributed load is changed to 5000 lb/ft.

Solution 534



$$M_{\text{live-load}} = 45,000(9) + \frac{1}{2}(9)(45,000)$$

$$M_{\text{live-load}} = 607,500 \text{ lb}\cdot\text{ft}$$

$$S_{\text{required}} \geq \frac{M_{\text{live-load}}}{(f_b)_{\max}} \geq \frac{607,500(12)}{20,000}$$

$$S_{\text{required}} \geq 364.5 \text{ in}^3$$

From Appendix B, Table B-7 Properties of Wide-Flange Sections (W Shapes): US Customary Units, of text book:

Designation	Section Modulus
W12 $\times$ 279	393 in ³
W14 $\times$ 233	375 in ³
W24 $\times$ 146	371 in ³
W27 $\times$ 146	411 in ³
W30 $\times$ 132	380 in ³
W33 $\times$ 130	406 in ³

Use W33 $\times$ 130 with $S = 406 \text{ in}^3$

Checking:

$$S_{\text{resisting}} \geq S_{\text{live-load}} + S_{\text{dead-load}}$$

$$S_{\text{live-load}} = 364.5 \text{ in}^3$$

$$S_{\text{dead-load}} = \frac{\frac{1}{8}(130)(36^2)(12)}{20,000}$$

$$= 12.636 \text{ in}^3$$

$$S_{\text{live-load}} + S_{\text{dead-load}} = 364.5 + 12.636 = 377.136 \text{ in}^3$$

$$(S_{\text{resisting}} = 406 \text{ in}^3) > 377.136 \text{ in}^3 \text{ (ok!)}$$

Actual bending moment:

$$M = M_{\text{live-load}} + M_{\text{dead-load}}$$

$$M = 607,500 + \frac{1}{8}(130)(36^2)$$

$$M = 628,560 \text{ lb}\cdot\text{ft}$$

Actual stress:

$$(f_b)_{\max} = \frac{M}{S} = \frac{628,560(12)}{406}$$

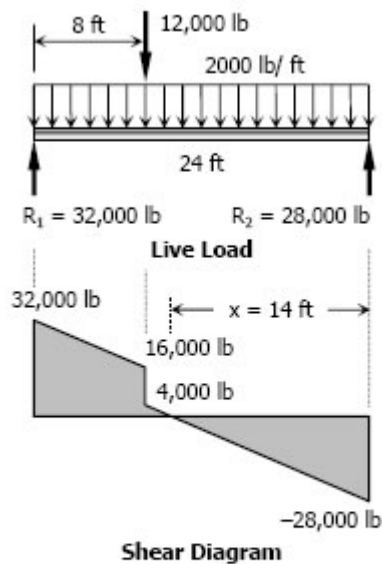
$$(f_b)_{\max} = 18,578.13 \text{ psi}$$

$$(f_b)_{\max} = 18.58 \text{ ksi}$$

Problem 535

A simply supported beam 24 ft long carries a uniformly distributed load of 2000 lb/ft over its entire length and a concentrated load of 12 kips at 8 ft from left end. If the allowable stress is 18 ksi, select the lightest suitable W shape. What is the actual maximum stress in the selected beam?

Solution 535



$$\Sigma M_{R2} = 0$$

$$24R_1 = 16(12\,000) + 2000(24)(12)$$

$$R_1 = 32,000 \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$24R_2 = 8(12\,000) + 2000(24)(12)$$

$$R_2 = 28,000 \text{ lb}$$

$$\frac{x}{28000} = \frac{16 - x}{4000}$$

$$4000x = 28\,000(16 - x)$$

$$32\,000x = 448\,000$$

$$x = 14 \text{ ft}$$

$$M_{\text{live-load}} = \frac{1}{2}x(28\,000) = \frac{1}{2}(14)(28\,000)$$

$$M_{\text{live-load}} = 196,000 \text{ lb}\cdot\text{ft}$$

$$S_{\text{required}} \geq \frac{M_{\text{live-load}}}{(f_b)_{\text{max}}} \geq \frac{196\,000(12)}{18\,000}$$

$$S_{\text{required}} \geq 130.67 \text{ in}^3$$

From Appendix B, Table B-7 Properties of Wide-Flange Sections (W Shapes): US Customary Units, of text book:

Designation	Section Modulus
W12 × 96	131 in ³
W14 × 90	143 in ³
W16 × 77	134 in ³
W18 × 76	146 in ³
W21 × 68	140 in ³
W24 × 62	131 in ³

Try W24 × 62 with $S = 131 \text{ in}^3$

Checking:

$$S_{\text{resisting}} \geq S_{\text{live-load}} + S_{\text{dead-load}}$$

$$S_{\text{live-load}} = 130.67 \text{ in}^3$$

$$\frac{y}{2} = \frac{744}{12}; y = 124 \text{ lb}$$

At critical section:

$$\begin{aligned} M_{\text{dead-load}} &= \frac{1}{2} (744 + 124)(10) \\ &= 4340 \text{ lb}\cdot\text{ft} \end{aligned}$$

$$S_{\text{dead-load}} = \frac{4340(12)}{18000} = 2.89 \text{ in}^3$$

$$\begin{aligned} S_{\text{live-load}} + S_{\text{dead-load}} &= 130.67 + 2.89 \\ &= 133.56 \text{ in}^3 \end{aligned}$$

$$(S_{\text{resisting}} = 131 \text{ in}^3) < 133.56 \text{ in}^3 \text{ (not ok!)}$$

Try W21 × 68 with $S = 140 \text{ in}^3$

Checking:

$$S_{\text{resisting}} \geq S_{\text{live-load}} + S_{\text{dead-load}}$$

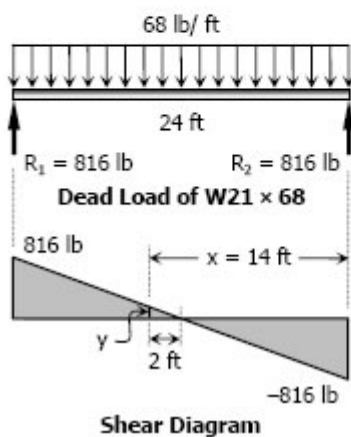
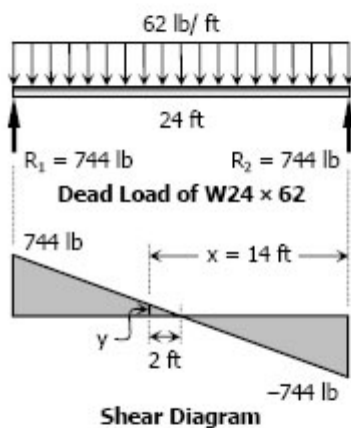
$$S_{\text{live-load}} = 130.67 \text{ in}^3$$

$$\frac{y}{2} = \frac{816}{12}; y = 136 \text{ lb}$$

At critical section:

$$\begin{aligned} M_{\text{dead-load}} &= \frac{1}{2} (816 + 136)(10) \\ &= 4760 \text{ lb}\cdot\text{ft} \end{aligned}$$

$$S_{\text{dead-load}} = \frac{4760(12)}{18000} = 3.17 \text{ in}^3$$



$$\begin{aligned}
 S_{\text{live-load}} + S_{\text{dead-load}} &= 130.67 + 3.17 \\
 &= 133.84 \text{ in}^3 \\
 (S_{\text{resisting}} = 140 \text{ in}^3) &> 133.84 \text{ in}^3 \text{ (ok!)}
 \end{aligned}$$

Use W21 x 68

Actual bending moment:

$$\begin{aligned}
 M &= M_{\text{live-load}} + M_{\text{dead-load}} = 196,000 + 4,760 \\
 M &= 200,760 \text{ lb-ft}
 \end{aligned}$$

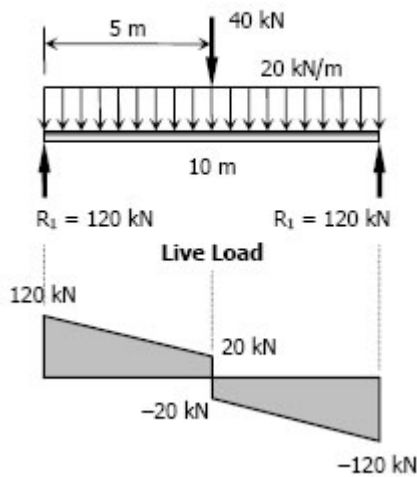
Actual stress:

$$\begin{aligned}
 (f_b)_{\text{max}} &= \frac{M}{S} = \frac{200,760(12)}{140} \\
 (f_b)_{\text{max}} &= 17,208 \text{ psi} \\
 (f_b)_{\text{max}} &= 17.208 \text{ ksi}
 \end{aligned}$$

Problem 536

A simply supported beam 10 m long carries a uniformly distributed load of 20 kN/m over its entire length and a concentrated load of 40 kN at midspan. If the allowable stress is 120 MPa, determine the lightest W shape beam that can be used.

Solution 536



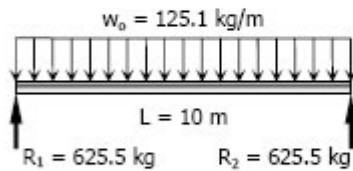
$$\begin{aligned}
 M_{\text{live-load}} &= \frac{1}{2} (120 + 20)(5) \\
 &= 350 \text{ kN-m}
 \end{aligned}$$

$$\begin{aligned}
 S_{\text{required}} &\geq \frac{M_{\text{live-load}}}{(f_b)_{\text{max}}} \geq \frac{350(1000^2)}{120} \\
 &\geq 2,916.67 \times 10^3 \text{ mm}^3
 \end{aligned}$$

From Appendix B, Table B-2
Properties of Wide-Flange Sections
(W Shapes): SI Units, of text book:

Designation	Section Modulus
W310 x 202	$3,050 \times 10^3 \text{ mm}^3$
W360 x 179	$3,120 \times 10^3 \text{ mm}^3$
W460 x 144	$3,080 \times 10^3 \text{ mm}^3$
W530 x 138	$3,140 \times 10^3 \text{ mm}^3$
W610 x 125	$3,220 \times 10^3 \text{ mm}^3$
W690 x 125	$3,500 \times 10^3 \text{ mm}^3$

W610 × 125 has a theoretical mass of 125.1 kg/m while W690 × 125 has a theoretical mass of 125.6 kg/m. Thus, use W610 × 125 with $S = 3,220 \times 10^3 \text{ mm}^3$.



Dead Load
 $M_{\max} = 1/8 w_o L^2$

Checking:

$$S_{\text{resisting}} \geq S_{\text{live-load}} + S_{\text{dead-load}}$$

$$S_{\text{live-load}} = 2,916.67 \times 10^3 \text{ mm}^3$$

$$S_{\text{dead-load}} = \frac{\frac{1}{8}(125.1)(9.81)(10^2)(1000)}{120}$$

$$= 127.84 \times 10^3 \text{ mm}^3$$

$$S_{\text{live-load}} + S_{\text{dead-load}}$$

$$= (2,916.67 \times 10^3) + (127.84 \times 10^3)$$

$$= 3,044.51 \times 10^3 \text{ mm}^3$$

$$(S_{\text{resisting}} = 3,220 \times 10^3 \text{ mm}^3) > 3,044.4 \times 10^3 \text{ mm}^3 \text{ (ok!)}$$

Floor Framing

In floor framing, the subfloor is supported by light beams called floor joists or simply joists which in turn supported by heavier beams called girders then girders pass the load to columns. Typically, joist act as simply supported beam carrying a uniform load of magnitude p over an area of sL ,

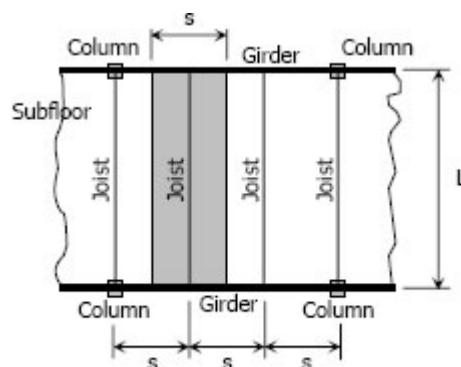
where

p = floor load per unit area

L = length (or span) of joist

s = center to center spacing of joists and

$w_o = sp$ = intensity of distributed load in joist.



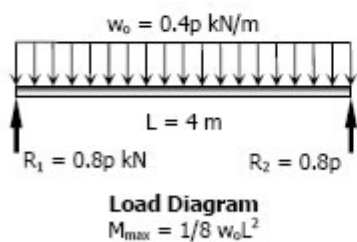
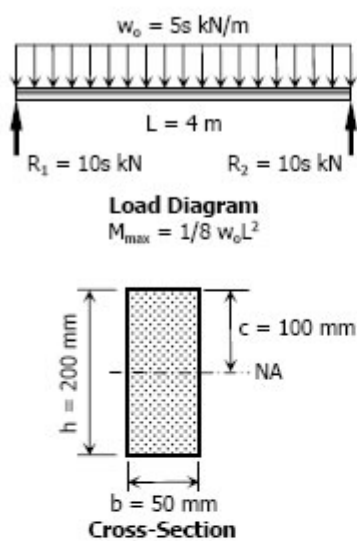
Typical Floor Framing Plan

Solved Problems in Floor Framing

Problem 538

Floor joists 50 mm wide by 200 mm high, simply supported on a 4-m span, carry a floor loaded at 5 kN/m^2 . Compute the center-line spacing between joists to develop a bending stress of 8 MPa. What safe floor load could be carried on a center-line spacing of 0.40 m?

Solution 538



Part 1:

$$(f_b)_{\max} = \frac{Mc}{I}$$

where: $(f_b)_{\max} = 8 \text{ MPa}$

$$M = \frac{1}{8} (5s)(4^2)$$

$$= 10s \text{ kN}\cdot\text{m}$$

$$c = h/2$$

$$= 200/2$$

$$= 100 \text{ mm}$$

$$I = \frac{bh^3}{12}$$

$$= \frac{50(200^3)}{12}$$

$$= 33.33 \times 10^6 \text{ mm}^4$$

$$8 = \frac{10s(100)(1000^2)}{33.33 \times 10^6}$$

$$s = 0.267 \text{ m}$$

Part 2:

$$(f_b)_{\max} = \frac{Mc}{I}$$

where: $M = \frac{1}{8} w_o L^2$

$$= \frac{1}{8} (0.4p)(4^2)$$

$$= 0.8p$$

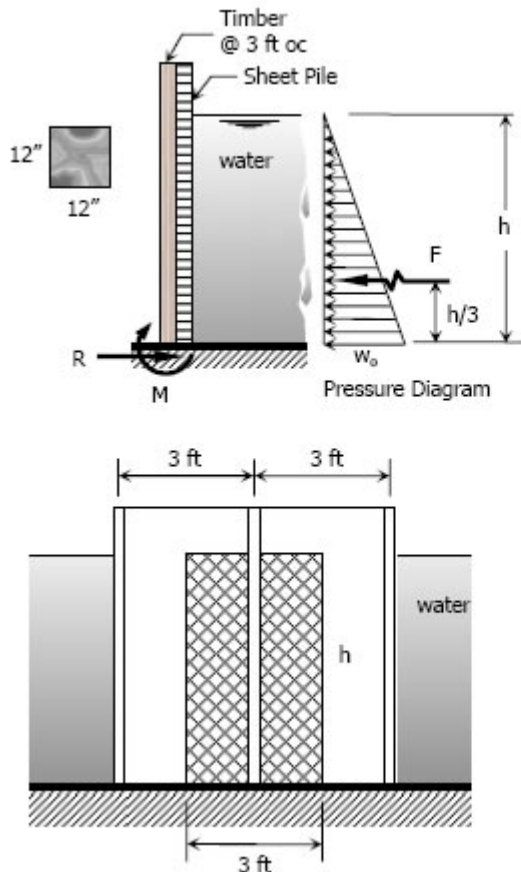
$$8 = \frac{0.8p(100)(1000^2)}{33.33 \times 10^6}$$

$$p = 3.33 \text{ kN/m}^2$$

Problem 539

Timbers 12 inches by 12 inches, spaced 3 feet apart on centers, are driven into the ground and act as cantilever beams to back-up the sheet piling of a coffer dam. What is the maximum safe height of water behind the dam if water weighs $\gamma = 62.5 \text{ lb/ft}^3$ and $(f_b)_{\max} = 1200 \text{ psi}$?

Solution 539



$$w_o = 62.5h \text{ lb/ft}^2$$

$$F = \frac{1}{2} w_o h (3)$$

$$F = \frac{1}{2} (62.5h) h$$

$$F = 93.75h^2 \text{ lb}$$

$$M = \left(\frac{1}{3} h\right) F$$

$$M = \frac{1}{3} h (93.75h^2)$$

$$M = 31.25h^3 \text{ lb}\cdot\text{ft}$$

$$(f_b)_{\max} = \frac{Mc}{I}$$

$$1200 = \frac{31.25h^3 (12)(12/2)}{\frac{12(12^3)}{12}}$$

$$h = 9.73 \text{ ft}$$

Problem 540

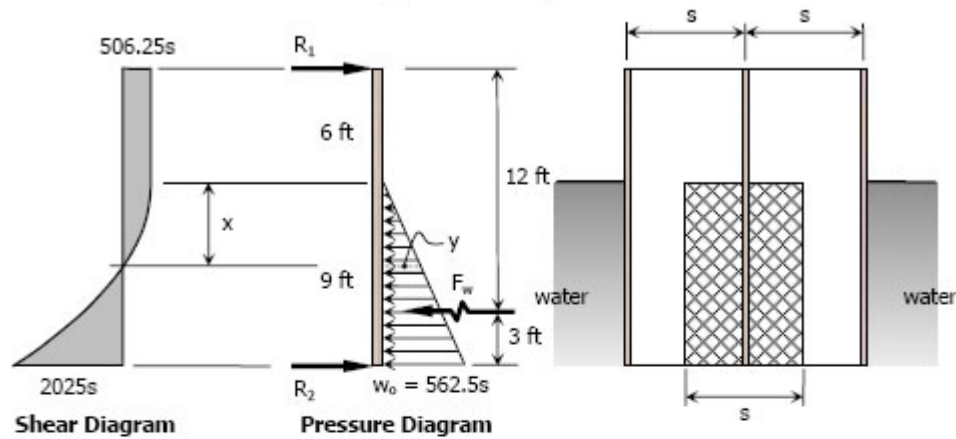
Timbers 8 inches wide by 12 inches deep and 15 feet long, supported at top and bottom, back up a dam restraining water 9 feet deep. Water weighs $\gamma = 62.5 \text{ lb/ft}^3$. (a) Compute the center-line spacing of the timbers to cause $f_b = 1000 \text{ psi}$. (b) Will this spacing be safe if the maximum f_b , $(f_b)_{\max} = 1600 \text{ psi}$, and the water reaches its maximum depth of 15 ft?

Solution 540

Part (a)

$$w_o = 62.5(9s)$$

$$w_o = 562.5s \text{ lb/ft}$$



$$F_w = \frac{1}{2} w_o(9)$$

$$F_w = \frac{1}{2} (562.5s)(9)$$

$$F_w = 2531.25s \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$15R_2 = 12F_w$$

$$15R_2 = 12(2531.25s)$$

$$R_2 = 2025s$$

$$\Sigma M_{R2} = 0$$

$$15R_1 = 3F_w$$

$$15R_1 = 3(2531.25s)$$

$$R_1 = 506.25s$$

Location of Maximum Moment

$$\frac{y}{x} = \frac{562.5s}{9}$$

$$y = 62.5s$$

$$506.25s - \frac{1}{2} xy = 0$$

$$506.25s - \frac{1}{2} x (62.5sx) = 0$$

$$x^2 = 16.2$$

$$x = 4.02 \text{ ft}$$

Maximum Moment

$$M = (506.25s)(6) + \frac{2}{3}(s)(506.25s)$$

$$M = 3037.5s + 337.5(4.02s)$$

$$M = 4394.25s$$

Required Spacing

$$(f_b)_{\max} = \frac{Mc}{I}$$

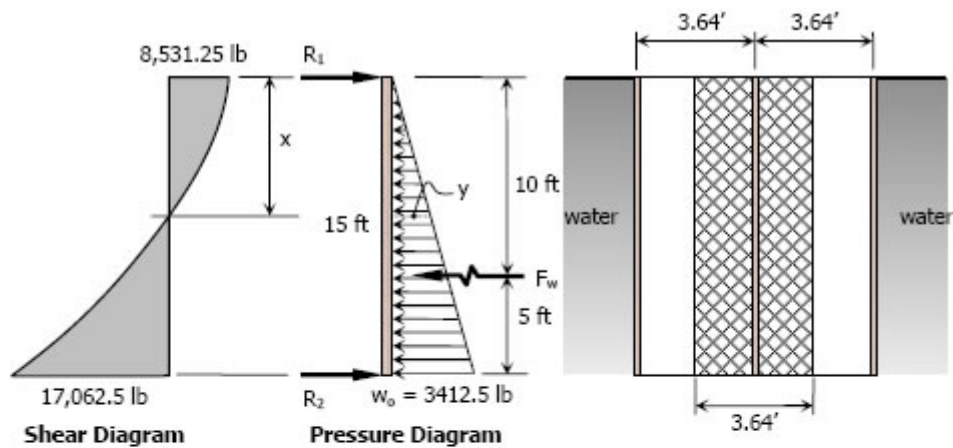
$$1000 = \frac{4394.25s(12)(12/2)}{\frac{8(12^3)}{12}}$$

$$s = 3.64 \text{ ft}$$

Part (b)

$$w_o = 62.5(15)(3.64)$$

$$w_o = 3412.5 \text{ lb/ft}$$



$$F_w = \frac{1}{2} w_o(15)$$

$$F_w = \frac{1}{2} (3412.5)(15)$$

$$F_w = 25,593.75 \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$15R_2 = 10F_w$$

$$15R_2 = 10(25,593.75)$$

$$R_2 = 17,062.5 \text{ lb}$$

$$\begin{aligned}\Sigma M_{R2} &= 0 \\ 15R_1 &= 5F_w \\ 15R_1 &= 5(25\,593.75) \\ R_1 &= 8,531.25 \text{ lb}\end{aligned}$$

Location of Maximum Moment (Shear = 0)

$$\begin{aligned}\frac{y}{x} &= \frac{3412.5}{15} \\ y &= 227.5x\end{aligned}$$

$$\begin{aligned}8531.25 - \frac{1}{2}xy &= 0 \\ 8531.25 - \frac{1}{2}x(227.5x) &= 0 \\ x^2 &= 75 \\ x &= 8.66 \text{ ft}\end{aligned}$$

Maximum Moment

$$\begin{aligned}M &= \frac{2}{3}x(8531.25) \\ M &= \frac{2}{3}(8.66)(8531.25) \\ M &= 49,255.19 \text{ lb}\cdot\text{ft}\end{aligned}$$

Actual Stress

$$\begin{aligned}f_b &= \frac{Mc}{I} \\ f_b &= \frac{(49\,255.19)(12)(12/2)}{\frac{8(12^3)}{12}} \\ f_b &= 3,078.36 \text{ psi} > 1600 \text{ psi}\end{aligned}$$

Therefore, the 3.64 ft spacing of timbers is **not safe** when water reaches its maximum depth of 15 ft.

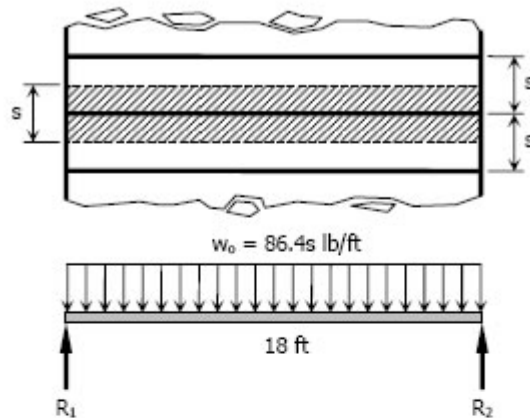
Problem 541

The 18-ft long floor beams in a building are simply supported at their ends and carry a floor load of 0.6 lb/in^2 . If the beams have $W10 \times 30$ sections, determine the center-line spacing using an allowable flexural stress of 18 ksi.

Solution 541

$$w_o = (0.6 \text{ lb/in}^2)(12 \text{ in/ft})^2(s \text{ ft})$$

$$w_o = 86.4s \text{ lb/ft}$$



$$R_1 = R_2 = \frac{1}{2} (86.4s)(18)$$

$$= 777.6s \text{ lb}$$

$$M_{\max} = 777.6s (9) - 86.4s (9)(4.5)$$

$$= 3499.2s \text{ lb-ft}$$

From Table B-7 in Appendix B of textbook:

Properties of Wide-Flange Sections (W-Shapes): US Customary Units.

Designation	Section Modulus, S
$W10 \times 30$	32.4 in^3

$$f_b = \frac{M}{S}$$

$$18,000 = \frac{3499.2s (12)}{32.4}$$

$$s = 13.9 \text{ ft}$$

Problem 542

Select the lightest W shape sections that can be used for the beams and girders in Illustrative Problem 537 of text book if the allowable flexural stress is 120 MPa. Neglect the weights of the members.

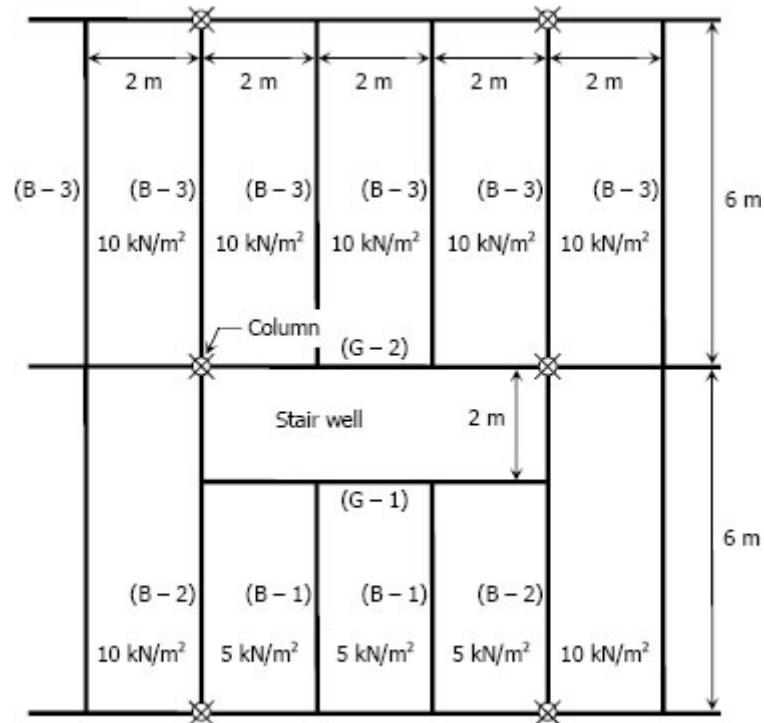


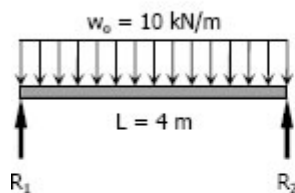
Figure in Illustrative Problem 537

Solution 542

For Beams (B - 1)

$$\begin{aligned}\text{Total Load, } W &= 5(2 \times 4) \\ &= 40 \text{ kN}\end{aligned}$$

$$\begin{aligned}\text{Distributed Load, } w_o &= W/L = 40/4 \\ &= 10 \text{ kN/m}\end{aligned}$$



$$\begin{aligned}R_1 &= R_2 = \frac{1}{2} W = \frac{1}{2} (40) \\ &= 20 \text{ kN}\end{aligned}$$

$$\begin{aligned}M_{\max} &= R_1(L/2) - 10(L/2)(L/4) \\ &= 20(4/2) - 10(4/2)(4/4) \\ &= 20 \text{ kN}\cdot\text{m}\end{aligned}$$

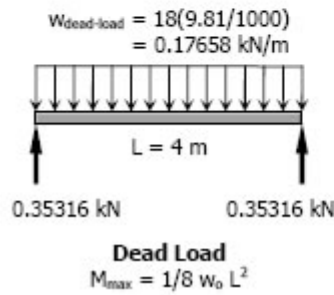
$$S_{\text{required}} = \frac{M}{f_b} = \frac{20(1000^2)}{120}$$

$$S_{\text{required}} = 166\,666.67 \text{ mm}^3$$

From Appendix B, Table B-2 Properties of Wide-Flange Sections (W Shapes): SI Units, of text book:

Designation	Section Modulus
W130 × 28	$167 \times 10^3 \text{ mm}^3$
W150 × 24	$168 \times 10^3 \text{ mm}^3$
W200 × 22	$194 \times 10^3 \text{ mm}^3$
W250 × 18	$179 \times 10^3 \text{ mm}^3$

Consider W250 × 18 with $S = 179 \times 10^3 \text{ mm}^3$



Checking:

$$S_{\text{resisting}} \geq S_{\text{live-load}} + S_{\text{dead-load}}$$

$$S_{\text{resisting}} = 179 \times 10^3 \text{ mm}^3$$

$$S_{\text{live-load}} = 166\,666.67 \text{ mm}^3$$

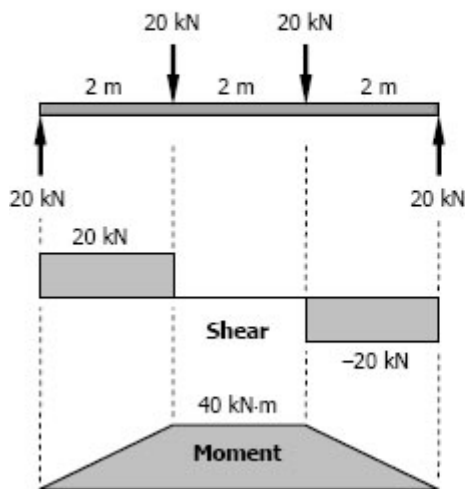
$$S_{\text{dead-load}} = \frac{M_{\text{dead-load}}}{f_b} = \frac{[\frac{1}{8}(0.17658)(4^2)]1000}{120}$$

$$= 2.943 \text{ mm}^3$$

$$179 \times 10^3 \geq 166\,666.67 + 2.943$$

$$179 \times 10^3 \geq 166.67 \times 10^3 \quad (\text{ok!})$$

Use W250 × 18 for B-1.



For Girder (G - 1)

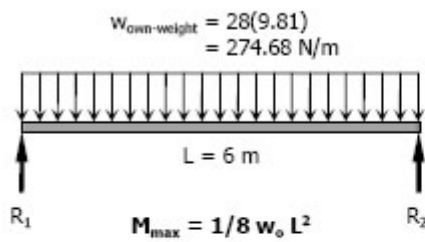
$$S_{\text{live-load}} = \frac{M}{f_b} = \frac{40(1000^2)}{120}$$

$$= 333.33 \times 10^3 \text{ mm}^3$$

From Appendix B, Table B-2 Properties of Wide-Flange Sections (W Shapes): SI Units, of text book:

Designation	Section Modulus
W200 × 36	$342 \times 10^3 \text{ mm}^3$
W250 × 33	$379 \times 10^3 \text{ mm}^3$
W310 × 28	$351 \times 10^3 \text{ mm}^3$

Consider W310 × 28 with
 $S = 351 \times 10^3 \text{ mm}^3$



Checking:

$$S_{\text{supplied}} \geq S_{\text{required}} + S_{\text{own-weight}}$$

$$S_{\text{supplied}} = 1790 \times 10^3 \text{ mm}^3$$

$$S_{\text{required}} = 333.33 \times 10^3 \text{ mm}^3$$

$$\begin{aligned}
 S_{\text{own-weight}} &= \frac{M_{\text{own-weight}}}{f_b} \\
 &= \frac{[\frac{1}{8}(274.68)(6^2)](1000)}{120} \\
 &= 10\,300.5 \text{ mm}^3
 \end{aligned}$$

$$1790 \times 10^3 \geq (333.33 \times 10^3) + 10\,300.5$$

$$1790 \times 10^3 \geq 343.63 \times 10^3 \quad (\text{ok!})$$

Use W310 $\times$ 28 for G-1.

For Beams (B - 2)

$$\Sigma M_{R2} = 0$$

$$6R_1 = 20(4) + 10(2)(5) + 15(4)(2)$$

$$R_1 = 50 \text{ kN}$$

$$\Sigma M_{R1} = 0$$

$$6R_2 = 20(2) + 10(2)(1) + 15(4)(4)$$

$$R_2 = 50 \text{ kN}$$

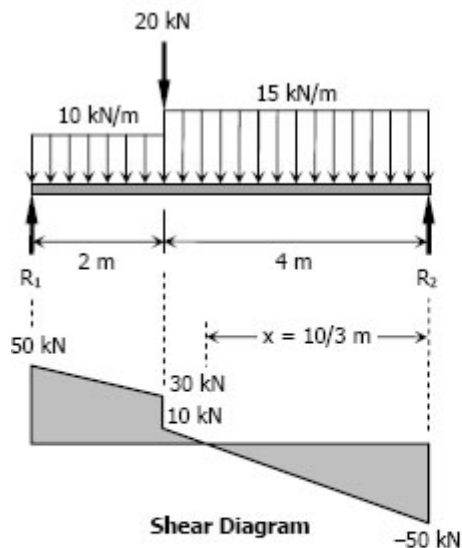
Location of Maximum Moment

$$\frac{x}{50} = \frac{4-x}{10}$$

$$10x = 200 - 50x$$

$$x = \frac{10}{3} \text{ m}$$

$$\begin{aligned}
 M_{\max} &= \frac{1}{2} \left(\frac{10}{3} \right) (50) \\
 &= \frac{250}{3} \text{ kN}\cdot\text{m}
 \end{aligned}$$

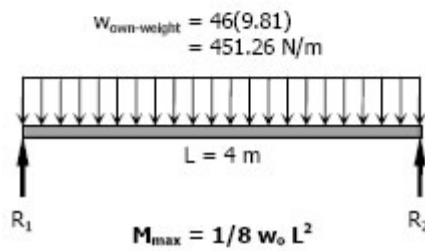


$$\begin{aligned}
 S_{\text{required}} &= \frac{M}{f_b} = \frac{\frac{250}{3}(1000^2)}{120} \\
 &= 695 \times 10^3 \text{ mm}^3
 \end{aligned}$$

From Appendix B, Table B-2 Properties of Wide Flange Sections (W Shapes): SI Units, of text book:

Designation	Section Modulus
W200 × 71	$709 \times 10^3 \text{ mm}^3$
W250 × 67	$806 \times 10^3 \text{ mm}^3$
W310 × 52	$747 \times 10^3 \text{ mm}^3$
W360 × 51	$796 \times 10^3 \text{ mm}^3$
W410 × 46	$773 \times 10^3 \text{ mm}^3$

Consider W410 × 46 with $S = 773 \times 10^3 \text{ mm}^3$



Checking:

$$S_{\text{supplied}} \geq S_{\text{required}} + S_{\text{own-weight}}$$

$$S_{\text{supplied}} = 773 \times 10^3 \text{ mm}^3$$

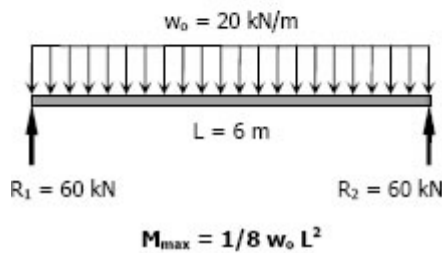
$$S_{\text{required}} = 695 \times 10^3 \text{ mm}^3$$

$$\begin{aligned}
 S_{\text{own-weight}} &= \frac{M_{\text{own-weight}}}{f_b} \\
 &= \frac{\frac{1}{8}(451.26)(4^2)(1000)}{120} \\
 &= 7521 \text{ mm}^3
 \end{aligned}$$

$$\begin{aligned}
 S_{\text{required}} + S_{\text{own-weight}} &= (695 \times 10^3) + 7521 \\
 &= 702\,521 \text{ mm}^3
 \end{aligned}$$

$$(S_{\text{supplied}} = 773 \times 10^3 \text{ mm}^3) > 702\,521 \text{ mm}^3 \quad (\text{ok!})$$

Use W410 × 46 for B-2



For Beams (B - 3)

$$\begin{aligned}
 M_{\text{max}} &= \frac{1}{8} (20)(6^2) \\
 &= 90 \text{ kN}\cdot\text{m}
 \end{aligned}$$

$$\begin{aligned}
 S_{\text{required}} &= \frac{M_{\text{max}}}{f_b} \\
 &= \frac{90(1000^2)}{120} \\
 &= 750 \times 10^3 \text{ mm}^3
 \end{aligned}$$

From Appendix B, Table B-2 Properties of Wide Flange Sections (W Shapes): SI Units, of text book:

Designation	Section Modulus
W200 × 86	$853 \times 10^3 \text{ mm}^3$
W250 × 67	$806 \times 10^3 \text{ mm}^3$
W310 × 60	$849 \times 10^3 \text{ mm}^3$
W360 × 51	$796 \times 10^3 \text{ mm}^3$
W410 × 46	$773 \times 10^3 \text{ mm}^3$

Consider W410 × 46 with $S = 773 \times 10^3 \text{ mm}^3$

From the Checking of B - 2

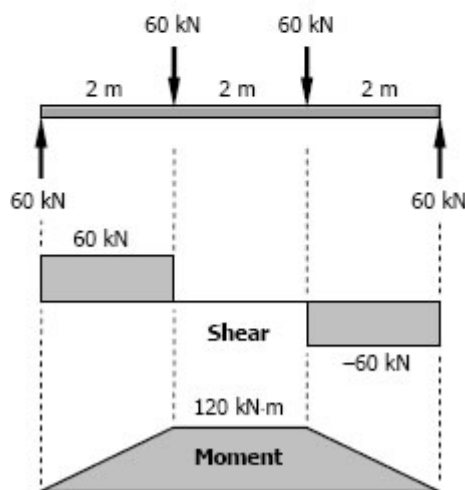
$$S_{\text{own-weight}} = 7521 \text{ mm}^3$$

$$S_{\text{required}} + S_{\text{own-weight}} = (750 \times 10^3) + 7521 \\ = 757\,521 \text{ mm}^3$$

$$(S_{\text{supplied}} = 773 \times 10^3 \text{ mm}^3) > 757\,521 \text{ mm}^3 \quad (\text{ok!})$$

Use W410 × 46 for B-3

This section is the same to B- 2



For Girders G - 2

$$S_{\text{required}} = \frac{M}{f_b} = \frac{120(1000^2)}{120} \\ = 1000 \times 10^3 \text{ mm}^3$$

From Appendix B, Table B-2 Properties of Wide-Flange Sections (W Shapes): SI Units, of text book:

Designation	Section Modulus
W250 × 89	$1100 \times 10^3 \text{ mm}^3$
W310 × 74	$1060 \times 10^3 \text{ mm}^3$
W360 × 64	$1030 \times 10^3 \text{ mm}^3$
W410 × 60	$1060 \times 10^3 \text{ mm}^3$
W460 × 60	$1120 \times 10^3 \text{ mm}^3$

There are two options, both exceeds the required S of $1000 \times 10^3 \text{ mm}^3$. One is W410 × 60 with theoretical mass of 59.5 kg/m and the other is W460 × 60 with

theoretical mass of 59.6 kg/m. For economic reason, we prefer W410 × 60.

Checking:

$$S_{\text{supplied}} \geq S_{\text{required}} + S_{\text{own-weight}}$$

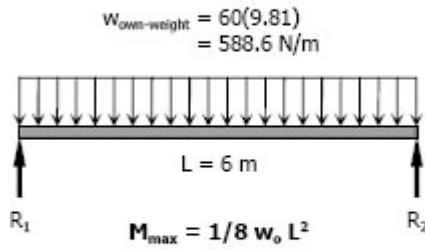
$$S_{\text{supplied}} = 1060 \times 10^3 \text{ mm}^3$$

$$S_{\text{required}} = 1000 \times 10^3 \text{ mm}^3$$

$$S_{\text{own-weight}} = \frac{M_{\text{own-weight}}}{f_b}$$

$$= \frac{\frac{1}{8}(588.6)(6^2)(1000)}{120}$$

$$= 22\,072.5 \text{ mm}^3$$



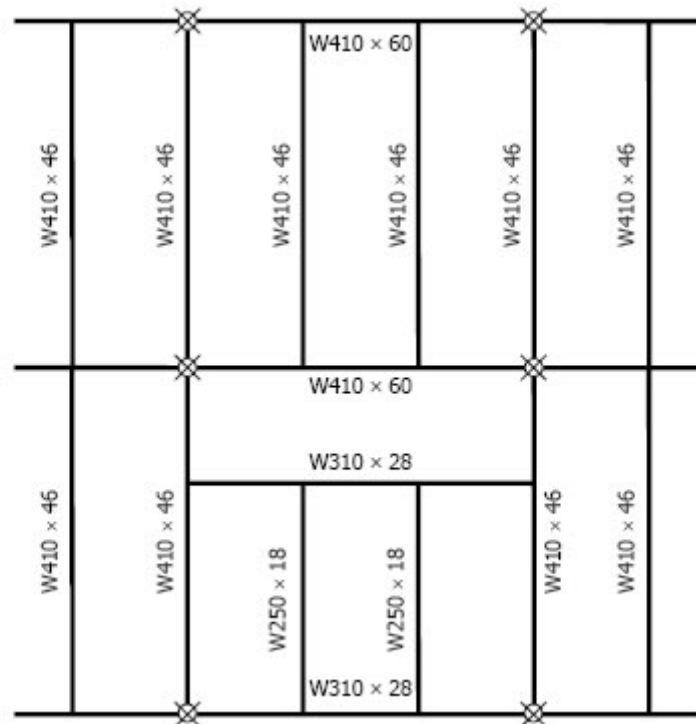
$$S_{\text{required}} + S_{\text{own-weight}} = (1000 \times 10^3) + 22\,072.5$$

$$= 1\,022\,072.5 \text{ mm}^3$$

$$(S_{\text{supplied}} = 1060 \times 10^3 \text{ mm}^3) > 1\,022\,072.5 \text{ mm}^3 \text{ (ok!)}$$

Use W410 × 60 for G-2

Summary:



Problem 543

A portion of the floor plan of a building is shown in Fig. P-543. The total loading (including live and dead loads) in each bay is as shown. Select the lightest suitable W if the allowable flexural stress is 120 MPa.

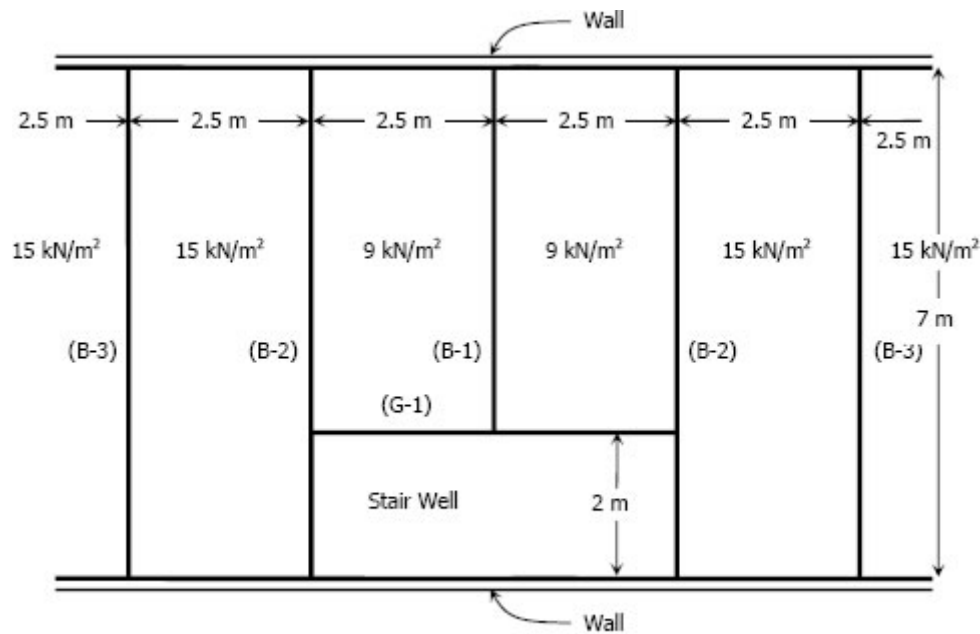
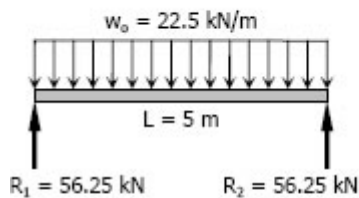


Figure P-543

Solution 543

For Member B - 1

$$\begin{aligned}
 S_{\text{required}} &= \frac{M}{f_b} \\
 &= \frac{\frac{1}{8}(22.5)(5^2)(1000^2)}{120} \\
 &= 586 \times 10^3 \text{ mm}^3
 \end{aligned}$$



Member B - 1
 $M_{\text{max}} = \frac{1}{8} w_o L^2$

From Appendix B, Table B-2 Properties of Wide-Flange Sections (W Shapes): SI Units, of text book:

Use W410 × 39 with $S = 634 \times 10^3 \text{ mm}^3$ for member B - 1.

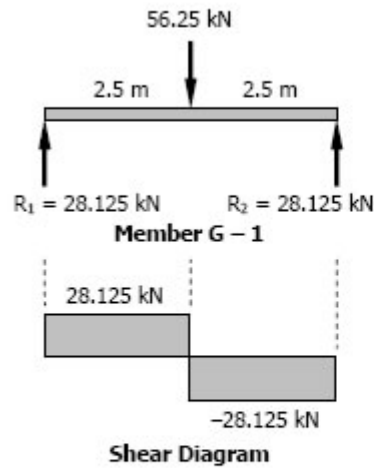
For Member G - 1

$$M = 2.5(28.125) \\ = 70.3125 \text{ kN}\cdot\text{m}$$

$$S_{\text{required}} = \frac{M}{f_b} \\ = \frac{70.3125(1000^2)}{120} \\ = 586 \times 10^3 \text{ mm}^3$$

From Appendix B, Table B-2 Properties of Wide-Flange Sections (W Shapes): SI Units, of text book:

Use W410 x 39 with $S = 634 \times 10^3 \text{ mm}^3$ for member G - 1.



For Member B - 2:

$$\Sigma M_{R2} = 0 \\ 7R_1 = 28.125(5) + 18.75(2)(6) \\ + 30(5)(2.5) \\ R_1 = 105.804 \text{ kN}$$

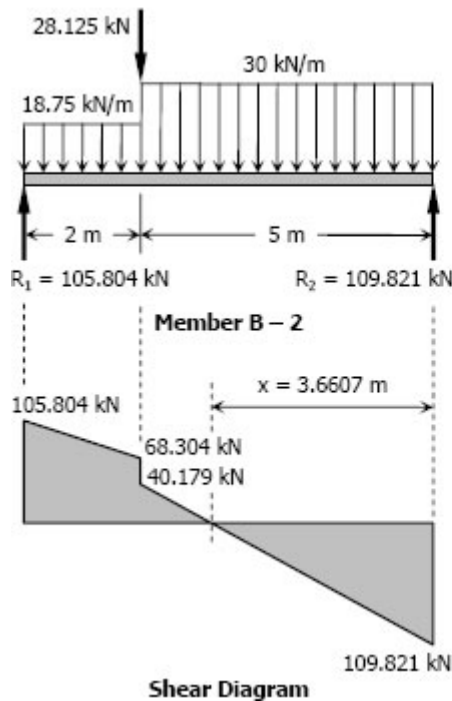
$$\Sigma M_{R1} = 0 \\ 7R_2 = 28.125(2) + 18.75(2)(1) \\ + 30(5)(4.5) \\ R_2 = 109.821 \text{ kN}$$

Location of Maximum Moment:

$$\frac{x}{109.821} = \frac{5-x}{40.179} \\ 40.179x = 549.105 - 109.821x \\ x = 3.6607 \text{ m}$$

Maximum Moment

$$M = \frac{1}{2} (3.6607)(109.821) \\ = 201.01 \text{ kN}\cdot\text{m}$$



$$S_{\text{required}} = \frac{M}{f_b} = \frac{201.01(1000^2)}{120}$$

$$S_{\text{required}} = 1\,675 \times 10^3 \text{ mm}^3$$

From Appendix B, Table B-2 Properties of Wide-Flange Sections (W Shapes): SI Units, of text book:

Use W610 × 82 with $S = 1\,870 \times 10^3 \text{ mm}^3$ for member B - 2.

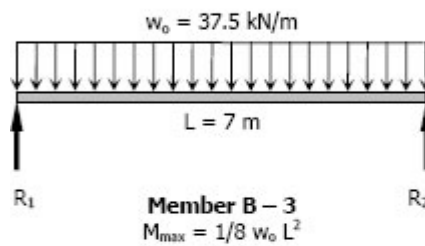
For Member B - 3

$$S_{\text{required}} = \frac{M}{f_b} = \frac{\frac{1}{8}(37.5)(7^2)(1000^2)}{120}$$

$$S_{\text{required}} = 1\,914 \times 10^3 \text{ mm}^3$$

From Appendix B, Table B-2 Properties of Wide-Flange Sections (W Shapes): SI Units, of text book:

Use W610 × 92 with $S = 2\,140 \times 10^3 \text{ mm}^3$ for member B - 3.

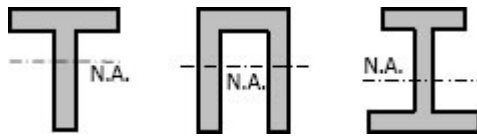


Summary:

W610 × 92	W610 × 82	W410 × 39	W610 × 82	W610 × 92
		W410 × 39		

Unsymmetrical Beams

Flexural Stress varies directly linearly with distance from the neutral axis. Thus for a symmetrical section such as wide flange, the compressive and tensile stresses will be the same. This will be desirable if the material is both equally strong in tension and compression. However, there are materials, such as cast iron, which are strong in compression than in tension. It is therefore desirable to use a beam with unsymmetrical cross section giving more area in the compression part making the stronger fiber located at a greater distance from the neutral axis than the weaker fiber. Some of these sections are shown below.



The proportioning of these sections is such that the ratio of the distance of the neutral axis from the outermost fibers in tension and in compression is the same as the ratio of the allowable stresses in tension and in compression. Thus, the allowable stresses are reached simultaneously.

In this section, the following notation will be use:

f_{bt} = flexure stress of fiber in tension

f_{bc} = flexure stress of fiber in compression

N.A. = neutral axis

y_t = distance of fiber in tension from N.A.

y_c = distance of fiber in compression from N.A.

M_r = resisting moment

M_c = resisting moment in compression

M_t = resisting moment in tension

Solved Problems in Unsymmetrical Beams

Problem 548

The inverted T section of a 4-m simply supported beam has the properties shown in Fig. P-548. The beam carries a uniformly distributed load of intensity w_o over its entire length. Determine w_o if $f_{bt} \leq 40$ MPa and $f_{bc} \leq 80$ MPa.

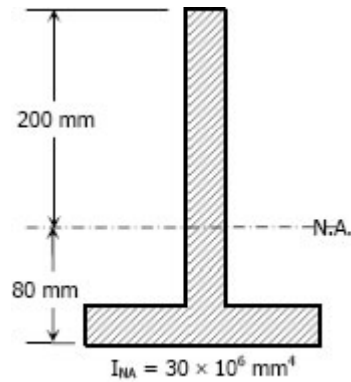
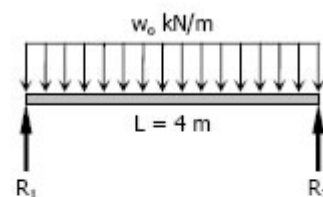


Figure P-548

Solution 548

$$\begin{aligned} M_{\max} &= \frac{1}{8} w_o L^2 \\ &= \frac{1}{8} w_o (4^2) \\ &= 2w_o \end{aligned}$$

$$M_r = \frac{f_b I}{y}$$



Member B – 1
 $M_{\max} = 1/8 w_o L^2$

$$\begin{aligned} M_t &= \frac{40(30 \times 10^6)}{80} \\ &= 15\,000\,000 \text{ N}\cdot\text{mm} \\ &= 15 \text{ kN}\cdot\text{mm} \end{aligned}$$

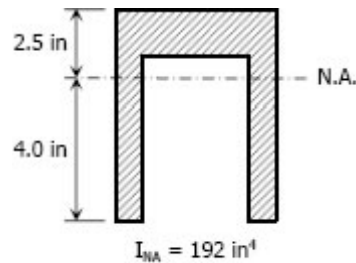
$$\begin{aligned} M_c &= \frac{80(30 \times 10^6)}{200} \\ &= 12\,000\,000 \text{ N}\cdot\text{mm} \\ &= 12 \text{ kN}\cdot\text{mm} \end{aligned}$$

The section is stronger in tension and weaker in compression, so compression governs in selecting the maximum moment.

$$\begin{aligned} M_{\max} &= M_c \\ 2w_o &= 12 \\ w_o &= 6 \text{ kN/m} \end{aligned}$$

Problem 549

A beam with cross-section shown in Fig. P-549 is loaded in such a way that the maximum moments are $+1.0P$ lb·ft and $-1.5P$ lb·ft, where P is the applied load in pounds. Determine the maximum safe value of P if the working stresses are 4 ksi in tension and 10 ksi in compression.

**Figure P-549****Solution 549**

At $M = +1.0P$ lb·ft the upper fiber is in compression while the lower fiber is in tension.

$$M = M_c$$

$$M = \frac{f_b I}{y}$$

For fibers in compression (upper fiber):

$$M_c = \frac{10(192)(1000)}{2.5}$$

$$1.0P = 768\,000 \text{ lb·in}$$

$$1.0P = 64\,000 \text{ lb·ft}$$

$$P = 64\,000 \text{ lb}$$

For fibers in tension (lower fiber):

$$M_c = \frac{4(192)(1000)}{4}$$

$$1.0P = 192\,000 \text{ lb·in}$$

$$1.0P = 16\,000 \text{ lb·ft}$$

$$P = 16\,000 \text{ lb}$$

At $M = -1.5P$ lb·ft, the upper fiber is in tension while the lower fiber is in compression.

$$M = M_c$$

$$M = \frac{f_b I}{y}$$

For fibers in compression (lower fiber):

$$M_c = \frac{10(192)(1000)}{4}$$

$$1.5P = 480\,000 \text{ lb-in}$$

$$1.5P = 40\,000 \text{ lb-ft}$$

$$P = 26\,666.67 \text{ lb}$$

For fibers in tension (upper fiber):

$$M_c = \frac{4(192)(1000)}{2.5}$$

$$1.5P = 307\,200 \text{ lb-in}$$

$$1.5P = 25\,600 \text{ lb-ft}$$

$$P = 17\,066.67 \text{ lb}$$

The safe load $P = 16\,000 \text{ lb}$

Problem 550

Resolve Prob. 549 if the maximum moments are $+2.5P \text{ lb-ft}$ and $-5.0P \text{ lb-ft}$.

Solution 550

At $M = +2.5P$

$$M_c = \frac{10(192)(1000)}{2.5} \rightarrow \text{upper fiber}$$

$$2.5P = 768\,000 \text{ lb-in}$$

$$2.5P = 64\,000 \text{ lb-ft}$$

$$P = 25\,600 \text{ lb}$$

$$M_t = \frac{4(192)(1000)}{4} \rightarrow \text{lower fiber}$$

$$2.5P = 192\,000 \text{ lb-in}$$

$$2.5P = 16\,000 \text{ lb-ft}$$

$$P = 6400 \text{ lb}$$

At $M = -5.0P \text{ lb-ft}$

$$M_c = \frac{10(192)(1000)}{4} \rightarrow \text{lower fiber}$$

$$5.0P = 480\,000 \text{ lb-in}$$

$$5.0P = 40\,000 \text{ lb-ft}$$

$$P = 8000 \text{ lb}$$

$$M_t = \frac{4(192)(1000)}{2.5} \rightarrow \text{upper fiber}$$

$$5.0P = 307\,200 \text{ lb-in}$$

$$5.0P = 25\,600 \text{ lb-ft}$$

$$P = 5120 \text{ lb}$$

Use $P = 5120 \text{ lb}$

Problem 551

Find the maximum tensile and compressive flexure stresses for the cantilever beam shown in Fig. P-551.

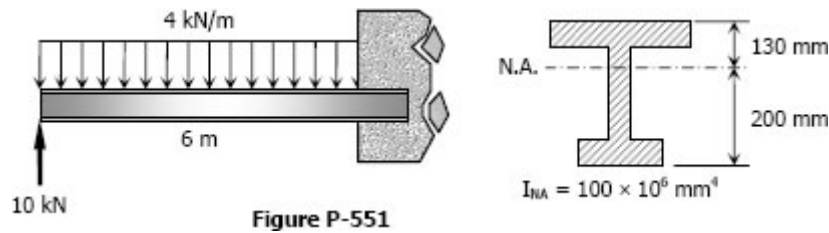
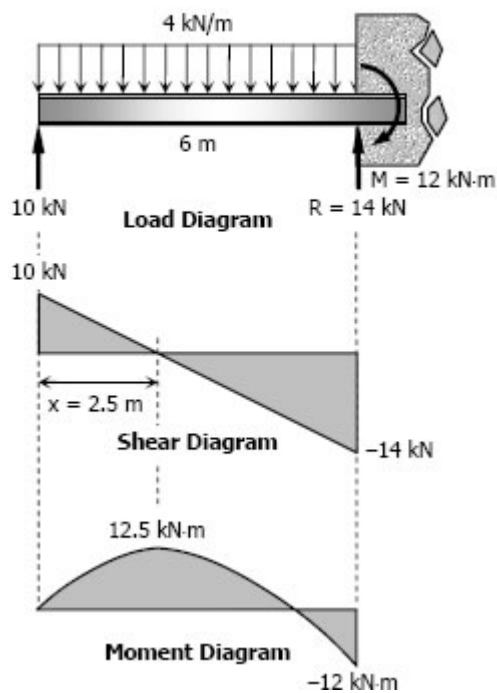


Figure P-551

Solution 551

$$M = 4(6)(3) - 10(6) = 12 \text{ kN}\cdot\text{m}$$

$$R = 4(6) - 10 = 14 \text{ kN}$$

$$\frac{x}{10} = \frac{6-x}{14}$$

$$14x = 60 - 10x$$

$$x = 2.5 \text{ m}$$

$$f_b = \frac{My}{I}$$

At $M = +12.5 \text{ kN}\cdot\text{m}$

$$f_{bc} = \frac{12.5(130)(1000^2)}{100 \times 10^6}$$

$$= 16.25 \text{ MPa} \rightarrow \text{upper fiber}$$

$$f_{bt} = \frac{12.5(200)(1000^2)}{100 \times 10^6}$$

$$= 25 \text{ MPa} \rightarrow \text{lower fiber}$$

At $M = -12 \text{ kN}\cdot\text{m}$

$$f_{bc} = \frac{12(200)(1000^2)}{100 \times 10^6}$$

$$= 24 \text{ MPa} \rightarrow \text{lower fiber}$$

$$f_{bt} = \frac{12(130)(1000^2)}{100 \times 10^6}$$

$$= 15.6 \text{ MPa} \rightarrow \text{upper fiber}$$

Maximum flexure stresses:

$$f_{bc} = 24 \text{ MPa at the fixed end}$$

$$f_{bt} = 25 \text{ MPa at } 2.5 \text{ m from the free end}$$

Problem 552

A cantilever beam carries the force and couple shown in Fig. P-552. Determine the maximum tensile and compressive bending stresses developed in the beam.

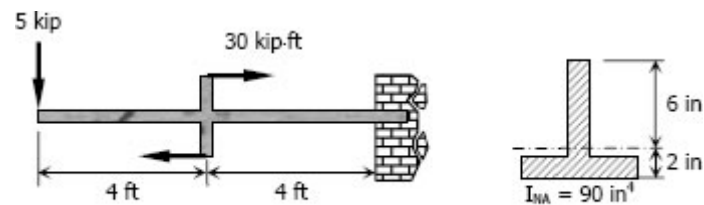
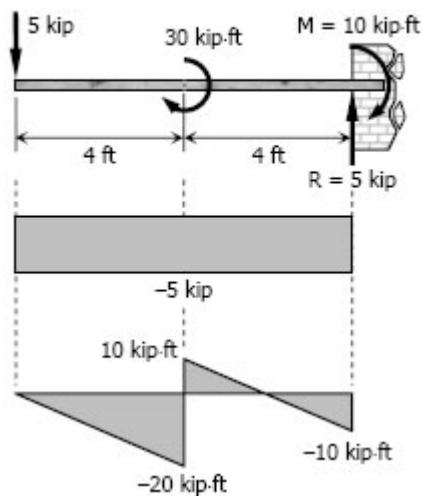


Figure P-552

Solution 552

$$R = 5 \text{ kip}$$

$$M = 5(8) - 30$$

$$= 10 \text{ kip-ft}$$

$$f_b = \frac{My}{I}$$

At $M = +10 \text{ kip-ft}$ of moment diagram

$$f_{bc} = \frac{10(6)(12)}{90}$$

$$= 8 \text{ ksi} \quad \rightarrow \text{upper fiber}$$

$$f_{bt} = \frac{10(2)(12)}{90}$$

$$= 2.67 \text{ ksi} \quad \rightarrow \text{lower fiber}$$

At $M = -20 \text{ kip-ft}$ of moment diagram

$$f_{bc} = \frac{20(2)(12)}{90}$$

$$= 5.33 \text{ ksi} \quad \rightarrow \text{lower fiber}$$

$$f_{bt} = \frac{20(6)(12)}{90}$$

$$= 16 \text{ ksi} \quad \rightarrow \text{upper fiber}$$

Maximum bending stresses:

$$f_{bc} = 8 \text{ ksi}$$

$$f_{bt} = 16 \text{ ksi}$$

Problem 553

Determine the maximum tensile and compressive bending stresses developed in the beam as shown in Fig. P-553.

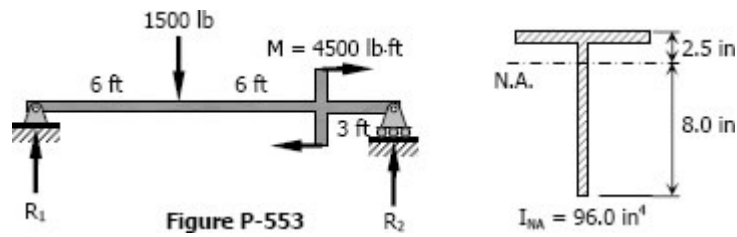
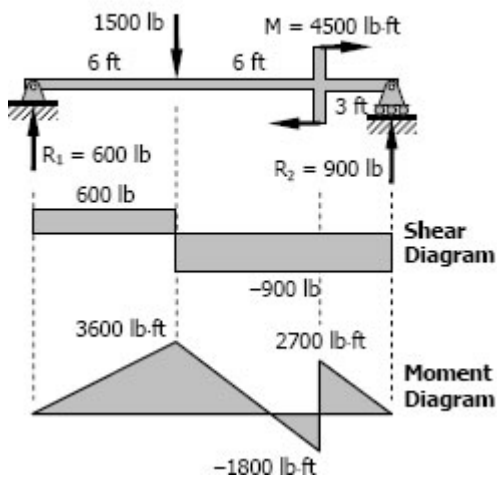


Figure P-553

Solution 553



$$\begin{aligned}\Sigma M_{R2} &= 0 \\ 15R_1 + 4500 &= 1500(9) \\ R_1 &= 600 \text{ lb}\end{aligned}$$

$$\begin{aligned}\Sigma M_{R1} &= 0 \\ 15R_2 &= 1500(6) + 4500 \\ R_2 &= 900 \text{ lb}\end{aligned}$$

$$f_b = \frac{My}{I}$$

$$\text{At } M = +3600 \text{ lb-ft}$$

$$\begin{aligned}f_{bc} &= \frac{3600(2.5)(12)}{96.0} \\ &= 1125 \text{ psi} \rightarrow \text{upper fiber}\end{aligned}$$

$$f_{bt} = \frac{3600(8.0)(12)}{96.0}$$

$$= 3600 \text{ psi} \rightarrow \text{lower fiber}$$

$$\text{At } M = -1800 \text{ lb-ft}$$

$$f_{bc} = \frac{1800(8.0)(12)}{96.0}$$

$$= 1800 \text{ psi} \rightarrow \text{lower fiber}$$

$$f_{bt} = \frac{1800(2.5)(12)}{96.0}$$

$$= 562.5 \text{ psi} \rightarrow \text{upper fiber}$$

Maximum flexure stresses

$$f_{bc} = 1800 \text{ psi}$$

$$f_{bt} = 3600 \text{ psi}$$

Problem 554

Determine the maximum tensile and compressive stresses developed in the overhanging beam shown in Fig. P-554. The cross-section is an inverted T with the given properties.

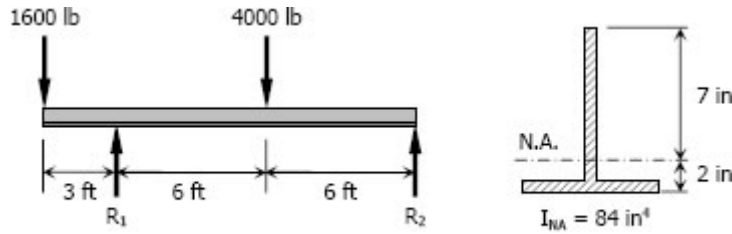
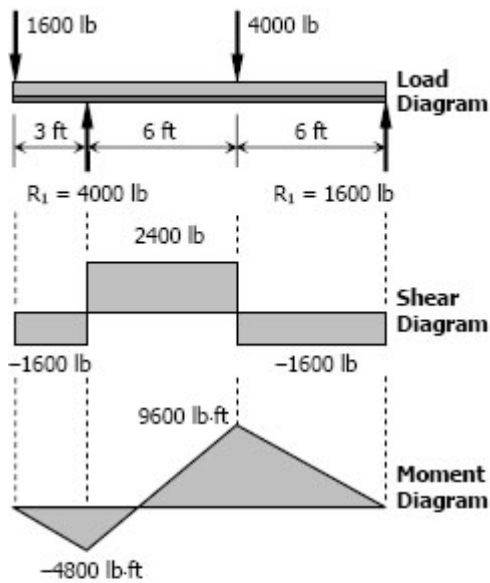


Figure P-554

Solution 554

$$\begin{aligned}\Sigma M_{R2} &= 0 \\ 12R_1 &= 1600(15) + 4000(6) \\ R_1 &= 4000 \text{ lb}\end{aligned}$$

$$\begin{aligned}\Sigma M_{R1} &= 0 \\ 12R_2 + 1600(3) &= 4000(6) \\ R_2 &= 1600 \text{ lb}\end{aligned}$$



$$f_b = \frac{My}{I}$$

At $M = -4800 \text{ lb-ft}$

$$\begin{aligned}f_{bc} &= \frac{4800(2)(12)}{84} \\ &= 1371.43 \text{ psi} \rightarrow \text{lower fiber}\end{aligned}$$

$$\begin{aligned}f_{bt} &= \frac{4800(7)(12)}{84} \\ &= 4800 \text{ psi} \rightarrow \text{upper fiber}\end{aligned}$$

At $M = +9600 \text{ lb-ft}$

$$\begin{aligned}f_{bc} &= \frac{9600(7)(12)}{84} \\ &= 9600 \text{ psi} \rightarrow \text{upper fiber}\end{aligned}$$

$$\begin{aligned}f_{bt} &= \frac{9600(2)(12)}{84} \\ &= 2742.86 \text{ psi} \rightarrow \text{lower fiber}\end{aligned}$$

Maximum flexure stress:

$$\begin{aligned}f_{bc} &= 9600 \text{ psi} \\ f_{bt} &= 4800 \text{ psi}\end{aligned}$$

Problem 555

A beam carries a concentrated load W and a total uniformly distributed load of $4W$ as shown in Fig. P-555. What safe value of W can be applied if $f_{bc} \leq 100$ MPa and $f_{bt} \leq 60$ MPa? Can a greater load be applied if the section is inverted? Explain.

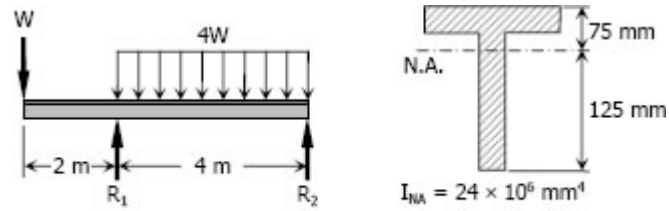
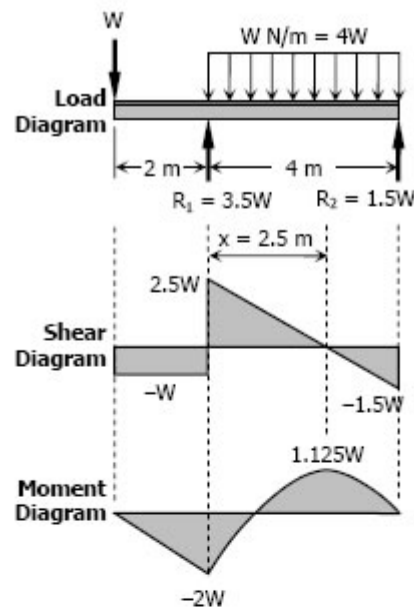


Figure P-555

Solution 555



$$\begin{aligned} [\Sigma M_{R2} = 0] \quad & 4R_1 = 6W + 4W(2) \\ & R_1 = 3.5W \\ [\Sigma M_{R1} = 0] \quad & 4R_2 + 2W = 4W(2) \\ & R_2 = 1.5W \end{aligned}$$

$$\begin{aligned} \frac{x}{2.5W} &= \frac{4-x}{1.5W} \\ 1.5Wx &= 10W - 2.5Wx \\ x &= 2.5 \text{ m} \end{aligned}$$

$$f_b = \frac{My}{I}$$

At $M = -2W$

For lower fiber, $f_{bc} \leq 100 \text{ MPa}$

$$100 = \frac{2W(125)(1000)}{24 \times 10^6}$$

$$W = 9600 \text{ N}$$

For upper fiber, $f_{bt} \leq 60 \text{ MPa}$

$$60 = \frac{2W(75)(1000)}{24 \times 10^6}$$

$$W = 9600 \text{ N}$$

At $M = 1.125W$

For upper fiber, $f_{bc} \leq 100 \text{ MPa}$

$$100 = \frac{1.125W(75)(1000)}{24 \times 10^6}$$

$$W = 28,444.44 \text{ N}$$

For lower fiber, $f_{bt} \leq 60 \text{ MPa}$

$$60 = \frac{1.125W(125)(1000)}{24 \times 10^6}$$

$$W = 10,240 \text{ N}$$

For safe load W , use $W = 9600 \text{ N}$

Discussion:

At $W = 9600 \text{ N}$, the allowable f_b in tension and compression are reached simultaneously when $M = -2W$. This is the same even if the section is inverted. Therefore, no load can be applied greater than $W = 9600 \text{ N}$.

Problem 556

A T beam supports the three concentrated loads shown in Fig. P-556. Prove that the NA is 3.5 in. above the bottom and that $I_{NA} = 97.0 \text{ in}^4$. Then use these values to determine the maximum value of P so that $f_{bt} \leq 4 \text{ ksi}$ and $f_{bc} \leq 10 \text{ ksi}$.

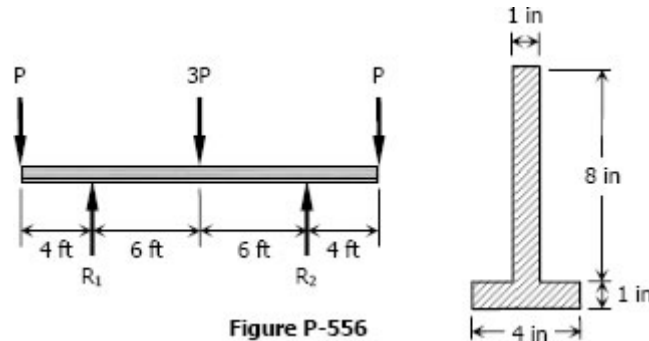
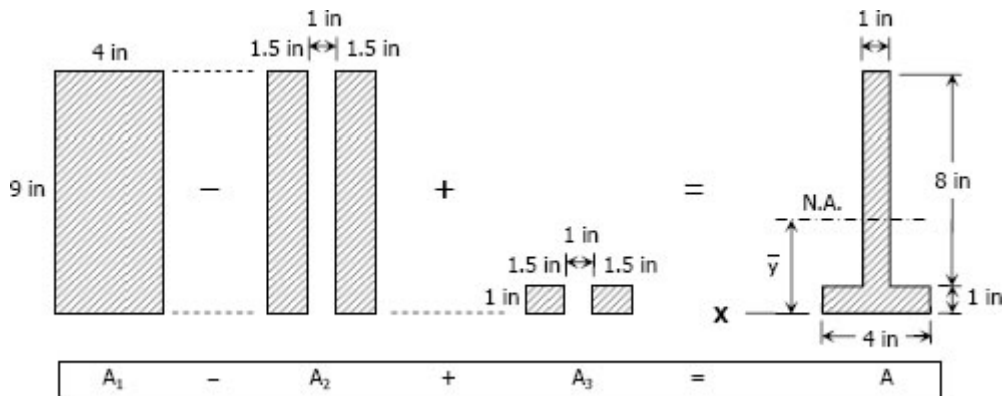


Figure P-556

Solution 556



$$A_1 = 9(4) = 36 \text{ in}^2$$

$$A_2 = 9(1.5)(2) = 27 \text{ in}^2$$

$$A_3 = 1(1.5)(2) = 3 \text{ in}^2$$

$$A = A_1 - A_2 + A_3$$

$$= 36 - 27 + 3$$

$$= 12 \text{ in}^2$$

$$A \bar{y} = \sum A_n y$$

$$12 \bar{y} = 36(4.5) - 27(4.5) + 3(0.5)$$

$$\bar{y} = 3.5 \text{ in} \quad (\text{ok!})$$

$$I_x = \sum \left(\frac{bh^3}{3} \right)_n$$

$$I_x = \frac{4(9^3)}{3} - 2 \cdot \frac{1.5(9^3)}{3} + 2 \cdot \frac{1.5(1^3)}{3}$$

$$I_x = 244 \text{ in}^4$$

By transfer formula for moment of inertia:

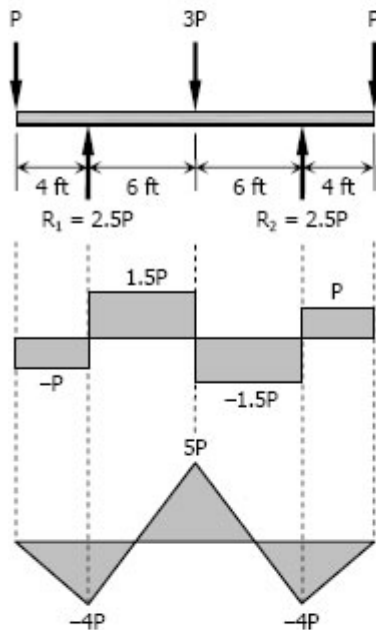
$$I_x = I_{NA} + Ad^2$$

$$244 = I_{NA} + 12(3.5)^2$$

$$I_{NA} = 97 \text{ in}^4 \quad (\text{ok!})$$

By symmetry:

$$R_1 = R_2 = 2.5P$$



$$f_b = \frac{My}{I}$$

At $M = -4P \text{ lb}\cdot\text{ft}$

Lower fiber is in compression:

$$10,000 = \frac{4P(3.5)(12)}{97}$$

$$P = 5773.81 \text{ lb}$$

Upper fiber is in tension:

$$4000 = \frac{4P(9 - 3.5)(12)}{97}$$

$$P = 1469.7 \text{ lb}$$

At $M = 5P \text{ lb}\cdot\text{ft}$

Lower fiber is in tension:

$$4000 = \frac{4P(3.5)(12)}{97}$$

$$P = 2309.52 \text{ lb}$$

Upper fiber is in compression

$$10,000 = \frac{4P(9 - 3.5)(12)}{97}$$

$$P = 3674.24 \text{ lb}$$

For safe value of P , use $P = 1469.7 \text{ lb}$

Problem 557

A cast-iron beam 10 m long and supported as shown in Fig. P-557 carries a uniformly distributed load of intensity w_0 (including its own weight). The allowable stresses are $f_{bt} \leq 20$ MPa and $f_{bc} \leq 80$ MPa. Determine the maximum safe value of w_0 if $x = 1.0$ m.

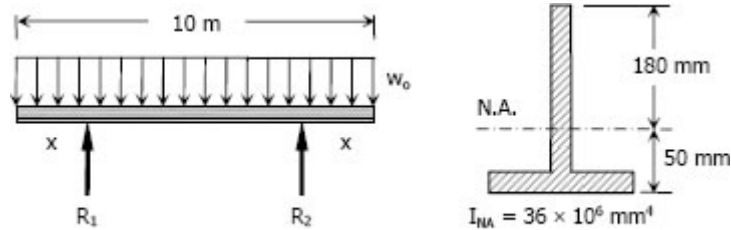
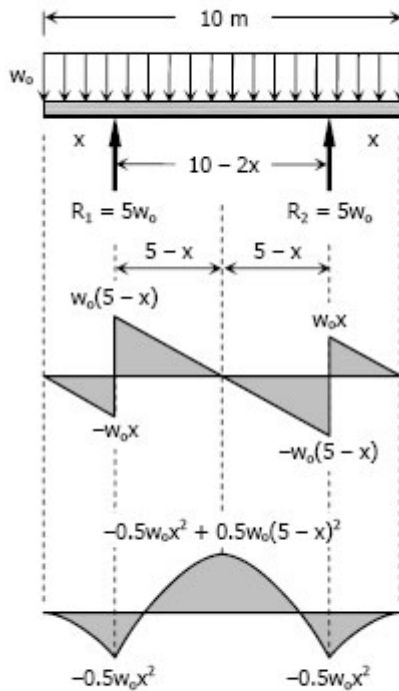


Figure P-557 and P-558

Solution 557

By symmetry:

$$R_1 = R_2 = \frac{1}{2} (10w_0) = 5w_0$$



$$f_b = \frac{My}{I}$$

At $M = -0.5w_0x^2$ N·m

when $x = 1$ m, $M = -0.5w_0$ N·m

For fiber in compression (lower)

$$80 = \frac{0.5w_0(50)(1000)}{36 \times 10^6}$$

$$w_0 = 115\,200 \text{ N/m}$$

$$w_0 = 115.2 \text{ kN/m}$$

For fiber in tension (upper)

$$20 = \frac{0.5w_0(180)(1000)}{36 \times 10^6}$$

$$w_0 = 8000 \text{ N/m}$$

$$w_0 = 8 \text{ kN/m}$$

At $M = -0.5w_0x^2 + 0.5w_0(5-x)^2$ N·m

when $x = 1$ m, $M = 7.5w_0$ N·m

For fiber in compression (upper)

$$80 = \frac{7.5w_0(180)(1000)}{36 \times 10^6}$$

$$w_0 = 2133.33 \text{ N/m}$$

$$w_0 = 2.13 \text{ kN/m}$$

For fiber in tension (lower)

$$20 = \frac{7.5w_0(50)(1000)}{36 \times 10^6}$$

$$w_0 = 1920 \text{ N/m}$$

$$w_0 = 1.92 \text{ kN/m}$$

For safe load w_0 , use $w_0 = 1.92 \text{ kN/m}$

Problem 558

In Prob. 557, find the values of x and w_o so that w_o is a maximum.

Solution 558

From Solution 557, tension governs at both positive and negative maximum moments.

At $M = -0.5w_o x^2$ N·m:

$$20 = \frac{0.5w_o x^2 (180)(1000)}{36 \times 10^6}$$

$$w_o = 8000/x^2$$

At $M = -0.5w_o x^2 + 0.5w_o(5-x)^2$ N·m:

$$20 = \frac{[-0.5w_o x^2 + 0.5w_o(5-x)^2](50)(1000)}{36 \times 10^6}$$

$$14\,400 = -0.5w_o x^2 + 0.5w_o(5-x)^2$$

$$28\,800 = -w_o x^2 + w_o(5-x)^2$$

$$28\,800 = -w_o x^2 + w_o(25 - 10x + x^2)$$

$$28\,800 = -w_o x^2 + (25 - 10x)w_o + w_o x^2$$

$$28\,800 = (25 - 10x)w_o$$

$$28\,800 = (25 - 10x)(8\,000/x^2)$$

$$(28\,800x^2 / 8\,000) = 25 - 10x$$

$$\frac{18}{5}x^2 = 25 - 10x$$

$$18x^2 = 125 - 50x$$

$$18x^2 + 50x - 125 = 0$$

$$x = 1.59 \text{ m and } -4.37 \text{ (meaningless)}$$

$$\text{use } x = 1.59 \text{ m}$$

$$w_o = 8000 / 1.59^2$$

$$w_o = 3164.43 \text{ N/m}$$

$$w_o = 3.16 \text{ kN/m}$$

Design for Flexure and Shear

To determine the load capacity or the size of beam section, it must satisfy the allowable stresses in both flexure (bending) and shear. Shearing stress usually governs in the design of short beams that are heavily loaded, while flexure is usually the governing stress for long beams. In material comparison, timber is low in shear strength than that of steel.

For any cross-sectional shape, flexure and shear are given in the following formulas:

Flexure Formula

$$f_b = \frac{Mc}{I}$$

Horizontal Shear Stress

$$f_v = \frac{VQ}{Ib}$$

For rectangular beam, the following defines for flexure and shear:

Flexure formula for rectangular beam

$$f_b = \frac{6M}{bd^2}$$

Horizontal shear stress for rectangular beam

$$f_v = \frac{3V}{2bd}$$

Where

f_b = flexure stress

f_v = bending stress

M = maximum moment applied to the beam

V = maximum shear applied to the beam

I = moment of inertia about the neutral axis

Q = moment of area

b = breadth

d = depth

Solution to Problem 580 | Design for Flexure and Shear

Problem 580

A rectangular beam of width b and height h carries a central concentrated load P on a simply

supported span of length L . Express the maximum f_v in terms of maximum f_b .

Solution 580

From the figure:

$$V_{max} = \frac{1}{2}P$$

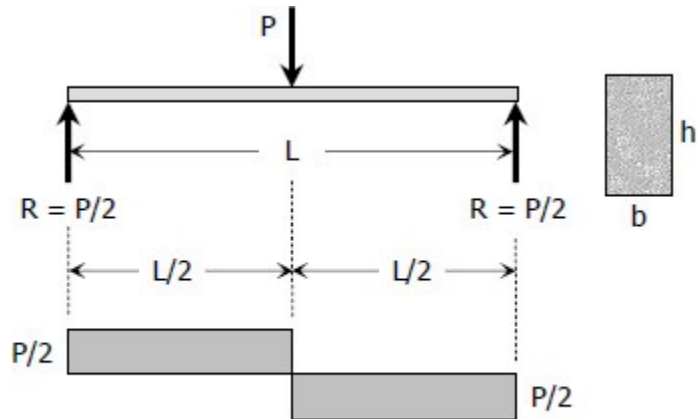
$$M_{max} = \frac{1}{2}L \left(\frac{1}{2}P \right) = \frac{1}{4}PL$$

From flexure formula:

$$f_b = \frac{6M}{bh^2} = \frac{6\left(\frac{1}{4}PL\right)}{bh^2}$$

$$f_b = \frac{3PL}{2bh^2}$$

$$bh = \frac{3PL}{2f_b h}$$



Shear Diagram

From shear stress formula:

$$f_b = \frac{3V}{2bh} = \frac{3\left(\frac{1}{2}P\right)}{2\left(\frac{3PL}{2f_b h}\right)}$$

$$f_v = \frac{f_b h}{2L} \quad \text{answer}$$

Problem 581

A laminated beam is composed of five planks, each 6 in. by 2 in., glued together to form a section 6 in. wide by 10 in. high. The allowable shear stress in the glue is 90 psi, the allowable shear stress in the wood is 120 psi, and the allowable flexural stress in the wood is 1200 psi. Determine the maximum uniformly distributed load that can be carried by the beam on a 6-ft simple span.

Solution 581

Maximum moment for simple beam

$$M_{max} = \frac{1}{8}w_o L^2$$

$$M_{max} = \frac{1}{8}w_o (6^2)$$

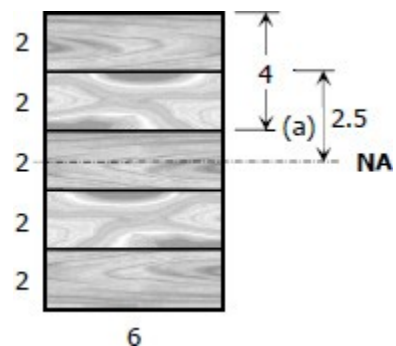
$$M_{max} = 4.5w_o \text{ lb} \cdot \text{ft}$$

Maximum shear for simple beam

$$V_{max} = \frac{1}{2}w_o L$$

$$V_{max} = \frac{1}{2}w_o (6)$$

$$V_{max} = 3w_o \text{ lb}$$



For bending stress of wood

$$f_b = \frac{6M}{bd^2}$$

$$120 = \frac{6(4.5w_o)(12)}{6(10^3)}$$

$$w_o = 2222.22 \text{ lb/ft}$$

For shear stress of wood

$$(f_v)_{wood} = \frac{3V}{2bd}$$

$$120 = \frac{3(3w_o)}{2(6)(10)}$$

$$w_o = 1600 \text{ lb/ft}$$

For shear stress in the glued joint

$$(f_v)_{glue} = \frac{VQ}{Ib}$$

Where:

$$Q = 6(4)(2.5) = 60 \text{ in}^3$$

$$I = \frac{bd^3}{12} = \frac{6(10^3)}{12} = 500 \text{ in}^4$$

$$b = 6 \text{ in}$$

Thus,

$$90 = \frac{3w_o(60)}{500(6)}$$

$$w_o = 1250 \text{ lb/ft}$$

Use $w_o = 1250 \text{ lb/ft}$ for safe value of uniformly distributed load.

answer

Problem 582

Find the cross-sectional dimensions of the smallest square beam that can be loaded as shown in Fig. P-582 if $f_v \leq 1.0 \text{ MPa}$ and $f_b \leq 8 \text{ MPa}$.

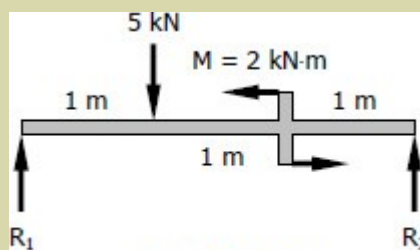


Figure P-582

Solution 582

$$\Sigma M_{R2} = 0$$

$$3R_1 = 5(2) + 2$$

$$R_1 = 4 \text{ kN}$$

$$\Sigma M_{R2} = 0$$

$$3R_2 + 2 = 5(1)$$

$$R_2 = 1 \text{ kN}$$

Based on bending stress (square $b = d$):

$$f_b = \frac{6M}{bd^2}$$

$$8 = \frac{6(4)(1000^2)}{d^3}$$

$$d = 144.22 \text{ mm}$$

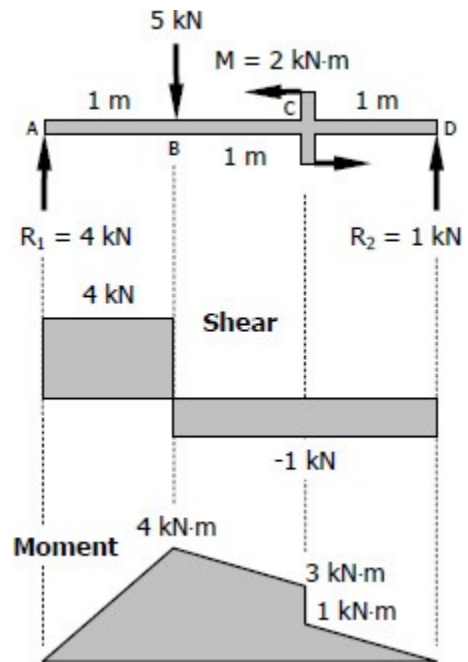
Based on shear stress (square $b = d$):

$$f_v = \frac{3V}{2bd}$$

$$1 = \frac{3(4)(1000)}{2d^2}$$

$$d = 77.46 \text{ mm}$$

Use **145 mm × 145 mm** square beam *answer*



Problem 584

A wide-flange section having the dimensions shown in Fig. P-584 supports a distributed load of w_0 lb/ft on a simple span of length L ft. Determine the ratio of the maximum flexural stress to the maximum shear stress.

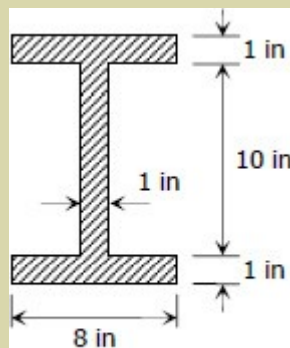


Figure P-584

Solution 584

Bending stress:

$$f_b = \frac{Mc}{I}$$

where:

$$M = \frac{1}{8}w_o L^2 \text{ lb} \cdot \text{ft}$$

$$c = 12/2 = 6 \text{ in}$$

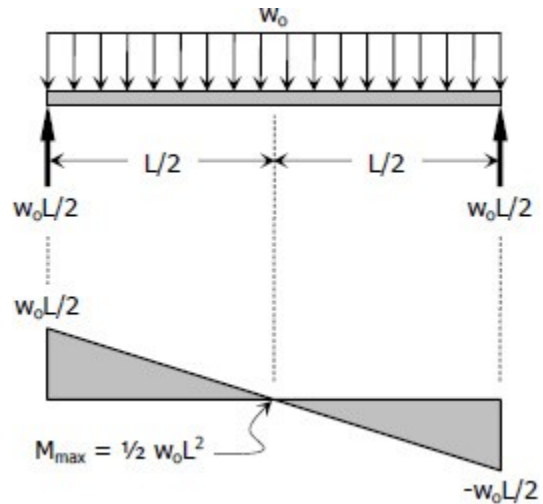
$$I = \frac{8(12^3)}{12} - \frac{7(10^3)}{12}$$

$$I = \frac{1706}{3} \text{ in}^4$$

Thus,

$$f_b = \frac{\frac{1}{8}w_o L^2 (12)(6)}{\frac{1706}{3}}$$

$$f_b = \frac{27}{1706}w_o L^2$$



Shear stress:

$$f_v = \frac{VQ}{Ib}$$

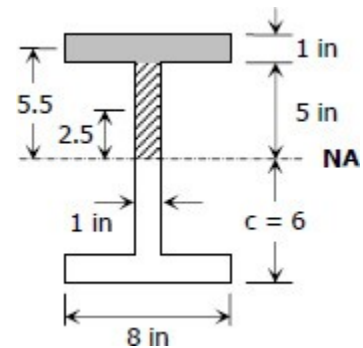
where:

$$V = \frac{1}{2}w_o L$$

$$Q = 8(1)(5.5) + 5(1)(2.5) = 56.5 \text{ in}^3$$

$$I = \frac{1706}{3} \text{ in}^4 \quad (\text{see computation above})$$

$$b = 1 \text{ in}$$



Thus,

$$f_v = \frac{(\frac{1}{2}w_o L)(56.5)}{\frac{1706}{3}(1)}$$

$$f_v = \frac{339}{6824}w_o L$$

Ratio (flexural stress : shear stress)

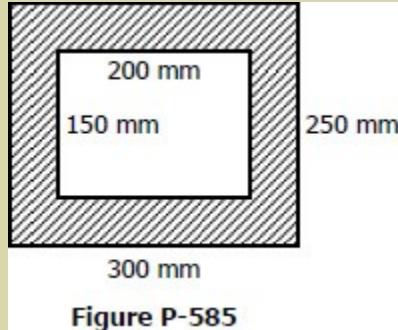
$$\text{Ratio} = \frac{\frac{27}{1706}w_o L^2}{\frac{339}{6824}w_o L}$$

$$\text{Ratio} = \frac{36}{113}L$$

$$\text{Ratio} = 0.3186L \quad \text{answer}$$

Problem 585

A simply supported beam of length L carries a uniformly distributed load of 6000 N/m and has the cross section shown in Fig. P-585. Find L to cause a maximum flexural stress of 16 MPa . What maximum shearing stress is then developed?



Solution 585

Flexural Stress

$$f_b = \frac{Mc}{I}$$

Where

$$f_b = 16 \text{ MPa}$$

$$M = \frac{1}{8} w_o L^2 = \frac{1}{8} (6000) L^2 = 750 L^2 \text{ N}\cdot\text{m}$$

$$c = \frac{1}{2}(250) = 125 \text{ mm}$$

$$I = \frac{300(250^3)}{12} - \frac{200(150^3)}{12} = 334\,375\,000 \text{ mm}^4$$

Thus,

$$16 = \frac{750 L^2 (1000) (125)}{334\,375\,000}$$

$$L = 7.55 \text{ m} \quad \text{answer}$$

Shearing Stress

$$f_v = \frac{VQ}{Ib}$$

Where

$$V = \frac{1}{2} w_o L = \frac{1}{2} (6000) (7.55) = 22\,650 \text{ N}$$

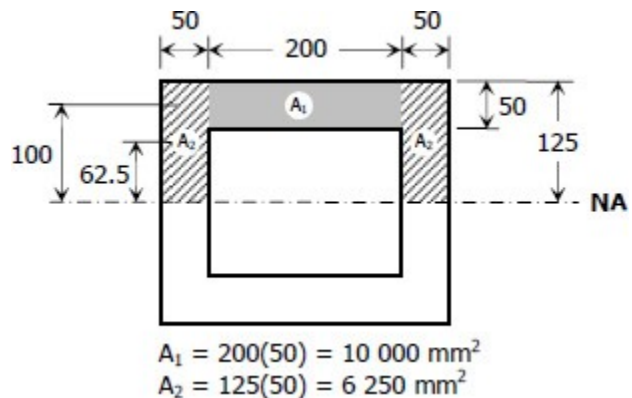
$$Q = 10\,000(100) + 2(6\,250)(62.5)$$

$$Q = 1\,781\,250 \text{ mm}^3$$

$$b = 2(50) = 100 \text{ mm}$$

Thus,

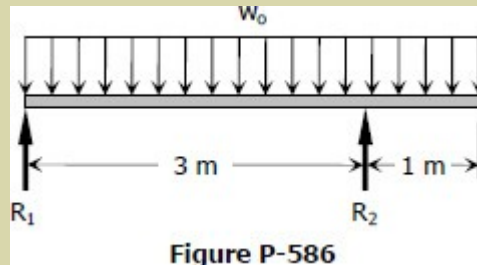
$$f_v = \frac{22\,650 (1\,781\,250)}{334\,375\,000 (100)}$$



$$f_v = 1.21 \text{ MPa} \quad \text{answer}$$

Problem 586

The distributed load shown in Fig. P-586 is supported by a box beam having the same cross-section as that in [Prob. 585](#). Determine the maximum value of w_o that will not exceed a flexural stress of 10 MPa or a shearing stress of 1.0 MPa.



Solution 586

$$\Sigma M_{R2} = 0$$

$$3R_1 = 4w_o(1)$$

$$R_1 = \frac{4}{3}w_o$$

$$\Sigma M_{R1} = 0$$

$$3R_2 = 4w_o(2)$$

$$R_2 = \frac{8}{3}w_o$$

From shear diagram

$$\frac{x}{\frac{4}{3}w_o} = \frac{3-x}{\frac{5}{3}w_o}$$

$$\frac{5}{3}x = 4 - \frac{4}{3}x$$

$$x = \frac{4}{3} \text{ m}$$

Based on allowable bending stress

$$f_b = \frac{Mc}{I}$$

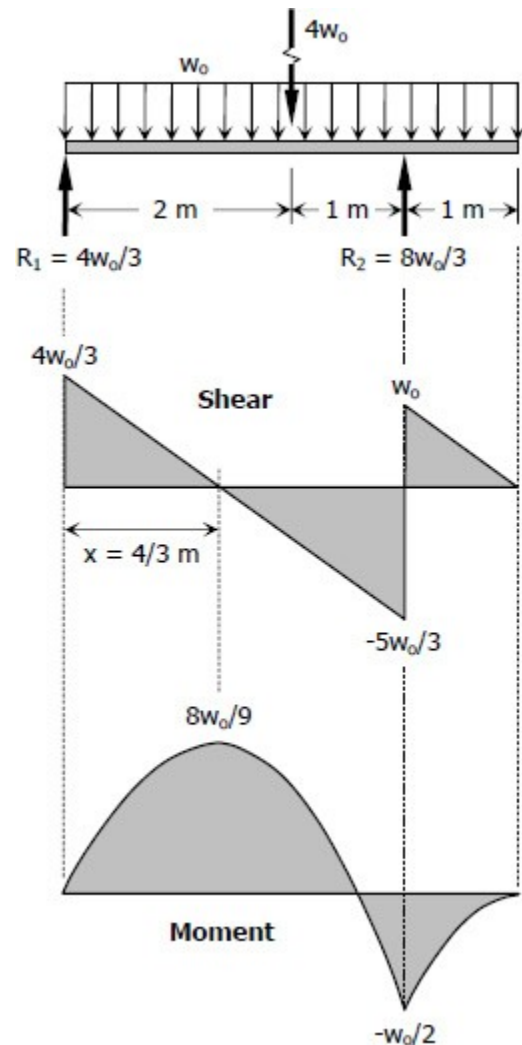
Where (From [Solution 585](#)):

$$c = 125 \text{ mm}$$

$$I = 334\,375\,000 \text{ mm}^4$$

Thus,

$$10 = \frac{\frac{8}{9}w_o(1000^2)(125)}{334\,375\,000}$$



$$w_o = 30.09 \text{ kN/m}$$

Based on allowable shear stress

$$f_v = \frac{VQ}{Ib}$$

Where (From [Solution 585](#)):

$$Q = 1\,781\,250 \text{ mm}^3$$

$$I = 334\,375\,000 \text{ mm}^4$$

$$b = 100 \text{ mm}$$

Thus,

$$1 = \frac{\frac{5}{3}w_o(1000)(1\,781\,250)}{334\,375\,000(100)}$$

$$w_o = 11.26 \text{ kN/m}$$

For safe value of w_o , use $w_o = 11.26 \text{ kN/m}$ *answer*

Problem 587

A beam carries two concentrated loads P and triangular load of $3P$ as shown in Fig. P-587. The beam section is the same as that in Fig. P-577 on [this page](#). Determine the safe value of P if $f_b \leq 1200 \text{ psi}$ and $f_v \leq 200 \text{ psi}$.

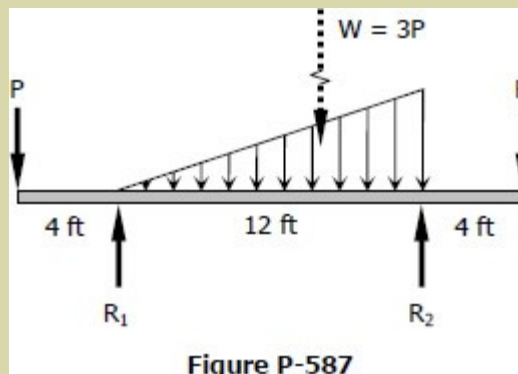


Figure P-587

Solution 587

$$\Sigma M_{R2} = 0$$

$$12R_1 + 4P = 16P + 4(3P)$$

$$R_1 = 2P$$

$$\Sigma M_{R1} = 0$$

$$12R_2 + 4P = 16P + 8(3P)$$

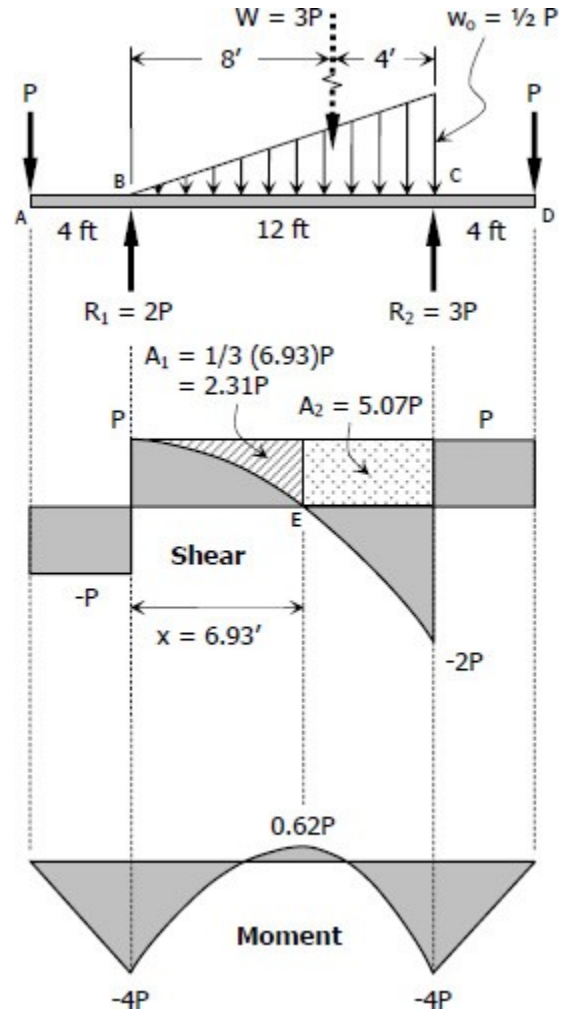
$$R_2 = 3P$$

$$W = \frac{1}{2}(12)w_o = 3P$$

$$w_o = \frac{1}{2}P$$

To draw the Shear Diagram

1. $V_A = -P$ lb
2. $V_B = V_A + \text{Area in load diagram}$
 $V_B = -P + 0 = -P$ lb
 $V_{B2} = V_B + R_1 = -P + 2P = P$ lb
3. $V_C = V_{B2} + \text{Area in load diagram}$
 $V_C = P - \frac{1}{2}(12)(\frac{1}{2}P) = -2P$ lb
 $V_{C2} = V_C + R_2 = -2P + 3P = P$ lb
4. $V_D = V_{C2} + \text{Area in load diagram}$
 $V_D = P + 0 = P$
 $V_{D2} = V_D - P = P - P = 0$
5. Shear at AB and CD are rectangular.
6. Shear at BC is parabolic (2nd degree curve).
7. Location of zero shear:
 By squared property of parabola
 $x^2 / P = 122 / 3P$
 $x = 6.93$ ft
 $12 - x = 5.07$ ft



To draw the Moment Diagram

1. $M_A = 0$
2. $M_B = M_A + \text{Area in shear diagram}$
 $M_B = 0 - 4P = -4P$ lb·ft
3. $M_E = M_B + \text{Area in shear diagram}$
 $M_E = -4P + \frac{2}{3}(6.93)(P) = 0.62P$ lb·ft
4. $M_C = M_E + \text{Area in shear diagram}$
 $M_C = 0.62P - [\frac{1}{3}(12)3P - 2.31P - 5.07P]$
 $M_C = -4P$ lb·ft

$$5. \quad M_D = M_C + \text{Area in shear diagram}$$

$$M_D = -4P + 4P = 0$$

6. The moment diagram at AB and CD are straight lines (1st degree curves) while at BC is 3rd degree curve.

Based on allowable bending stress

$$f_b = \frac{Mc}{I}$$

Where (From [Solution 577](#))

$$c = 6 \text{ in}$$

$$I = 350.67 \text{ in}^4$$

$$\text{Thus,}$$

$$1200 = \frac{4P(12)(6)}{350.67}$$

$$P = 1461.125 \text{ lb}$$

Based on allowable shear stress

$$f_v = \frac{VQ}{Ib}$$

Where (From [Solution 577](#))

$$Q = 35.5 \text{ in}^3$$

$$I = 350.67 \text{ in}^4$$

$$b = 0.75 \text{ in}$$

$$\text{Thus,}$$

$$200 = \frac{2P(35.5)}{350.67(0.75)}$$

$$P = 740.85 \text{ lb}$$

For safe value of P, use **P = 740.85 lb.** *answer*

Problem 588

The distributed load shown in Fig. P-588 is supported by a wide-flange section of the given dimensions. Determine the maximum value of w_0 that will not exceed a flexural stress of 10 MPa or a shearing stress of 1.0 MPa.

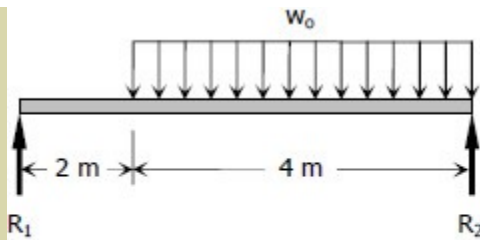
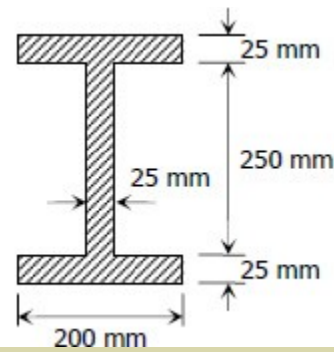


Figure P-588



Solution 588

$$\Sigma M_{R2} = 0$$

$$6R_1 = 4w_o(2)$$

$$R_1 = \frac{4}{3}w_o \text{ N}$$

$$\Sigma M_{R1} = 0$$

$$6R_2 = 4w_o(4)$$

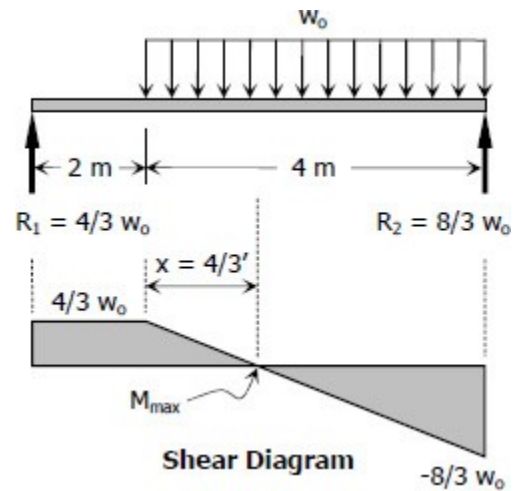
$$R_2 = \frac{8}{3}w_o \text{ N}$$

From the shear diagram

$$\frac{x}{\frac{4}{3}w_o} = \frac{4-x}{\frac{8}{3}w_o}$$

$$2x = 4 - x$$

$$x = \frac{4}{3} \text{ m}$$



Maximum moment = sum of area in Shear Diagram at the left of point of zero shear

$$M_{max} = 2\left(\frac{4}{3}w_o\right) + \frac{1}{2}\left(\frac{4}{3}\right)\left(\frac{4}{3}w_o\right)$$

$$M_{max} = \frac{32}{9}w_o \text{ N} \cdot \text{m}$$

Based on allowable flexural stress

$$f_b = \frac{Mc}{I}$$

Where

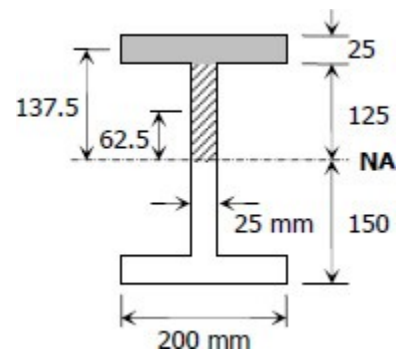
$$c = 150 \text{ mm}$$

$$I = 200(300^3)/12 - 175(250^3)/12$$

$$I = 222\,135\,416.67 \text{ mm}^4$$

Thus,

$$10 = \frac{\frac{32}{9}w_o(1000)(150)}{222\,135\,416.67}$$



$$w_o = 4165.04 \text{ N/m}$$

Based on allowable shear stress

$$f_v = \frac{VQ}{Ib}$$

Where

$$Q = 200(25)(137.5) + 125(25)(62.5)$$

$$Q = 882\,812.5 \text{ mm}^3$$

$$I = 222\,135\,416.67 \text{ mm}^4$$

$$b = 25 \text{ mm}$$

Thus,

$$1.0 = \frac{\frac{8}{3}w_o(882\,812.5)}{222\,135\,416.67(25)}$$

$$w_o = 94.36 \text{ N/m}$$

For safe value of w_o , use $w_o = 94.36 \text{ N/m}$. *answer*

Problem 589

A channel section carries a concentrated loads W and a total distributed load of $4W$ as shown in Fig.

P-589. Verify that the NA is 2.17 in. above the bottom and that $I_{NA} = 62 \text{ in}^4$. Use these values to determine the maximum value of W that will not exceed allowable stresses in tension of 6,000 psi, in compression of 10,000 psi, or in shear of 8,000 psi.

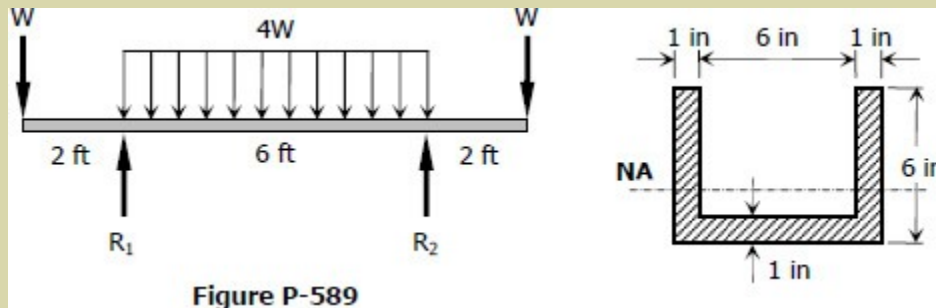


Figure P-589

Solution 589

Based on allowable bending stress:

By symmetry

$$R_1 = R_2 = \frac{1}{2}(W + 4W + W)$$

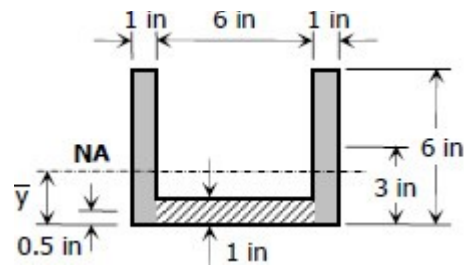
$$R_1 = R_2 = 3W$$

$$6w_o = 4W$$

$$w_o = \frac{2}{3}W$$

$$A = 3(6)(1)$$

$$A = 18 \text{ in}^2$$

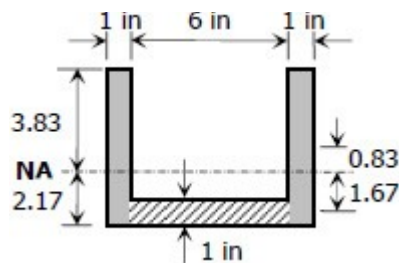


$$A\bar{y} = \Sigma ay$$

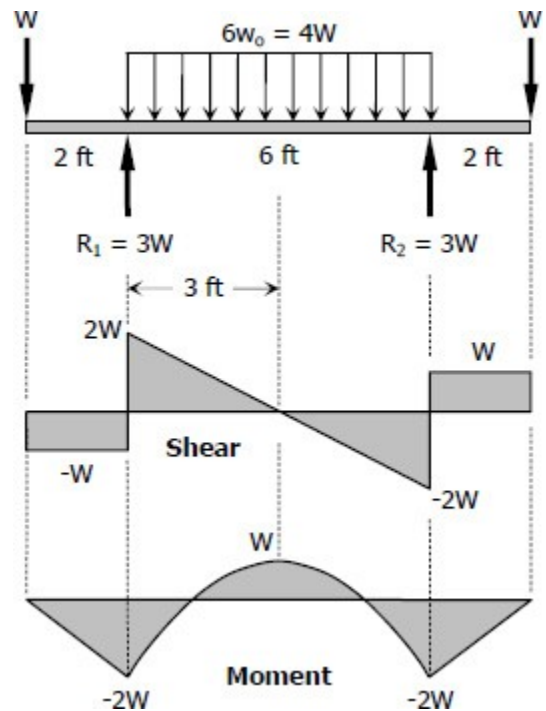
$$18\bar{y} = 2[6(1)(3)] + 6(1)(0.5)$$

$$\bar{y} = 2.17 \text{ in} \quad (\text{okay!})$$

By transfer formula for moment of inertia



$$I_{NA} = 2 \left[\frac{1(6^3)}{12} + 6(0.83^2) \right] + \left[\frac{6(1^3)}{12} + 6(1.67^2) \right]$$



$$I_{NA} = 61.5002 \text{ in}^4 \quad (\text{okay!})$$

$$f_b = \frac{Mc}{I}$$

For $M = -2W \text{ lb}\cdot\text{ft}$

Top fiber in tension

$$6000 = \frac{2W(12)(3.83)}{62}$$

$$W = 4045 \text{ lb}$$

Bottom fiber in compression

$$10\,000 = \frac{2W(12)(2.17)}{62}$$

$$W = 11\,905 \text{ lb}$$

For $M = W \text{ lb}\cdot\text{ft}$

Top fiber in compression

$$10\,000 = \frac{W(12)(3.83)}{62}$$

$$W = 13\,490 \text{ lb}$$

Bottom fiber in tension

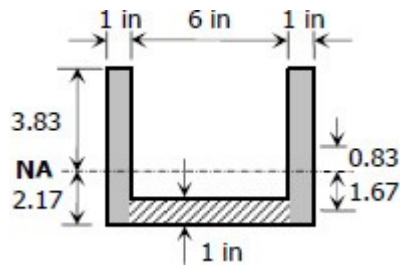
$$6000 = \frac{W(12)(2.17)}{62}$$

$$W = 14\,286 \text{ lb}$$

Based on allowable shear stress:

$$f_v = \frac{VQ}{Ib}$$

Where



$$V = 2W$$

$$Q_{NA} = 2[3.83(1)(3.83/2)] = 14.6689 \text{ in}^3$$

$$I = 62 \text{ in}^4$$

$$b = 2 \text{ in}$$

Thus,

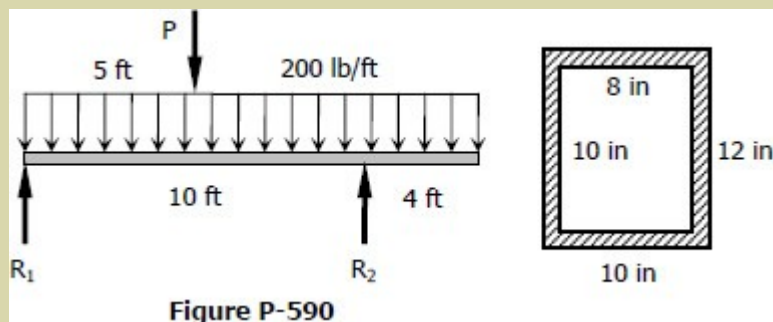
$$8000 = \frac{2W(14.6689)}{62(2)}$$

$$W = 33\,813 \text{ lb}$$

For safe value of W , use **$W = 4045 \text{ lb}$** *answer*

Problem 590

A box beam carries a distributed load of 200 lb/ft and a concentrated load P as shown in Fig. P-590. Determine the maximum value of P if $f_b \leq 1200 \text{ psi}$ and $f_v \leq 150 \text{ psi}$.



Solution 590

$$\Sigma M_{R2} = 0$$

$$10R_1 = 5P + 3(2800)$$

$$R_1 = 0.5P + 840$$

$$\Sigma M_{R1} = 0$$

$$10R_2 = 5P + 7(2800)$$

$$R_2 = 0.5P + 1960$$

$$M_B = \frac{1}{2}[(0.5P + 840) + (0.5P - 160)](5)$$

$$M_B = 2.5P + 1700 \text{ lb} \cdot \text{ft}$$

$$M_C = M_B - \frac{1}{2}[(0.5P + 160) + (0.5P + 1160)](5)$$

$$M_C = M_B - (2.5P + 1320)$$

$$M_C = (2.5P + 1700) - (2.5P + 3300)$$

$$M_C = -1600 \text{ lb} \cdot \text{ft}$$

Check M_C from the overhang segment

$$M_C = -\frac{1}{2}(4)(800)$$

$$M_G = -1600 \text{ lb} \cdot \text{ft} \quad (\text{okay!})$$

Based on allowable bending stress

$$f_b = \frac{Mc}{I}$$

Where

$$M = 2.5P + 1700 \text{ lb}\cdot\text{ft}$$

$$c = 12/2 = 6 \text{ in}$$

$$I = 10(12^3)/12 - 8(10^3)/12 = 773.33 \text{ in}^4$$

Thus,

$$1200 = \frac{(2.5P + 1700)(12)(6)}{773.33}$$

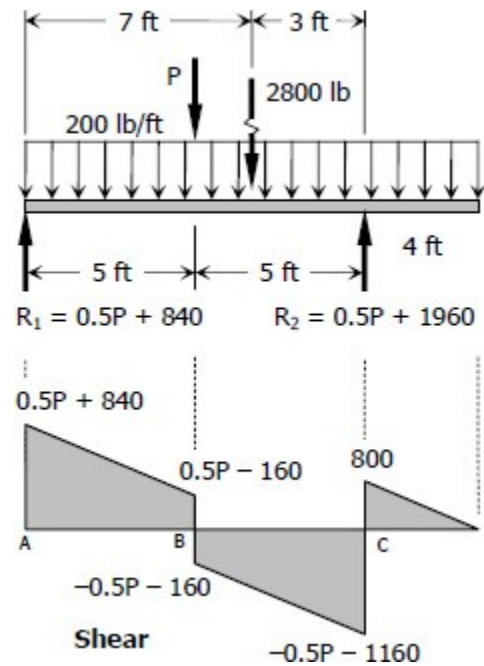
$$P = 4475.56 \text{ lb}$$

Based on allowable shear stress

$$f_v = \frac{VQ}{Ib}$$

Where

$$V = 0.5P + 1160 \text{ lb}$$



$$Q = 10(1)(5.5) + 2 [5(1)(2.5)] = 80 \text{ in}^3$$

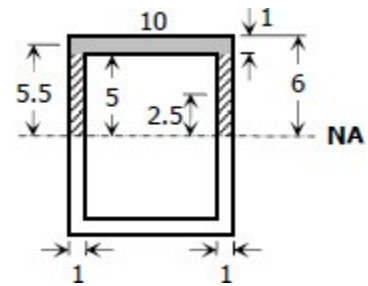
$$b = 2 \text{ in}$$

Thus,

$$150 = \frac{(0.5P + 1160)(80)}{773.33(2)}$$

$$P = 3480 \text{ lb}$$

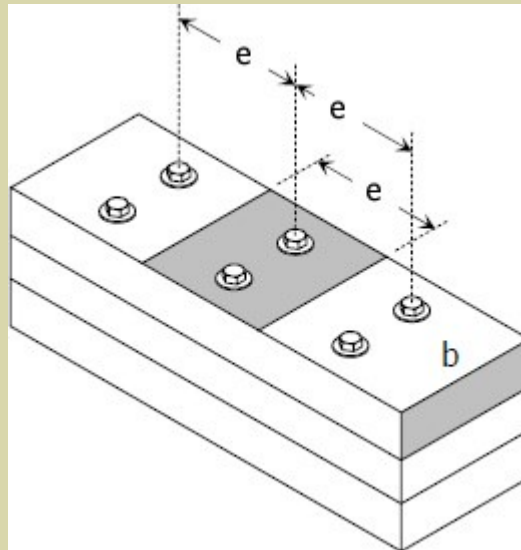
For safe value of P, use **P = 3480 lb** *answer*

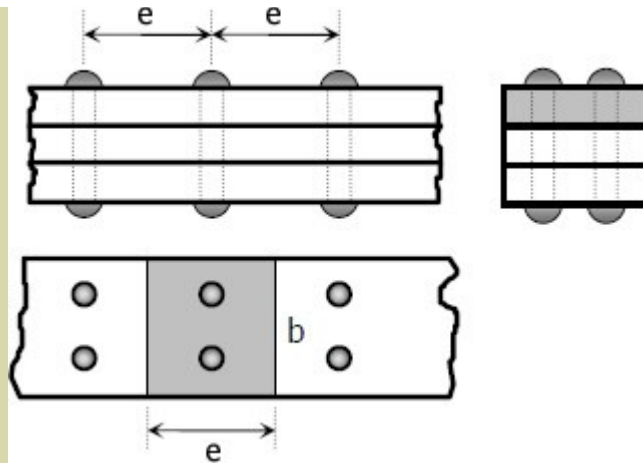


Spacing of Rivets or Bolts in Built-Up Beams

When two or more thin layers of beams are fastened together with a bolt or a rivet so that they act as a unit to gain more strength, it is necessary to design the size or spacing of these bolts or rivets so that it can carry the shearing force acting between each adjacent layers.

Consider the beam shown in the figure.





The shearing stress at the contact surface between the two planks is

$$f_v = \frac{VQ}{Ib}$$

The effective area covered by each bolt group has a length equal to the spacing of the bolts. The total shearing force F acting between the two surfaces must be equal to the total shearing force R produced by the bolts.

$$F = f_v(be) = \frac{VQ}{Ib}be = \frac{VQ}{I}e$$

then,

$$R = \frac{VQ}{I}e$$

where R is the total shearing force to be resisted by the bolts and is equal to the allowable shearing stress $\times$ area $\times$ number of bolts in the group. R should be taken at the contact surface nearest the neutral axis where the shearing stress is greatest. The spacing of bolts, e , is also called pitch.

Problem 592

A wide flange section is formed by bolting together three planks, each 80 mm by 200 mm, arranged as shown in Fig. P-592. If each bolt can withstand a shearing force of 8 kN, determine the pitch if the beam is loaded so as to cause a maximum shearing stress of 1.4 MPa.

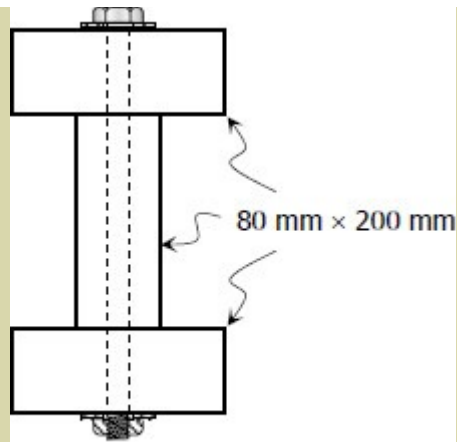


Figure P-592

Solution 592

$$(f_v)_{max} = \frac{VQ_{NA}}{Ib}$$

Where

$$(f_v)_{max} = 1.4 \text{ MPa}$$

$$Q_{NA} = 200(80)(140) + 100(80)(50)$$

$$Q_{NA} = 2\,640\,000 \text{ mm}^3$$

$$I = \frac{200(360^3)}{12} - \frac{120(200^3)}{12}$$

$$I = 697\,600\,000 \text{ mm}^4$$

$$b = 80 \text{ mm}$$

Thus,

$$1.4 = \frac{V(2\,640\,000)}{697\,600\,000(80)}$$

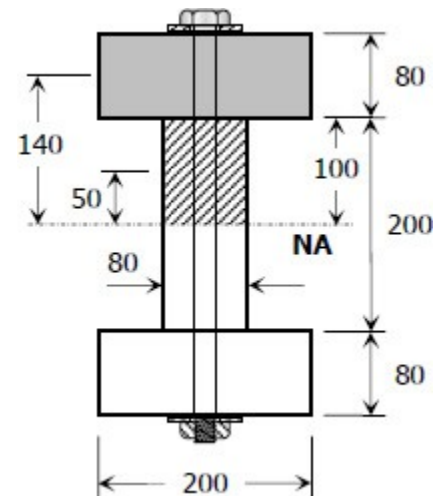
$$V = 29\,595.15 \text{ N}$$

Spacing of bolts

$$R = \frac{VQ_{flange}}{I}e$$

$$8(1000) = \frac{29\,595.15 [200(80)(140)]}{697\,600\,000}e$$

$$e = 84.18 \text{ mm} \quad \text{answer}$$



Problem 593

A box beam, built up as shown in Fig. P-593, is secured by screws spaced 5 in. apart. The beam supports a concentrated load P at the third point of a simply supported span 12 ft long. Determine

the maximum value of P that will not exceed $f_v = 120$ psi in the beam or a shearing force of 300 lb in the screws. What is the maximum flexural stress in the beam?

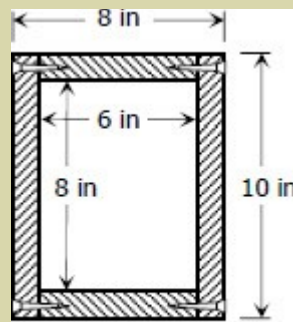


Figure P-539

Solution 593

$$\Sigma M_{R2} = 0$$

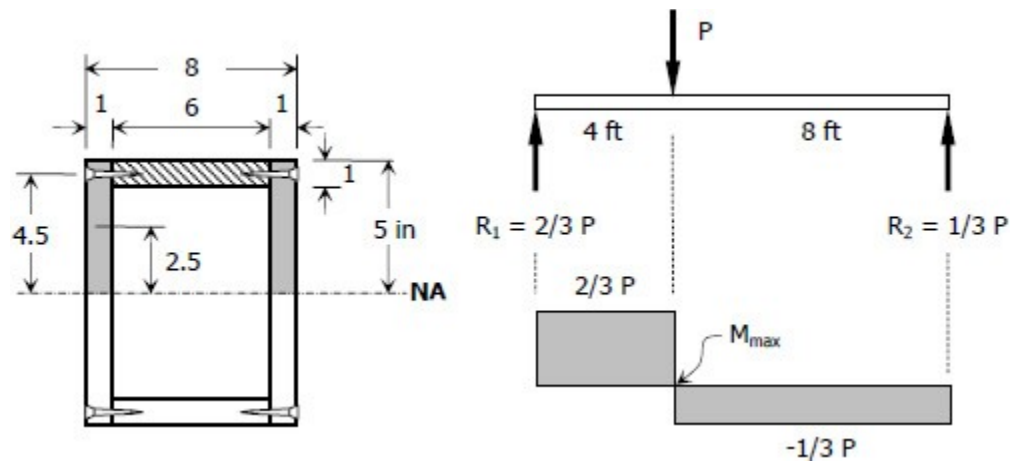
$$12R_1 = 8P$$

$$R_1 = \frac{2}{3}P$$

$$\Sigma M_{R1} = 0$$

$$12R_2 = 4P$$

$$R_2 = \frac{1}{3}P$$



$$M_{max} = 4\left(\frac{2}{3}P\right) = \frac{8}{3}P \text{ lb} \cdot \text{ft}$$

Based on allowable shearing force of beam

$$f_v = \frac{VQ_{NA}}{Ib}$$

Where:

$$V = 2/3 P$$

$$Q_{NA} = 6(1)(4.5) + 2 [5(1)(2.5)] = 52 \text{ in}^3$$

$$I = 8(10^3)/12 - 6(8^3)/12 = 410.67 \text{ in}^4$$

$$b = 2 \text{ in}$$

$$f_v = 120 \text{ psi}$$

Thus,

$$120 = \frac{\frac{2}{3}P(52)}{410.67(2)}$$

$$P = 3843.1 \text{ lb}$$

Based on allowable shearing force of the screws

$$R = \frac{VQ_{screw}}{I}e$$

Where:

$$R = 2(300) = 600 \text{ lb}$$

$$V = 2/3 P$$

$$Q_{screw} = 6(1)(4.5) = 27 \text{ in}^3$$

$$I = 410.67 \text{ in}^4$$

$$e = 5 \text{ in}$$

Thus,

$$600 = \frac{\frac{2}{3}P(27)}{410.67}(5)$$

$$P = 2737.8 \text{ lb}$$

For safe value of P, use **P = 2737.8 lb.** *answer*

Bending stress:

$$f_b = \frac{Mc}{I} = \frac{\frac{3}{8}(2737.8)(12)(5)}{410.67}$$

$$f_b = 150 \text{ psi} \quad \text{answer}$$

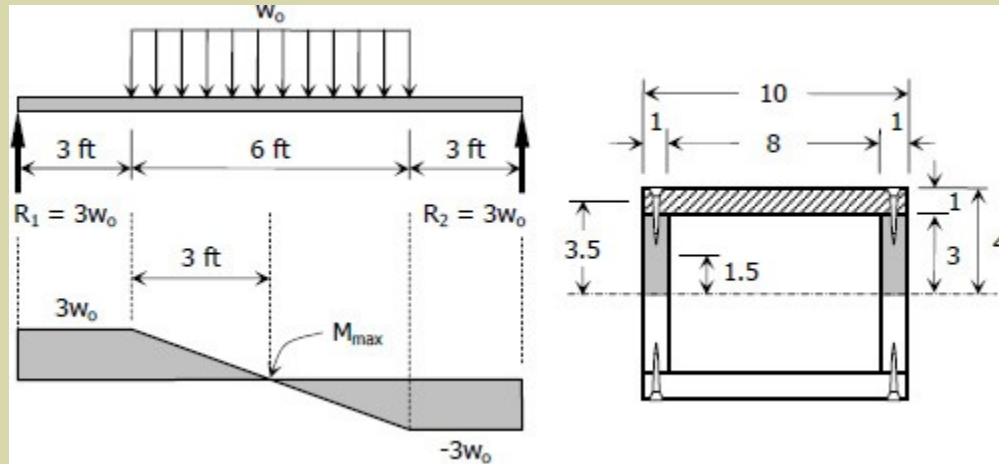
Problem 594

A distributed load of w_o lb/ft is applied over a middle 6 ft of a simply supported span 12 ft long. The beam section is that in [Prob. 593](#), but used here so that the 8-in dimension is vertical. Determine the maximum value of w_o if $f_b \leq 1200$ psi, $f_v \leq 120$ psi, and the screws have a shear strength of 200 lb and a pitch of 2 in.

Solution 594

$$M_{max} = 3(3w_o) + \frac{1}{2}(3)(3w_o)$$

$$M_{max} = 13.5w_o \text{ lb} \cdot \text{ft}$$



$$I = \frac{10(8^3)}{12} - \frac{8(6^3)}{12}$$

$$I = 848/3 \text{ in}^3$$

Based on allowable flexure stress:

$$f_b = \frac{Mc}{I}$$

$$1200 = \frac{13.5w_o(12)(4)}{848/3}$$

$$w_o = 523.46 \text{ lb/ft}$$

Based on shear stress of wood:

$$f_v = \frac{VQ_{NA}}{Ib}$$

$$120 = \frac{3w_o [10(1)(3.5) + 3(1)(1.5)(2)]}{(848/3)(2)}$$

$$w_o = 513.94 \text{ lb/ft}$$

Based on shear strength of screws:

$$R = \frac{VQ_{screws}e}{I}$$

$$2(200) = \frac{3w_o [10(1)(3.5)]}{848/3} (2)$$

$$w_o = 538.41 \text{ lb/ft}$$

For safe value of w_o , use $w_o = 513.94 \text{ lb/ft}$. *answer*

Problem 595

A concentrated load P is carried at midspan of a simply supported 12-ft span. The beam is made of 2-in. by 6-in. pieces screwed together, as shown in Fig. P-595. If the maximum flexural stress developed is 1400 psi, find the maximum shearing stress and the pitch of the screws if each screw can resist 200 lb.

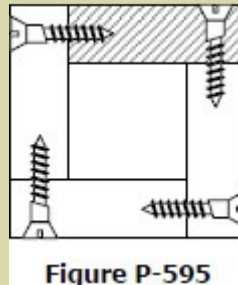


Figure P-595

Solution 595

For concentrated load P at midspan of a simply supported beam of span $L = 12 \text{ ft}$.

$$V_{max} = \frac{1}{2}P$$

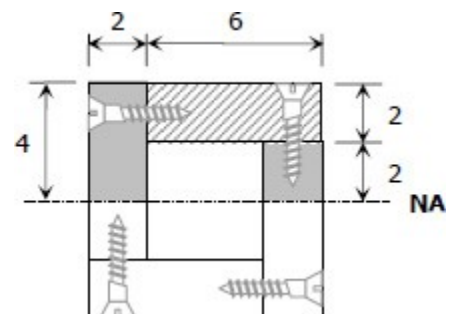
$$M_{max} = \frac{1}{4}PL = \frac{1}{4}P(12) = 3P$$

From the cross section shown:

$$I = \frac{8(8^3)}{12} - \frac{4(4^3)}{12} = 320 \text{ in}^4$$

$$Q_{NA} = 4(2)(2) + 6(2)(3) + 2(2)(1) = 56 \text{ in}^3$$

$$Q_{screw} = 6(2)(3) = 36 \text{ in}^3$$



From bending stress

$$f_b = \frac{Mc}{I}$$

$$1400 = \frac{3P(12)(4)}{320}$$

$$P = 3111.11 \text{ lb}$$

Maximum shear stress

$$f_v = \frac{VQ_{NA}}{Ib} = \frac{\frac{1}{2}(3111.11)(56)}{320(4)}$$

$$f_v = 68.06 \text{ psi} \quad \text{answer}$$

From strength of screws

$$R = \frac{VQ_{screws}}{I}e$$

$$2(200) = \frac{\frac{1}{2}(3111.11)(36)}{320}s$$

$$s = 2.28 \text{ in} \quad \text{answer}$$

Problem 596

Three planks 4 in by 6 in., arranged as shown in Fig. P-596 and secured by bolts spaced 1 ft apart, are used to support a concentrated load P at the center of a simply supported span 12 ft long. If P causes a maximum flexural stress of 1200 psi, determine the bolt diameters, assuming that the shear between the planks is transmitted by friction only. The bolts are tightened to a tension of 20 ksi and the coefficient of friction between the planks is 0.40.

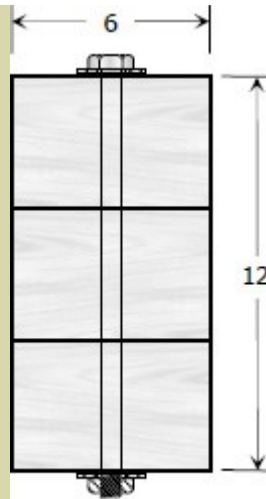


Figure P-596

Solution 596

$$M_{max} = \frac{1}{4}PL = \frac{1}{4}P(12) = 3P \text{ lb} \cdot \text{ft}$$

$$V_{max} = \frac{1}{2}P$$

$$I = \frac{1}{12}(6)(12^3) = 864 \text{ in}^4$$

From allowable flexural stress

$$f_b = \frac{Mc}{I}$$

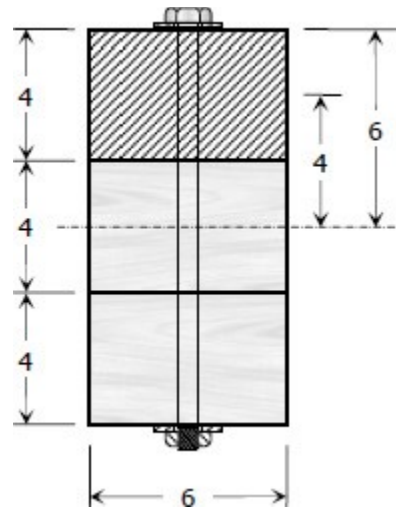
$$1200 = \frac{3P(12)(6)}{864}$$

$$P = 4800 \text{ lb}$$

Strength of bolt

$$R = \frac{VQ_{1st \text{ plank}}}{I}e = \frac{\frac{1}{2}(4800) \left[4(6)(4) \right] (12)}{864}$$

$$R = 3200 \text{ lb}$$



Normal force

$$R = \mu N$$

$$3200 = 0.40N$$

$$N = 8000 \text{ lb}$$

From tensile stress of bolt:

$$\sigma = \frac{\text{Force}}{\text{Area}}$$

$$20\,000 = \frac{8000}{\frac{1}{4}\pi d^2}$$

$$d = 0.7136 \text{ in} \quad \text{answer}$$

Problem 597

A plate and angle girder similar to that shown in Fig. 5-32 is fabricated by riveting the short legs of four $125 \times 75 \times 13$ mm angles to a web plate 1000 mm by 10 mm to form a section 1020 mm deep. Cover plates, each $300 \text{ mm} \times 10 \text{ mm}$, are then riveted to the flange angles making the overall height 1040 mm. The moment of inertia of the entire section about the NA is $I = 4770 \times 10^6 \text{ mm}^4$. Using the allowable stresses specified in Illustrative Problem 591, determine the rivet pitch for 22-mm rivets, attaching the angles to the web plate at a section where $V = 450 \text{ kN}$.

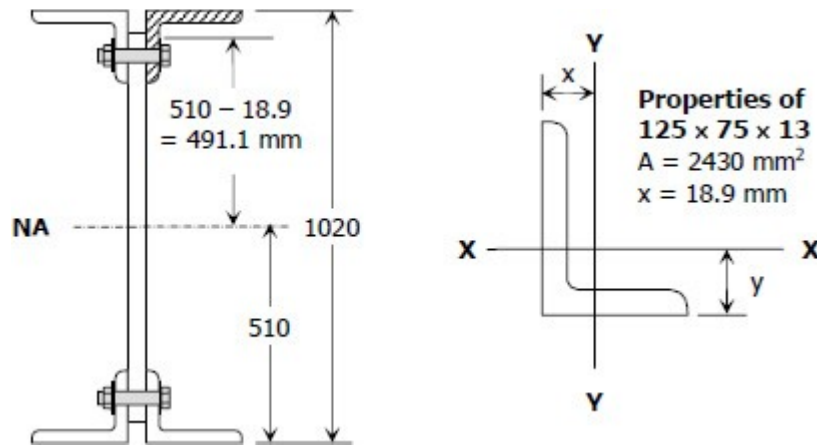
Solution 597

From Illustrative Problem 591

$\tau = 100 \text{ MPa}$ shear stress

$\sigma_b = 220 \text{ MPa}$ bearing stress for single shear rivet

$\sigma_b = 280 \text{ MPa}$ bearing stress for double shear rivet



Rivet capacity in terms of shear (double shear)

$$R_s = 2(A_s \tau) = 2 \left[\frac{1}{4} \pi (22^2) (100) \right]$$

$$R_s = 24\,200\pi \text{ N} = 24.2\pi \text{ kN} = 76.03 \text{ kN}$$

Rivet capacity in terms of bearing (use $\sigma_b = 280 \text{ MPa}$)

$$R_b = A_b \sigma_b = [22(10)](280)$$

$$R_b = 61\,600 \text{ N} = 61.6 \text{ kN}$$

Use $R = 61.6 \text{ kN}$ for safe value of R

From the strength of rivet

$$R = \frac{VQ_{angle}e}{I}$$

$$61.6 = \frac{450 [2(2430)(491.1)]}{4770 \times 10^6} e$$

$$e = 273.58 \text{ mm} \quad \text{answer}$$

Problem 598

As shown in Fig. P-598, two C380 x 60 channels are riveted together by pairs of 19-mm rivets spaced 200 mm apart along the length of the beam. What maximum vertical shear V can be applied to the section without exceeding the stresses given in Illustrative Problem 591?

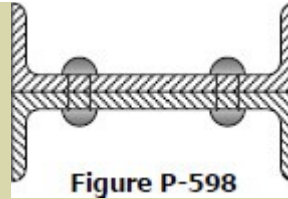


Figure P-598

Solution 598

From Illustrative Problem 591

$\tau = 100$ MPa shear stress

$\sigma_b = 220$ MPa bearing stress for single shear rivet

$\sigma_b = 280$ MPa bearing stress for double shear rivet

By transfer formula for moment of inertia

$$I = \bar{I} + Ad^2$$

$$I_{NA} = 2 \left[(3.84 \times 10^6) + 7570(19.7^2) \right]$$

$$I_{NA} = 13\,555\,682.6 \text{ mm}^4$$

Rivet capacity in shear (single shear)

$$R_s = 2(A_s \tau) = 2 \left[\frac{1}{4} \pi (19^2) (100) \right]$$

$$R_s = 18\,050\pi \text{ N} = 56.705 \text{ kN}$$

Rivet capacity in bearing (use $\sigma_b = 220$ MPa)

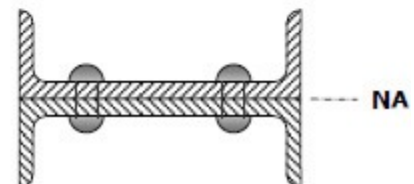
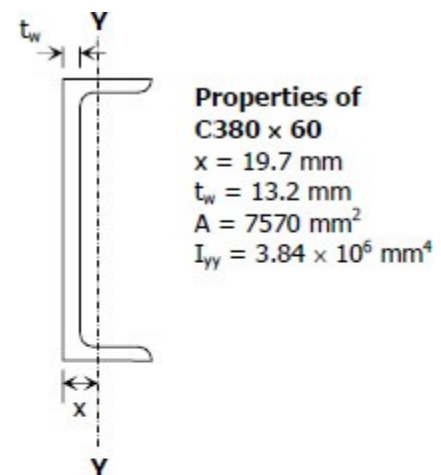
$$R_b = 2(A_b \sigma_b) = 2 \left[19(13.2)(220) \right]$$

$$R_b = 110\,352 \text{ N} = 110.352 \text{ kN}$$

Use $R = 56.705$ kN for safe value of R

From strength of rivets

$$R = \frac{VQ_{NA}}{I} e$$



$$56.705 = \frac{V [7570(19.7)]}{13555682.6} (200)$$

$$V = 25.77 \text{ kN} \quad \text{answer}$$

Problem 599

A beam is formed by bolting together two W200 × 100 sections as shown in Fig. P-599. It is used to support a uniformly distributed load of 30 kN/m (including the weight of the beam) on a simply supported span of 10 m. Compute the maximum flexural stress and the pitch between bolts that have a shearing strength of 30 kN.

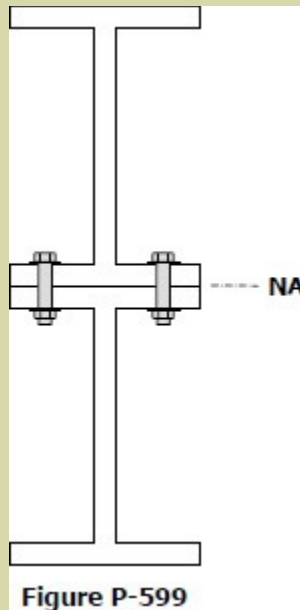


Figure P-599

Solution 599

Properties of W200 × 100

$$A = 12\,700 \text{ mm}^2$$

$$t_f = 23.7 \text{ mm}$$

$$d = 229 \text{ mm}$$

$$I_{xx} = 113 \times 10^6 \text{ mm}^4$$

Maximum moment

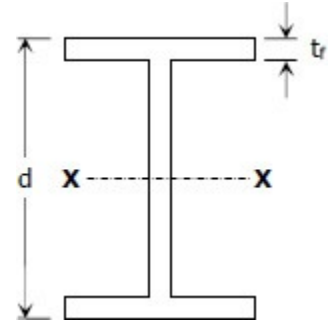
$$M_{max} = \frac{w_o L^2}{8} = \frac{30(10^2)}{8}$$

$$M_{max} = 375 \text{ kN} \cdot \text{m}$$

Maximum shear

$$V_{max} = \frac{w_o L}{2} = \frac{30(10)}{2}$$

$$V_{max} = 150 \text{ kN}$$



By transfer formula for moment of inertia

$$I_{NA} = 2 [(113 \times 10^6) + (12700)(229/2)^2]$$

$$I_{NA} = 559\,000\,350 \text{ mm}^4$$

Maximum flexural stress

$$(f_b)_{max} = \frac{Mc}{I} = \frac{375(1000^2)(229)}{559\,000\,350}$$

$$(f_b)_{max} = 153.62 \text{ MPa} \quad \text{answer}$$

Bolt pitch

$$R = \frac{VQ_{NA}}{I}e$$

$$2(30) = \frac{150 [12700(229/2)]}{559\,000\,350}e$$

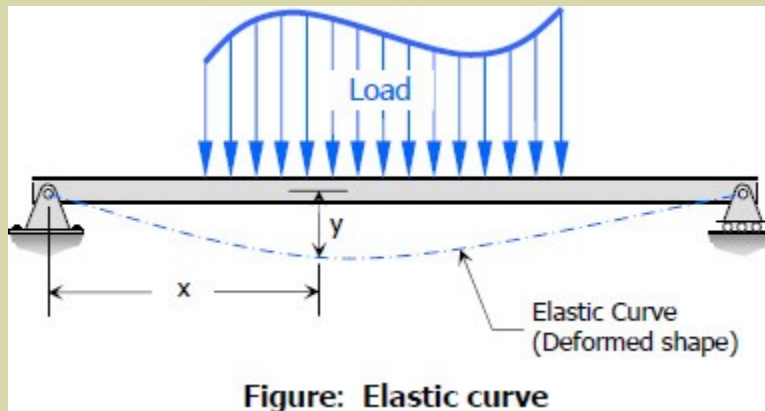
$$e = 153.77 \text{ mm} \quad \text{answer}$$

Chapter 06 - Beam Deflections

Deflection of Beams

The deformation of a beam is usually expressed in terms of its deflection from its original unloaded position. The deflection is measured from the original neutral surface of the beam to the neutral surface of the deformed beam. The configuration assumed by the deformed neutral surface is known

as the elastic curve of the beam.



Methods of Determining Beam Deflections

Numerous methods are available for the determination of beam deflections. These methods include:

1. [Double-integration method](#)
2. [Area-moment method](#)
3. [Strain-energy method \(Castigliano's Theorem\)](#)
4. [Conjugate-beam method](#)
5. [Method of superposition](#)

Of these methods, the first two are the ones that are commonly used

Double Integration Method | Beam Deflections

The double integration method is a powerful tool in solving deflection and slope of a beam at any point because we will be able to get the equation of the elastic curve.

In calculus, the radius of curvature of a curve $y = f(x)$ is given by

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|}$$

In the derivation of [flexure formula](#), the radius of curvature of a beam is given as

$$\rho = \frac{EI}{M}$$

Deflection of beams is so small, such that the slope of the elastic curve dy/dx is very small, and squaring this expression the value becomes practically negligible, hence

$$\rho = \frac{1}{d^2y/dx^2} = \frac{1}{y''}$$

Thus, $EI / M = 1 / y''$

$$y'' = \frac{M}{EI} = \frac{1}{EI}M$$

If EI is constant, the equation may be written as:

$$EI y'' = M$$

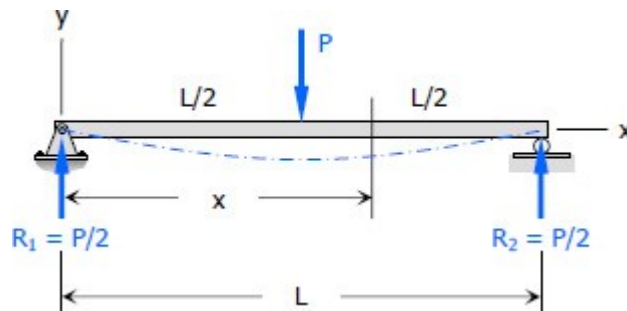
where x and y are the coordinates shown in the figure of the [elastic curve of the beam under load](#), y is the deflection of the beam at any distance x . E is the modulus of elasticity of the beam, I represent the moment of inertia about the neutral axis, and M represents the bending moment at a distance x from the end of the beam. The product EI is called the **flexural rigidity** of the beam.

The first integration y' yields the slope of the elastic curve and the second integration y gives the deflection of the beam at any distance x . The resulting solution must contain two constants of integration since $EI y'' = M$ is of second order. These two constants must be evaluated from known conditions concerning the slope deflection at certain points of the beam. For instance, in the case of a simply supported beam with rigid supports, at $x = 0$ and $x = L$, the deflection $y = 0$, and in locating the point of maximum deflection, we simply set the slope of the elastic curve y' to zero.

Problem 605

Determine the maximum deflection δ in a simply supported beam of length L carrying a concentrated load P at midspan.

Solution 605



$$EI y'' = \frac{1}{2}Px - P\langle x - \frac{1}{2}L \rangle$$

$$EI y' = \frac{1}{4}Px^2 - \frac{1}{2}P\langle x - \frac{1}{2}L \rangle^2 + C_1$$

$$EI y = \frac{1}{12}Px^3 - \frac{1}{6}P\langle x - \frac{1}{2}L \rangle^3 + C_1x + C_2$$

At $x = 0$, $y = 0$, therefore, $C_2 = 0$

At $x = L$, $y = 0$

$$0 = \frac{1}{12}PL^3 - \frac{1}{6}P\langle L - \frac{1}{2}L \rangle^3 + C_1L$$

$$0 = \frac{1}{12}PL^3 - \frac{1}{48}PL^3 + C_1L$$

$$C_1 = -\frac{1}{16}PL^2$$

Thus,

$$EI y = \frac{1}{12}Px^3 - \frac{1}{6}P\langle x - \frac{1}{2}L \rangle^3 - \frac{1}{16}PL^2x$$

Maximum deflection will occur at $x = \frac{1}{2}L$ (midspan)

$$EI y_{max} = \frac{1}{12}P(\frac{1}{2}L)^3 - \frac{1}{6}P(\frac{1}{2}L - \frac{1}{2}L)^3 - \frac{1}{16}PL^2(\frac{1}{2}L)$$

$$EI y_{max} = \frac{1}{96}PL^3 - 0 - \frac{1}{32}PL^3$$

$$y_{max} = -\frac{PL^3}{48EI}$$

The negative sign indicates that the deflection is below the undeformed neutral axis.

Therefore,

$$\delta_{max} = \frac{PL^3}{48EI} \quad \text{answer}$$

Problem 606

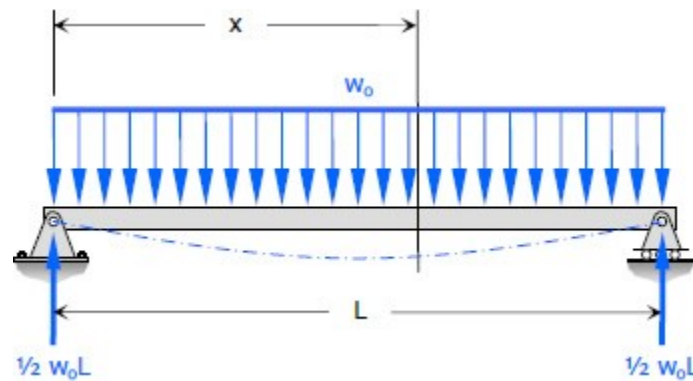
Determine the maximum deflection δ in a simply supported beam of length L carrying a uniformly distributed load of intensity w_o applied over its entire length.

Solution 606

From the figure below

$$EI y'' = \frac{1}{2}w_o Lx - w_o x\left(\frac{1}{2}x\right)$$

$$EI y'' = \frac{1}{2}w_o Lx - \frac{1}{2}w_o x^2$$



$$EI y' = \frac{1}{4}w_o Lx^2 - \frac{1}{6}w_o x^3 + C_1$$

$$EI y = \frac{1}{12}w_o Lx^3 - \frac{1}{24}w_o x^4 + C_1 x + C_2$$

At $x = 0$, $y = 0$, therefore $C_2 = 0$

At $x = L$, $y = 0$

$$0 = \frac{1}{12}w_o L^4 - \frac{1}{24}w_o L^4 + C_1 L$$

$$C_1 = -\frac{1}{24}w_o L^3$$

Therefore,

$$EI y = \frac{1}{12}w_o Lx^3 - \frac{1}{24}w_o x^4 - \frac{1}{24}w_o L^3 x$$

Maximum deflection will occur at $x = \frac{1}{2} L$ (midspan)

$$EI y_{max} = \frac{1}{12}w_o L\left(\frac{1}{2}L\right)^3 - \frac{1}{24}w_o\left(\frac{1}{2}L\right)^4 - \frac{1}{24}w_o L^3\left(\frac{1}{2}L\right)$$

$$EI y_{max} = \frac{1}{96}w_o L^4 - \frac{1}{384}w_o L^4 - \frac{1}{48}w_o L^4$$

$$EI y_{max} = -\frac{5}{384}w_o L^4$$

$$\delta_{max} = \frac{5w_o L^4}{384EI} \quad \text{answer}$$

Taking $W = w_o L$:

$$\delta_{max} = \frac{5(w_o L)(L^3)}{384EI}$$

$$\delta_{max} = \frac{5WL^3}{384EI} \quad \text{answer}$$

Problem 607

Determine the maximum value of EIy for the cantilever beam loaded as shown in [Fig. P-607](#). Take the origin at the wall.

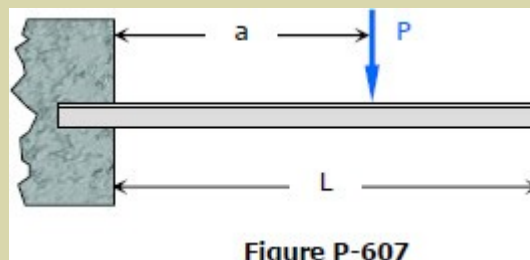


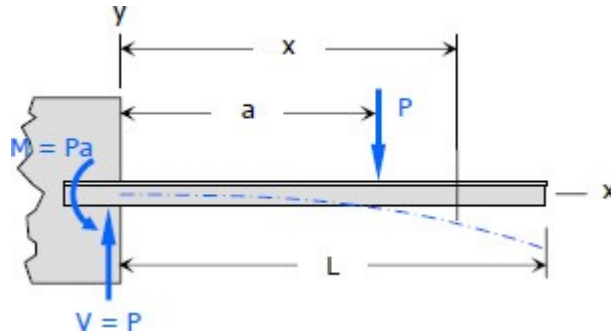
Figure P-607

Solution 607

$$EI y'' = -Pa + Px - P\langle x - a \rangle$$

$$EI y' = -Pax + \frac{1}{2}Px^2 - \frac{1}{2}P\langle x - a \rangle^2 + C_1$$

$$EI y = -\frac{1}{2}Pax^2 + \frac{1}{6}Px^3 - \frac{1}{6}P\langle x - a \rangle^3 + C_1x + C_2$$



At $x = 0$, $y' = 0$, therefore $C_1 = 0$

At $x = 0$, $y = 0$, therefore $C_2 = 0$

Therefore,

$$EI y = -\frac{1}{2}Pax^2 + \frac{1}{6}Px^3 - \frac{1}{6}P\langle x - a \rangle^3$$

The maximum value of $EI y$ is at $x = L$ (free end)

$$EI y_{max} = -\frac{1}{2}PaL^2 + \frac{1}{6}PL^3 - \frac{1}{6}P(L - a)^3$$

$$EI y_{max} = -\frac{1}{2}PaL^2 + \frac{1}{6}PL^3 - \frac{1}{6}P(L^3 - 3L^2a + 3La^2 - a^3)$$

$$EI y_{max} = -\frac{1}{2}PaL^2 + \frac{1}{6}PL^3 - \frac{1}{6}PL^3 + \frac{1}{2}PL^2a - \frac{1}{2}PLa^2 + \frac{1}{6}Pa^3$$

$$EI y_{max} = -\frac{1}{2}PLa^2 + \frac{1}{6}Pa^3$$

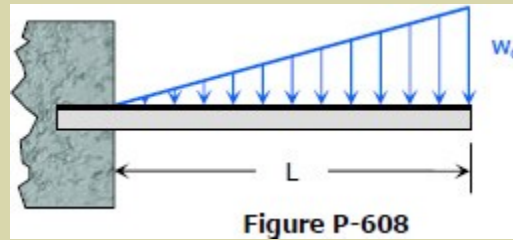
$$EI y_{max} = -\frac{1}{2}PLa^2 + \frac{1}{6}Pa^3$$

$$EI y_{max} = -\frac{1}{6}Pa^2(3L - a) \quad \text{answer}$$

Problem 608

Find the equation of the elastic curve for the cantilever beam shown in [Fig. P-608](#); it carries a load

that varies from zero at the wall to w_o at the free end. Take the origin at the wall.

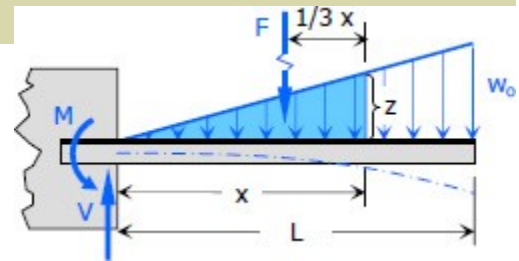


Solution 608

$$V = \frac{1}{2}w_oL$$

$$M = \frac{1}{2}w_oL\left(\frac{2}{3}L\right)$$

$$M = \frac{1}{3}w_oL^2$$



By ratio and proportion

$$\frac{z}{x} = \frac{w_o}{L}$$

$$z = \frac{w_o}{L}x$$

$$F = \frac{1}{2}xz$$

$$F = \frac{1}{2}x\left(\frac{w_o}{L}x\right)$$

$$F = \frac{w_o}{2L}x^2$$

$$EI y'' = -M + Vx - F\left(\frac{1}{3}x\right)$$

$$EI y'' = -\frac{1}{3}w_oL^2 + \frac{1}{2}w_oLx - \frac{1}{3}x\left(\frac{w_o}{2L}x^2\right)$$

$$EI y'' = -\frac{w_oL^2}{3} + \frac{w_oL}{2}x - \frac{w_o}{6L}x^3$$

$$EI y' = -\frac{w_o L^2}{3}x + \frac{w_o L}{4}x^2 - \frac{w_o}{24L}x^4 + C_1$$

$$EI y = -\frac{w_o L^2}{6}x^2 + \frac{w_o L}{12}x^3 - \frac{w_o}{120L}x^5 + C_1 x + C_2$$

At $x = 0$, $y' = 0$, therefore $C_1 = 0$

At $x = 0$, $y = 0$, therefore $C_2 = 0$

Therefore, the equation of the elastic curve is

$$EI y = -\frac{w_o L^2}{6}x^2 + \frac{w_o L}{12}x^3 - \frac{w_o}{120L}x^5 \quad \text{answer}$$

Problem 609

As shown in [Fig. P-609](#), a simply supported beam carries two symmetrically placed concentrated loads. Compute the maximum deflection δ .

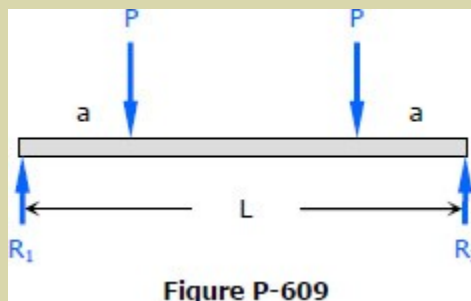


Figure P-609

Solution 609

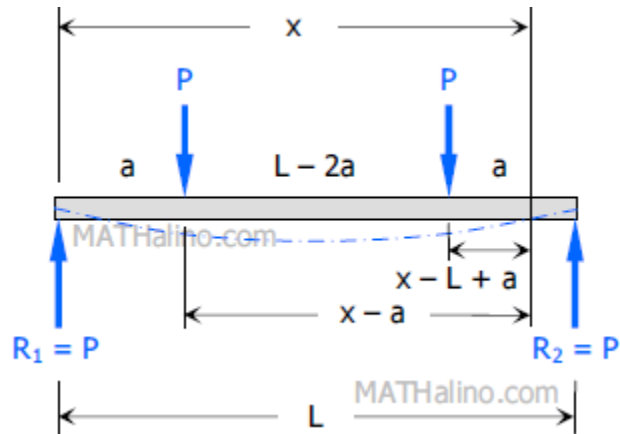
By symmetry

$$R_1 = R_2 = P$$

$$EI y'' = Px - P\langle x - a \rangle - P\langle x - L + a \rangle$$

$$EI y' = \frac{1}{2}Px^2 - \frac{1}{2}P\langle x - a \rangle^2 - \frac{1}{2}P\langle x - L + a \rangle^2 + C_1$$

$$EI y = \frac{1}{6}Px^3 - \frac{1}{6}P\langle x - a \rangle^3 - \frac{1}{6}P\langle x - L + a \rangle^3 + C_1 x + C_2$$



At $x = 0, y = 0$, therefore $C_2 = 0$

At $x = L, y = 0$

$$0 = \frac{1}{6}PL^3 - \frac{1}{6}P(L-a)^3 + C_1L$$

$$0 = PL^3 - P(L^3 - 3L^2a + 3La^2 - a^3) - Pa^3 + 6C_1L$$

$$0 = PL^3 - PL^3 + 3PL^2a - 3PLa^2 + Pa^3 - Pa^3 + 6C_1L$$

$$0 = 3PL^2a - 3PLa^2 + 6C_1L$$

$$0 = 3PLa(L-a) + 6C_1L$$

$$C_1 = -\frac{1}{2}Pa(L-a)$$

Therefore,

$$EI y = \frac{1}{6}Px^3 - \frac{1}{6}P(x-a)^3 - \frac{1}{6}P(x-L+a)^3 - \frac{1}{2}Pa(L-a)x$$

Maximum deflection will occur at $x = \frac{1}{2}L$ (midspan)

$$EI y_{max} = \frac{1}{6}P(\frac{1}{2}L)^3 - \frac{1}{6}P(\frac{1}{2}L-a)^3 - \frac{1}{2}Pa(L-a)(\frac{1}{2}L)$$

$$EI y_{max} = \frac{1}{48}PL^3 - \frac{1}{6}P[\frac{1}{2}(L-2a)]^3 - \frac{1}{4}PL^2a + \frac{1}{4}PLa^2$$

$$EI y_{max} = \frac{1}{48}PL^3 - \frac{1}{48}P[L^3 - 3L^2(2a) + 3L(2a)^2 - (2a)^3] - \frac{1}{4}PL^2a + \frac{1}{4}PLa^2$$

$$EI y_{max} = \frac{1}{48}PL^3 - \frac{1}{48}PL^3 + \frac{1}{8}PL^2a - \frac{1}{4}PLa^2 + \frac{1}{6}Pa^3 - \frac{1}{4}PL^2a + \frac{1}{4}PLa^2$$

$$EI y_{max} = -\frac{1}{8}PL^2a + \frac{1}{6}Pa^3$$

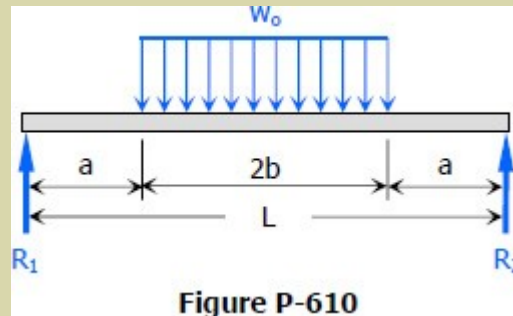
$$EI y_{max} = -\frac{1}{24}Pa(3L^2 - 4a^2)$$

$$y_{max} = -\frac{Pa}{24EI}(3L^2 - 4a^2)$$

$$\delta_{max} = \frac{Pa}{24EI}(3L^2 - 4a^2) \quad \text{answer}$$

Problem 610

The simply supported beam shown in [Fig. P-610](#) carries a uniform load of intensity w_o symmetrically distributed over part of its length. Determine the maximum deflection δ and check your result by letting $a = 0$ and comparing with the answer to [Problem 606](#).



Solution 610

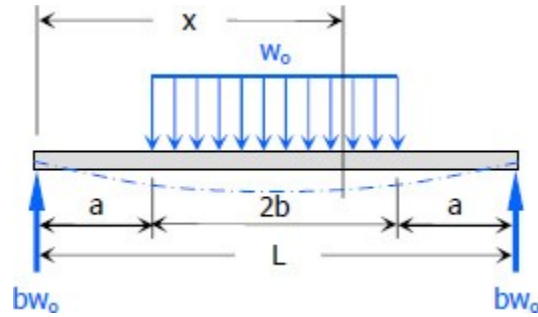
By symmetry

$$R_1 = R_2 = bw_o$$

$$EI y'' = bw_o x - \frac{1}{2}w_o \langle x - a \rangle^2$$

$$EI y' = \frac{1}{2}bw_o x^2 - \frac{1}{6}w_o \langle x - a \rangle^3 + C_1$$

$$EI y = \frac{1}{6}bw_o x^3 - \frac{1}{24}w_o \langle x - a \rangle^4 + C_1 x + C_2$$



At $x = 0, y = 0$, therefore $C_2 = 0$

At $x = a + b, y' = 0$

$$0 = \frac{1}{2}bw_o(a+b)^2 - \frac{1}{6}w_ob^3 + C_1$$

$$C_1 = \frac{1}{6}w_ob^3 - \frac{1}{2}bw_o(a+b)^2$$

Therefore,

$$EI y = \frac{1}{6}bw_o x^3 - \frac{1}{24}w_o(x-a)^4 + \frac{1}{6}w_ob^3 x - \frac{1}{2}bw_o(a+b)^2 x$$

Maximum deflection will occur at $x = a + b$ (midspan)

$$EI y_{max} = \frac{1}{6}bw_o(a+b)^3 - \frac{1}{24}w_ob^4 + \frac{1}{6}w_ob^3(a+b) - \frac{1}{2}bw_o(a+b)^3$$

$$EI y_{max} = -\frac{1}{3}bw_o(a+b)^3 - \frac{1}{24}w_ob^4 + \frac{1}{6}w_ob^3(a+b)$$

$$EI y_{max} = -\frac{1}{24}w_ob[8(a+b)^3 + b^3 - 4b^2(a+b)]$$

Therefore,

$$\delta_{max} = \frac{w_ob}{24EI} [8(a+b)^3 + b^3 - 4b^2(a+b)] \quad \text{answer}$$

Checking:

When $a = 0, 2b = L$, thus $b = \frac{1}{2}L$

$$\delta_{max} = \frac{w_o(\frac{1}{2}L)}{24EI} [8(0 + \frac{1}{2}L)^3 + (\frac{1}{2}L)^3 - 4(\frac{1}{2}L)^2(0 + \frac{1}{2}L)]$$

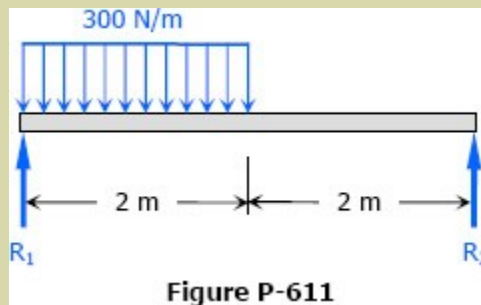
$$\delta_{max} = \frac{w_oL}{48EI} [L^3 + \frac{1}{8}L^3 - \frac{1}{2}L^3]$$

$$\delta_{max} = \frac{w_o L}{48EI} \left[\frac{5}{8} L^3 \right]$$

$$\delta_{max} = \frac{5w_o L^4}{384EI} \quad (\text{okay!})$$

Problem 611

Compute the value of $EI \delta$ at midspan for the beam loaded as shown in Fig. P-611. If $E = 10$ GPa, what value of I is required to limit the midspan deflection to $1/360$ of the span?



Solution 611

$$\Sigma M_{R2} = 0$$

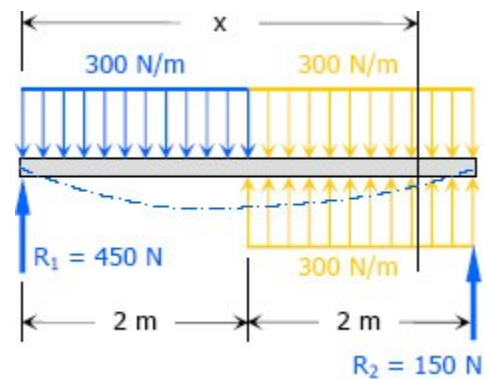
$$4R_1 = 300(2)(3)$$

$$R_1 = 450 \text{ N}$$

$$\Sigma M_{R1} = 0$$

$$4R_2 = 300(2)(1)$$

$$R_2 = 150 \text{ N}$$



$$EI y'' = 450x - \frac{1}{2}(300)x^2 + \frac{1}{2}(300)\langle x - 2 \rangle^2$$

$$EI y'' = 450x - 150x^2 + 150\langle x - 2 \rangle^2$$

$$EI y' = 225x^2 - 50x^3 + 50\langle x - 2 \rangle^3 + C_1$$

$$EI y = 75x^3 - 12.5x^4 + 12.5\langle x - 2 \rangle^4 + C_1x + C_2$$

At $x = 0$, $y = 0$, therefore $C_2 = 0$

At $x = 4$ m, $y = 0$

$$0 = 75(4^3) - 12.5(4^4) + 12.5(4 - 2)^4 + 4C_1$$

$$C_1 = -450 \text{ N} \cdot \text{m}^2$$

Therefore,

$$EI y = 75x^3 - 12.5x^4 + 12.5\langle x - 2 \rangle^4 - 450x$$

At $x = 2$ m (midspan)

$$EI y_{midspan} = 75(2^3) - 12.5(2^4) + 12.5(2 - 2)^4 - 450(2)$$

$$EI y_{midspan} = -500 \text{ N} \cdot \text{m}^3$$

$$EI \delta_{midspan} = 500 \text{ N} \cdot \text{m}^3$$

Maximum midspan deflection

$$\delta_{midspan} = \frac{1}{360} L = \frac{1}{360} (4) = \frac{1}{90} \text{ m}$$

$$\delta_{midspan} = \frac{100}{9} \text{ mm}$$

Thus,

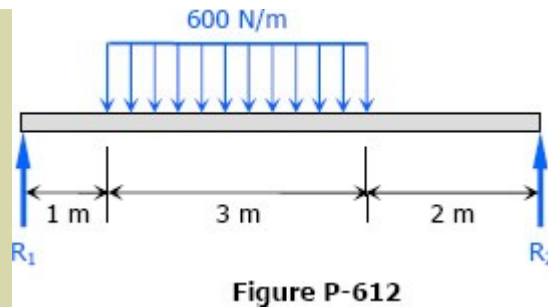
$$10\,000 I \left(\frac{100}{9} \right) = 500(1000^3)$$

$$I = 4\,500\,000 \text{ mm}^4$$

$$I = 4.5 \times 10^6 \text{ mm}^4 \quad \text{answer}$$

Problem 612

Compute the midspan value of $EI \delta$ for the beam loaded as shown in Fig. P-612.



Solution 612

$$\Sigma M_{R_2} = 0$$

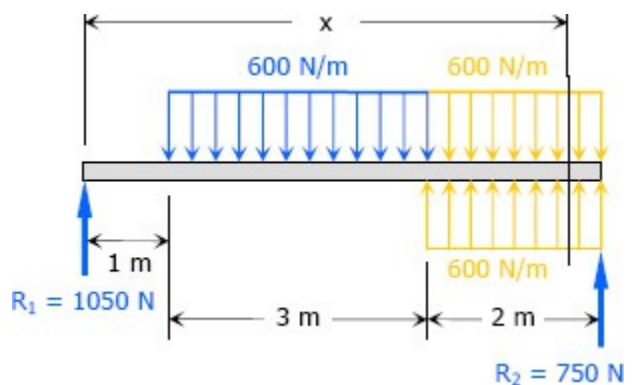
$$6R_1 = 600(3)(3.5)$$

$$R_1 = 1050 \text{ N}$$

$$\Sigma M_{R_1} = 0$$

$$6R_2 = 600(3)(2.5)$$

$$R_2 = 750 \text{ N}$$



$$EI y'' = 1050x - \frac{1}{2}(600)\langle x - 1 \rangle^2 + \frac{1}{2}(600)\langle x - 4 \rangle^2$$

$$EI y'' = 1050x - 300\langle x - 1 \rangle^2 + 300\langle x - 4 \rangle^2$$

$$EI y' = 525x^2 - 100\langle x - 1 \rangle^3 + 100\langle x - 4 \rangle^3 + C_1$$

$$EI y = 175x^3 - 25\langle x - 1 \rangle^4 + 25\langle x - 4 \rangle^4 + C_1x + C_2$$

At $x = 0$, $y = 0$, therefore $C_2 = 0$

At $x = 6 \text{ m}$, $y = 0$

$$0 = 175(6^3) - 25(6 - 1)^4 + 25(6 - 4)^4 + 6C_1$$

$$C_1 = -3762.5 \text{ N} \cdot \text{m}^2$$

Therefore,

$$EI y = 175x^3 - 25\langle x - 1 \rangle^4 + 25\langle x - 4 \rangle^4 - 3762.5x$$

At midspan, $x = 3$ m

$$EI y_{midspan} = 175(3^3) - 25(3 - 1)^4 - 3762.5(3)$$

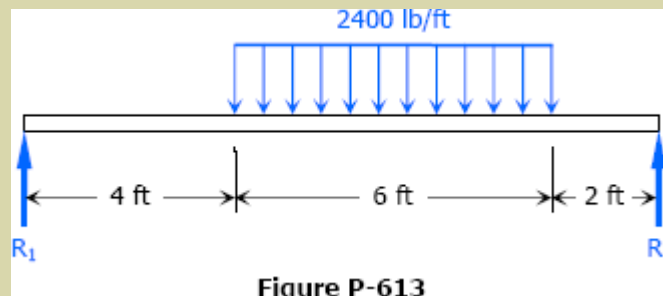
$$EI y_{midspan} = -6962.5 \text{ N} \cdot \text{m}^3$$

Thus,

$$EI \delta_{midspan} = 6962.5 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

Problem 613

If $E = 29 \times 10^6$ psi, what value of I is required to limit the midspan deflection to $1/360$ of the span for the beam in [Fig. P-613](#)?



Solution 613

$$\Sigma M_{R2} = 0$$

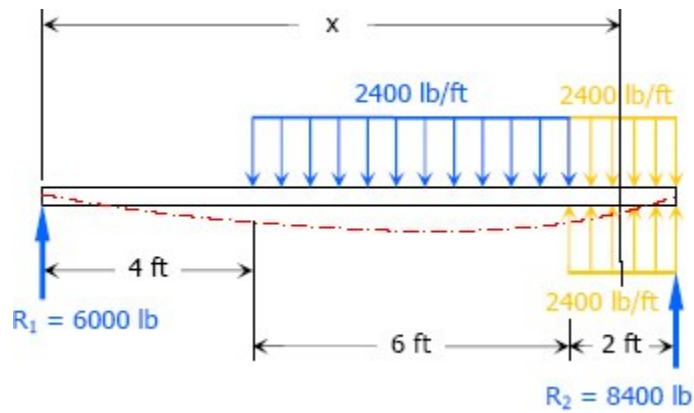
$$12R_1 = 2400(6)(5)$$

$$R_1 = 6000 \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$12R_2 = 2400(6)(7)$$

$$R_2 = 8400 \text{ lb}$$



$$EI y'' = 6000x - \frac{1}{2}(2400) \langle x - 4 \rangle^2 + \frac{1}{2}(2400) \langle x - 10 \rangle^2$$

$$EI y'' = 6000x - 1200 \langle x - 4 \rangle^2 + 1200 \langle x - 10 \rangle^2$$

$$EI y' = 3000x^2 - 400 \langle x - 4 \rangle^3 + 400 \langle x - 10 \rangle^3 + C_1$$

$$EI y = 1000x^3 - 100 \langle x - 4 \rangle^4 + 100 \langle x - 10 \rangle^4 + C_1x + C_2$$

At $x = 0$, $y = 0$, therefore $C_2 = 0$

At $x = 12$ ft, $y = 0$

$$0 = 1000(12^3) - 100(12 - 4)^4 + 100(12 - 10)^4 + 12C_1$$

$$C_1 = -110\,000 \text{ lb} \cdot \text{ft}$$

Therefore

$$EI y = 1000x^3 - 100 \langle x - 4 \rangle^4 + 100 \langle x - 10 \rangle^4 - 110\,000x$$

$$E = 29 \times 10^6 \text{ psi}$$

$$L = 12 \text{ ft}$$

At midspan, $x = 6$ ft

$$y = -1/360 (12) = -1/30 \text{ ft} = -2/5 \text{ in}$$

Thus,

$$EI y = 1000x^3 - 100 \langle x - 4 \rangle^4 + 100 \langle x - 10 \rangle^4 - 110\,000x$$

$$(29 \times 10^6)I\left(-\frac{2}{5}\right) = [1000(6^3) + 100(2^4) - 110\,000(6)](12^3)$$

$$I = 65.9 \text{ in}^4 \quad \text{answer}$$

Problem 614

For the beam loaded as shown in Fig. P-614, calculate the slope of the elastic curve over the right support.

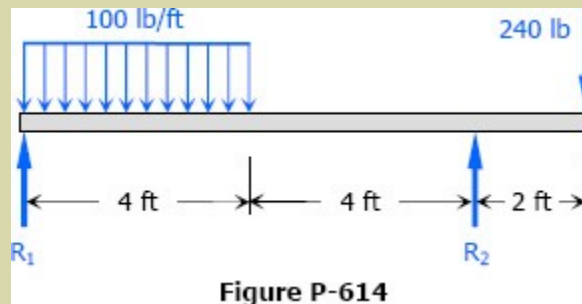


Figure P-614

Solution 614

$$\Sigma M_{R2} = 0$$

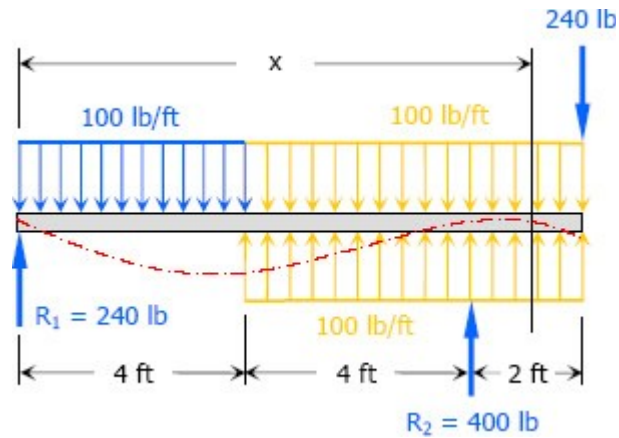
$$8R_1 + 240(2) = 100(4)(6)$$

$$R_1 = 240 \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$8R_2 = 240(10) + 100(4)(2)$$

$$R_2 = 400 \text{ lb}$$



$$EI y'' = 240x - \frac{1}{2}(100)x^2 + \frac{1}{2}(100) \langle x - 4 \rangle^2 + 400 \langle x - 8 \rangle$$

$$EI y'' = 240x - 50x^2 + 50 \langle x - 4 \rangle^2 + 400 \langle x - 8 \rangle$$

$$EI y' = 120x^2 - \frac{50}{3}x^3 + \frac{50}{3} \langle x - 4 \rangle^3 + 200 \langle x - 8 \rangle^2 + C_1$$

$$EI y = 40x^3 - \frac{25}{6}x^4 + \frac{25}{6} \langle x - 4 \rangle^4 + \frac{200}{3} \langle x - 8 \rangle^3 + C_1x + C_2$$

At $x = 0$, $y = 0$, therefore $C_2 = 0$

At $x = 8$ ft, $y = 0$

$$0 = 40(8^3) - \frac{25}{6}(8^4) + \frac{25}{6}(4^4) + 8C_1$$

$$C_1 = -560 \text{ lb} \cdot \text{ft}^2$$

Thus,

$$EI y' = 120x^2 - \frac{50}{3}x^3 + \frac{50}{3} \langle x - 4 \rangle^3 + 200 \langle x - 8 \rangle^2 - 560$$

At the right support, $x = 8$ ft

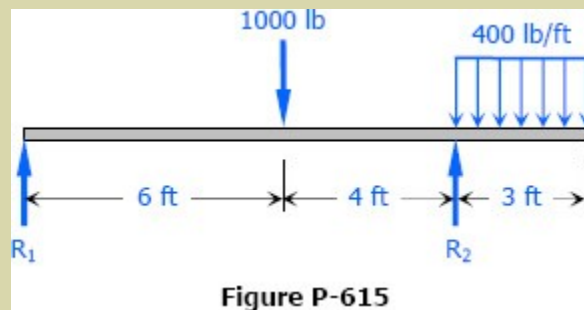
$$EI y' = 120(8^2) - \frac{50}{3}(8^3) + \frac{50}{3}(4^3) - 560$$

$$EI y' = -\frac{1040}{3} \text{ lb} \cdot \text{ft}^2$$

$$y' = -\frac{1040}{3EI} \text{ lb} \cdot \text{ft}^2 \quad \text{answer}$$

Problem 615

Compute the value of $EI y$ at the right end of the overhanging beam shown in Fig. P-615.

**Solution 615**

$$\Sigma M_{R_2} = 0$$

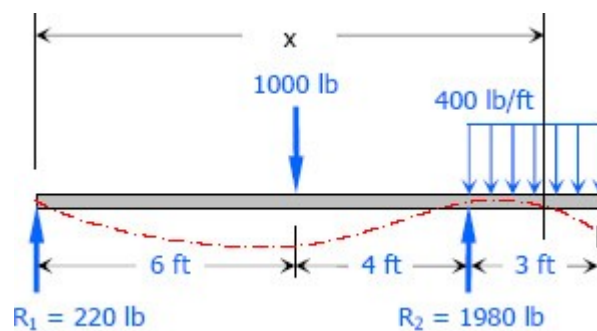
$$10R_1 + 400(3)(1.5) = 1000(4)$$

$$R_1 = 220 \text{ lb}$$

$$\Sigma M_{R_1} = 0$$

$$10R_2 = 400(3)(11.5) + 1000(6)$$

$$R_2 = 1980 \text{ lb}$$



$$EI y'' = 220x - 1000 \langle x - 6 \rangle + 1980 \langle x - 10 \rangle - \frac{1}{2}(400) \langle x - 10 \rangle^2$$

$$EI y'' = 220x - 1000 \langle x - 6 \rangle + 1980 \langle x - 10 \rangle - 200 \langle x - 10 \rangle^2$$

$$EI y' = 110x^2 - 500 \langle x - 6 \rangle^2 + 990 \langle x - 10 \rangle^2 - \frac{200}{3} \langle x - 10 \rangle^3 + C_1$$

$$EI y = \frac{110}{3}x^3 - \frac{500}{3} \langle x - 6 \rangle^3 + 330 \langle x - 10 \rangle^3 - \frac{50}{3} \langle x - 10 \rangle^4 + C_1 x + C_2$$

At $x = 0$, $y = 0$, therefore $C_2 = 0$

At $x = 10$ ft, $y = 0$

$$0 = (110/3)(10^3) - (500/3)(4^3) + 10C_1$$

$$C_1 = -2600 \text{ lb} \cdot \text{ft}^2$$

Therefore,

$$EI y = \frac{110}{3}x^3 - \frac{500}{3} \langle x - 6 \rangle^3 + 330 \langle x - 10 \rangle^3 - \frac{50}{3} \langle x - 10 \rangle^4 - 2600x$$

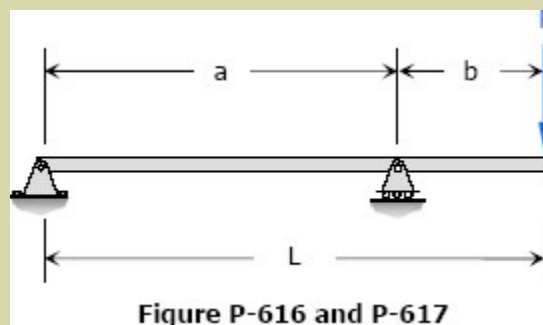
At the right end of the beam, $x = 13$ ft

$$EI y = \frac{110}{3}(13^3) - \frac{500}{3}(7^3) + 330(3^3) - \frac{50}{3}(3^4) - 2600(13)$$

$$EI y = -2850 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

Problem 616

For the beam loaded as shown in Fig. P-616, determine (a) the deflection and slope under the load P and (b) the maximum deflection between the supports.



Solution 616

$$\Sigma M_{R2} = 0$$

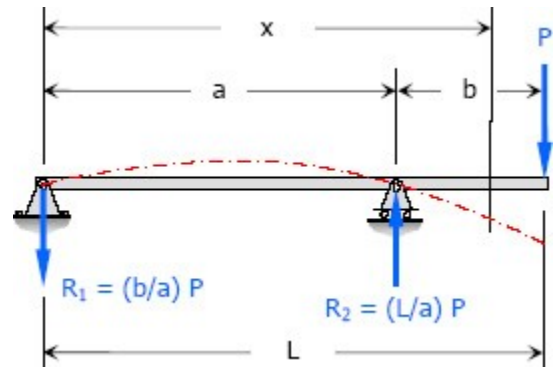
$$aR_1 = bP$$

$$R_1 = \frac{b}{a}P$$

$$\Sigma M_{R1} = 0$$

$$aR_2 = PL$$

$$R_2 = \frac{L}{a}P$$



$$EI y'' = -\frac{b}{a}Px + \frac{L}{a}P \langle x - a \rangle$$

$$EI y' = -\frac{b}{2a}Px^2 + \frac{L}{2a}P \langle x - a \rangle^2 + C_1$$

$$EI y = -\frac{b}{6a}Px^3 + \frac{L}{6a}P \langle x - a \rangle^3 + C_1x + C_2$$

At $x = 0, y = 0$, therefore $C_2 = 0$

At $x = a, y = 0$

$$0 = -\left[\frac{b}{6a} \right] Pa^3 + aC_1$$

$$C_1 = \frac{ab}{6}P$$

Therefore,

$$EI y' = -\frac{b}{2a}Px^2 + \frac{L}{2a}P \langle x - a \rangle^2 + \frac{ab}{6}P$$

$$EI y = -\frac{b}{6a}Px^3 + \frac{L}{6a}P \langle x - a \rangle^3 + \frac{ab}{6}Px$$

Part (a): Slope and deflection under the load P

Slope under the load P: (note $x = a + b = L$)

$$EI y' = -\frac{b}{2a}P(a+b)^2 + \frac{a+b}{2a}Pb^2 + \frac{ab}{6}P$$

$$EI y' = -\frac{b}{2a}P(a^2 + 2ab + b^2) + \frac{ab^2 + b^3}{2a}P + \frac{ab}{6}P$$

$$EI y' = -\frac{ab}{2}P - b^2P - \frac{b^3}{2a}P + \frac{b^2}{2}P\frac{b^3}{2a}P + \frac{ab}{6}P$$

$$EI y' = -\frac{1}{2}b^2P - \frac{1}{3}abP$$

$$EI y' = -\frac{1}{6}b(3b + 2a)P$$

$$EI y' = -\frac{1}{6}b[2(a + b) + b]P$$

$$EI y' = -\frac{1}{6}b(2L + b)P \quad \text{answer}$$

Deflection under the load P: (note $x = a + b = L$)

$$EI y = -\frac{b}{6a}P(a+b)^3 + \frac{a+b}{6a}P(b^3) + \frac{ab}{6}P(a+b)$$

$$EI y = -\frac{b}{6a}P(a^3 + 3a^2b + 3ab^2 + b^3) + \frac{ab^3 + b^4}{6a}P + \frac{ab}{6}P(a+b)$$

$$EI y = -\frac{a^2b}{6}P - \frac{ab^2}{2}P - \frac{b^3}{2}P - \frac{b^4}{6a}P + \frac{b^3}{6}P + \frac{b^4}{6a}P + \frac{a^2b}{6}P + \frac{ab^2}{6}P$$

$$EI y = -\frac{1}{3}ab^2P - \frac{1}{3}b^3P$$

$$EI y = -\frac{1}{3}(a+b)b^2P$$

$$EI y = -\frac{1}{3}Lb^2P \quad \text{answer}$$

Part (b): Maximum deflection between the supports

The maximum deflection between the supports will occur at the point where $y' = 0$.

$$EI y' = -\frac{b}{2a}Px^2 + \frac{L}{2a}P(x-a)^2 + \frac{ab}{6}P$$

At $y' = 0$, $(x - a)$ do not exist thus,

$$0 = -\frac{b}{2a}Px^2 + \frac{ab}{6}P$$

$$x^2 = \frac{1}{3}a^2$$

$$x = \frac{1}{\sqrt{3}}a$$

$$\text{At } x = \frac{1}{\sqrt{3}}a,$$

$$EI y_{max} = -\frac{b}{6a}P\left(\frac{1}{\sqrt{3}}a\right)^3 + \frac{ab}{6}P\left(\frac{1}{\sqrt{3}}a\right)$$

$$EI y_{max} = -\frac{a^2b}{6(3\sqrt{3})}P + \frac{a^2b}{6\sqrt{3}}P$$

$$EI y_{max} = \frac{a^2b}{6\sqrt{3}}P \left(-\frac{1}{3} + 1\right)$$

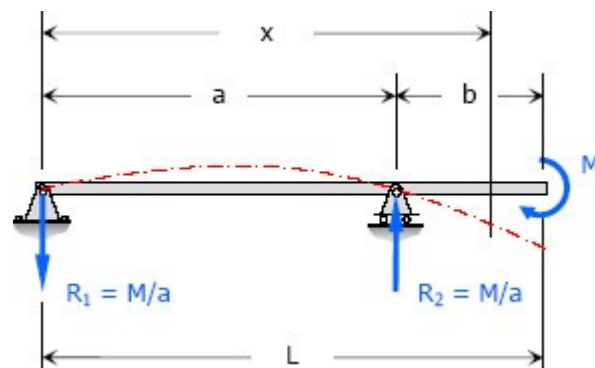
$$EI y_{max} = \frac{a^2b}{6\sqrt{3}}P \left(\frac{2}{3}\right)$$

$$EI y_{max} = \frac{a^2b}{9\sqrt{3}}P \quad \text{answer}$$

Problem 617

Replace the load P in [Prob. 616](#) by a clockwise couple M applied at the right end and determine the slope and deflection at the right end.

Solution 617



$$EI y'' = -\frac{M}{a}x + \frac{M}{a} \langle x - a \rangle$$

$$EI y' = -\frac{M}{2a}x^2 + \frac{M}{2a}\langle x - a \rangle^2 + C_1$$

$$EI y = -\frac{M}{6a}x^3 + \frac{M}{6a}\langle x - a \rangle^3 + C_1x + C_2$$

At $x = 0, y = 0$, therefore $C_2 = 0$

At $x = a, y = 0$

$$0 = -(M/6a)(a^3) + aC_1$$

$$C_1 = Ma/6$$

Therefore,

$$EI y' = -\frac{M}{2a}x^2 + \frac{M}{2a}\langle x - a \rangle^2 + \frac{Ma}{6}$$

$$EI y = -\frac{M}{6a}x^3 + \frac{M}{6a}\langle x - a \rangle^3 + \frac{Ma}{6}x$$

Slope at $x = a + b$

$$EI y' = -\frac{M}{2a}(a + b)^2 + \frac{M}{2a}(b^2) + \frac{Ma}{6}$$

$$EI y' = -\frac{M}{2a}(a^2 + 2ab + b^2) + \frac{M}{2a}(b^2) + \frac{Ma}{6}$$

$$EI y' = -\frac{1}{2}Ma - Mb - \frac{Mb^2}{2a} + \frac{Mb^2}{2a} + \frac{1}{6}Ma$$

$$EI y' = -\frac{1}{3}Ma - Mb$$

$$EI y' = -\frac{1}{3}M(a + 3b)$$

$$EI y' = -\frac{1}{3}M[(a + b) + 2b]$$

$$EI y' = -\frac{1}{3}M(L + 2b) \quad \text{answer}$$

Deflection at $x = a + b$

$$EI y = -\frac{M}{6a}(a+b)^3 + \frac{M}{6a}(b^3) + \frac{Ma}{6}(a+b)$$

$$EI y = -\frac{M}{6a}(a^3 + 3a^2b + 3ab^2 + b^3) + \frac{Mb^3}{6a} + \frac{Ma}{6}(a+b)$$

$$EI y = -\frac{Ma^2}{6} - \frac{Mab}{2} - \frac{Mb^2}{2} - \frac{Mb^3}{6a} + \frac{Mb^3}{6a} + \frac{Ma^2}{6} + \frac{Mab}{6}$$

$$EI y = -\frac{1}{3}Mab - \frac{1}{2}Mb^2$$

$$EI y = -\frac{1}{6}Mb(2a + 3b)$$

$$EI y = -\frac{1}{6}Mb[2(a+b) + b]$$

$$EI y = -\frac{1}{6}Mb(2L + b)$$

$$EI \delta = \frac{1}{6}Mb(2L + b) \quad \text{answer}$$

Problem 618

A simply supported beam carries a couple M applied as shown in Fig. P-618. Determine the equation of the elastic curve and the deflection at the point of application of the couple. Then letting $a = L$ and $a = 0$, compare your solution of the elastic curve with cases 11 and 12 in the [Summary of Beam Loadings](#).

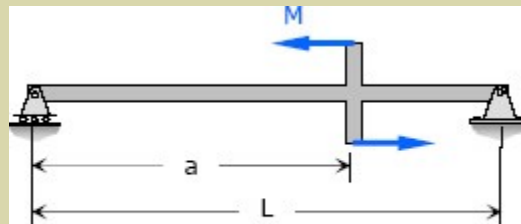


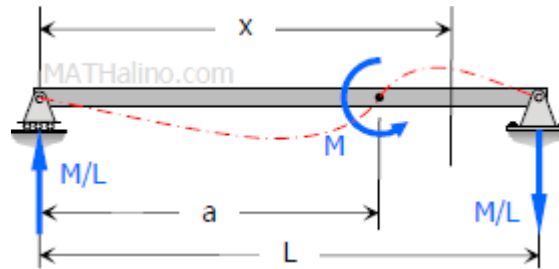
Figure P-618

Solution 618

$$EI y'' = \frac{M}{L}x - M \langle x - a \rangle^0$$

$$EI y' = \frac{M}{2L}x^2 - M \langle x - a \rangle + C_1$$

$$EI y = \frac{M}{6L} x^3 - \frac{1}{2} M \langle x - a \rangle^2 + C_1 x + C_2$$



At $x = 0, y = 0$, therefore $C_2 = 0$

At $x = L, y = 0$

$$0 = \frac{1}{6} ML^2 - \frac{1}{2} M(L - a)^2 + C_1 L$$

$$0 = \frac{1}{6} ML^2 - \frac{1}{2} M(L^2 - 2La + a^2) + C_1 L$$

$$0 = \frac{1}{6} ML^2 - \frac{1}{2} ML^2 + MLa - \frac{1}{2} Ma^2 + C_1 L$$

$$0 = -\frac{1}{3} ML^2 + MLa - \frac{1}{2} Ma^2 + C_1 L$$

$$C_1 L = \frac{1}{3} ML^2 - MLa + \frac{1}{2} Ma^2$$

$$C_1 = \frac{1}{3} ML - Ma + \frac{Ma^2}{2L}$$

Therefore,

$$EI y = \frac{M}{6L} x^3 - \frac{1}{2} M \langle x - a \rangle^2 + \left(\frac{1}{3} ML - Ma + \frac{Ma^2}{2L} \right) x \quad \text{answer}$$

At $x = a$

$$EI y = \frac{Ma^3}{6L} + \left(\frac{1}{3} ML - Ma + \frac{Ma^2}{2L} \right) a$$

$$EI y = \frac{2Ma^3}{3L} + \frac{1}{3} MLa - Ma^2$$

$$EI y = \frac{Ma}{3L}(2a^2 + L^2 - 3La)$$

$$EI y = \frac{Ma}{3L}(L^2 - 3La + 2a^2) \quad \text{answer}$$

When $a = 0$ (moment load is at the left support):

$$EI y = \frac{M}{6L}x^3 - \frac{1}{2}M \langle x - a \rangle^2 + \left(\frac{1}{3}ML - Ma + \frac{Ma^2}{2L} \right) x$$

$$EI y = \frac{M}{6L}x^3 - \frac{1}{2}Mx^2 + \frac{1}{3}MLx$$

$$EI y = \frac{Mx}{6L}(x^2 - 3Lx + 2L^2)$$

$$EI y = \frac{Mx}{6L}(2L^2 - 3Lx + x^2)$$

$$EI y = \frac{Mx}{6L}(L - x)(2L - x) \quad \text{answer}$$

When $a = L$ (moment load is at the right support):

$$EI y = \frac{M}{6L}x^3 - \frac{1}{2}M \langle x - a \rangle^2 + \left(\frac{1}{3}ML - Ma + \frac{Ma^2}{2L} \right) x$$

$$EI y = \frac{M}{6L}x^3 + \left(\frac{1}{3}ML - ML + \frac{1}{2}ML \right) x$$

$$EI y = \frac{M}{6L}x^3 - \frac{1}{6}MLx$$

$$EI y = \frac{Mx^3 - ML^2x}{6L}$$

$$EI y = \frac{-Mx(-x^2 + L^2)}{6L}$$

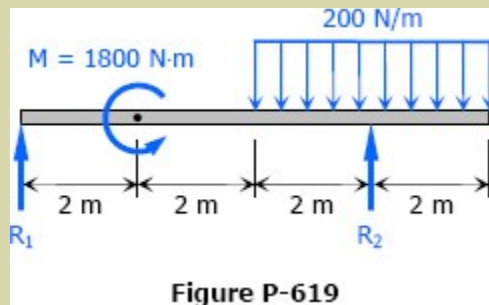
$$EI y = \frac{-MLx(L^2 - x^2)}{6L^2}$$

$$EI y = -\frac{MLx}{6L^2}(L^2 - x^2)$$

$$EI y = -\frac{MLx}{6} \left(1 - \frac{x^2}{L^2} \right) \quad \text{answer}$$

Problem 619

Determine the value of $EI y$ midway between the supports for the beam loaded as shown in Fig. P-619.



Solution 619

$$\Sigma M_{R2} = 0$$

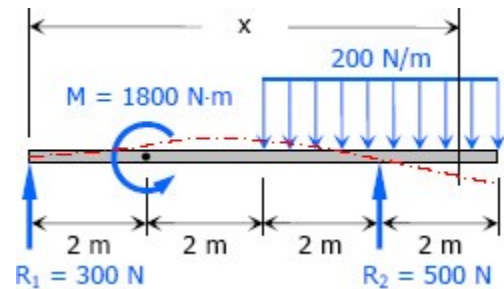
$$6R_1 + 200(4)(0) = 1800$$

$$R_1 = 300 \text{ N}$$

$$\Sigma M_{R1} = 0$$

$$6R_2 + 1800 = 200(4)(6)$$

$$R_2 = 500 \text{ N}$$



$$EI y'' = 300x - 1800 \langle x - 2 \rangle^0 + 500 \langle x - 6 \rangle - \frac{1}{2}(200) \langle x - 4 \rangle^2$$

$$EI y'' = 300x - 1800 \langle x - 2 \rangle^0 + 500 \langle x - 6 \rangle - 100 \langle x - 4 \rangle^2$$

$$EI y' = 150x^2 - 1800 \langle x - 2 \rangle + 250 \langle x - 6 \rangle^2 - \frac{100}{3} \langle x - 4 \rangle^3 + C_1$$

$$EI y = 50x^3 - 900 \langle x - 2 \rangle^2 + \frac{250}{3} \langle x - 6 \rangle^3 - \frac{25}{3} \langle x - 4 \rangle^4 + C_1 x + C_2$$

At $x = 0, y = 0$, therefore $C_2 = 0$

At $x = 6 \text{ m}, y = 0$

$$0 = 50(6^3) - 900(4^2) - (25/3)(2^4) + 6C_1$$

$$C_1 = 5600/9 \text{ N}\cdot\text{m}^3$$

Therefore,

$$EI y = 50x^3 - 900 \langle x - 2 \rangle^2 + \frac{250}{3} \langle x - 6 \rangle^3 - \frac{25}{3} \langle x - 4 \rangle^4 + \frac{5600}{9}x$$

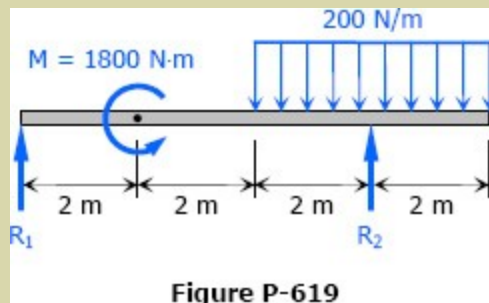
At $x = 3 \text{ m}$

$$EI y = 50(3^3) - 900(1^2) + \frac{5600}{9}(3)$$

$$EI y = \frac{6950}{3} \text{ N}\cdot\text{m}^3 \quad \text{answer}$$

Problem 619

Determine the value of $EI y$ midway between the supports for the beam loaded as shown in Fig. P-619.



Solution 619

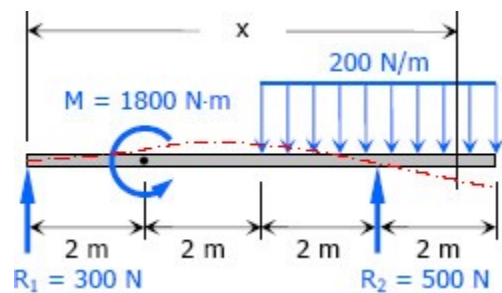
$$\Sigma M_{R2} = 0$$

$$6R_1 + 200(4)(0) = 1800$$

$$R_1 = 300 \text{ N}$$

$$\Sigma M_{R1} = 0$$

$$6R_2 + 1800 = 200(4)(6)$$



$$R_2 = 500 \text{ N}$$

$$EI y'' = 300x - 1800 \langle x - 2 \rangle^0 + 500 \langle x - 6 \rangle - \frac{1}{2}(200) \langle x - 4 \rangle^2$$

$$EI y'' = 300x - 1800 \langle x - 2 \rangle^0 + 500 \langle x - 6 \rangle - 100 \langle x - 4 \rangle^2$$

$$EI y' = 150x^2 - 1800 \langle x - 2 \rangle + 250 \langle x - 6 \rangle^2 - \frac{100}{3} \langle x - 4 \rangle^3 + C_1$$

$$EI y = 50x^3 - 900 \langle x - 2 \rangle^2 + \frac{250}{3} \langle x - 6 \rangle^3 - \frac{25}{3} \langle x - 4 \rangle^4 + C_1 x + C_2$$

At $x = 0, y = 0$, therefore $C_2 = 0$

At $x = 6 \text{ m}, y = 0$

$$0 = 50(6^3) - 900(4^2) - \frac{25}{3}(2^4) + 6C_1$$

$$C_1 = 5600/9 \text{ N}\cdot\text{m}^3$$

Therefore,

$$EI y = 50x^3 - 900 \langle x - 2 \rangle^2 + \frac{250}{3} \langle x - 6 \rangle^3 - \frac{25}{3} \langle x - 4 \rangle^4 + \frac{5600}{9}x$$

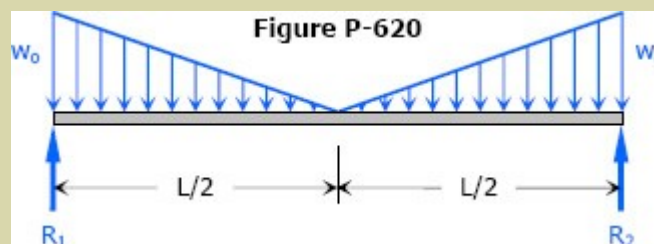
At $x = 3 \text{ m}$

$$EI y = 50(3^3) - 900(1^2) + \frac{5600}{9}(3)$$

$$EI y = \frac{6950}{3} \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

Problem 620

Find the midspan deflection δ for the beam shown in Fig. P-620, carrying two triangularly distributed loads. (*Hint:* For convenience, select the origin of the axes at the midspan position of the elastic curve.)

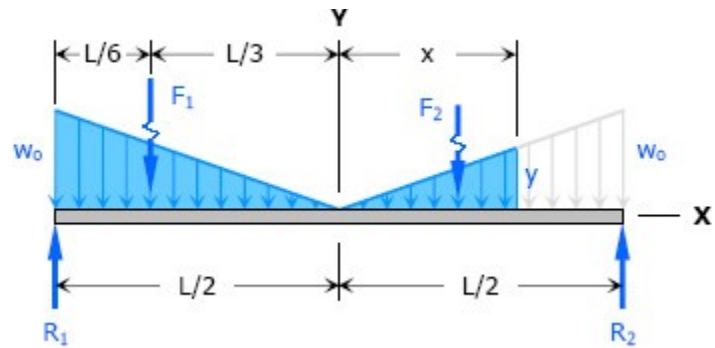


Solution 620

By ratio and proportion:

$$\frac{y}{x} = \frac{w_o}{L/2}$$

$$y = \frac{2w_o}{L}x$$



By symmetry:

$$R_1 = R_2 = \frac{1}{2}(L/2)(w_o) = \frac{1}{4}w_oL$$

$$F_1 = \frac{1}{2}(L/2)(w_o) = \frac{1}{4}w_oL$$

$$F_2 = \frac{1}{2}xy = \frac{1}{2}x \left(\frac{2w_o}{L}x \right) = \frac{w_o}{L}x^2$$

$$EI y'' = R_1(x + \frac{1}{2}L) - F_1(x + \frac{1}{3}L) - F_2(\frac{1}{3}x)$$

$$EI y'' = \frac{1}{4}w_oL(x + \frac{1}{2}L) - \frac{1}{4}w_oL(x + \frac{1}{3}L) - \frac{w_o}{L}x^2(\frac{1}{3}x)$$

$$EI y'' = \frac{1}{4}w_oLx + \frac{1}{8}w_oL^2 - \frac{1}{4}w_oLx - \frac{1}{12}w_oL^2 - \frac{w_o}{3L}x^3$$

$$EI y'' = \frac{1}{24}w_oL^2 - \frac{w_o}{3L}x^3$$

$$EI y' = \frac{1}{24}w_oL^2x - \frac{w_o}{12L}x^4 + C_1$$

$$EI y = \frac{1}{48}w_oL^2x^2 - \frac{w_o}{60L}x^5 + C_1x + C_2$$

At $x = 0$, $y' = 0$, therefore $C_1 = 0$

At $x = \frac{1}{2}L$, $y = 0$

$$0 = (1/48)w_oL^2(\frac{1}{2}L)^2 - (w_o/60L)(\frac{1}{2}L)^5 + C_2$$

$$0 = (1/192)w_oL^4 - (1/1920)w_oL^4 + C_2$$

$$C_2 = -(3/640)w_oL^4$$

Therefore,

$$EI y = \frac{1}{48} w_o L^2 x^2 - \frac{w_o}{60L} x^5 - \frac{3}{640} w_o L^4$$

At $x = 0$ (midspan)

$$EI y_{max} = -\frac{3}{640} w_o L^4 = -\frac{3}{640} w_o L^4 \times \frac{3}{3}$$

$$EI y_{max} = -\frac{9}{1920} w_o L^4$$

Thus,

$$\delta_{midspan} = \frac{9w_o L^4}{1920EI} \quad \text{answer}$$

Problem 621

Determine the value of $EI\delta$ midway between the supports for the beam shown in Fig. P-621. Check your result by letting $a = 0$ and comparing with [Prob. 606](#). (Apply the hint given in [Prob. 620](#).)

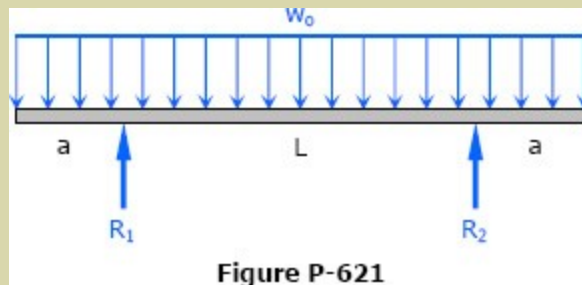
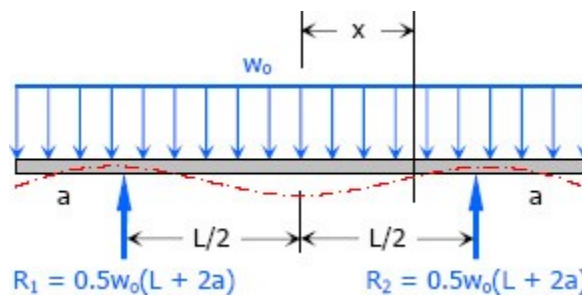


Figure P-621

Solution 621

By symmetry

$$R_1 = R_2 = \frac{1}{2} w_o (L + 2a)$$



$$EI y'' = \left[\frac{1}{2} w_o (L + 2a) \right] \left(x + \frac{1}{2} L \right) - \frac{1}{2} w_o \left(x + \frac{1}{2} L + a \right)^2$$

$$EI y'' = \frac{1}{2} w_o (L + 2a) x + \frac{1}{4} w_o (L + 2a) L - \frac{1}{2} w_o \left[x^2 + 2x \left(\frac{1}{2} L + a \right) + \left(\frac{1}{2} L + a \right)^2 \right]$$

$$EI y'' = \frac{1}{2} w_o (L + 2a) x + \frac{1}{4} w_o (L + 2a) L - \frac{1}{2} w_o x^2 - w_o \left(\frac{1}{2} L + a \right) x - \frac{1}{2} w_o \left(\frac{1}{2} L + a \right)^2$$

$$EI y'' = \frac{1}{2} w_o (L + 2a) x + \frac{1}{4} w_o (L + 2a) L - \frac{1}{2} w_o x^2 - \frac{1}{2} w_o (L + 2a) x - \frac{1}{8} w_o (L + 2a)^2$$

$$EI y'' = \frac{1}{4} w_o (L + 2a) L - \frac{1}{2} w_o x^2 - \frac{1}{8} w_o (L + 2a)^2$$

$$EI y' = \frac{1}{4} w_o (L + 2a) L x - \frac{1}{6} w_o x^3 - \frac{1}{8} w_o (L + 2a)^2 x + C_1$$

$$EI y = \frac{1}{8} w_o (L + 2a) L x^2 - \frac{1}{24} w_o x^4 - \frac{1}{16} w_o (L + 2a)^2 x^2 + C_1 x + C_2$$

At $x = 0$, $y' = 0$, therefore $C_1 = 0$

At $x = \frac{1}{2}L$, $y = 0$

$$0 = \frac{1}{8} w_o (L + 2a) L \left(\frac{1}{2} L \right)^2 - \frac{1}{24} w_o \left(\frac{1}{2} L \right)^4 - \frac{1}{16} w_o (L + 2a)^2 \left(\frac{1}{2} L \right)^2 + C_2$$

$$0 = \frac{1}{32} w_o (L + 2a) L^3 - \frac{1}{384} w_o L^4 - \frac{1}{64} w_o (L + 2a)^2 L^2 + C_2$$

$$0 = \frac{1}{32} w_o L^4 - \frac{1}{16} w_o L^3 a - \frac{1}{384} w_o L^4 - \frac{1}{64} w_o (L^2 + 4La + 4a^2) L^2 + C_2$$

$$0 = \frac{1}{32} w_o L^4 - \frac{1}{16} w_o L^3 a - \frac{1}{384} w_o L^4 - \frac{1}{64} w_o L^4 - \frac{1}{16} w_o L^3 a - \frac{1}{16} w_o L^2 a^2 + C_2$$

$$0 = \frac{5}{384} w_o L^4 - \frac{1}{16} w_o L^2 a^2 + C_2$$

$$C_2 = \frac{1}{16} w_o L^2 a^2 - \frac{5}{384} w_o L^4$$

$$C_2 = \frac{1}{384} w_o L^2 (24a^2 - 5L^2)$$

Therefore,

$$EI y = \frac{1}{8} w_o (L + 2a) L x^2 - \frac{1}{24} w_o x^4 - \frac{1}{16} w_o (L + 2a)^2 x^2 + \frac{1}{384} w_o L^2 (24a^2 - 5L^2)$$

At $x = 0$ (midspan)

$$EI y = \frac{1}{384} w_o L^2 (24a^2 - 5L^2) \quad \text{answer}$$

At $x = 0$ when $a = 0$

$$EI y_{max} = \frac{1}{384} w_o L^2 (0 - 5L^2)$$

$$EI y_{max} = -\frac{5w_o L^4}{384}$$

Thus,

$$\delta_{max} = \frac{5w_o L^4}{384EI} \quad \text{answer}$$

Moment Diagrams by Parts

The [moment-area method](#) of finding the deflection of a beam will demand the accurate computation of the area of a moment diagram, as well as the moment of such area about any axis. To pave its way, this section will deal on how to draw moment diagrams by parts and to calculate the moment of such diagrams about a specified axis.

Basic Principles

1. The bending moment caused by all forces to the left or to the right of any section is equal to the respective algebraic sum of the bending moments at that section caused by each load acting separately.

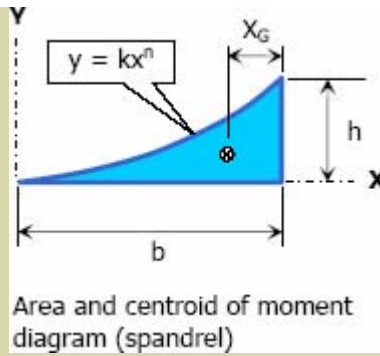
$$M = (\Sigma M)_L = (\Sigma M)_R$$

2. The moment of a load about a specified axis is always defined by the equation of a spandrel

$$y = kx^n$$

where n is the degree of power of x .

The graph of the above equation is as shown below



and the area and location of centroid are defined as follows.

$$A = \frac{1}{n+1}bh$$

$$X_G = \frac{1}{n+2}b$$

Cantilever Loadings

A = area of moment diagram

M_x = moment about a section of distance x

barred x = location of centroid

Degree = degree power of the moment diagram

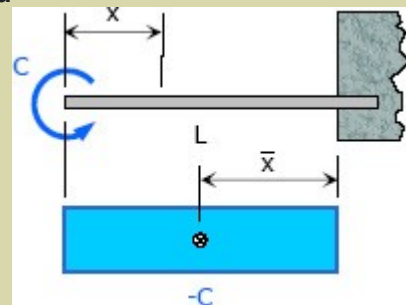
Couple or Moment Load

$$A = -CL$$

$$M_x = -C$$

$$\bar{x} = \frac{1}{2}L$$

Degree: zero



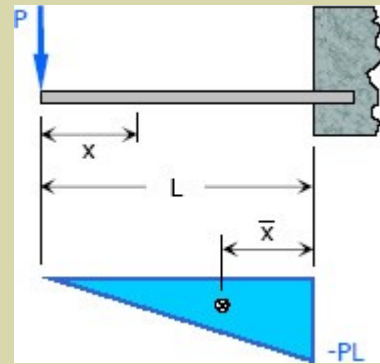
Concentrated Load

$$A = -\frac{1}{2}PL^2$$

$$M_x = -Px$$

$$\bar{x} = \frac{1}{3}L$$

Degree: first



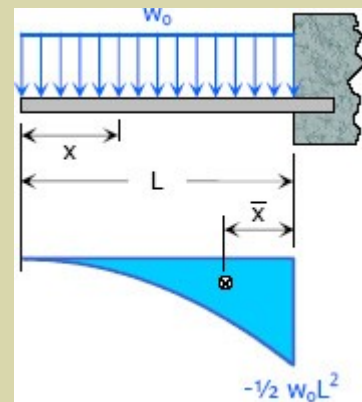
Uniformly Distributed Load

$$A = -\frac{1}{6}w_oL^3$$

$$M_x = -\frac{1}{2}w_o x^2$$

$$\bar{x} = \frac{1}{4}L$$

Degree: second



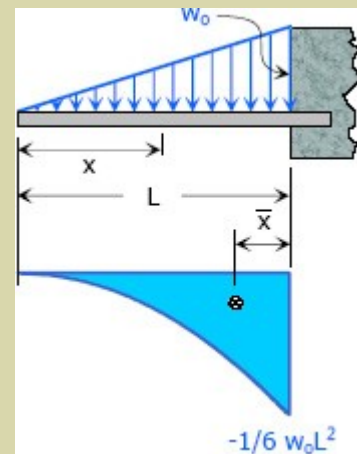
Uniformly Varying Load

$$A = -\frac{1}{24}w_oL^3$$

$$M_x = -\frac{w_o}{6L}x^3$$

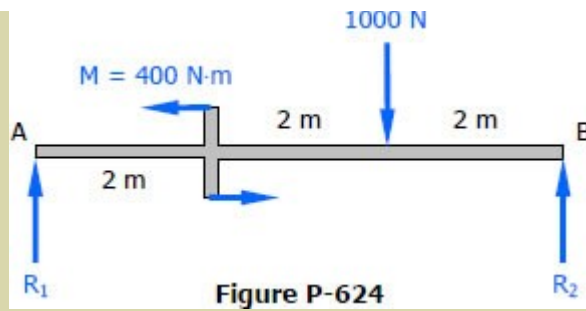
$$\bar{x} = \frac{1}{5}L$$

Degree: third



Problem 624

For the beam loaded as shown in Fig. P-624, compute the moment of area of the M diagrams between the reactions about both the left and the right reaction.



Solution 624

$$\Sigma M_{R2} = 0$$

$$6R_1 = 400 + 1000(2)$$

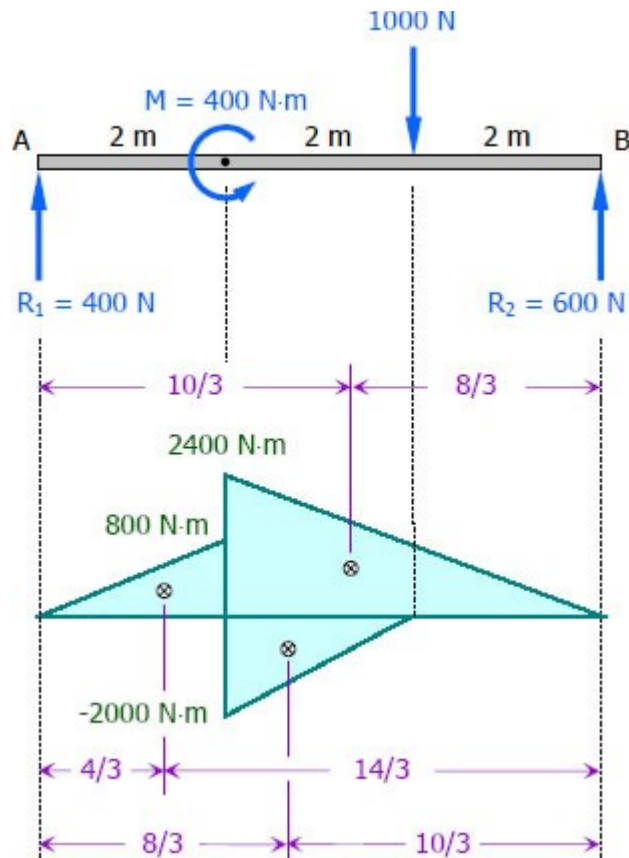
$$R_1 = 400 \text{ N}$$

$$\Sigma M_{R1} = 0$$

$$6R_2 + 400 = 1000(2)$$

$$R_2 = 600 \text{ N}$$

Moment diagram by parts can be drawn in different ways; three are shown below.

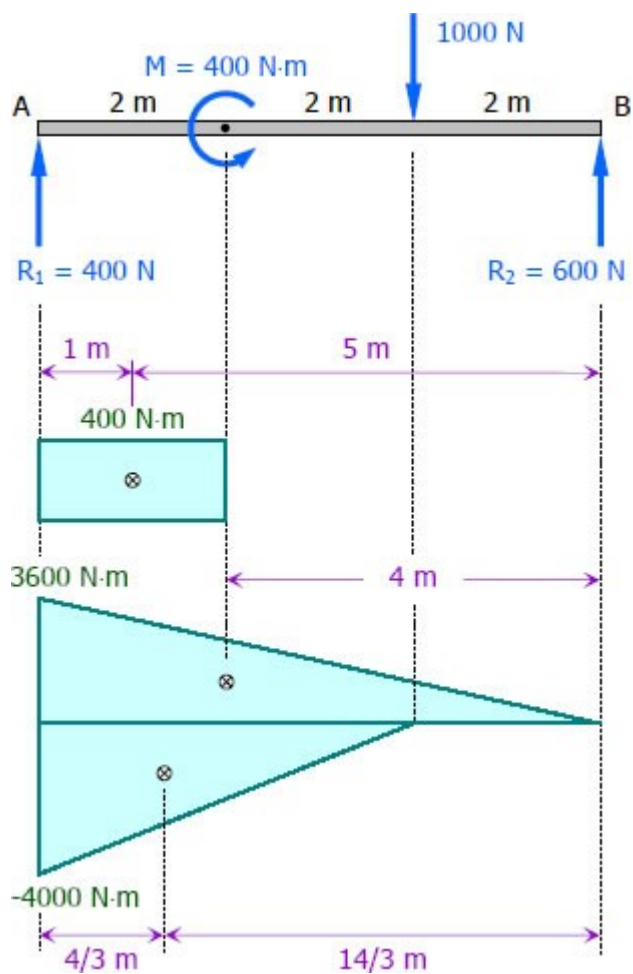


$$(Area_{AB})\bar{X}_A = \frac{1}{2}(2)(800)\left(\frac{4}{3}\right) + \frac{1}{2}(4)(2400)\left(\frac{10}{3}\right) - \frac{1}{2}(2)(2000)\left(\frac{8}{3}\right)$$

$$(Area_{AB})\bar{X}_A = 11\,733.33 \text{ N}\cdot\text{m}^3$$

$$(Area_{AB})\bar{X}_B = \frac{1}{2}(2)(800)\left(\frac{14}{3}\right) + \frac{1}{2}(4)(2400)\left(\frac{8}{3}\right) - \frac{1}{2}(2)(2000)\left(\frac{10}{3}\right)$$

$$(Area_{AB})\bar{X}_B = 9\,866.67 \text{ N}\cdot\text{m}^3$$

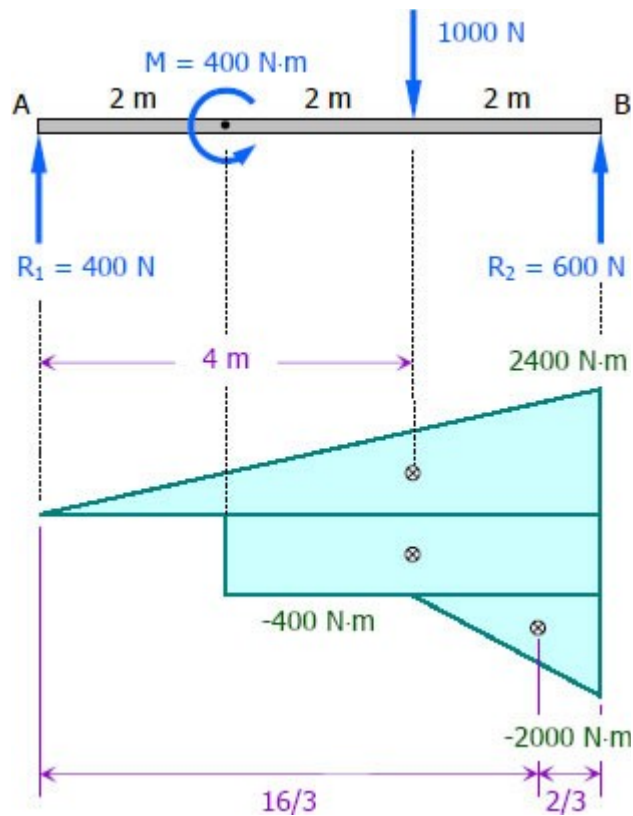


$$(Area_{AB})\bar{X}_A = 400(2)(1) + \frac{1}{2}(6)(3600)(2) - \frac{1}{2}(4)(4000)(\frac{4}{3})$$

$$(Area_{AB})\bar{X}_A = 11\,733.33 \text{ N} \cdot \text{m}^3$$

$$(Area_{AB})\bar{X}_B = 400(2)(5) + \frac{1}{2}(6)(3600)(4) - \frac{1}{2}(4)(4000)(\frac{14}{3})$$

$$(Area_{AB})\bar{X}_B = 9\,866.67 \text{ N} \cdot \text{m}^3$$



$$(Area_{AB}) \bar{X}_A = \frac{1}{2}(6)(2400)(4) - 400(4)(4) - \frac{1}{2}(2)(2000)(\frac{16}{3})$$

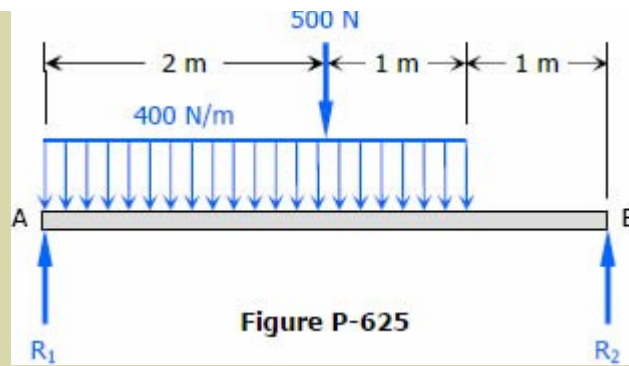
$$(Area_{AB}) \bar{X}_A = 11\,733.33 \text{ N} \cdot \text{m}^3$$

$$(Area_{AB}) \bar{X}_B = \frac{1}{2}(6)(2400)(2) - 400(4)(2) - \frac{1}{2}(2)(2000)(\frac{2}{3})$$

$$(Area_{AB}) \bar{X}_B = 9\,866.67 \text{ N} \cdot \text{m}^3$$

Problem 625

For the beam loaded as shown in Fig. P-625, compute the moment of area of the M diagrams between the reactions about both the left and the right reaction. (Hint: Draw the moment diagram by parts from right to left.)



Solution 625

$$\Sigma M_{R2} = 0$$

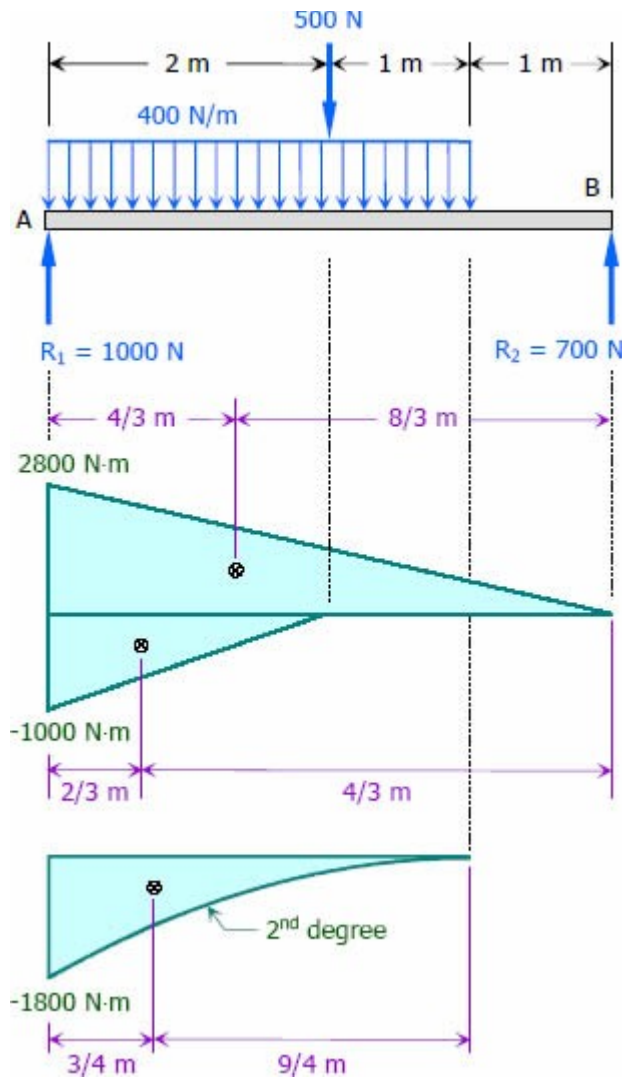
$$4R_1 = 400(3)(2.5) + 500(2)$$

$$R_1 = 1000 \text{ N}$$

$$\Sigma M_{R1} = 0$$

$$4R_2 = 400(3)(1.5) + 500(2)$$

$$R_2 = 700 \text{ N}$$



$$(Area_{AB}) \bar{X}_A = \frac{1}{2}(4)(2800)\left(\frac{4}{3}\right) - \frac{1}{2}(2)(1000)\left(\frac{2}{3}\right) - \frac{1}{3}(3)(1800)\left(\frac{3}{4}\right)$$

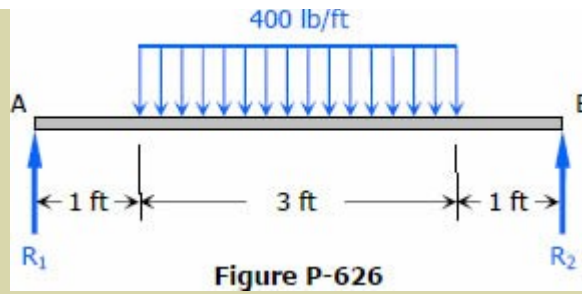
$$(Area_{AB}) \bar{X}_A = 5450 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

$$(Area_{AB}) \bar{X}_B = \frac{1}{2}(4)(2800)\left(\frac{8}{3}\right) - \frac{1}{2}(2)(1000)\left(\frac{4}{3}\right) - \frac{1}{3}(3)(1800)\left(\frac{9}{4} + 1\right)$$

$$(Area_{AB}) \bar{X}_B = 7750 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

Problem 626

For the beam loaded as shown in Fig. P-626, compute the moment of area of the M diagrams between the reactions about both the left and the right reaction.



Solution 626

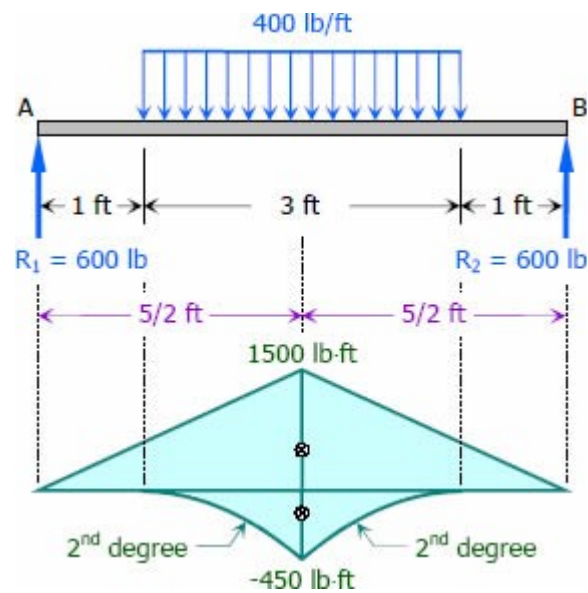
By symmetry

$$R_1 = R_2 = \frac{1}{2}(400)(3)$$

$$R_1 = R_2 = 600 \text{ lb}$$

and

$$(Area_{AB}) \bar{X}_A = (Area_{AB}) \bar{X}_B$$



$$(Area_{AB}) \bar{X}_A = \frac{1}{2}(5)(1500)\left(\frac{5}{2}\right) - \frac{1}{3}(3)(450)\left(\frac{5}{2}\right) \quad \text{answer}$$

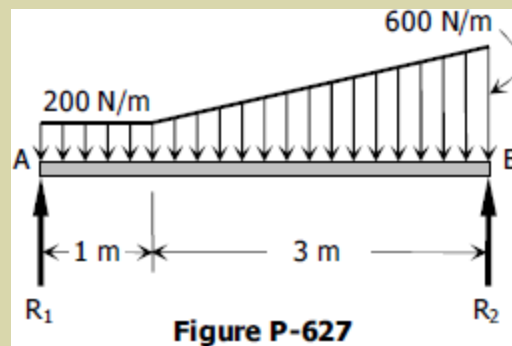
$$(\text{Area}_{AB}) \bar{X}_A = 8\,250 \text{ lb} \cdot \text{ft}^3$$

Thus,

$$(\text{Area}_{AB}) \bar{X}_B = 8\,250 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

Problem 627

For the beam loaded as shown in Fig. P-627 compute the moment of area of the M diagrams between the reactions about both the left and the right reaction. (Hint: Resolve the trapezoidal loading into a uniformly distributed load and a uniformly varying load.)



Solution 627

$$\Sigma M_{R2} = 0$$

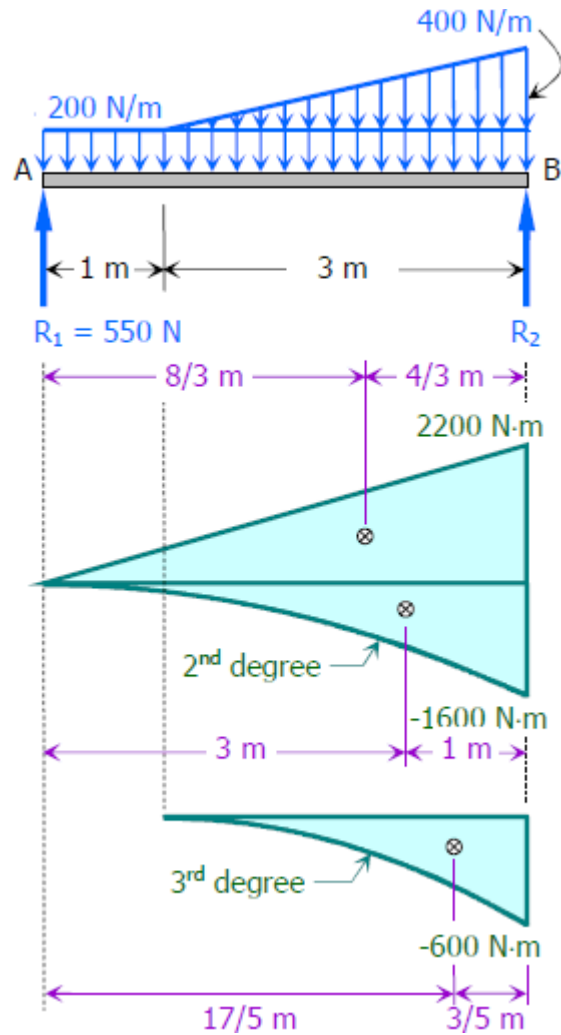
$$4R_1 = 200(4)(2) + \frac{1}{2}(3)(400)(1)$$

$$R_1 = 550 \text{ N}$$

$$\Sigma M_{R1} = 0$$

$$4R_2 = 200(4)(2) + \frac{1}{2}(3)(400)(3)$$

$$R_2 = 850 \text{ N}$$



$$(Area_{AB}) \bar{X}_A = \frac{1}{2}(4)(2200)\left(\frac{8}{3}\right) - \frac{1}{3}(4)(1600)(3) - \frac{1}{4}(3)(600)\left(\frac{17}{5}\right)$$

$$(Area_{AB}) \bar{X}_A = 3803.33 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

$$(Area_{AB}) \bar{X}_B = \frac{1}{2}(4)(2200)\left(\frac{4}{3}\right) - \frac{1}{3}(4)(1600)(1) - \frac{1}{4}(3)(600)\left(\frac{3}{5}\right)$$

$$(Area_{AB}) \bar{X}_B = 3463.33 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

Problem 628

For the beam loaded with uniformly varying load and a couple as shown in Fig. P-628 compute the moment of area of the M diagrams between the reactions about both the left and the right reaction.

Solution 628

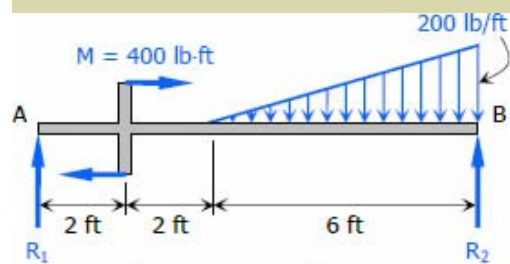


Figure P-628 and P-629

$$\Sigma M_{R2} = 0$$

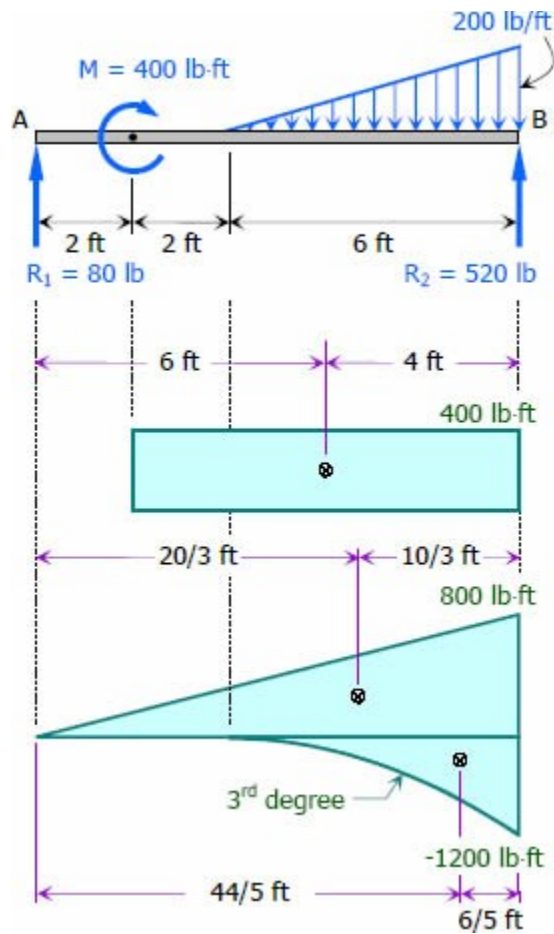
$$10R_1 + 400 = \frac{1}{2}(6)(200)(2)$$

$$R_1 = 80 \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$10R_2 = 400 + \frac{1}{2}(6)(200)(8)$$

$$R_2 = 520 \text{ lb}$$



$$(Area_{AB}) \bar{X}_A = 400(8)(6) + \frac{1}{2}(10)(800)\left(\frac{20}{3}\right) - \frac{1}{4}(6)(1200)\left(\frac{44}{5}\right)$$

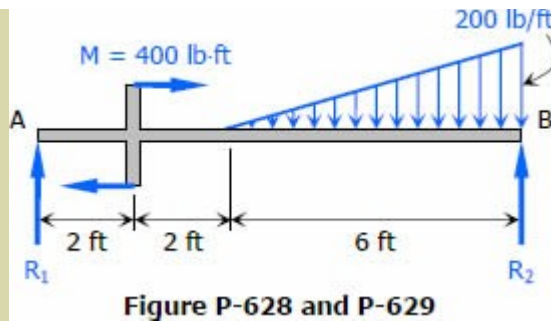
$$(Area_{AB}) \bar{X}_A = 30\,026.67 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

$$(Area_{AB}) \bar{X}_B = 400(8)(4) + \frac{1}{2}(10)(800)\left(\frac{10}{3}\right) - \frac{1}{4}(6)(1200)\left(\frac{6}{5}\right)$$

$$(Area_{AB}) \bar{X}_B = 23\,973.33 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

Problem 629

Solve [Prob. 628](#) if the sense of the couple is counterclockwise instead of clockwise as shown in Fig. P-628.



Solution 629

$$\Sigma M_{R2} = 0$$

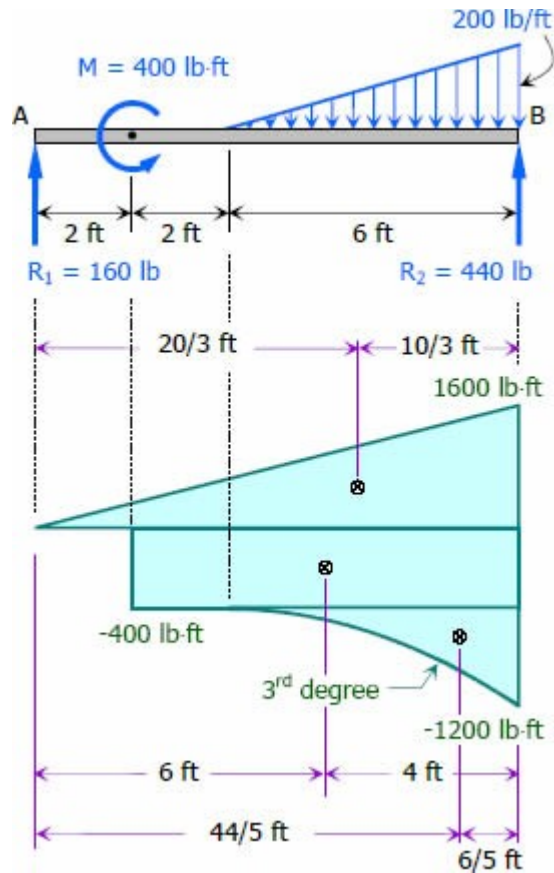
$$10R_1 = 400 + \frac{1}{2}(6)(200)(2)$$

$$R_1 = 160 \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$10R_2 + 400 = \frac{1}{2}(6)(200)(8)$$

$$R_2 = 440 \text{ lb}$$



$$(Area_{AB}) \bar{X}_A = \frac{1}{2}(10)(1600)\left(\frac{20}{3}\right) - 400(8)(6) - \frac{1}{4}(6)(1200)\left(\frac{44}{5}\right)$$

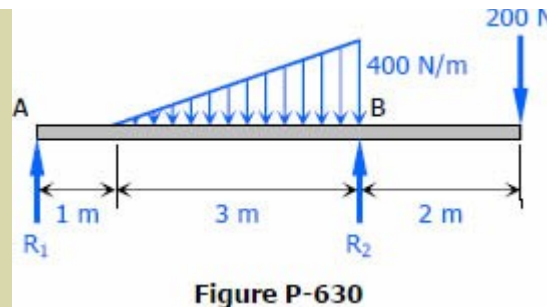
$$(Area_{AB}) \bar{X}_A = 18\,293.33 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

$$(Area_{AB}) \bar{X}_B = \frac{1}{2}(10)(1600)\left(\frac{10}{3}\right) - 400(8)(4) - \frac{1}{4}(6)(1200)\left(\frac{6}{5}\right)$$

$$(Area_{AB}) \bar{X}_B = 11\,706.67 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

Problem 630

For the beam loaded as shown in Fig. P-630, compute the value of $(Area_{AB})\bar{X}_A$. From the result determine whether the tangent drawn to the elastic curve at B slopes up or down to the right. (Hint: Refer to the [deviation equations](#) and [rules of sign](#).)



Solution 630

$$\Sigma M_{R2} = 0$$

$$4R_1 + 200(2) = \frac{1}{2}(3)(400)(1)$$

$$R_1 = 50 \text{ N}$$

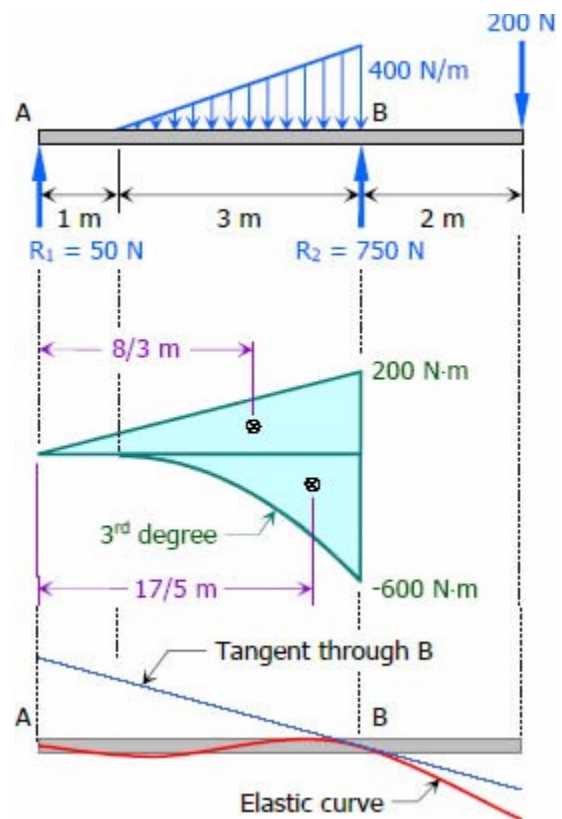
$$\Sigma M_{R1} = 0$$

$$4R_2 = 200(6) + \frac{1}{2}(3)(400)(3)$$

$$R_2 = 750 \text{ N}$$

$$(\text{Area}_{AB}) \bar{X}_A = \frac{1}{2}(4)(200)\left(\frac{8}{3}\right) - \frac{1}{4}(3)(600)\left(\frac{17}{5}\right)$$

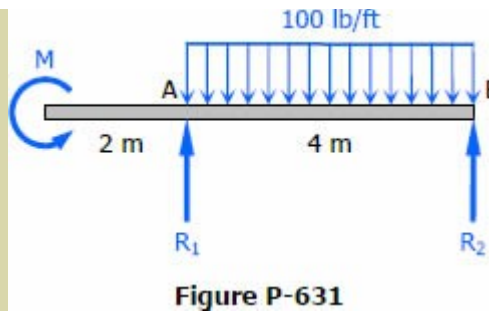
$$(\text{Area}_{AB}) \bar{X}_A = -463.33 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$



The value of $(\text{Area}_{AB}) \bar{X}_A$ is negative; therefore point A is below the tangent through B, thus **the tangent through B slopes downward to the right**. See the approximate elastic curve shown to the right and refer to the [rules of sign](#) for more information.

Problem 631

Determine the value of the couple M for the beam loaded as shown in Fig. P-631 so that the moment of area about A of the M diagram between A and B will be zero. What is the physical significance of this result?



Solution 631

[HideClick here to show or hide the solution](#)

$$\Sigma M_A = 0$$

$$4R_2 + M = 100(4)(2)$$

$$R_2 = 200 - \frac{1}{4}M$$

$$\Sigma M_B = 0$$

$$4R_1 = 100(4)(2) + M$$

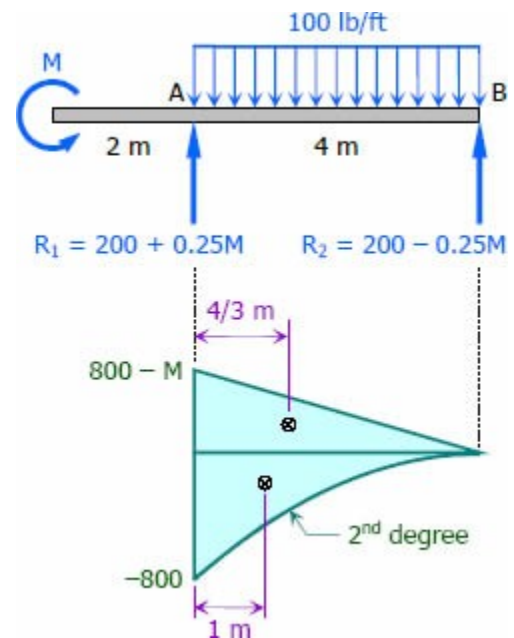
$$R_1 = 200 + \frac{1}{4}M$$

$$(Area_{AB}) \bar{X}_A = 0$$

$$\frac{1}{2}(4)(800 - M)\left(\frac{4}{3}\right) - \frac{1}{3}(4)(800)(1) = 0$$

$$\frac{8}{3}(800 - M) = \frac{3200}{3}$$

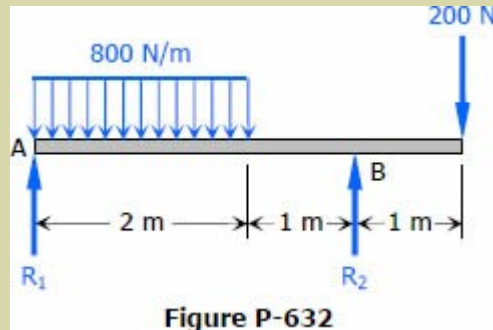
$$M = 400 \text{ lb} \cdot \text{ft} \quad \text{answer}$$



The uniform load over span AB will cause segment AB to deflect downward. The moment load equal to 400 lb·ft applied at the free end will cause the slope through B to be horizontal making the deviation of A from the tangent through B equal to zero. The downward deflection therefore due to uniform load will be countered by the moment load.

Problem 632

For the beam loaded as shown in Fig. P-632, compute the value of $(\text{Area}_{AB})_{\text{barred}(X)_A}$. From this result, is the tangent drawn to the elastic curve at B directed up or down to the right? (Hint: Refer to the [deviation equations and rules of sign.](#))



Solution 632

$$\Sigma M_B = 0$$

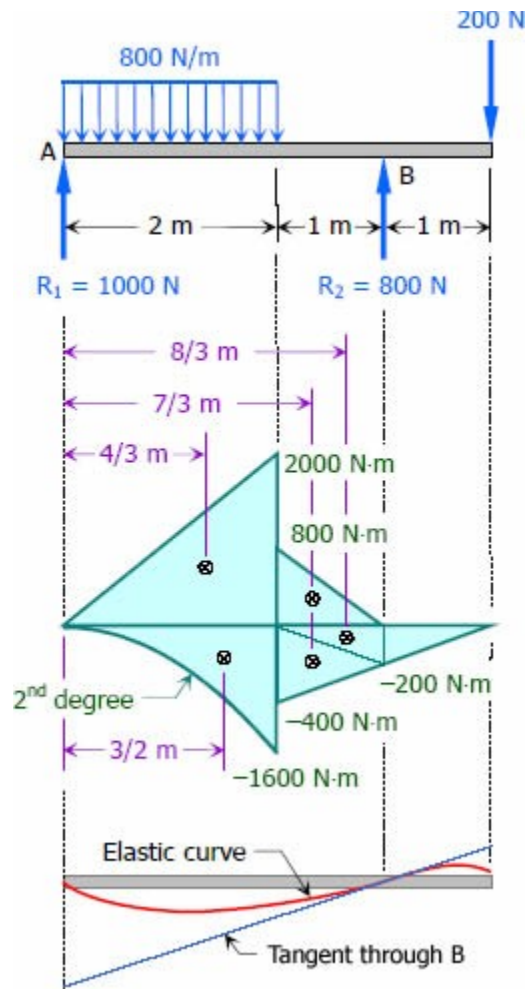
$$3R_1 + 200(1) = 800(2)(2)$$

$$R_1 = 1000 \text{ N}$$

$$\Sigma M_A = 0$$

$$3R_2 = 200(4) + 800(2)(1)$$

$$R_2 = 800 \text{ N}$$



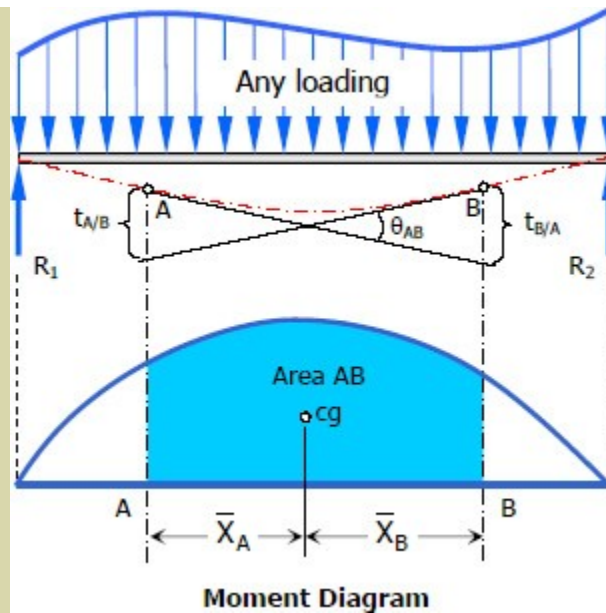
$$(Area_{AB}) \bar{X}_A = \frac{1}{2}(2)(2000)\left(\frac{4}{3}\right) + \frac{1}{2}(1)(800)\left(\frac{7}{3}\right) - \frac{1}{3}(2)(1600)\left(\frac{3}{2}\right) - \frac{1}{2}(1)(400)\left(\frac{7}{3}\right) - \frac{1}{2}(1)(200)\left(\frac{8}{3}\right)$$

$$(Area_{AB}) \bar{X}_A = 1\,266.67 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

The value of $(Area_{AB}) \bar{X}_A$ is positive, therefore point A is above the tangent through B, thus **the tangent through B is upward to the right**. See the approximate elastic curve shown above and refer to the [rules of sign](#) for more information.

Area-Moment Method | Beam Deflections

Another method of determining the slopes and deflections in beams is the area-moment method, which involves the area of the moment diagram.



Theorems of Area-Moment Method

Theorem I

The change in slope between the tangents drawn to the elastic curve at any two points A and B is equal to the product of $1/EI$ multiplied by the area of the moment diagram between these two points.

$$\theta_{AB} = \frac{1}{EI} (Area_{AB})$$

Theorem II

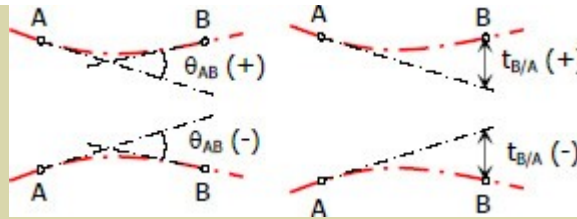
The deviation of any point B relative to the tangent drawn to the elastic curve at any other point A, in a direction perpendicular to the original position of the beam, is equal to the product of $1/EI$ multiplied by the moment of an area about B of that part of the moment diagram between points A and B.

$$t_{B/A} = \frac{1}{EI} (Area_{AB}) \cdot \bar{X}_B$$

and

$$t_{A/B} = \frac{1}{EI} (Area_{AB}) \cdot \bar{X}_A$$

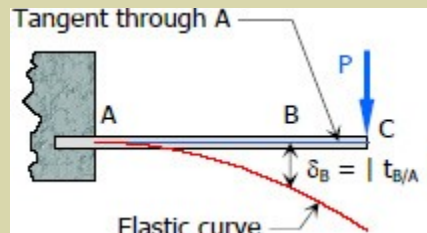
Rules of Sign



1. The deviation at any point is positive if the point lies above the tangent, negative if the point is below the tangent.
2. Measured from left tangent, if θ is counterclockwise, the change of slope is positive, negative if θ is clockwise.

Deflection of Cantilever Beams | Area-Moment Method

Generally, the tangential deviation t is not equal to the beam deflection. In cantilever beams, however, the tangent drawn to the elastic curve at the wall is horizontal and coincidence therefore with the neutral axis of the beam. The tangential deviation in this case is equal to the deflection of the beam as shown below.



From the figure above, the deflection at B denoted as δ_B is equal to the deviation of B from a tangent line through A denoted as $t_{B/A}$. This is because the tangent line through A lies with the neutral axis of the beam.

Problem 636

The cantilever beam shown in Fig. P-636 has a rectangular cross-section 50 mm wide by h mm high. Find the height h if the maximum deflection is not to exceed 10 mm. Use $E = 10$ GPa.

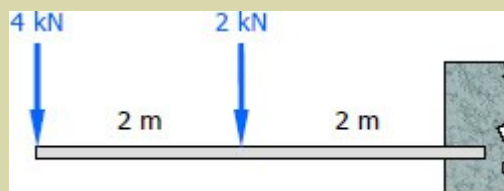
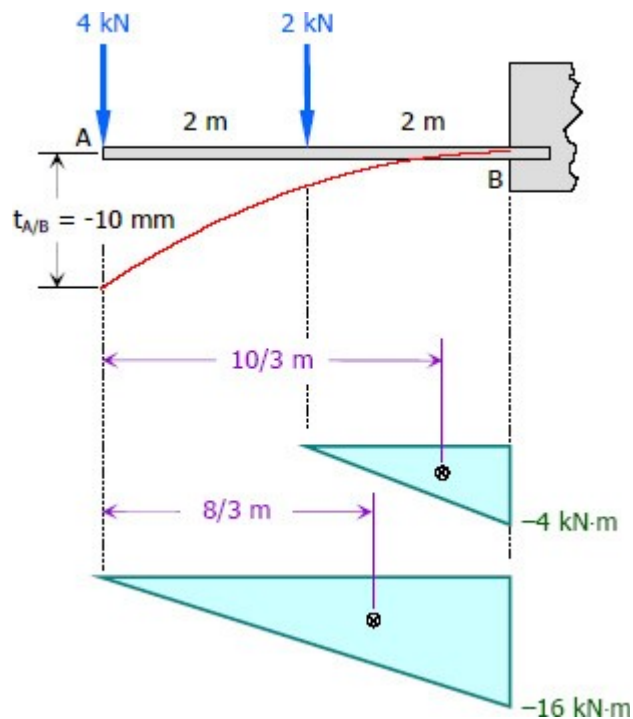


Figure P-636

Solution 636



$$t_{A/B} = \frac{1}{EI} (\text{Area}_{AB}) \bar{X}_A$$

$$-10 = \frac{1}{10\,000 \left(\frac{50h^3}{12} \right)} \left[-\frac{1}{2}(2)(4)\left(\frac{10}{3}\right) - \frac{1}{2}(4)(16)\left(\frac{8}{3}\right) \right] (1000^4)$$

$$-10 = \frac{3}{125\,000h^3} \left[-\frac{296}{3} \right] (1000^4)$$

$$h^3 = \frac{-296(1000^4)}{125\,000(-10)}$$

$$h = 618.67 \text{ mm} \quad \text{answer}$$

Problem 637

For the beam loaded as shown in Fig. P-637, determine the deflection 6 ft from the wall. Use $E = 1.5 \times 10^6 \text{ psi}$ and $I = 40 \text{ in}^4$.

Solution 637

$$t_{B/C} = \frac{1}{EI} \left[\frac{1}{2}(6)(3840)(2) - 6(2560)(3) - \frac{1}{3}(6)(1440)(1.5) \right] (12^3)$$

$$t_{B/C} = \frac{1}{EI} [27\,360] (12^3)$$

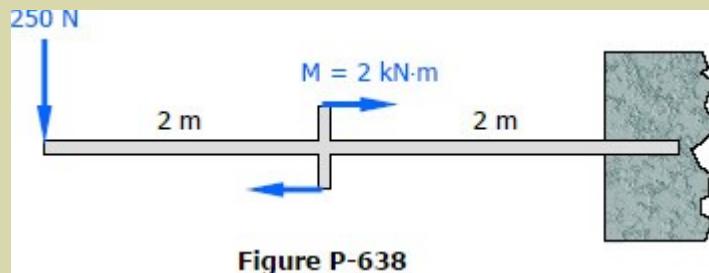
$$t_{B/C} = \frac{1}{(1.5 \times 10^6)(40)} [27\,360] (12^3)$$

$$t_{B/C} = -0.787968 \text{ in}$$

Thus, $\delta_B = |t_{B/C}| = 0.787968 \text{ in}$ *answer*

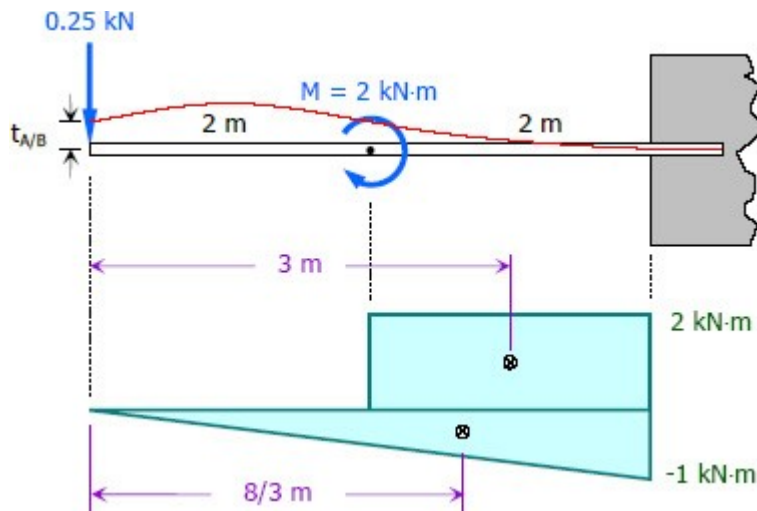
Problem 638

For the cantilever beam shown in Fig. P-638, determine the value of $EI\delta$ at the left end. Is this deflection upward or downward?



Solution 638

[HideClick here to show or hide the solution](#)



$$EI t_{A/B} = (Area_{AB}) \bar{X}_A$$

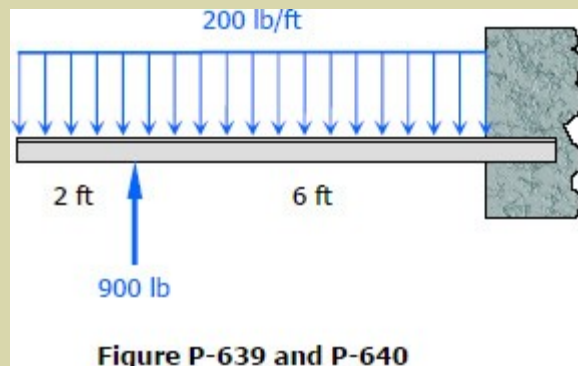
$$EI t_{A/B} = 2(2)(3) - \frac{1}{2}(4)(1)\left(\frac{8}{3}\right)$$

$$EI t_{A/B} = \frac{20}{3} = 6.67 \text{ kN} \cdot \text{m}^3$$

$$\therefore EI \delta = 6.67 \text{ kN} \cdot \text{m}^3 \text{ upward} \quad \text{answer}$$

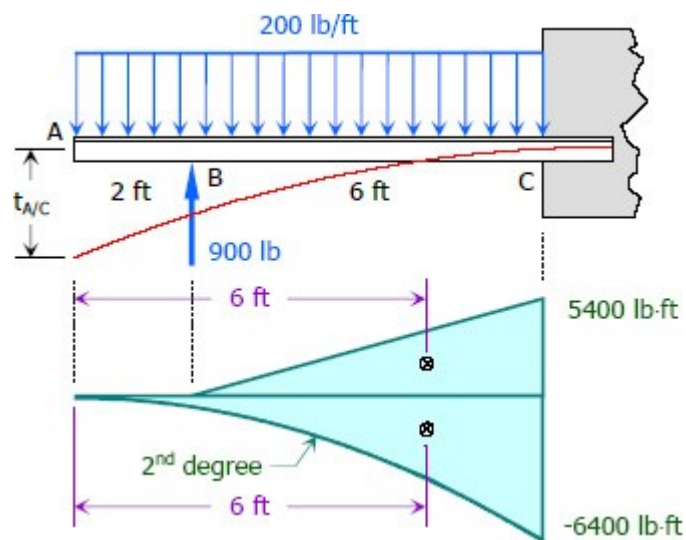
Problem 639

The downward distributed load and an upward concentrated force act on the cantilever beam in Fig. P-639. Find the amount the free end deflects upward or downward if $E = 1.5 \times 10^6$ psi and $I = 60$ in⁴.



Solution 639

[Hide](#) [Click here to show or hide the solution](#)



$$t_{A/C} = \frac{1}{EI} (Area_{AB}) \bar{X}_A$$

$$t_{A/C} = \frac{1}{(1.5 \times 10^6)(60)} \left[\frac{1}{2}(6)(5400)(6) - \frac{1}{3}(8)(6400)(6) \right] (12^3)$$

$$t_{A/C} = -0.09984 \text{ in}$$

∴ The free end will move by 0.09984 inch downward. answer

Problem 640

Compute the value of δ at the concentrated load in [Prob. 639](#). Is the deflection upward downward?

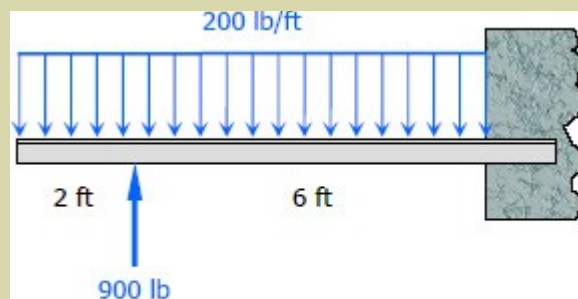


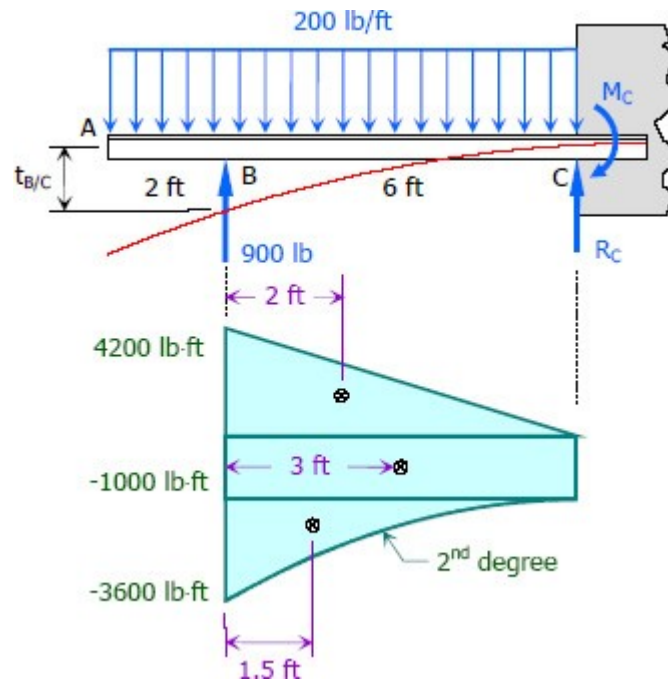
Figure P-639 and P-640

Solution 640

[HideClick here to show or hide the solution](#)

$$R_C = 200(8) - 900 = 700 \text{ lb}$$

$$M_C = 200(8)(4) - 900(6) = 1000 \text{ lb} \cdot \text{ft}$$



$$t_{B/C} = \frac{1}{EI} (Area_{BC}) \bar{X}_B$$

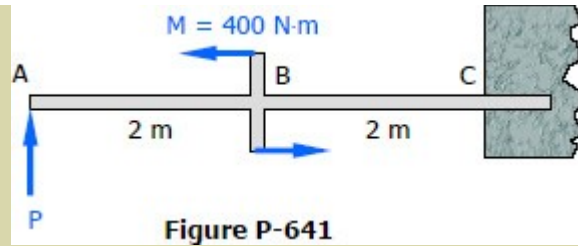
$$t_{B/C} = \frac{1}{(1.5 \times 10^6)(60)} \left[\frac{1}{2}(6)(4200)(2) - 1000(6)(3) - \frac{1}{3}(6)(3600)(15) \right] (12^3)$$

$$t_{B/C} = -0.06912 \text{ in}$$

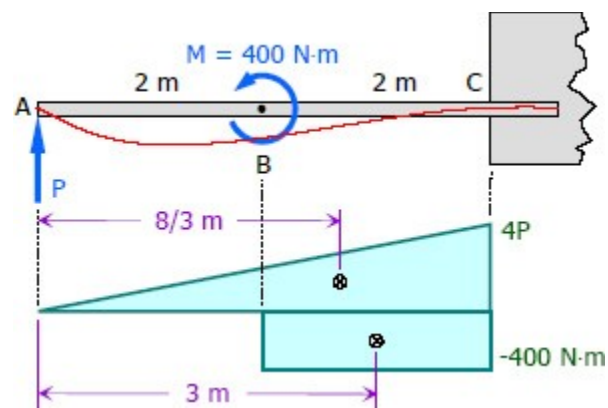
$\therefore \delta = 0.06912 \text{ inch downward}$ *answer*

Problem 641

For the cantilever beam shown in Fig. P-641, what will cause zero deflection at A?



Solution 641



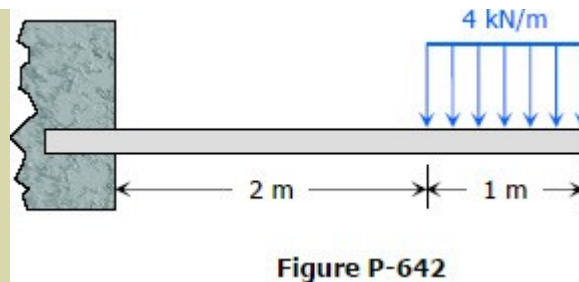
$$\frac{1}{EI} (Area_{AC}) \bar{X}_A = 0$$

$$\frac{1}{EI} \left[\frac{1}{2} (4) (4P) \left(\frac{8}{3} \right) - 2 (400) (3) \right] = 0$$

$$P = 112.5 \text{ N} \quad \text{answer}$$

Problem 642

Find the maximum deflection for the cantilever beam loaded as shown in Figure P-642 if the cross section is 50 mm wide by 150 mm high. Use $E = 69 \text{ GPa}$.

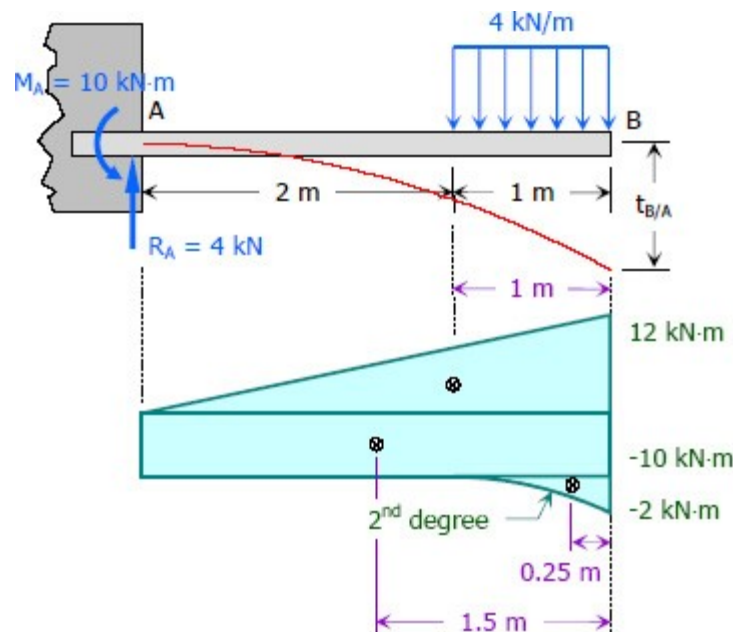


Solution 642

[Hide](#) [Click here to show or hide the solution](#)

$$R_A = 4(1) = 4 \text{ kN}$$

$$M_A = 4(1)(2.5) = 10 \text{ kN} \cdot \text{m}$$



$$t_{B/A} = \frac{1}{EI} (\text{Area}_{AB}) \bar{X}_B$$

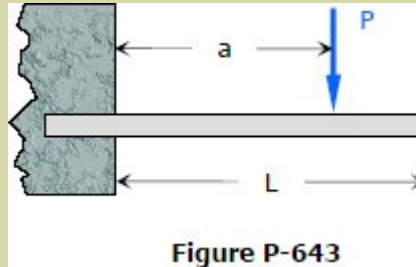
$$t_{B/A} = \frac{1}{69\,000 \left[\frac{50(150^3)}{12} \right]} \left[\frac{1}{2}(3)(12)(1) - 3(10)(1.5) - \frac{1}{3}(1)(2)(0.25) \right] (1000^4)$$

$$t_{B/A} = -28 \text{ mm}$$

$$\therefore \delta_{\max} = 28 \text{ mm} \quad \text{answer}$$

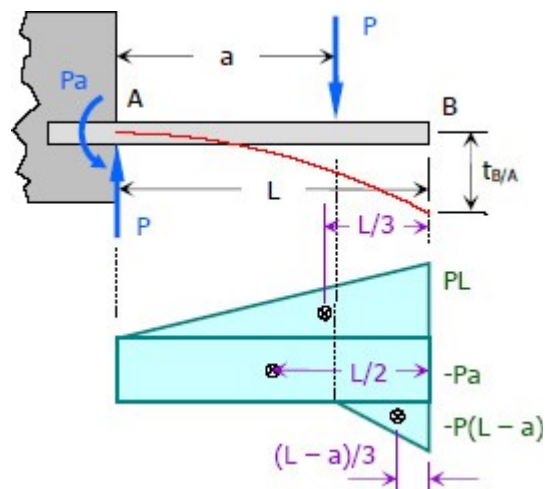
Problem 643

Find the maximum value of $EI\delta$ for the cantilever beam shown in Fig. P-643.



Solution 643

[HideClick here to show or hide the solution](#)



$$EI t_{B/A} = (\text{Area}_{AB}) \bar{X}_B$$

$$EI t_{B/A} = \frac{1}{2}L(PL)\left(\frac{1}{3}L\right) - PaL\left(\frac{1}{2}L\right) - \frac{1}{2}(L-a)P(L-a)\left[\frac{1}{3}(L-a)\right]$$

$$EI t_{B/A} = \frac{1}{6} PL^3 - \frac{1}{2} PL^2 a - \frac{1}{6} P(L-a)^3$$

$$EI t_{B/A} = \frac{1}{6} PL^3 - \frac{1}{2} PL^2 a - \frac{1}{6} P(L^3 - 3L^2 a + 3La^2 - a^3)$$

$$EI t_{B/A} = \frac{1}{6} PL^3 - \frac{1}{2} PL^2 a - \frac{1}{6} PL^3 + \frac{1}{2} PL^2 a - \frac{1}{2} PLa^2 + \frac{1}{6} Pa^3$$

$$EI t_{B/A} = -\frac{1}{2} PLa^2 + \frac{1}{6} Pa^3$$

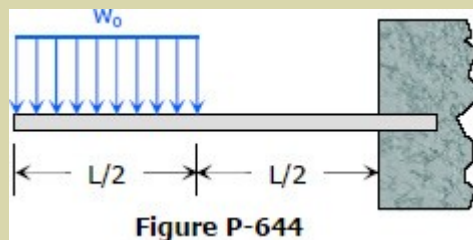
$$EI t_{B/A} = -\frac{1}{6} Pa^2 (3L - a)$$

Therefore

$$EI \delta_{max} = \frac{1}{6} Pa^2 (3L - a) \quad \text{answer}$$

Problem 644

Determine the maximum deflection for the beam loaded as shown in Fig. P-644.



Solution 644

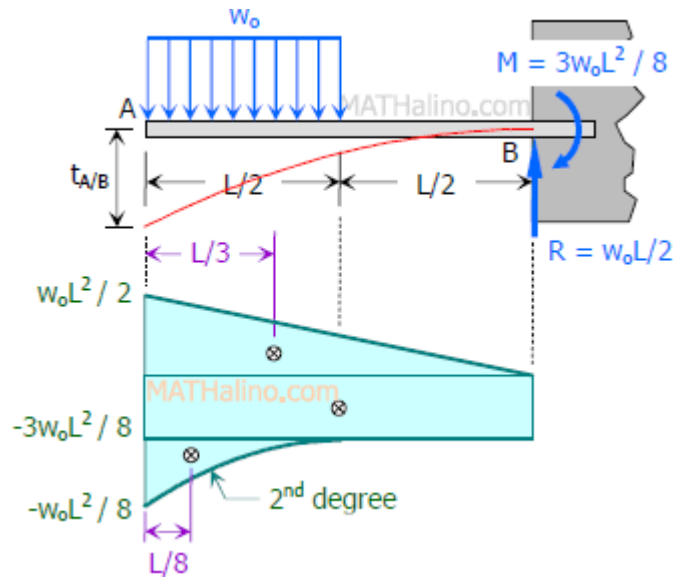
[HideClick here to show or hide the solution](#)

$$R = w_o \left(\frac{1}{2} L \right)$$

$$R = \frac{1}{2} w_o L$$

$$M = w_o \left(\frac{1}{2} L \right) \left(\frac{3}{4} L \right)$$

$$M = \frac{3}{8} w_o L^2$$



$$t_{A/B} = \frac{1}{EI} (Area_{AB}) \bar{X}_A$$

$$t_{A/B} = \frac{1}{EI} \left[\frac{1}{2}(L) \left(\frac{1}{2} w_o L^2 \right) \left(\frac{1}{3} L \right) - \frac{3}{8} w_o L^2 (L) \left(\frac{1}{2} L \right) - \frac{1}{3} \left(\frac{1}{8} w_o L^2 \right) \left(\frac{1}{2} L \right) \left(\frac{1}{8} L \right) \right]$$

$$t_{A/B} = \frac{1}{EI} \left[\frac{1}{12} w_o L^4 - \frac{3}{16} w_o L^4 - \frac{1}{384} w_o L^4 \right]$$

$$t_{A/B} = \frac{1}{EI} \left[-\frac{41}{384} w_o L^4 \right]$$

$$t_{A/B} = -\frac{41 w_o L^4}{384 EI}$$

Therefore

$$\delta_{max} = \frac{41 w_o L^4}{384 EI} \quad \text{answer}$$

Problem 645

Compute the deflection and slope at a section 3 m from the wall for the beam shown in Fig. P-645. Assume that $E = 10 \text{ GPa}$ and $I = 30 \times 10^6 \text{ mm}^4$.

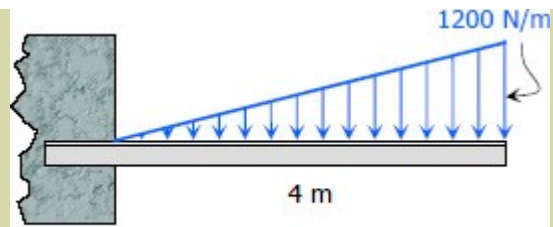


Figure P-645

Solution 645

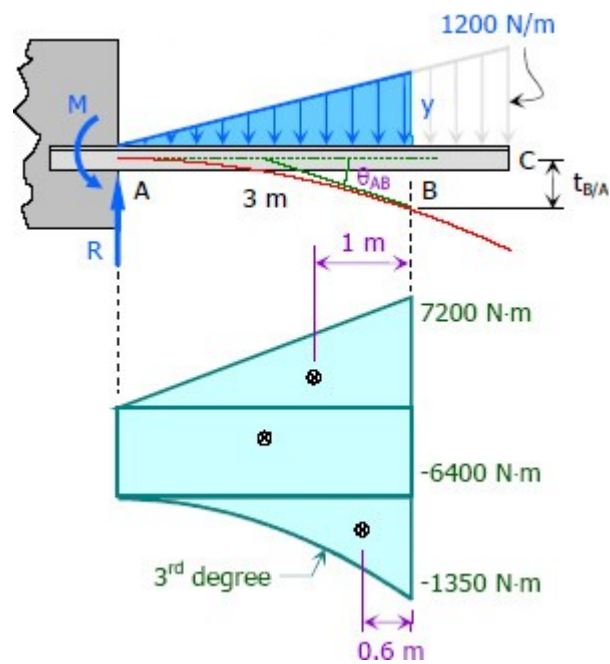
[Hide](#)[Click here to show or hide the solution](#)

$$R = \frac{1}{2}(4)(1200)$$

$$R = 2400 \text{ N}$$

$$M = \frac{1}{2}(4)(1200)\left(\frac{8}{3}\right)$$

$$M = 6400 \text{ N} \cdot \text{m}$$



$$\frac{y}{3} = \frac{1200}{4}$$

$$y = 900 \text{ N/m}$$

$$t_{B/A} = \frac{1}{EI} (\text{Area}_{AB}) \bar{X}_B$$

$$t_{B/A} = \frac{1}{EI} \left[\frac{1}{2}(3)(7200)(1) - 3(6400)(1.5) - \frac{1}{4}(3)(1350)(0.6) \right] (1000^3)$$

$$t_{B/A} = \frac{1}{10000(30 \times 10^6)} [-18607.5] (1000^3)$$

$$t_{B/A} = -62.025 \text{ mm}$$

Therefore:

$$\delta_B = 62.025 \text{ mm} \quad \text{answer}$$

$$\theta_{AB} = \frac{1}{EI} (\text{Area}_{AB})$$

$$\theta_{AB} = \frac{1}{EI} \left[\frac{1}{2}(3)(7200) - 3(6400) - \frac{1}{4}(3)(1350) \right] (1000^2)$$

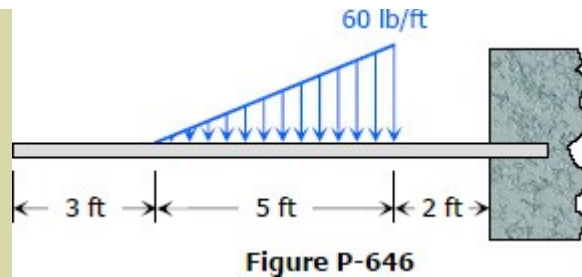
$$\theta_{AB} = \frac{1}{10000(30 \times 10^6)} [-9412.5] (1000^2)$$

$$\theta_{AB} = -0.031375 \text{ radian}$$

$$\theta_{AB} = 1.798 \text{ degree} \quad \text{answer}$$

Problem 646

For the beam shown in Fig. P-646, determine the value of I that will limit the maximum deflection to 0.50 in. Assume that $E = 1.5 \times 10^6$ psi.

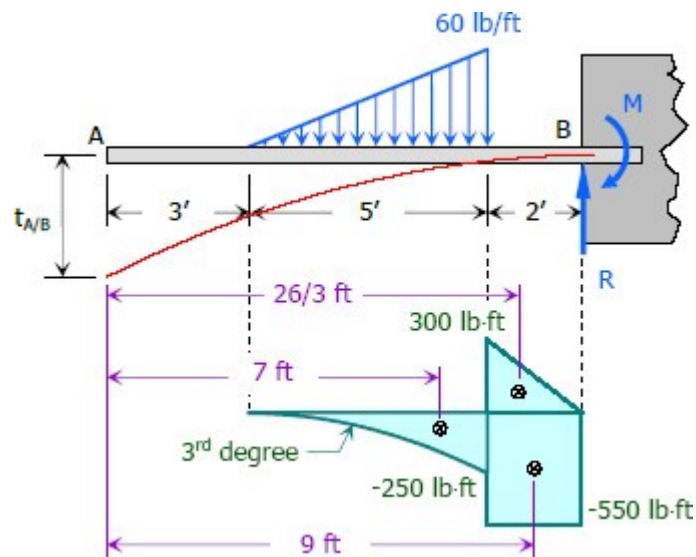


Solution 646

[Hide](#) Click here to show or hide the solution

$$M = \frac{1}{2}(5)(60)\left(2 + \frac{5}{3}\right) = 550 \text{ lb} \cdot \text{ft}$$

$$R = \frac{1}{2}(5)(60) = 150 \text{ lb}$$



$$t_{A/B} = \frac{1}{EI} (\text{Area}_{AB}) \bar{X}_A$$

$$-5 = \frac{1}{EI} \left[\frac{1}{2}(300)(2)\left(\frac{26}{3}\right) - 550(2)(9) - \frac{1}{4}(5)(250)(7) \right] (12^3)$$

$$-5 = \frac{1}{(1.5 \times 10^6)I} (-16394400)$$

$$I = 2.18592 \text{ in}^4$$

Problem 647

Find the maximum value of $EI\delta$ for the beam shown in Fig. P-647.

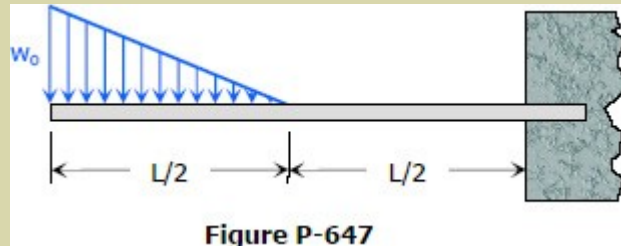


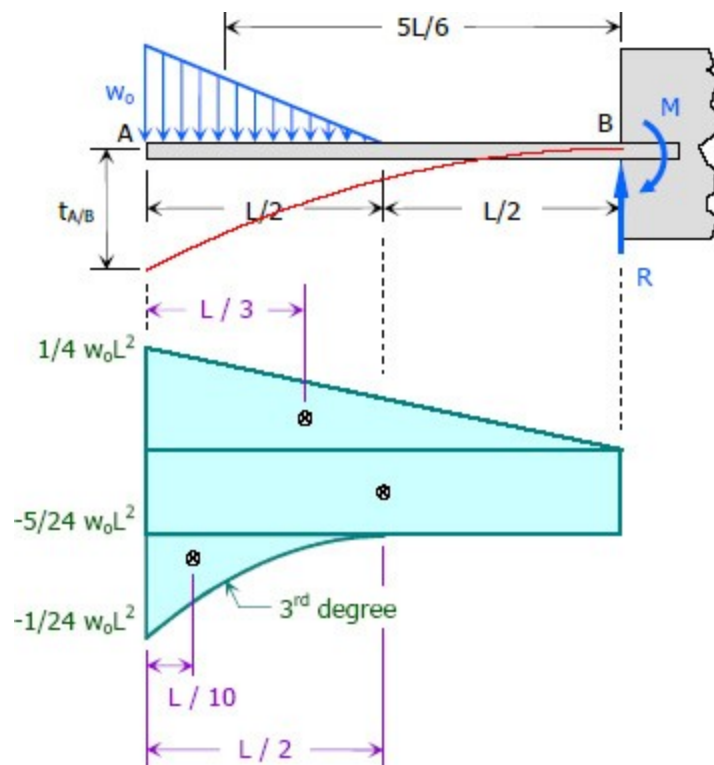
Figure P-647

Solution 647

[HideClick here to show or hide the solution](#)

$$R = \frac{1}{2} \left(\frac{1}{2} L \right) (w_0) = \frac{1}{4} w_0 L$$

$$M = \frac{1}{2} \left(\frac{1}{2} L \right) (w_0) \left(\frac{5}{6} L \right) = \frac{5}{24} w_0 L^2$$



$$EI t_{A/B} = (Area_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{2}L(\frac{1}{4}w_o L^2)(\frac{1}{3}L) - L(\frac{5}{24}w_o L^2)(\frac{1}{2}L) - \frac{1}{4}(\frac{1}{2}L)(\frac{1}{24}w_o L^2)(\frac{1}{10}L)$$

$$EI t_{A/B} = \frac{1}{24}w_o L^4 - \frac{5}{48}w_o L^4 - \frac{1}{1920}w_o L^4$$

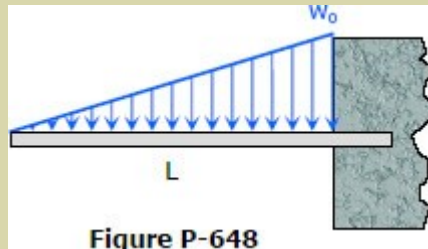
$$EI t_{A/B} = -\frac{121}{1920}w_o L^4$$

Therefore

$$EI \delta_{max} = \frac{121}{1920}w_o L^4 \quad \text{answer}$$

Problem 648

For the cantilever beam loaded as shown in Fig. P-648, determine the deflection at a distance x from the support.



Solution 648

[HideClick here to show or hide the solution](#)

$$\frac{y}{x} = \frac{w_o}{L}$$

$$y = \frac{w_o}{L} x$$

$$M = \frac{1}{2} L (w_o) \left(\frac{1}{3} L \right) = \frac{1}{6} w_o L^2$$

$$R = \frac{1}{2} w_o L$$

Moments about B:

Triangular force to the left of B:

$$M_1 = -\frac{1}{2} (L-x) (w_o - y) \left(\frac{1}{3} \right) (L-x)$$

$$M_1 = -\frac{1}{6} (L-x)^2 \left(w_o - \frac{w_o x}{L} \right)$$

$$M_1 = -\frac{w_o (L-x)^3}{6L}$$

Triangular upward force:

$$M_2 = \frac{1}{2} (xy) \left(\frac{1}{3} x \right) = \frac{1}{6} x^2 \frac{w_o x}{L}$$

$$M_2 = \frac{w_o x^3}{6L}$$

Rectangle (w_o by x):

$$M_3 = -w_o x \left(\frac{1}{2} x \right) = -\frac{1}{2} w_o x^2$$

Reactions R and M:

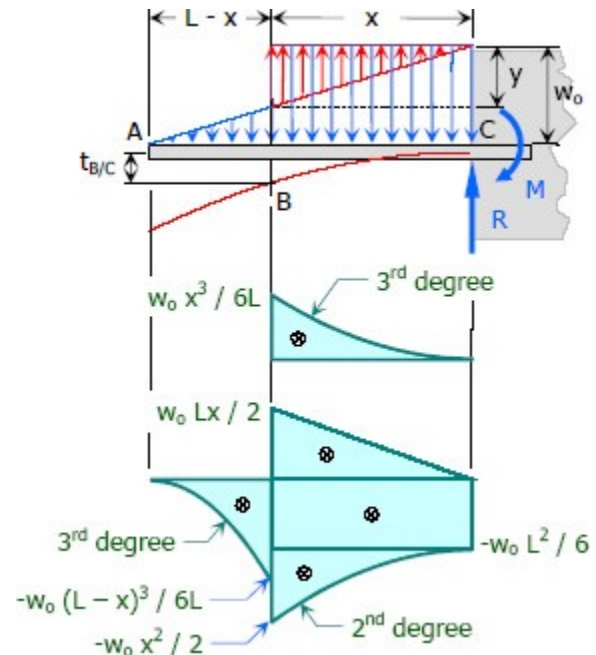
$$M_4 = Rx = \frac{1}{2} w_o Lx$$

$$M_5 = -M = -\frac{1}{6} w_o L^2$$

Deviation at B with the tangent line through C

$$EI t_{B/C} = (Area_{BC}) \bar{X}_B$$

$$EI t_{B/C} = \frac{1}{4} x \left(\frac{w_o x^3}{6L} \right) \left(\frac{1}{5} x \right) + \frac{1}{2} x \left(\frac{1}{2} w_o Lx \right) \left(\frac{1}{3} x \right) - \left(\frac{1}{6} w_o L^2 \right) x \left(\frac{1}{2} x \right) - \frac{1}{3} x \left(\frac{1}{2} w_o x^2 \right) \left(\frac{1}{4} x \right)$$



$$EI t_{B/C} = \frac{w_o}{120L} x^5 + \frac{w_o L}{12} x^3 - \frac{w_o L^2}{12} x^2 - \frac{w_o}{24} x^4$$

$$EI t_{B/C} = \frac{w_o x^2}{120L} (x^3 + 10L^2 x - 10L^3 - 5Lx^2)$$

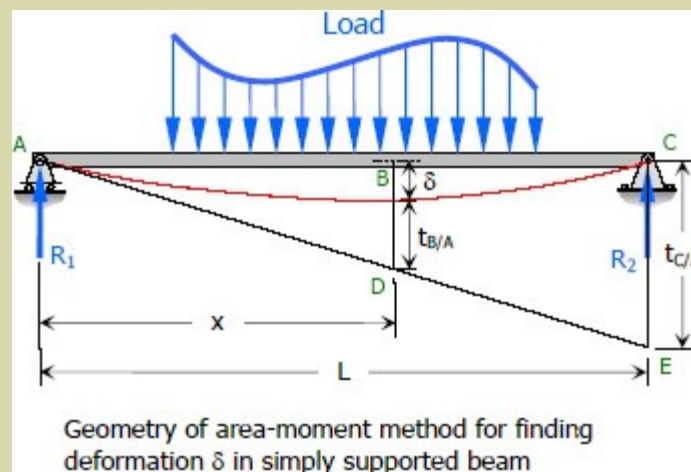
Therefore,

$$EI \delta = -\frac{w_o x^2}{120L} (x^3 + 10L^2 x - 10L^3 - 5Lx^2)$$

$$EI \delta = \frac{w_o x^2}{120L} (10L^3 - 10L^2 x + 5Lx^2 - x^3) \quad \text{answer}$$

-Deflections in Simply Supported Beams | Area-Moment Method

The deflection δ at some point B of a simply supported beam can be obtained by the following steps:



$$1. \text{ Compute } t_{C/A} = \frac{1}{EI} (\text{Area}_{AC}) \bar{X}_C$$

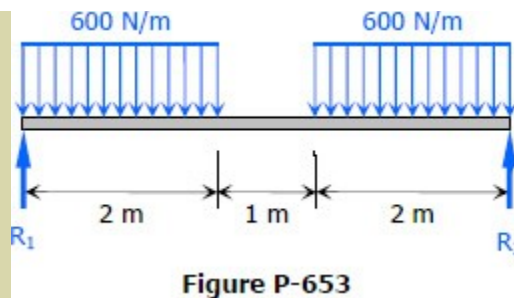
$$2. \text{ Compute } t_{B/A} = \frac{1}{EI} (\text{Area}_{AB}) \bar{X}_B$$

3. Solve δ by ratio and proportion (see figure above).

$$\frac{\delta + t_{B/A}}{x} = \frac{t_{C/A}}{L}$$

Problem 653

Compute the midspan value of $EI\delta$ for the beam shown in Fig. P-653. (Hint: Draw the M diagram by parts, starting from midspan toward the ends. Also take advantage of symmetry to note that the tangent drawn to the elastic curve at midspan is horizontal.)

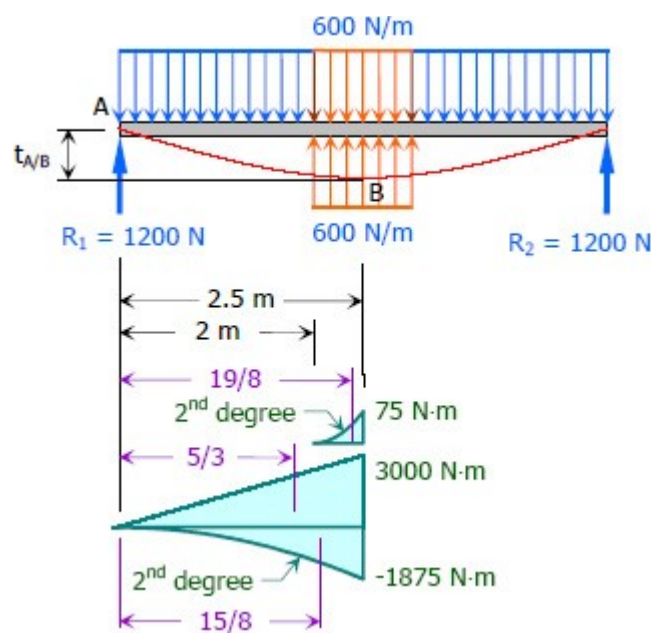


Solution 653

[HideClick here to show or hide the solution](#)

By symmetry:

$$R_1 = R_2 = 600(2) = 1200 \text{ N}$$



$$t_{A/B} = \frac{1}{EI} (Area_{AB}) \bar{X}_A$$

$$t_{A/B} = \frac{1}{EI} \left[\frac{1}{2} (2.5) (3000) \left(\frac{5}{3} \right) + \frac{1}{3} (0.5) (75) \left(\frac{19}{8} \right) - \frac{1}{3} (2.5) (1875) \left(\frac{15}{8} \right) \right]$$

$$t_{A/B} = \frac{3350}{EI}$$

From the figure

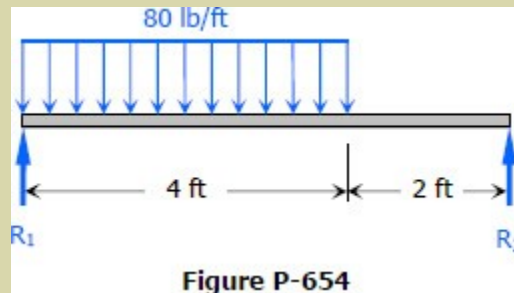
$$\delta_{midspan} = t_{A/B}$$

Thus

$$EI \delta_{midspan} = 3350 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

Problem 654

For the beam in Fig. P-654, find the value of $EI\delta$ at 2 ft from R_2 . (Hint: Draw the reference tangent to the elastic curve at R_2 .)



Solution 654

[HideClick here to show or hide the solution](#)

$$\Sigma M_{R2} = 0$$

$$6R_1 = 80(4)(4)$$

$$R_1 = \frac{640}{3} \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$6R_2 = 80(4)(2)$$

$$R_2 = \frac{320}{3} \text{ lb}$$

By ratio and proportion:

$$y = \frac{2}{6} \left(\frac{8960}{3EI} \right)$$

$$y = \frac{8960}{9EI}$$

$$\delta_B = y - t_{B/C}$$

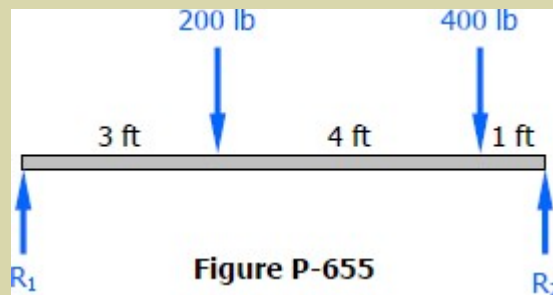
$$\delta_B = \frac{8960}{9EI} - \frac{1280}{9EI}$$

$$\delta_B = \frac{2560}{3EI}$$

$$EI \delta_B = \frac{2560}{3} \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

Problem 655

Find the value of $EI\delta$ under each concentrated load of the beam shown in Fig. P-655.



Solution 655

[HideClick here to show or hide the solution](#)

$$\Sigma M_{R2} = 0$$

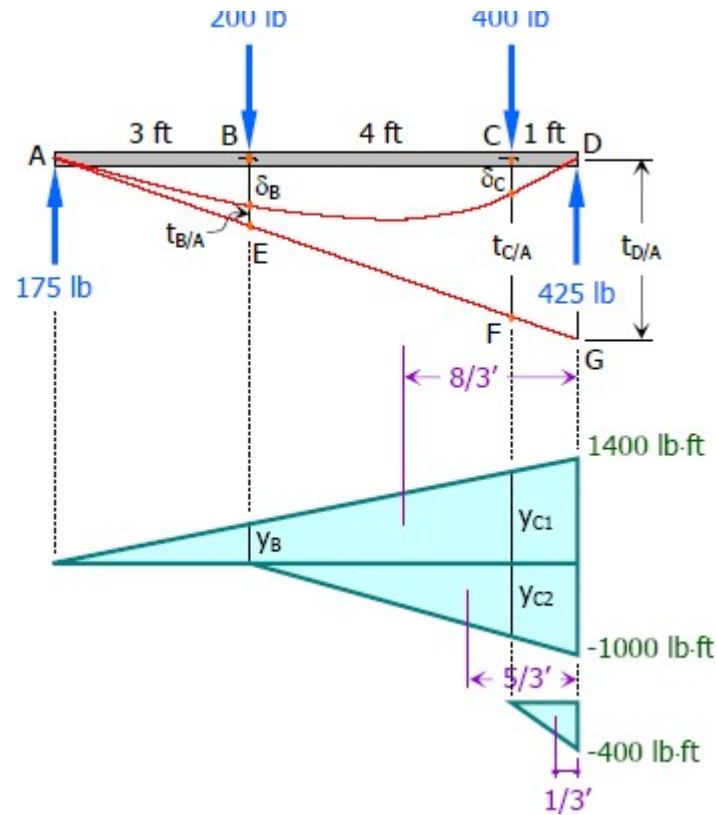
$$8R_1 = 200(5) + 400(1)$$

$$R_1 = 175 \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$8R_2 = 200(3) + 400(7)$$

$$R_2 = 425 \text{ lb}$$



$$\frac{y_{C1}}{7} = \frac{1400}{8}$$

$$y_{C1} = 1225 \text{ lb}$$

$$\frac{y_{C2}}{4} = \frac{-1000}{5}$$

$$y_{C2} = -800 \text{ lb}$$

$$\frac{y_B}{3} = \frac{1400}{8}$$

$$y_B = 525 \text{ lb}$$

$$EI t_{D/A} = (Area_{AD}) \bar{X}_D$$

$$EI t_{D/A} = \frac{1}{2}(8)(1400)(\frac{8}{3}) - \frac{1}{2}(5)(1000)(\frac{5}{3}) - \frac{1}{2}(1)(400)(\frac{1}{3})$$

$$EI t_{D/A} = 10\,700 \text{ lb} \cdot \text{ft}^3$$

$$EI t_{C/A} = (Area_{AC}) \bar{X}_C$$

$$EI t_{C/A} = \frac{1}{2}(7)(y_{C1})(\frac{7}{3}) - \frac{1}{2}(4)(y_{C2})(\frac{4}{3})$$

$$EI t_{C/A} = \frac{1}{2}(7)(1225)(\frac{7}{3}) - \frac{1}{2}(4)(800)(\frac{4}{3})$$

$$EI t_{C/A} = \frac{47\,225}{6} \text{ lb} \cdot \text{ft}^3$$

$$EI t_{B/A} = (Area_{AB}) \bar{X}_B$$

$$EI t_{C/A} = \frac{1}{2}(3)(y_B)(1)$$

$$EI t_{C/A} = \frac{1}{2}(3)(525)(1)$$

$$EI t_{C/A} = \frac{1575}{2} \text{ lb} \cdot \text{ft}^3$$

By ratio and proportion:

$$\frac{\bar{B}E}{3} = \frac{\bar{C}F}{7} = \frac{t_{D/A}}{8}$$

$$\bar{B}E = \frac{3}{8}t_{D/A} = \frac{3}{8}(10\,700) = \frac{8025}{2}$$

$$\bar{C}F = \frac{7}{8}t_{D/A} = \frac{7}{8}(10\,700) = \frac{18\,725}{2}$$

Deflections:

$$\delta_B = \bar{B}E - t_{B/A}$$

$$EI \delta_B = EI \bar{B}E - EI t_{B/A} = \frac{8025}{2} - \frac{1575}{2}$$

$$EI \delta_B = 3225 \text{ lb} \cdot \text{ft}^3 \quad \rightarrow \text{answer}$$

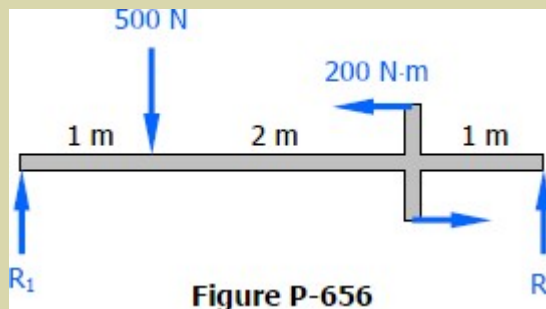
$$\delta_C = \bar{C}F - t_{C/A}$$

$$EI \delta_C = EI \bar{C}F - EI t_{C/A} = \frac{18\,725}{2} - \frac{47\,225}{6}$$

$$EI \delta_C = \frac{4475}{3} = 1491.67 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

Problem 656

Find the value of $EI\delta$ at the point of application of the 200 N·m couple in Fig. P-656.



Solution 656

[HideClick here to show or hide the solution](#)

$$\Sigma M_{R2} = 0$$

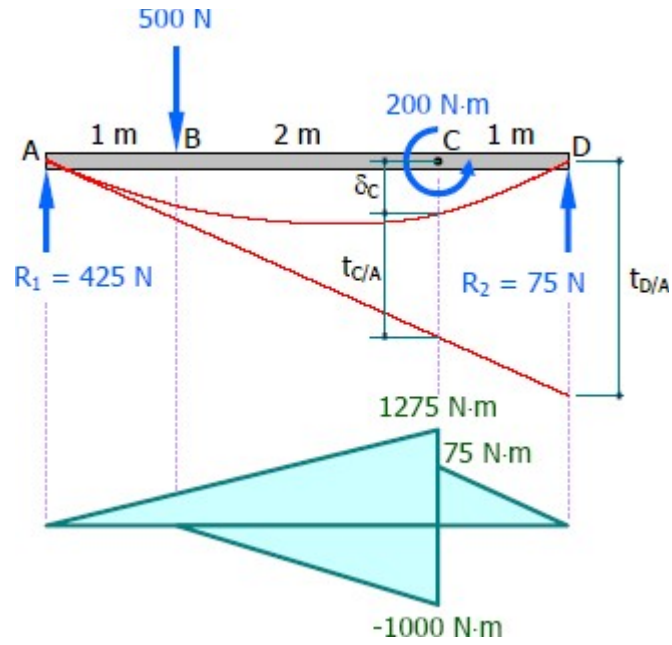
$$4R_1 = 500(3) + 200$$

$$R_1 = 425 \text{ N}$$

$$\Sigma M_{R1} = 0$$

$$4R_2 + 200 = 500(1)$$

$$R_2 = 75 \text{ N}$$



$$EI t_{D/A} = (\text{Area}_{AD}) \bar{X}_D$$

$$EI t_{D/A} = \frac{1}{2}(1)(75)\left(\frac{2}{3}\right) + \frac{1}{2}(3)(1275)(2) - \frac{1}{2}(2)(1000)\left(\frac{5}{3}\right)$$

$$EI t_{D/A} = \frac{6550}{3} \text{ N} \cdot \text{m}^3$$

$$EI t_{C/A} = (\text{Area}_{AC}) \bar{X}_C$$

$$EI t_{C/A} = \frac{1}{2}(3)(1275)(1) - \frac{1}{2}(2)(1000)\left(\frac{2}{3}\right)$$

$$EI t_{C/A} = \frac{7475}{6} \text{ N} \cdot \text{m}^3$$

$$\frac{\bar{C}E}{3} = \frac{t_{D/A}}{4}$$

$$\bar{C}E = \frac{3}{4} \left(\frac{6550}{3EI} \right) = \frac{3275}{2EI}$$

$$EI \bar{C}E = \frac{3275}{2} \text{ N} \cdot \text{m}^3$$

$$\delta_C = \bar{C}\bar{E} - t_{C/A}$$

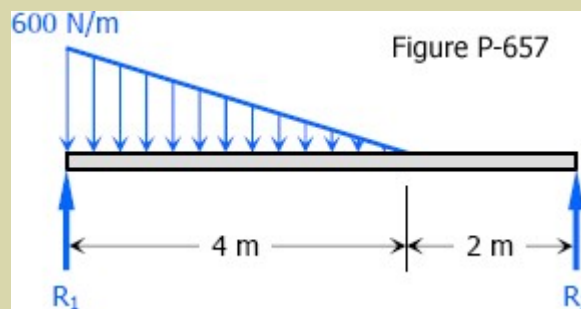
$$EI \delta_C = EI \bar{C}\bar{E} - EI t_{C/A}$$

$$EI \delta_C = \frac{3275}{2} - \frac{7475}{6} = \frac{1175}{3}$$

$$EI \delta_C = 391.67 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

Problem 657

Determine the midspan value of $EI\delta$ for the beam shown in Fig. P-657.



Solution 657

[HideClick here to show or hide the solution](#)

$$\Sigma M_{R1} = 0$$

$$6R_2 = \frac{1}{2}(4)(600)(\frac{4}{3})$$

$$R_2 = \frac{800}{3} \text{ N}$$

$$EI t_{A/B} = (Area_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{2}(6)(1600)(2) - \frac{1}{4}(4)(1600)(\frac{4}{5})$$

$$EI t_{A/B} = 8320 \text{ N} \cdot \text{m}^3$$

$$EI t_{M/B} = (Area_{MB}) \bar{X}_M$$

$$EI t_{M/B} = \frac{1}{2}(3)(800)(1) - \frac{1}{4}(1)(25)(\frac{1}{5})$$

$$EI t_{M/B} = 1198.75 \text{ N} \cdot \text{m}^3$$

By ratio and proportion:

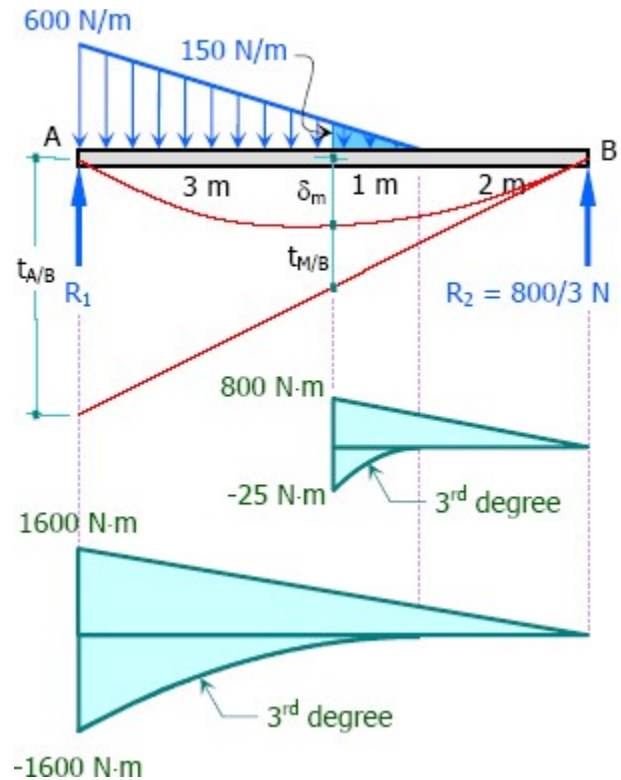
$$\frac{\delta_m + t_{M/B}}{3} = \frac{t_{A/B}}{6}$$

$$\delta_m + t_{M/B} = \frac{1}{2} t_{A/B}$$

$$EI \delta_m + EI t_{M/B} = EI \frac{1}{2} t_{A/B}$$

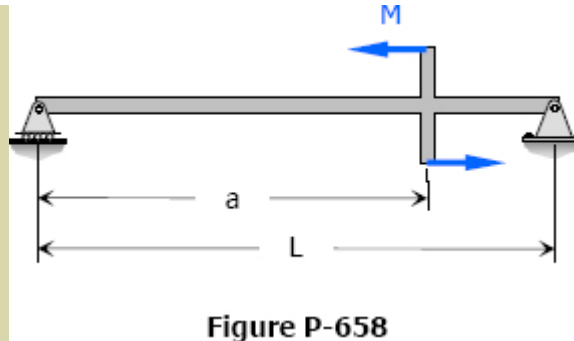
$$EI \delta_m + 1198.75 = EI \frac{1}{2} (8320)$$

$$EI \delta_m = 2961.25 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$



Problem 658

For the beam shown in Fig. P-658, find the value of $EI\delta$ at the point of application of the couple.



Solution 658

[HideClick here to show or hide the solution](#)

$$\frac{y}{a} = \frac{M}{L}$$

$$y = Ma/L$$

$$EI t_{B/A} = (\text{Area}_{AB}) \bar{X}_B$$

$$EI t_{B/A} = \frac{1}{2}(ay)\left(\frac{1}{3}a\right)$$

$$EI t_{B/A} = \frac{1}{6}a^2(Ma/L)$$

$$EI t_{B/A} = \frac{Ma^3}{6L}$$

$$EI t_{C/A} = (\text{Area}_{AC}) \bar{X}_C$$

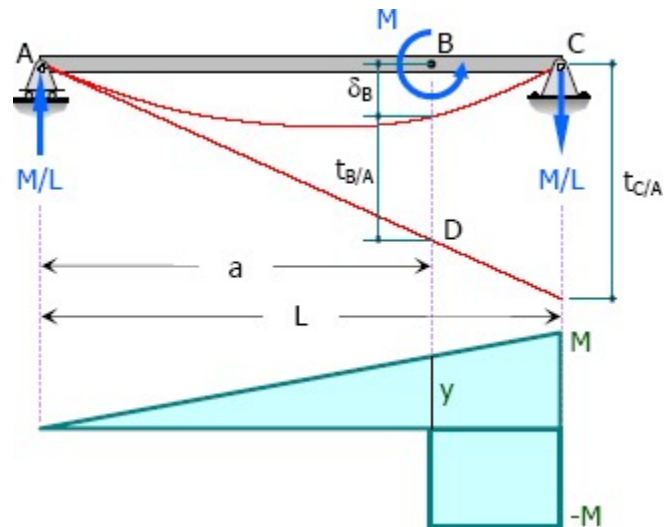
$$EI t_{C/A} = \frac{1}{2}(LM)\left(\frac{1}{3}L\right) - M(L-a)\left[\frac{1}{2}(L-a)\right]$$

$$EI t_{C/A} = \frac{1}{6}ML^2 - \frac{1}{2}M(L-a)^2$$

By ratio and proportion:

$$\frac{\delta_B + t_{B/A}}{a} = \frac{t_{C/A}}{L}$$

$$\delta_B = \frac{a}{L}t_{C/A} - t_{B/A}$$



$$EI \delta_B = \frac{a}{L} EI t_{C/A} - EI t_{B/A}$$

$$EI \delta_B = \frac{a}{L} \left[\frac{1}{6} ML^2 - \frac{1}{2} M(L-a)^2 \right] - \frac{Ma^3}{6L}$$

$$EI \delta_B = \frac{a}{L} \left[\frac{1}{6} ML^2 - \frac{1}{2} M(L-a)^2 - \frac{1}{6} Ma^2 \right]$$

$$EI \delta_B = \frac{Ma}{6L} [L^2 - 3(L-a)^2 - a^2]$$

$$EI \delta_B = \frac{Ma}{6L} [L^2 - 3(L^2 - 2La + a^2) - a^2]$$

$$EI \delta_B = \frac{Ma}{6L} [L^2 - 3L^2 + 6La - 3a^2 - a^2]$$

$$EI \delta_B = \frac{Ma}{6L} [-3L^2 + 6La - 4a^2]$$

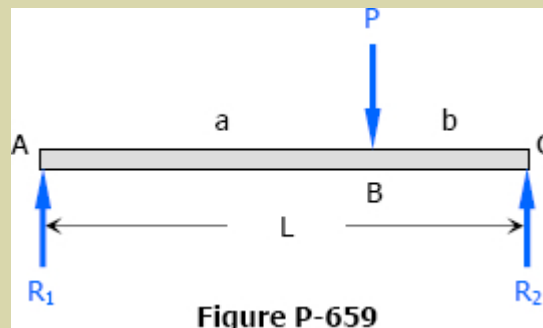
$$EI \delta_B = -\frac{Ma}{6L} (3L^2 - 6La + 4a^2)$$

The negative sign indicates that the deflection is opposite to the direction sketched in the figure. Thus,

$$EI \delta_B = \frac{Ma}{6L} (3L^2 - 6La + 4a^2) \quad \text{upward} \quad \text{answer}$$

Problem 659

A simple beam supports a concentrated load placed anywhere on the span, as shown in Fig. P-659. Measuring x from A, show that the maximum deflection occurs at $x = \sqrt{[(L^2 - b^2)/3]}$.



Solution 659

[Hide/Click here to show or hide the solution](#)

$$\Sigma M_{R2} = 0$$

$$LR_1 = Pb$$

$$R_1 = Pb/L$$

$$\Sigma M_{R1} = 0$$

$$LR_2 = Pa$$

$$R_2 = Pa/L$$

$$\frac{y}{x} = \frac{Pb}{L}$$

$$y = \frac{Pb}{L}x$$

$$t_{A/D} = \frac{1}{EI} (Area_{AD}) \bar{X}_A$$

$$t_{A/D} = \frac{1}{EI} \left[\frac{1}{2}xy \left(\frac{2}{3}x \right) \right]$$

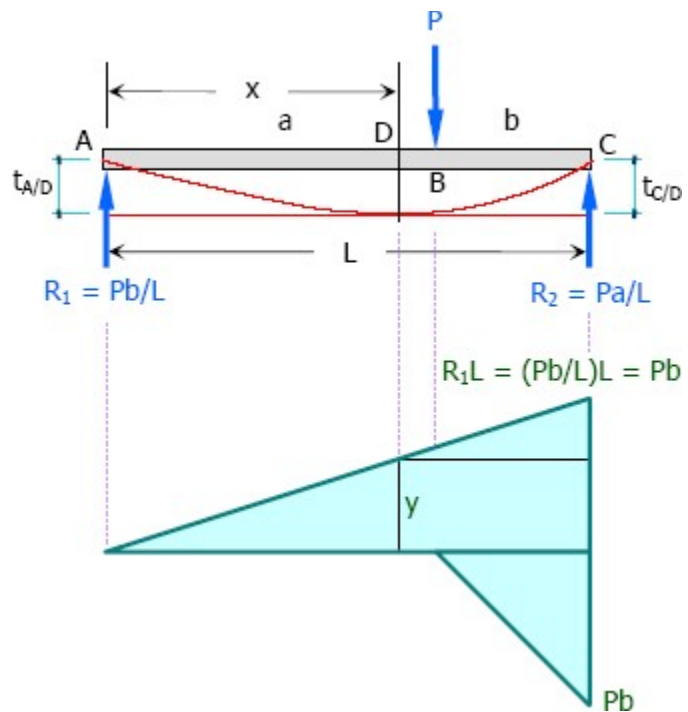
$$t_{A/D} = \frac{1}{EI} \left[\frac{1}{3}x^2y \right]$$

$$t_{A/D} = \frac{1}{EI} \left[\frac{1}{3}x^2 \left(\frac{Pb}{L}x \right) \right]$$

$$t_{A/D} = \frac{1}{EI} \frac{Pb}{3L} x^3$$

$$t_{C/D} = \frac{1}{EI} (Area_{CD}) \bar{X}_C$$

$$t_{C/D} = \frac{1}{EI} \left[\frac{1}{6}(L-x)^2(Pb-y) + \frac{1}{2}(L-x)^2y - \frac{1}{6}Pb^3 \right]$$



$$t_{C/D} = \frac{1}{EI} \left[\frac{1}{6}(L-x)^2 \left(Pb - \frac{Pb}{L}x \right) + \frac{1}{2}(L-x)^2 \left(\frac{Pb}{L}x \right) - \frac{1}{6}Pb^3 \right]$$

$$t_{C/D} = \frac{1}{EI} \left[\frac{1}{6}Pb(L-x)^2 \left(1 - \frac{x}{L} \right) + \frac{1}{2}Pb(L-x)^2 \left(\frac{x}{L} \right) - \frac{1}{6}Pb^3 \right]$$

$$t_{C/D} = \frac{1}{EI} \left[\frac{Pb}{6L}(L-x)^3 + \frac{Pb}{2L}(L-x)^2x - \frac{Pb^3}{6} \right]$$

From the figure:

$$t_{A/D} = t_{C/D}$$

$$\frac{1}{EI} \frac{Pb}{3L} x^3 = \frac{1}{EI} \left[\frac{Pb}{6L}(L-x)^3 + \frac{Pb}{2L}(L-x)^2x - \frac{Pb^3}{6} \right]$$

$$\frac{Pb}{3L} x^3 = \frac{Pb}{6L}(L-x)^3 + \frac{Pb}{2L}(L-x)^2x - \frac{Pb^3}{6}$$

$$\frac{2x^3}{L} = \frac{(L-x)^3}{L} + \frac{3(L-x)^2x}{L} - b^2$$

$$2x^3 = (L-x)^3 + 3(L-x)^2x - Lb^2$$

$$2x^3 = (L^3 - 3L^2x + 3Lx^2 - x^3) + 3(L^2 - 2Lx + x^2)x - Lb^2$$

$$2x^3 = L^3 - 3L^2x + 3Lx^2 - x^3 + 3L^2x - 6Lx^2 + 3x^3 - Lb^2$$

$$0 = L^3 - 3Lx^2 - Lb^2$$

$$0 = L^2 - 3x^2 - b^2$$

$$3x^2 = L^2 - b^2$$

$$x = \sqrt{\frac{L^2 - b^2}{3}} \quad (\text{okay!})$$

Problem 660

A simply supported beam is loaded by a couple M at its right end, as shown in Fig. P-660. Show that the maximum deflection occurs at $x = 0.577L$.



Figure P-660

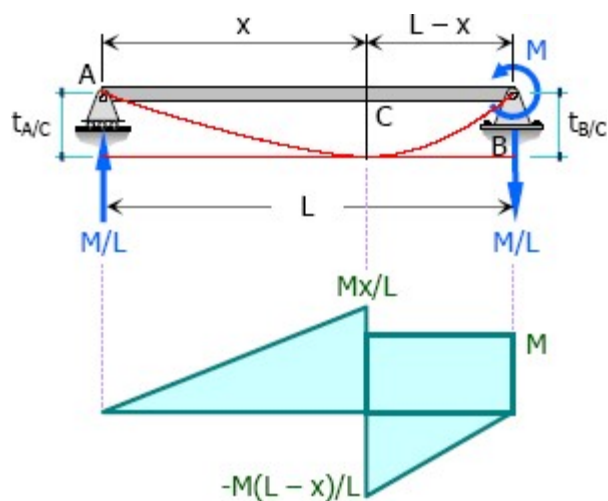
Solution 660

[Hide](#)[Click here to show or hide the solution](#)

$$EI t_{A/C} = (\text{Area}_{AB}) \bar{X}_A$$

$$EI t_{A/C} = \frac{1}{2}x \left(\frac{Mx}{L} \right) \left(\frac{2}{3}x \right)$$

$$EI t_{A/C} = \frac{Mx^3}{3L}$$



$$EI t_{B/C} = (\text{Area}_{BC}) \bar{X}_B$$

$$EI t_{B/C} = M(L-x) \frac{1}{2}(L-x) - \frac{1}{2}(L-x) \frac{M}{L} (L-x) \frac{2}{3}(L-x)$$

$$EI t_{B/C} = \frac{1}{2}M(L-x)^2 - \frac{M}{3L}(L-x)^3$$

$$EI t_{B/C} = \frac{M}{6L} (L-x)^2 [3L - 2(L-x)]$$

$$EI t_{B/C} = \frac{M}{6L} (L-x)^2 (L+2x)$$

$$EI t_{B/C} = \frac{M}{6} (L-x)^2 + \frac{Mx}{3L} (L-x)^2$$

$$EI t_{B/C} = \frac{M}{6} (L^2 - 2Lx + x^2) + \frac{Mx}{3L} (L^2 - 2Lx + x^2)$$

$$EI t_{B/C} = \frac{ML^2}{6} - \frac{MLx}{3} + \frac{Mx^2}{6} + \frac{MLx}{3} - \frac{2Mx^2}{3} + \frac{Mx^3}{3L}$$

$$EI t_{B/C} = \frac{ML^2}{6} - \frac{Mx^2}{2} + \frac{Mx^3}{3L}$$

From the figure

$$EI t_{A/C} = EI t_{B/C}$$

$$\frac{Mx^3}{3L} = \frac{ML^2}{6} - \frac{Mx^2}{2} + \frac{Mx^3}{3L}$$

$$\frac{Mx^2}{2} = \frac{ML^2}{6}$$

$$x^2 = \frac{1}{3} L^2$$

$$x = 0.577L \text{ (okay!)}$$

Problem 661

Compute the midspan deflection of the symmetrically loaded beam shown in Fig. P-661. Check your answer by letting $a = L/2$ and comparing with the answer to [Problem 609](#).

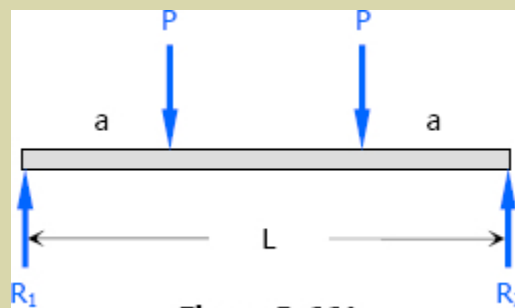
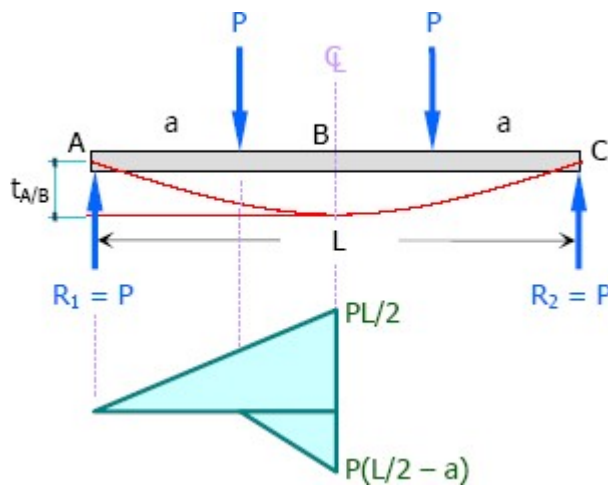


Figure P-661

Solution 661

[Hide](#) [Click here to show or hide the solution](#)

$$EI t_{A/B} = (\text{Area}_{AB}) \bar{X}_A$$



$$EI t_{A/B} = \frac{1}{2} \left(\frac{1}{2} L \right) \left(\frac{1}{2} PL \right) \left[\frac{2}{3} \left(\frac{1}{2} L \right) \right] - \frac{1}{2} \left(\frac{1}{2} L - a \right) P \left(\frac{1}{2} L - a \right) \left[a + \frac{2}{3} \left(\frac{1}{2} L - a \right) \right]$$

$$EI t_{A/B} = \frac{1}{24} PL^3 - \frac{1}{2} Pa \left(\frac{1}{2} L - a \right)^2 - \frac{1}{3} P \left(\frac{1}{2} L - a \right)^3$$

$$EI t_{A/B} = \frac{1}{24} PL^3 - \frac{1}{2} Pa \left(\frac{1}{4} L^2 - La + a^2 \right) - \frac{1}{3} P \left(\frac{1}{8} L^3 - \frac{3}{4} L^2 a + \frac{3}{2} La^2 - a^3 \right)$$

$$EI t_{A/B} = \frac{1}{24} PL^3 - \frac{1}{8} PL^2 a + \frac{1}{2} PL a^2 - \frac{1}{8} Pa^3 - \frac{1}{24} PL^3 + \frac{1}{4} PL^2 a - \frac{1}{2} PL a^2 + \frac{1}{3} Pa^3$$

$$EI t_{A/B} = \frac{1}{8} PL^2 a - \frac{1}{6} Pa^3$$

$$EI t_{A/B} = \frac{1}{24} Pa (3L^2 - 4a^2) \quad \text{answer}$$

When $a = \frac{1}{2}L$

$$EI t_{A/B} = \frac{1}{24} P \left(\frac{1}{2} L \right) \left[3L^2 - 4 \left(\frac{1}{2} L \right)^2 \right]$$

$$EI t_{A/B} = \frac{1}{48} PL [3L^2 - L^2]$$

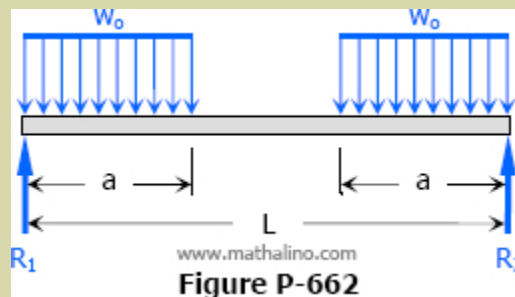
$$EI t_{A/B} = \frac{1}{48} PL(2L^2)$$

$$EI t_{A/B} = \frac{1}{24} PL^3 \rightarrow \text{answer}$$

$$EI t_{A/B} = 2\left(\frac{1}{48} PL^3\right) \text{ (okay!)}$$

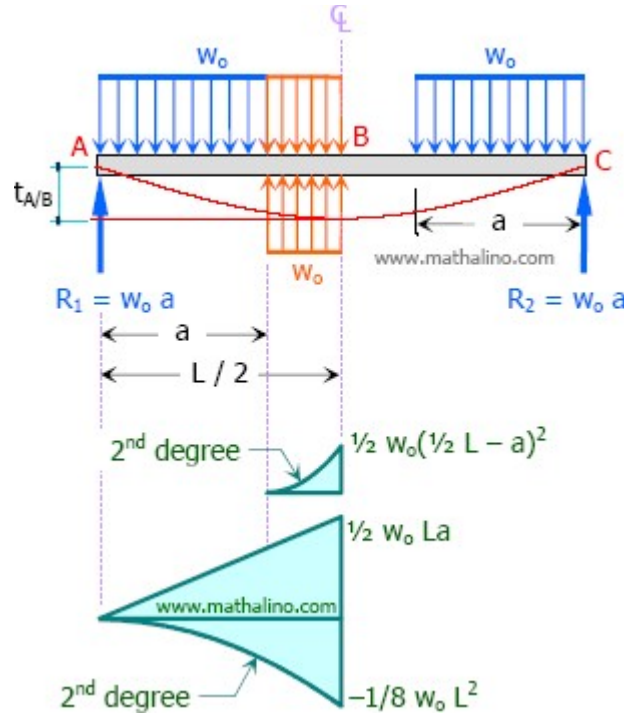
Problem 662

Determine the maximum deflection of the beam shown in Fig. P-662. Check your result by letting $a = L/2$ and comparing with case 8 in Table 6-2. Also, use your result to check the answer to [Prob. 653](#).



Solution 662

[Hide](#)[Click here to show or hide the solution](#)



$$EI t_{A/B} = (Area_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{3} \left(\frac{1}{2} L - a \right) \left[\frac{1}{2} w_o \left(\frac{1}{2} L - a \right)^2 \right] \left[a + \frac{3}{4} \left(\frac{1}{2} L - a \right) \right] + \frac{1}{2} \left(\frac{1}{2} L \right) \left(\frac{1}{2} w_o L a \right) \left(\frac{1}{3} L \right) - \frac{1}{3} \left(\frac{1}{2} L \right) \left(\frac{1}{8} w_o L^2 \right) \left(\frac{3}{8} L \right)$$

$$EI t_{A/B} = \frac{1}{6} w_o a \left(\frac{1}{2} L - a \right)^3 + \frac{1}{8} w_o \left(\frac{1}{2} L - a \right)^4 + \frac{1}{24} w_o L^3 a - \frac{1}{128} w_o L^4$$

$$EI t_{A/B} = \frac{1}{6} w_o a \left[\frac{1}{2} (L - 2a) \right]^3 + \frac{1}{8} w_o \left[\frac{1}{2} (L - 2a) \right]^4 + \frac{1}{24} w_o L^3 a - \frac{1}{128} w_o L^4$$

$$EI t_{A/B} = \frac{1}{48} w_o a (L - 2a)^3 + \frac{1}{128} w_o (L - 2a)^4 + \frac{1}{24} w_o L^3 a - \frac{1}{128} w_o L^4$$

$$EI t_{A/B} = \frac{1}{48} w_o a [L^3 - 3L^2(2a) + 3L(2a)^2 - (2a)^3] + \frac{1}{128} w_o [L^4 - 4L^3(2a) + 6L^2(2a)^2 - 4L(2a)^3 + (2a)^4] + \frac{1}{24} w_o L^3 a - \frac{1}{128} w_o L^4$$

$$EI t_{A/B} = \frac{1}{48} w_o L^3 a - \frac{1}{8} w_o L^2 a^2 + \frac{1}{4} w_o L a^3 - \frac{1}{6} w_o a^4 + \frac{1}{128} w_o L^4 - \frac{1}{16} w_o L^3 a + \frac{3}{16} w_o L^2 a^2 - \frac{1}{4} w_o L a^3 + \frac{1}{8} w_o a^4 + \frac{1}{24} w_o L^3 a - \frac{1}{128} w_o L^4$$

$$EI t_{A/B} = \frac{1}{16} w_o L^2 a^2 - \frac{1}{24} w_o a^4$$

$$EI t_{A/B} = \frac{1}{48} w_o a^2 (3L^2 - 2a^2) \quad \text{answer}$$

Check [Problem 653](#):

$w_o = 600 \text{ N/m}$; $L = 5 \text{ m}$; $a = 2 \text{ m}$

$$EI t_{A/B} = \frac{1}{48}(600)(2^2)[3(5^2) - 2(2^2)]$$

$$EI t_{A/B} = 3350 \text{ N} \cdot \text{m}^3 \quad (\text{okay!})$$

When $a = L/2$ (the load is over the entire span)

$$EI t_{A/B} = \frac{1}{48}w_o\left(\frac{1}{2}L\right)^2\left[3L^2 - 2\left(\frac{1}{2}L\right)^2\right]$$

$$EI t_{A/B} = \frac{1}{192}w_oL^2\left[3L^2 - \frac{1}{2}L^2\right]$$

$$EI t_{A/B} = \frac{1}{192}w_oL^2\left[\frac{5}{2}L^2\right]$$

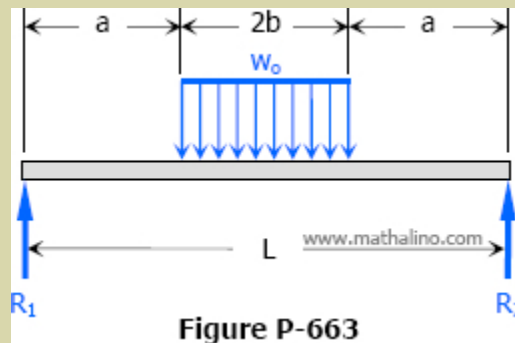
$$EI t_{A/B} = \frac{5}{384}w_oL^4$$

Therefore

$$\delta_{max} = \frac{5w_oL^4}{384EI}$$

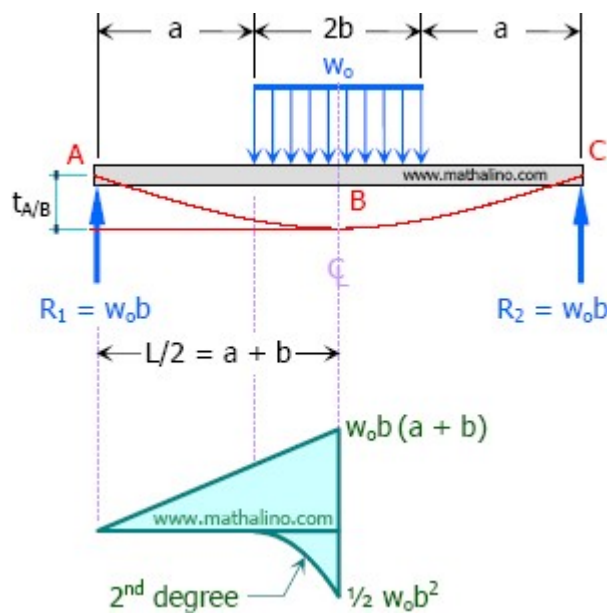
Problem 663

Determine the maximum deflection of the beam carrying a uniformly distributed load over the middle portion, as shown in Fig. P-663. Check your answer by letting $2b = L$.



Solution 663

[Hide](#)[Click here to show or hide the solution](#)



$$EI t_{A/B} = (Area_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{2}(a+b)[w_o b(a+b)][\frac{2}{3}(a+b)] - \frac{1}{3}b(\frac{1}{2}w_o b^2)(a + \frac{3}{4}b)$$

$$EI t_{A/B} = \frac{1}{3}w_o b(a+b)^3 - \frac{1}{6}w_o b^3(a + \frac{3}{4}b)$$

$$EI t_{A/B} = \frac{1}{3}w_o b(a+b)^3 - \frac{1}{24}w_o b^3(4a + 3b)$$

$$EI t_{A/B} = \frac{1}{3}w_o b(a+b)^3 - \frac{1}{24}w_o b^3[(2a + 2b) + (a+b) + a]$$

$$EI t_{A/B} = \frac{1}{3}w_o b(\frac{1}{2}L)^3 - \frac{1}{24}w_o b^3[L + \frac{1}{2}L + a]$$

$$EI t_{A/B} = \frac{1}{24}w_o L^3 b - \frac{1}{24}w_o b^3(\frac{3}{2}L + a)$$

$$EI t_{A/B} = \frac{1}{24}w_o L^3 b - \frac{1}{48}w_o b^3(3L + 2a)$$

$$EI t_{A/B} = \frac{1}{24}w_o L^3 b - \frac{1}{48}w_o b^3[3L + (L - 2b)]$$

$$EI t_{A/B} = \frac{1}{24}w_o L^3 b - \frac{1}{48}w_o b^3(4L - 2b)$$

$$EI t_{A/B} = \frac{1}{24}w_o L^3 b - \frac{1}{24}w_o b^3(2L - b)$$

$$EI t_{A/B} = \frac{1}{24}w_o b[L^3 - b^2(2L - b)]$$

$$EI t_{A/B} = \frac{1}{24} w_o b (L^3 - 2Lb^2 + b^3) \quad \text{answer}$$

When $2b = L$; $b = \frac{1}{2}L$

$$EI t_{A/B} = \frac{1}{24} w_o \left(\frac{1}{2}L\right) \left[L^3 - 2L\left(\frac{1}{2}L\right)^2 + \left(\frac{1}{2}L\right)^3 \right]$$

$$EI t_{A/B} = \frac{1}{48} w_o L \left[L^3 - \frac{1}{2}L^3 + \frac{1}{8}L^3 \right]$$

$$EI t_{A/B} = \frac{1}{48} w_o L \left[\frac{5}{8}L^3 \right]$$

$$EI t_{A/B} = \frac{5}{384} w_o L^4 \quad (\text{okay!})$$

Problem 664

The middle half of the beam shown in Fig. P-664 has a moment of inertia 1.5 times that of the rest of the beam. Find the midspan deflection. (Hint: Convert the M diagram into an M/EI diagram.)

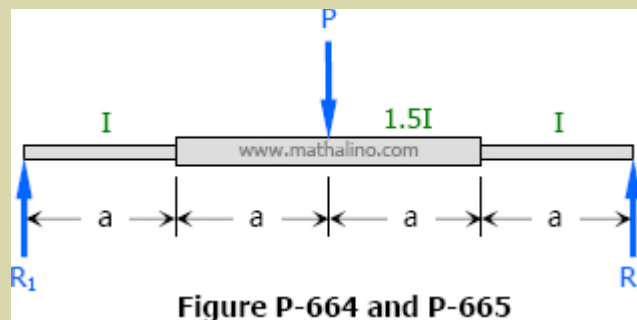
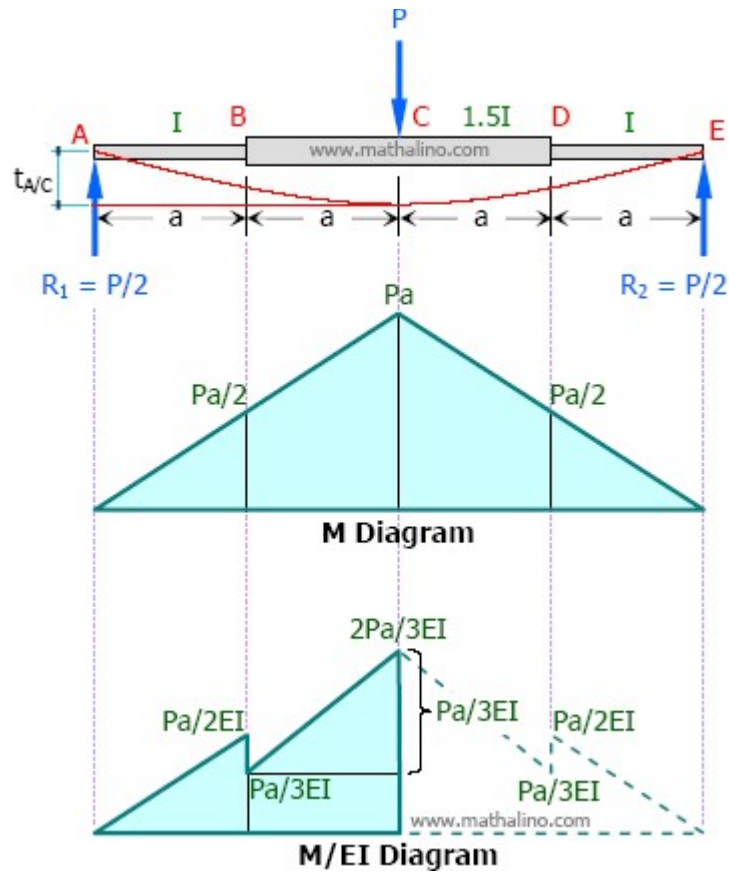


Figure P-664 and P-665

Solution 664

[HideClick here to show or hide the solution](#)



$$t_{A/C} = \frac{1}{EI} (\text{Area}_{AC}) \bar{X}_A$$

$$t_{A/C} = \frac{1}{2}a \left(\frac{Pa}{2EI} \right) \left(\frac{2}{3}a \right) + a \left(\frac{Pa}{3EI} \right) \left(\frac{3}{2}a \right) + \frac{1}{2}a \left(\frac{2Pa}{3EI} - \frac{Pa}{3EI} \right) \left(\frac{5}{3}a \right)$$

$$t_{A/C} = \frac{Pa^3}{6EI} + \frac{Pa^3}{2EI} + \frac{5Pa^3}{18EI}$$

$$t_{A/C} = \frac{17Pa^3}{18EI}$$

Therefore,

$$\delta_{midspan} = \frac{17Pa^3}{18EI} \quad \text{answer}$$

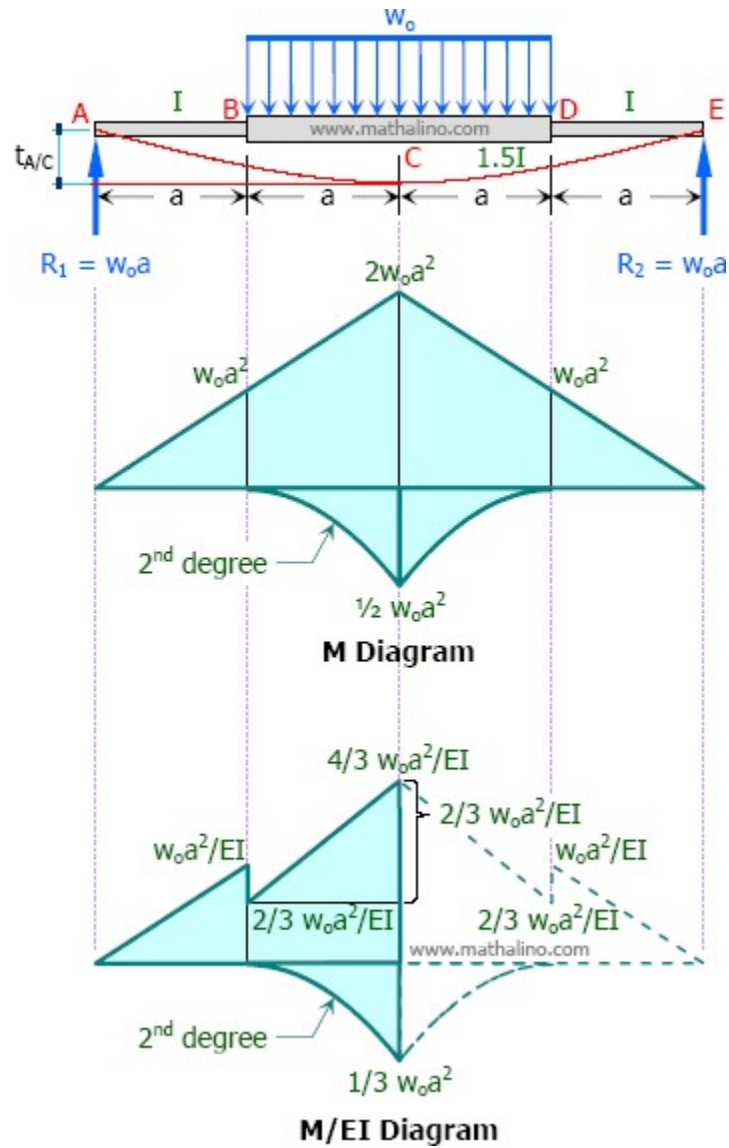
Problem 665

Replace the concentrated load in [Prob. 664](#) by a uniformly distributed load of intensity w_0 acting over

the middle half of the beam. Find the maximum deflection.

Solution 665

[Click here to show or hide the solution](#)



$$t_{A/C} = \frac{1}{EI} (\text{Area}_{AC}) \bar{X}_A$$

$$t_{A/C} = \frac{1}{2}a \left(\frac{w_o a^2}{EI} \right) \left(\frac{2}{3}a \right) + a \left(\frac{2w_o a^2}{3EI} \right) \left(\frac{3}{2}a \right) + \frac{1}{2}a \left(\frac{2w_o a^2}{3EI} \right) \left(\frac{5}{3}a \right) - \frac{1}{3}a \left(\frac{w_o a^2}{3EI} \right) \left(\frac{7}{4}a \right)$$

$$t_{A/C} = \frac{w_o a^4}{3EI} + \frac{w_o a^4}{EI} + \frac{5w_o a^4}{9EI} - \frac{7w_o a^4}{36EI}$$

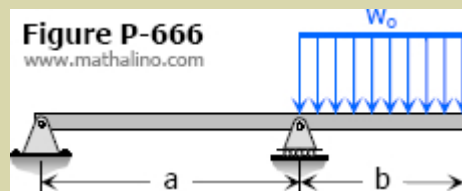
$$t_{A/C} = \frac{61w_o a^4}{36EI}$$

Therefore,

$$\delta_{midspan} = \frac{61w_o a^4}{36EI} \quad \text{answer}$$

Problem 666

Determine the value of $EI\delta$ at the right end of the overhanging beam shown in Fig. P-666.



Solution 666

[Click here to show or hide the solution](#)

$$\Sigma M_B = 0$$

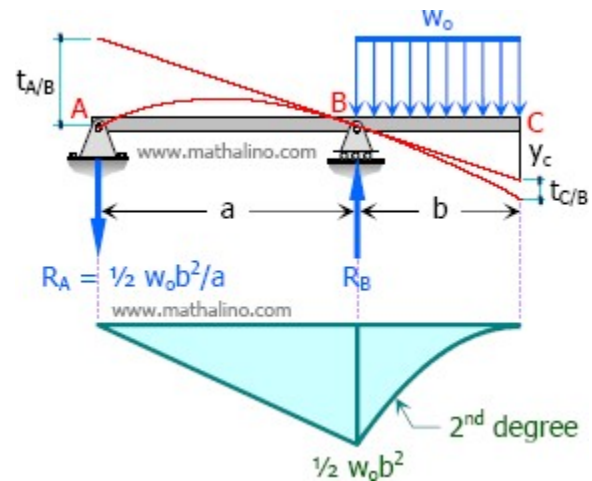
$$aR_A = w_o b \left(\frac{1}{2}b \right)$$

$$R_A = \frac{w_o b^2}{2a}$$

$$EI t_{A/B} = (\text{Area}_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{2}a \left(\frac{1}{2}w_o b^2 \right) \left(\frac{2}{3}a \right)$$

$$EI t_{A/B} = \frac{1}{6}w_o a^2 b^2$$



$$EI t_{C/B} = (Area_{BC}) \bar{X}_C$$

$$EI t_{C/B} = \frac{1}{3}b\left(\frac{1}{2}w_o b^2\right)\left(\frac{3}{4}b\right)$$

$$EI t_{C/B} = \frac{1}{8}w_o b^4$$

$$\frac{y_C}{b} = \frac{t_{A/B}}{a}$$

$$y_C = \frac{b}{a}t_{A/B}$$

$$EI y_C = \frac{b}{a}EI t_{A/B}$$

$$EI y_C = \frac{b}{a}\left(\frac{1}{6}w_o a^2 b^2\right)$$

$$EI y_C = \frac{1}{6}w_o a b^3$$

$$\delta_C = y_C + t_{C/B}$$

$$EI \delta_C = EI y_C + EI t_{C/B}$$

$$EI \delta_C = \frac{1}{6}w_o a b^3 + \frac{1}{8}w_o b^4$$

$$EI \delta_C = \frac{1}{24}w_o b^3(4a + 3b) \quad \text{answer}$$

Problem 667

Determine the value of $EI\delta$ at the right end of the overhanging beam shown in Fig. P-667. Is the deflection up or down?

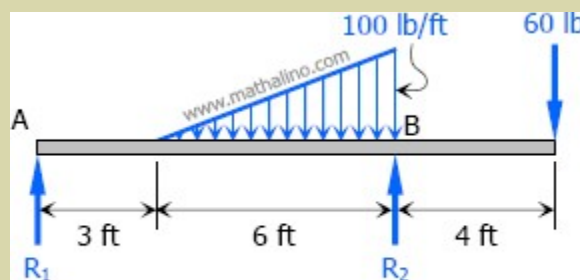


Figure P-667

Solution 667

[Click here to show or hide the solution](#)

$$\Sigma M_{R2} = 0]$$

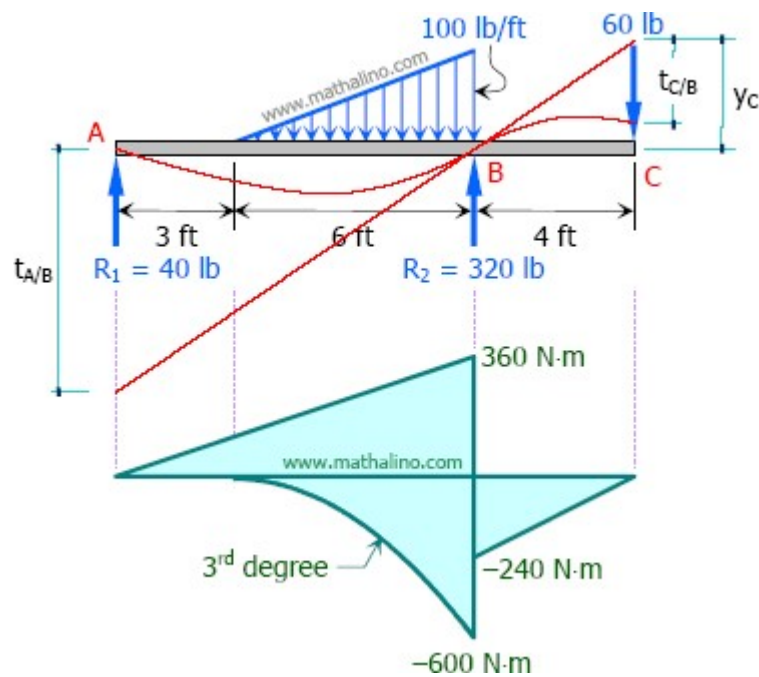
$$9R_1 + 4(60) = \frac{1}{2}(6)(100)(2)$$

$$R_1 = 40 \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$9R_2 = \frac{1}{2}(6)(100)(7) + 13(60)$$

$$R_2 = 320 \text{ lb}$$



$$EI t_{A/B} = (Area_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{2}(9)(360)(6) - \frac{1}{4}(6)(600)(\frac{39}{5})$$

$$EI t_{A/B} = 2700 \text{ N} \cdot \text{m}^3$$

$$EI t_{C/B} = (\text{Area}_{BC}) \bar{X}_C$$

$$EI t_{C/B} = -\frac{1}{2}(4)(240)\left(\frac{8}{3}\right)$$

$$EI t_{C/B} = -1280 \text{ N} \cdot \text{m}^3$$

The negative sign indicates that the elastic curve is below the tangent line. It is shown in the figure indicated as $t_{C/B}$. See [Rules of Sign for Area-Moment Method](#).

$$\frac{y_C}{4} = \frac{t_{A/B}}{9}$$

$$y_C = \frac{4}{9} t_{A/B}$$

$$EI y_C = \frac{4}{9} EI t_{A/B}$$

$$EI y_C = \frac{4}{9} (2700)$$

$$EI y_C = 1200 \text{ N} \cdot \text{m}^3$$

Since the absolute value of $EI t_{C/B}$ is greater than the absolute value of $EI y_C$, the elastic curve is below the undeformed neutral axis (NA) of the beam.

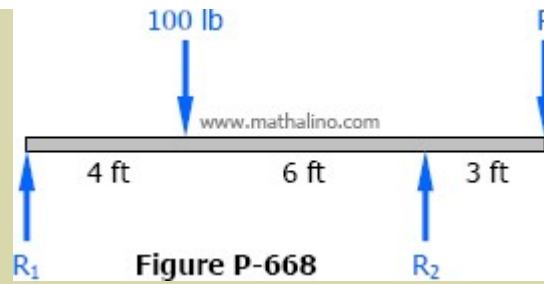
Therefore,

$$EI \delta_D = 1280 - 1200$$

$$EI \delta_D = 80 \text{ N} \cdot \text{m}^3 \quad \text{below C (deflection is down)} \quad \text{answer}$$

Problem 668

For the beam shown in Fig. P-668, compute the value of P that will cause the tangent to the elastic curve over support R_2 to be horizontal. What will then be the value of $EI\delta$ under the 100-lb load?

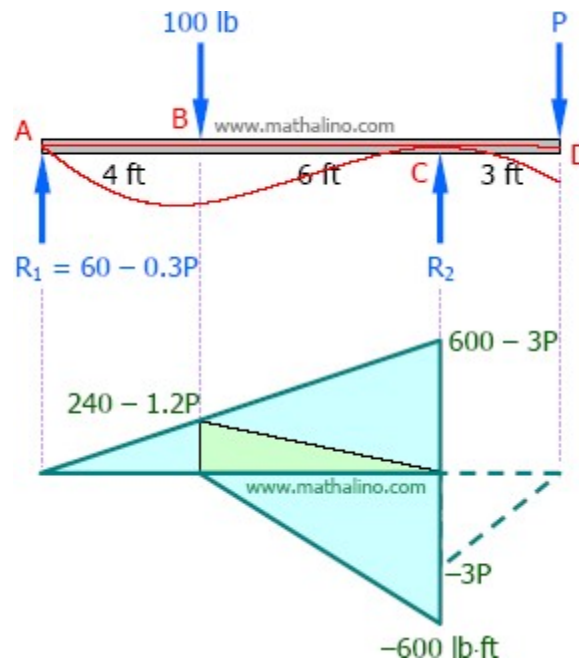


Solution 668

[HideClick here to show or hide the solution](#)

$$10R_1 + 3P = 6(100)$$

$$R_1 = 60 - 0.30P$$



$$EI t_{A/C} = 0$$

$$(Area_{AC}) \bar{X}_A = 0$$

$$\frac{1}{2}(10)(600 - 3P)\left(\frac{20}{3}\right) - \frac{1}{2}(6)(600)(8) = 0$$

$$P = 56 \text{ lb} \quad \text{answer}$$

Thus,

$$240 - 1.2P = 172.8 \text{ lb}$$

$$600 - 3P = 432 \text{ lb}$$

Under the 100-lb load:

$$EI t_{B/C} = (\text{Area}_{BC}) \bar{X}_B$$

$$EI t_{B/C} = \frac{1}{2}(6)(172.8)(2) + \frac{1}{2}(6)(432)(4) - \frac{1}{2}(6)(600)(4)$$

$$EI t_{B/C} = -979.2 \text{ lb} \cdot \text{ft}^3$$

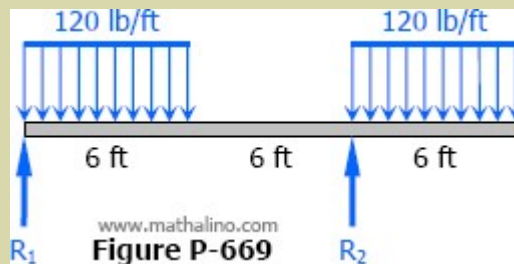
The negative sign indicates that the elastic curve is below the reference tangent.

Therefore,

$$EI \delta_B = 979.2 \text{ lb} \cdot \text{ft}^3 \quad \text{downward} \quad \text{answer}$$

Problem 669

Compute the value of $EI\delta$ midway between the supports of the beam shown in Fig. P-669.



Solution 669

[HideClick here to show or hide the solution](#)

$$\Sigma M_{R2} = 0$$

$$12R_1 + 3(6)(120) = 9(6)(120)$$

$$R_1 = 360 \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$12R_2 = 3(6)(120) + 15(6)(120)$$

$$R_2 = 1080 \text{ lb}$$

By ratio and proportion:

$$\frac{a}{6} = \frac{4320}{12}$$

$$a = 2160 \text{ lb} \cdot \text{ft}$$

By squared property of parabola:

$$\frac{b}{6^2} = \frac{-8640}{12^2}$$

$$b = -2160 \text{ lb} \cdot \text{ft}$$

$$EI t_{C/A} = (\text{Area}_{AC}) \bar{X}_C$$

$$EI t_{C/A} = \frac{1}{3}(6)(2160)\left(\frac{3}{2}\right) + \frac{1}{2}(12)(4320)(4) - \frac{1}{3}(12)(8640)(3)$$

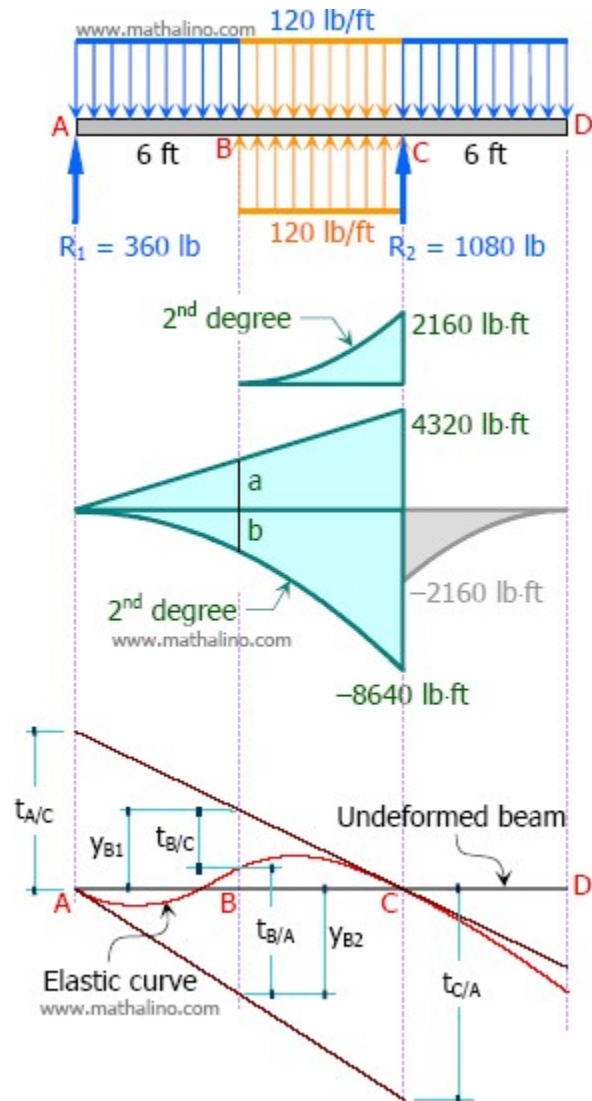
$$EI t_{C/A} = 6480 \text{ lb} \cdot \text{ft}^3$$

$$EI t_{B/A} = (\text{Area}_{AB}) \bar{X}_B$$

$$EI t_{B/A} = \frac{1}{2}(6a)(2) - \frac{1}{3}(6b)\left(\frac{3}{2}\right)$$

$$EI t_{B/A} = 6a - 3b$$

$$EI t_{B/A} = 6(2160) - 3(-2160)$$



$$EI t_{B/A} = 6480 \text{ lb} \cdot \text{ft}^3$$

With the values of $EI t_{C/A}$ and $EI t_{B/A}$, it is obvious that the elastic curve is above point B. The deflection at B (up or down) can also be determined by comparing the values of $t_{B/A}$ and y_{B2} .

By ratio and proportion:

$$\frac{y_{B2}}{6} = \frac{t_{C/A}}{12}$$

$$y_{B2} = \frac{1}{2} t_{C/A}$$

$$EI y_{B2} = \frac{1}{2} EI t_{C/A}$$

$$EI y_{B2} = \frac{1}{2} (6480)$$

$$EI y_{B2} = 3240 \text{ lb} \cdot \text{ft}^3$$

Since $t_{B/A}$ is greater than y_{B2} , the elastic curve is above point B as concluded previously.

Therefore,

$$EI \delta_B = EI t_{B/A} - EI y_{B2}$$

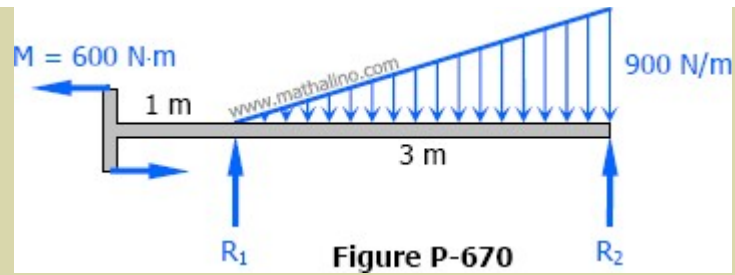
$$EI \delta_B = 6480 - 3240$$

$$EI \delta_B = 3240 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

You can also find the value $EI \delta_B$ by finding $t_{A/C}$, $t_{B/C}$, and y_{B1} . I encourage you to do it yourself.

Problem 670

Determine the value of $EI \delta$ at the left end of the overhanging beam shown in Fig. P-670.



Solution 670

[Hide](#) [Click here to show or hide the solution](#)

$$\Sigma M_{R2} = 0$$

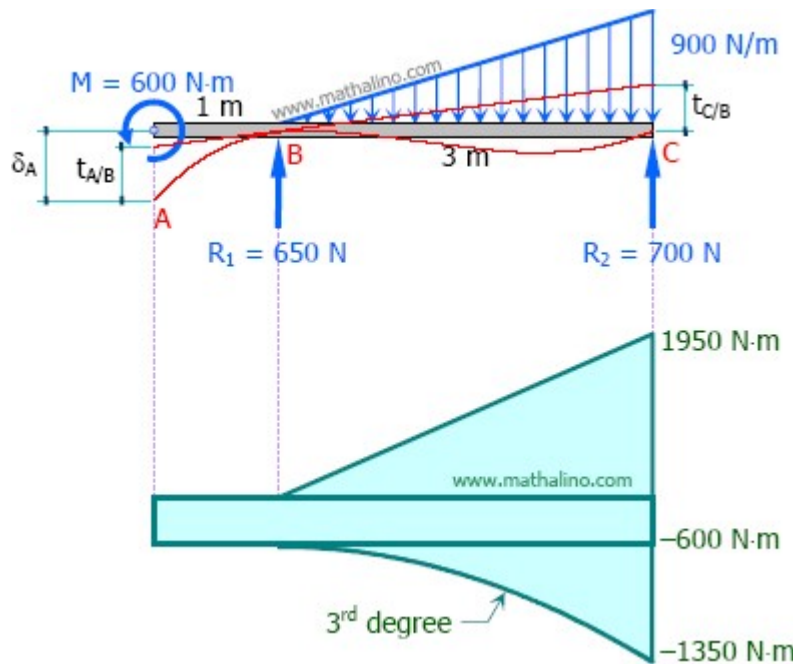
$$3R_1 = 600 + \frac{1}{2}(3)(900)(1)$$

$$R_1 = 650 \text{ N}$$

$$\Sigma M_{R1} = 0$$

$$3R_2 + 600 = \frac{1}{2}(3)(900)(2)$$

$$R_2 = 700 \text{ N}$$



$$EI t_{C/B} = (\text{Area}_{BC}) \bar{X}_C$$

$$EI t_{C/B} = \frac{1}{2}(3)(1950)(1) - 3(600)\left(\frac{3}{2}\right) - \frac{1}{4}(3)(1350)\left(\frac{3}{5}\right)$$

$$EI t_{C/B} = -382.5 \text{ N} \cdot \text{m}^3$$

$$EI t_{A/B} = (\text{Area}_{AB}) \bar{X}_A$$

$$EI t_{A/B} = -1(600)\left(\frac{1}{2}\right)$$

$$EI t_{A/B} = -300 \text{ N} \cdot \text{m}^3$$

The negative signs above indicates only the location of elastic curve relative to the reference tangent. It does not indicate magnitude. It shows that the elastic curve is below the reference tangent at points A and C.

By ratio and proportion

$$\frac{\delta_A - t_{A/B}}{1} = \frac{t_{C/B}}{3}$$

$$\delta_A = \frac{1}{3}t_{C/B} + t_{A/B}$$

$$EI \delta_A = \frac{1}{3}EI t_{C/B} + EI t_{A/B}$$

$$EI \delta_A = \frac{1}{3}(382.5) + 300$$

$$EI \delta_A = 427.5 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

Midspan Deflection | Deflections in Simply Supported Beams

In simply supported beams, the tangent drawn to the elastic curve at the point of maximum deflection is horizontal and parallel to the unloaded beam. It simply means that the deviation from unsetting supports to the horizontal tangent is equal to the maximum deflection. If the simple beam is symmetrically loaded, the maximum deflection will occur at the midspan.

Finding the midspan deflection of a symmetrically loaded simple beam is straightforward because its value is equal to the maximum deflection. In unsymmetrically loaded simple beam however, the midspan deflection is not equal to the maximum deflection. To deal with unsymmetrically loaded simple beam, we will add a symmetrically placed load for each load actually acting on the beam, making the beam symmetrically loaded. The effect of this transformation to symmetry will double the actual midspan deflection, making the actual midspan deflection equal to one-half of the midspan deflection of the transformed symmetrically loaded beam.

Problem 673

For the beam shown in Fig. P-673, show that the midspan deflection is $\delta = (Pb/48EI)(3L^2 - 4b^2)$.

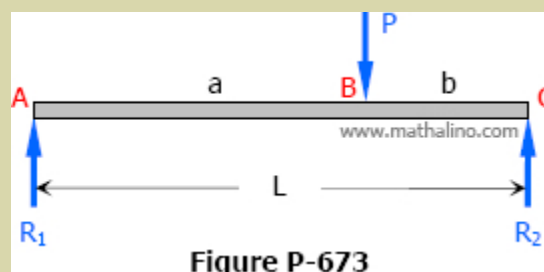
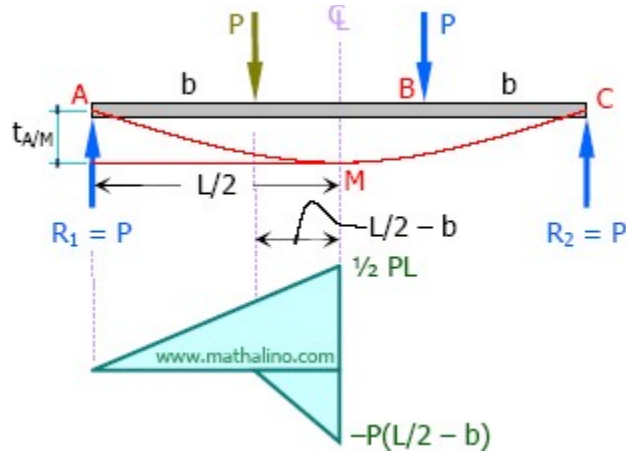


Figure P-673

Solution 673

[HideClick here to show or hide the solution](#)



$$EI t_{A/M} = (\text{Area}_{AM}) \bar{X}_A$$

$$EI t_{A/M} = \frac{1}{2}(\frac{1}{2}L)(\frac{1}{2}PL)[\frac{2}{3}(\frac{1}{2}L)] - \frac{1}{2}(\frac{1}{2}L - b)P(\frac{1}{2}L - b)[b + \frac{2}{3}(\frac{1}{2}L - b)]$$

$$EI t_{A/M} = \frac{1}{24}PL^3 - \frac{1}{2}Pb(\frac{1}{2}L - b)^2 - \frac{1}{3}P(\frac{1}{2}L - b)^3$$

$$EI t_{A/M} = \frac{1}{24}PL^3 - \frac{1}{2}Pb\left(\frac{L - 2b}{2}\right)^2 - \frac{1}{3}P\left(\frac{L - 2b}{2}\right)^3$$

$$EI t_{A/M} = \frac{1}{24}PL^3 - \frac{1}{8}Pb(L - 2b)^2 - \frac{1}{24}P(L - 2b)^3$$

$$EI t_{A/M} = \frac{1}{24}PL^3 - \frac{1}{8}Pb(L^2 - 4Lb + 4b^2) - \frac{1}{24}P(L^3 - 6L^2b + 12Lb^2 - 8b^3)$$

$$EI t_{A/M} = \frac{1}{24}PL^3 - \frac{1}{8}PL^2b + \frac{1}{2}PLb^2 - \frac{1}{2}Pb^3 - \frac{1}{24}PL^3 + \frac{1}{4}PL^2b - \frac{1}{2}PLb^2 + \frac{1}{3}Pb^3$$

$$EI t_{A/M} = \frac{1}{8}PL^2b - \frac{1}{6}Pb^3$$

$$EI t_{A/M} = \frac{1}{24}Pb(3L^2 - 4b^2)$$

$$t_{A/M} = \frac{Pb}{24EI}(3L^2 - 4b^2)$$

$$\delta_{midspan} = \frac{1}{2}t_{A/M}$$

$$\delta_{midspan} = \frac{Pb}{48EI}(3L^2 - 4b^2) \quad (\text{okay!})$$

Problem 674

Find the deflection midway between the supports for the overhanging beam shown in Fig. P-674.

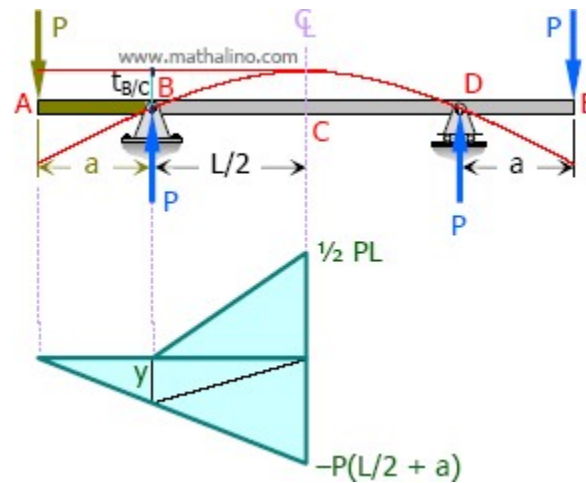


Solution 674

[Click here to show or hide the solution](#)

$$\frac{y}{a} = \frac{-P(\frac{1}{2}L + a)}{\frac{1}{2}L + a}$$

$$y = -Pa$$



$$EI t_{B/C} = (\text{Area}_{BC}) \bar{X}_B$$

$$EI t_{B/C} = \frac{1}{2}(\frac{1}{2}L)(\frac{1}{2}PL)[\frac{2}{3}(\frac{1}{2}L)] - \frac{1}{2}(\frac{1}{2}L)(y)[\frac{1}{3}(\frac{1}{2}L)] - \frac{1}{2}(\frac{1}{2}L)[P(\frac{1}{2}L+a)][\frac{2}{3}(\frac{1}{2}L)]$$

$$EI t_{B/C} = \frac{1}{24}PL^3 - \frac{1}{24}PL^2a - \frac{1}{12}PL^2(\frac{1}{2}L + a)$$

$$EI t_{B/C} = \frac{1}{24}PL^3 - \frac{1}{24}PL^2a - \frac{1}{24}PL^3 - \frac{1}{12}PL^2a$$

$$EI t_{B/C} = -\frac{1}{8}PL^2a$$

$$2\delta = t_{B/C}$$

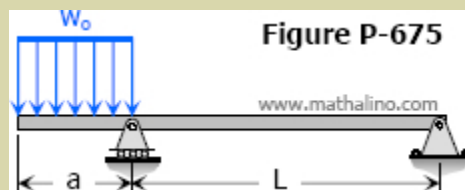
$$\delta = \frac{1}{2} \left[-\frac{1}{8EI} PL^2a \right]$$

$$\delta = -\frac{1}{16EI} PL^2a$$

$$\delta = \frac{1}{16EI} PL^2a \text{ upward} \quad \text{answer}$$

Problem 675

Repeat [Prob. 674](#) for the overhanging beam shown in [Fig. P-675](#).



Solution 675

[Hide](#) [Click here to show or hide the solution](#)

$$M = w_o a \left(\frac{1}{2} L \right) - w_o a \left(\frac{1}{2} L + \frac{1}{2} a \right)$$

$$M = \frac{1}{2}w_oLa - \frac{1}{2}w_oLa - \frac{1}{2}w_oa^2$$

$$M = -\frac{1}{2}w_oa^2$$

$$EI t_{B/C} = (\text{Area}_{BC}) \bar{X}_B$$

$$EI t_{B/C} = \frac{1}{2}L(-\frac{1}{2}w_oa^2)(\frac{1}{4}L)$$

$$EI t_{B/C} = -\frac{1}{16}w_oL^2a^2$$

$$t_{B/C} = -\frac{1}{16EI}w_oL^2a^2$$

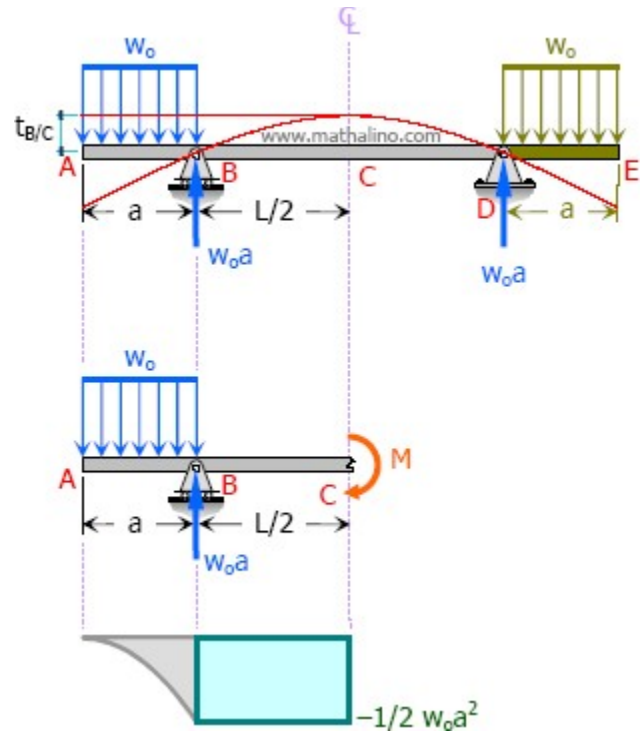
$$2\delta = t_{B/C}$$

$$2\delta = -\frac{1}{16EI}w_oL^2a^2$$

$$\delta = -\frac{1}{32EI}w_oL^2a^2$$

$$\delta = \frac{1}{32EI}w_oL^2a^2 \text{ upward}$$

answer



Problem 676

Determine the midspan deflection of the simply supported beam loaded by the couple shown in [Fig. P-676](#).

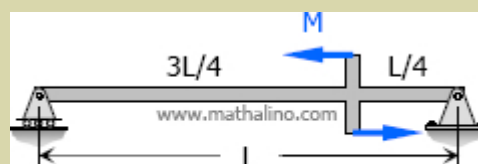


Figure P-676

Solution 676

[HideClick here to show or hide the solution](#)

$$EI t_{A/C} = (\text{Area}_{AC}) \bar{X}_A$$

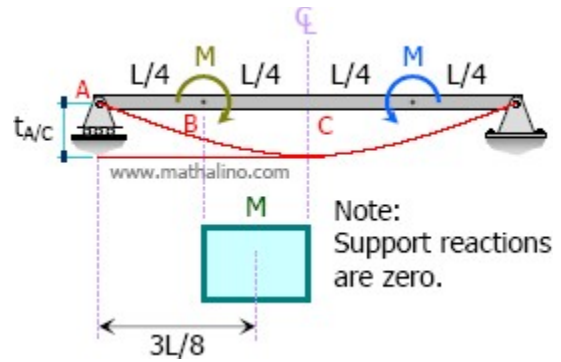
$$EI t_{A/C} = M \left(\frac{L}{4} \right) \left(\frac{3L}{8} \right)$$

$$t_{A/C} = \frac{3ML^2}{32EI}$$

$$\delta_{midspan} = \frac{1}{2} t_{A/C}$$

$$\delta_{midspan} = \frac{1}{2} \left(\frac{3ML^2}{32EI} \right)$$

$$\delta_{midspan} = \frac{3ML^2}{64EI} \quad \text{answer}$$



Problem 677

Determine the midspan deflection of the beam loaded as shown in Fig. P-677.

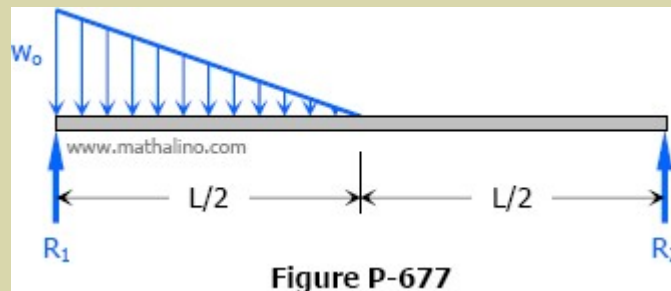
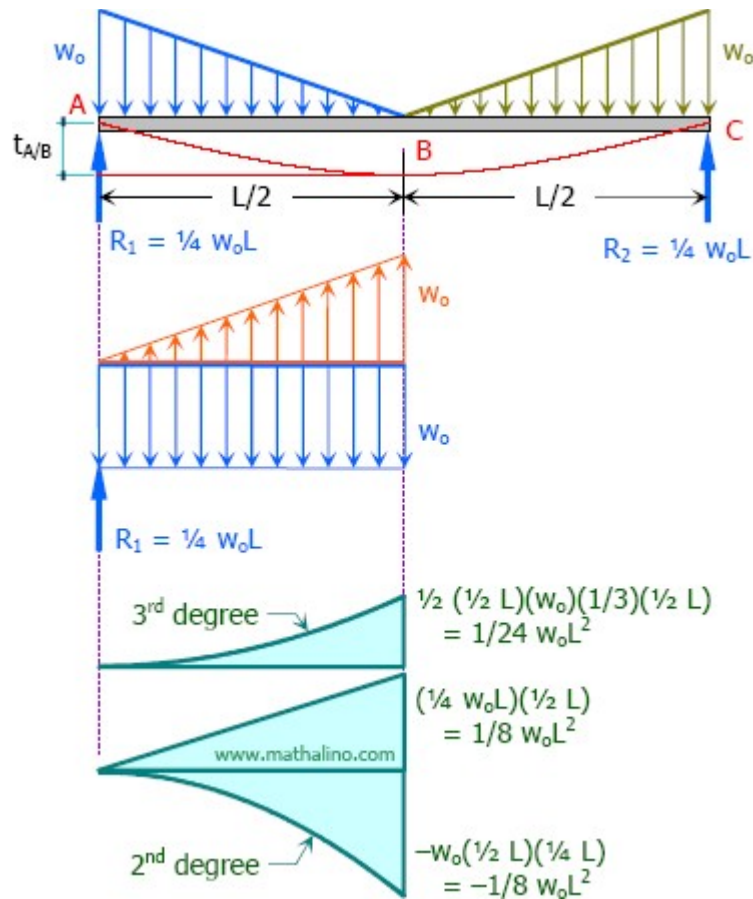


Figure P-677

Solution 677

[HideClick here to show or hide the solution](#)



$$EI t_{A/B} = (Area_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{4} \left(\frac{1}{2} L \right) \left(\frac{1}{24} w_o L^2 \right) \frac{4}{5} \left(\frac{1}{2} L \right) + \frac{1}{2} \left(\frac{1}{2} L \right) \left(\frac{1}{8} w_o L^2 \right) \frac{2}{3} \left(\frac{1}{2} L \right) - \frac{1}{3} \left(\frac{1}{2} L \right) \left(\frac{1}{8} w_o L^2 \right) \frac{3}{4} \left(\frac{1}{2} L \right)$$

$$EI t_{A/B} = \frac{1}{480} w_o L^4 + \frac{1}{96} w_o L^4 - \frac{1}{128} w_o L^4$$

$$EI t_{A/B} = \frac{3}{640} w_o L^4$$

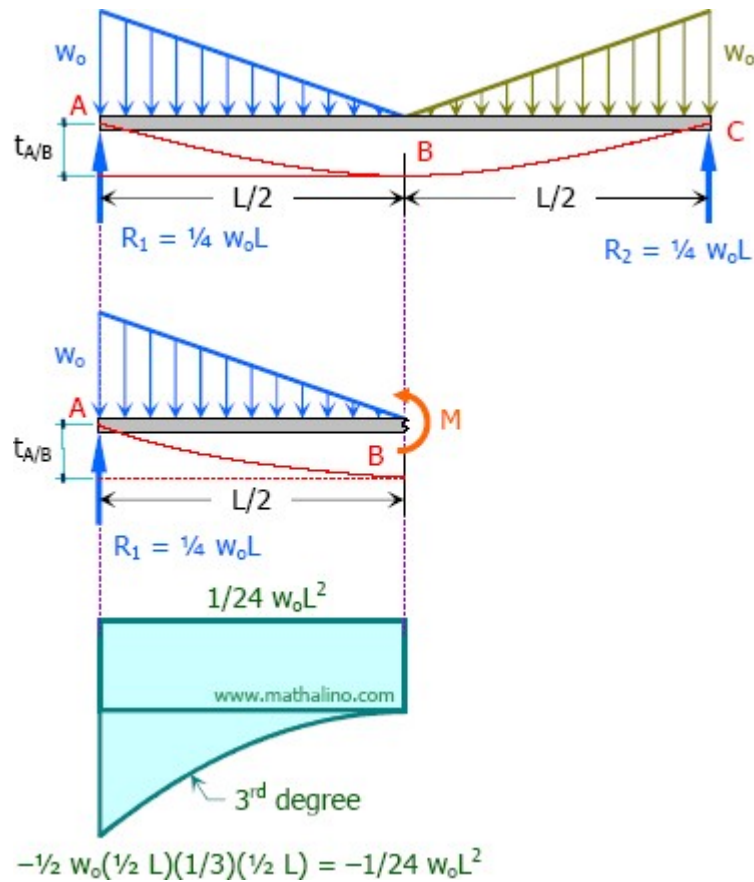
$$t_{A/B} = \frac{3 w_o L^4}{640 EI}$$

$$\delta_{midspan} = \frac{1}{2} t_{A/B}$$

$$\delta_{midspan} = \frac{1}{2} \left(\frac{3 w_o L^4}{640 EI} \right)$$

$$\delta_{midspan} = \frac{3w_o L^4}{1280EI} \quad \text{answer}$$

[HideAnother Solution](#)



$$M = \frac{1}{4} w_o L \left(\frac{1}{2} L \right) - \frac{1}{2} \left(\frac{1}{2} w_o L \right) \frac{2}{3} \left(\frac{1}{2} L \right)$$

$$M = \frac{1}{24} w_o L^2$$

$$EI t_{A/B} = (Area_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{2} L \left(\frac{1}{24} w_o L^2 \right) \frac{1}{2} \left(\frac{1}{2} L \right) - \frac{1}{4} \left(\frac{1}{2} L \right) \left(\frac{1}{24} w_o L^2 \right) \frac{1}{5} \left(\frac{1}{2} L \right)$$

$$EI t_{A/B} = \frac{1}{192} w_o L^4 - \frac{1}{1920} w_o L^4$$

$$EI t_{A/B} = \frac{3}{640} w_o L^4$$

$$t_{A/B} = \frac{3w_o L^4}{640EI} \quad (\text{okay!})$$

Problem 678

Determine the midspan value of $EI\delta$ for the beam shown in Fig. P-678.

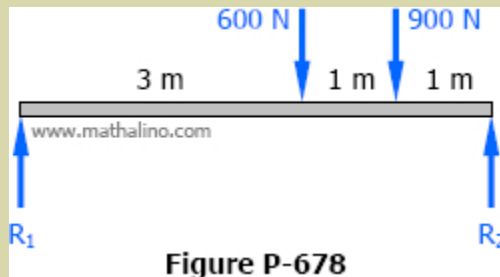
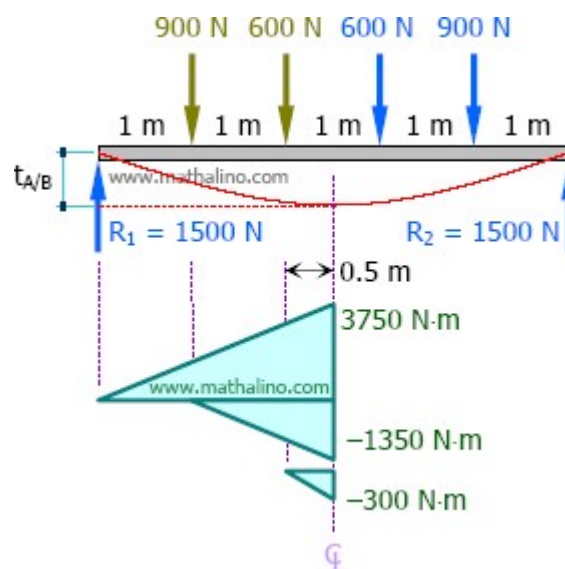


Figure P-678

Solution 678

[Click here to show or hide the solution](#)



$$EI t_{A/B} = (\text{Area}_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{2}(2.5)(3750)\left(\frac{5}{3}\right) - \frac{1}{2}(1.5)(1350)(2) - \frac{1}{2}(0.5)(300)\left(\frac{7}{3}\right)$$

$$EI t_{A/B} = 5612.5 \text{ N} \cdot \text{m}^3$$

$$EI \delta_{midspan} = \frac{1}{2}(EI t_{A/B})$$

$$EI \delta_{midspan} = \frac{1}{2}(5612.5)$$

$$EI \delta_{midspan} = 2806.25 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

Problem 679

Determine the midspan value of $EI\delta$ for the beam shown in [Fig. P-679](#) that carries a uniformly varying load over part of the span.

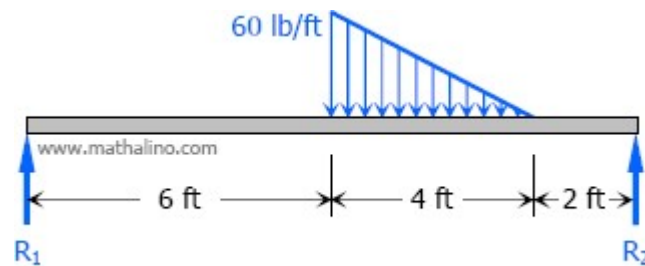
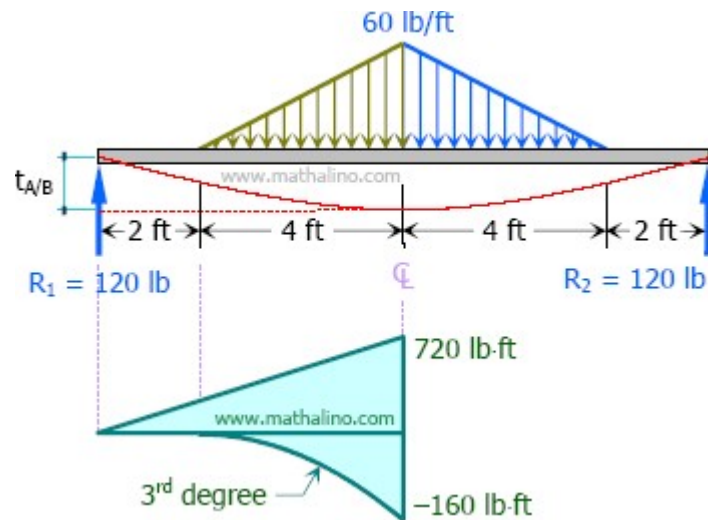


Figure P-679

Solution 679

[HideClick here to show or hide the solution](#)



$$EI t_{A/B} = (Area_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{2}(6)(720)(4) - \frac{1}{4}(4)(160)(\frac{26}{5})$$

$$EI t_{A/B} = 7808 \text{ lb} \cdot \text{ft}^3$$

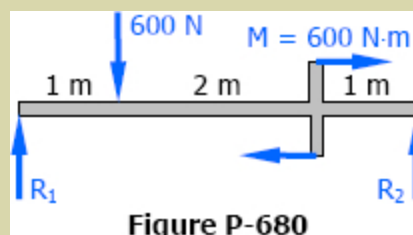
$$EI \delta = \frac{1}{2}(EI t_{A/B})$$

$$EI \delta = \frac{1}{2}(7808)$$

$$EI \delta = 3904 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

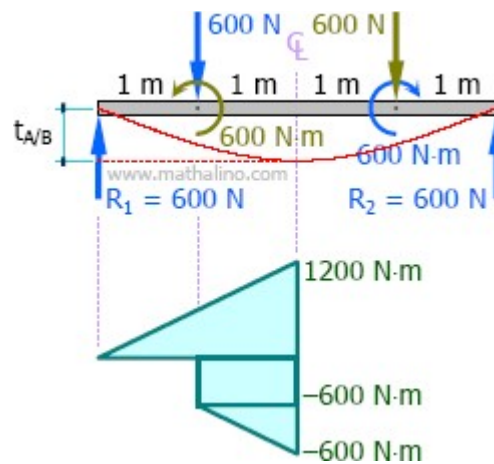
Problem 680

Determine the midspan value of $EI\delta$ for the beam loaded as shown in Fig. P-680.



Solution 680

[Hide](#)[Click here to show or hide the solution](#)



$$EI t_{A/B} = (\text{Area}_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{2}(2)(1200)\left(\frac{4}{3}\right) - 600(1)\left(\frac{3}{2}\right) - \frac{1}{2}(1)(600)\left(\frac{5}{3}\right)$$

$$EI t_{A/B} = 200 \text{ N} \cdot \text{m}^3$$

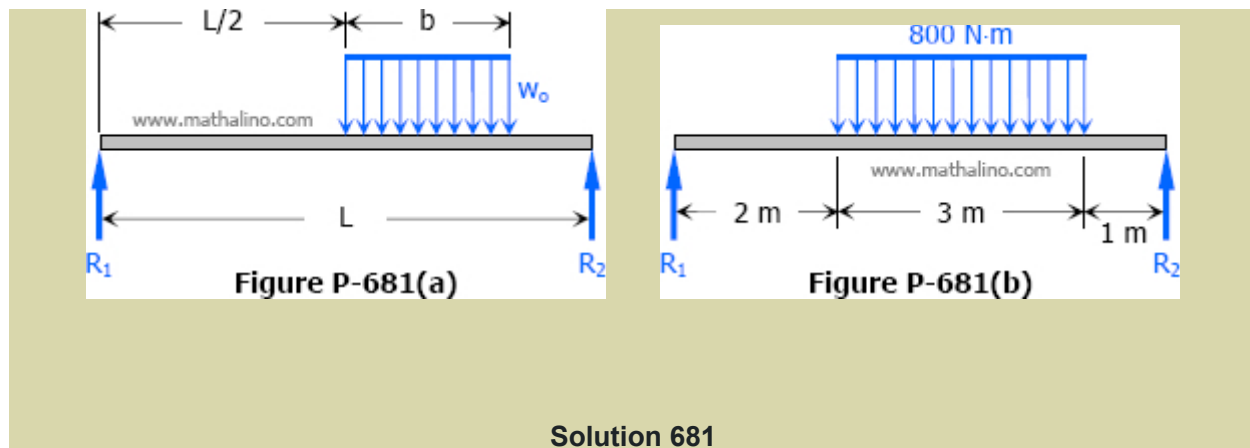
$$EI \delta = \frac{1}{2}(EI t_{A/B})$$

$$EI \delta = \frac{1}{2}(200)$$

$$EI \delta = 100 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

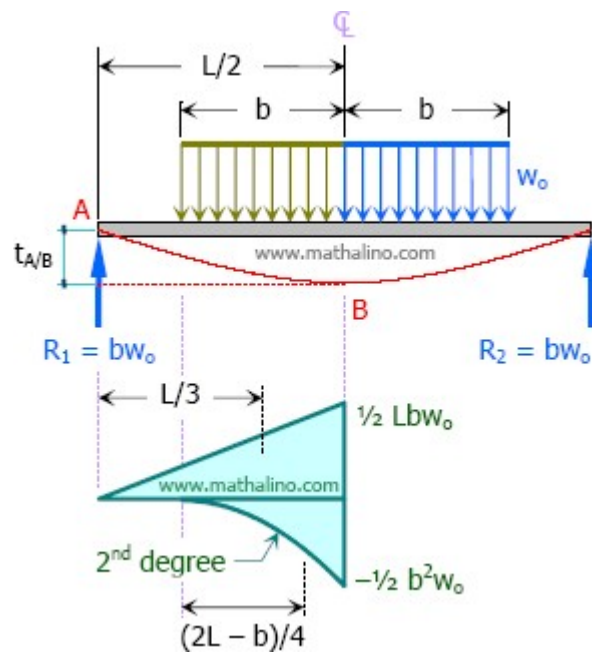
Problem 681

Show that the midspan value of $EI\delta$ is $(w_0 b/48)(L^3 - 2Lb^2 + b^3)$ for the beam in part (a) of Fig. P-681. Then use this result to find the midspan $EI\delta$ of the loading in part (b) by assuming the loading to exceed over two separate intervals that start from midspan and adding the results.



[HideClick here to show or hide the solution](#)

Part (a)



$$EI t_{A/B} = (\text{Area}_{AB}) \bar{X}_A$$

$$EI t_{A/B} = \frac{1}{2} \left(\frac{1}{2} L \right) \left(\frac{1}{2} L b w_o \right) \left(\frac{1}{3} L \right) - \frac{1}{3} (b) \left(\frac{1}{2} b^2 w_o \right) \frac{1}{4} (2L - b)$$

$$EI t_{A/B} = \frac{1}{24} L^3 b w_o - \frac{1}{12} L b^3 w_o + \frac{1}{24} b^4 w_o$$

$$EI t_{A/B} = \frac{1}{24} w_o b (L^3 - 2Lb^2 + b^3)$$

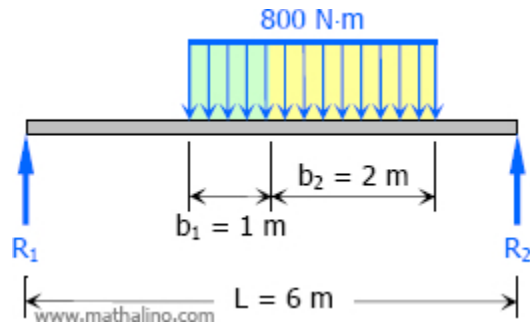
$$EI \delta = \frac{1}{2} (EI t_{A/B})$$

$$EI \delta = \frac{1}{2} \left[\frac{1}{24} w_o b (L^3 - 2Lb^2 + b^3) \right]$$

$$EI \delta = \frac{w_o b}{48} (L^3 - 2Lb^2 + b^3) \quad \text{answer}$$

Part (b)

$$EI \delta = EI \delta_1 + EI \delta_2$$



$$EI \delta = \frac{1}{48} w_o b_1 (L^3 - 2Lb_1^2 + b_1^3) + \frac{1}{48} w_o b_2 (L^3 - 2Lb_2^2 + b_2^3)$$

$$EI \delta = \frac{1}{48} (800)(1) [6^3 - 2(6)(1^2) + 1^3] + \frac{1}{48} (800)(2) [6^3 - 2(6)(2^2) + 2^3]$$

$$EI \delta = 3416.67 + 5866.67$$

$$EI \delta = 9283.34 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

Method of Superposition | Beam Deflection

The slope or deflection at any point on the beam is equal to the resultant of the slopes or deflections at that point caused by each of the load acting separately.

Rotation and Deflection for Common Loadings

Case 1: Concentrated load at the free end of cantilever beam

Maximum Moment

$$M = -PL$$

Slope at end

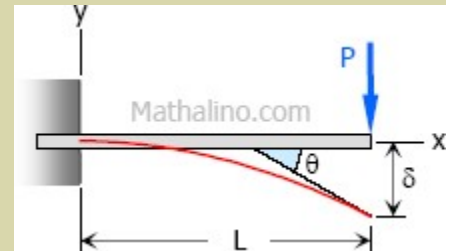
$$\theta = \frac{PL^2}{2EI}$$

Maximum deflection

$$\delta = \frac{PL^3}{3EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Px^2}{6}(3L - x)$$



Case 2: Concentrated load at any point on the span of cantilever beam

Maximum Moment

$$M = -Pa$$

Slope at end

$$\theta = \frac{Pa^2}{2EI}$$

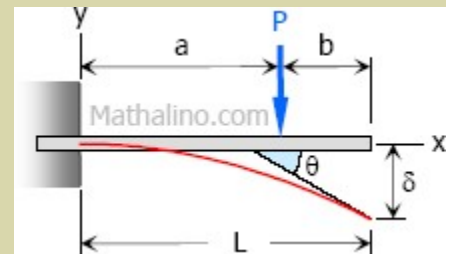
Maximum deflection

$$\delta = \frac{Pa^3}{6EI}(3L - a)$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Px^2}{6}(3a - x) \text{ for } 0 < x < a$$

$$EI y = \frac{Pa^2}{6}(3x - a) \text{ for } a < x < L$$



Case 3: Uniformly distributed load over the entire length of cantilever beam

Maximum Moment

$$M = -\frac{w_o L^2}{2}$$

Slope at end

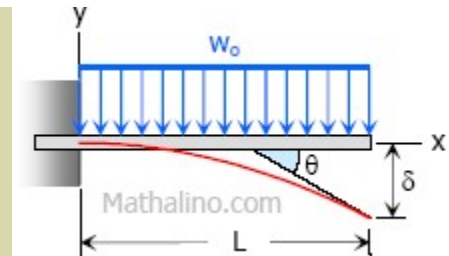
$$\theta = \frac{w_o L^3}{6EI}$$

Maximum deflection

$$\delta = \frac{w_o L^4}{8EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{w_o x^2}{120L} (6L^2 - 4Lx + x^2)$$



Case 4: Triangular load, full at the fixed end and zero at the free end, of cantilever beam

Maximum Moment

$$M = -\frac{w_o L^2}{6}$$

Slope at end

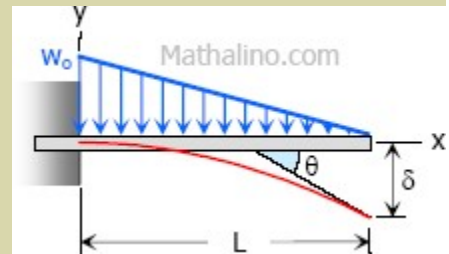
$$\theta = \frac{w_o L^3}{24EI}$$

Maximum deflection

$$\delta = \frac{w_o L^4}{30EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{w_o x^2}{120L} (10L^3 - 10L^2 x + 5Lx^2 - x^3)$$



Case 5: Moment load at the free end of cantilever beam

Maximum Moment

$$M = -M$$

Slope at end

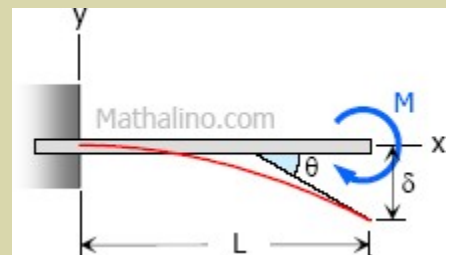
$$\theta = \frac{ML}{EI}$$

Maximum deflection

$$\delta = \frac{ML^2}{2EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Mx^2}{2}$$



Case 6: Concentrated load at the midspan of simple beam

Maximum Moment

$$M = \frac{PL}{4}$$

Slope at end

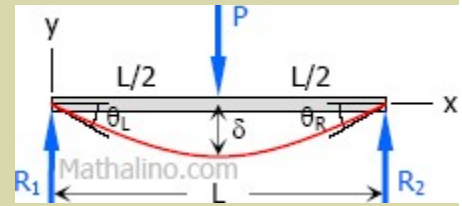
$$\theta_L = \theta_R = \frac{PL^2}{16EI}$$

Maximum deflection

$$\delta = \frac{PL^3}{48EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Px}{12} \left(\frac{3}{4}L^2 - x^2 \right) \text{ for } 0 < x < \frac{1}{2}L$$



Case 7: Concentrated load at any point on simple beam

Maximum Moment

$$M = \frac{Pab}{L} \text{ at } x = a$$

Slope at end

$$\theta_L = \frac{Pb(L^2 - b^2)}{6EIL}$$

$$\theta_R = \frac{Pa(L^2 - a^2)}{6EIL}$$

Maximum deflection

$$\delta = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EIL} \text{ at } x = \sqrt{\frac{L^2 - b^2}{3}}$$

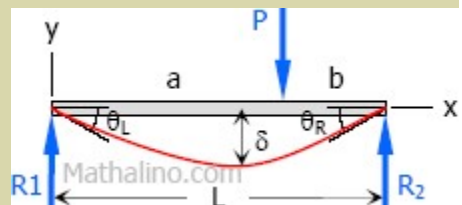
Deflection at the center (not maximum)

$$\delta = \frac{Pb}{48EI} (3L^2 - 4b^2) \text{ when } a > b$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Pbx}{6L} (L^2 - x^2 - b^2) \text{ for } 0 < x < a$$

$$EI y = \frac{Pb}{6L} \left[\frac{L}{b} (x - a)^3 + (L^2 - b^2)x - x^3 \right] \text{ for } a < x < L$$



**Case 8: Uniformly distributed load over the entire span
of simple beam**

Maximum Moment

$$M = \frac{w_o L^2}{8}$$

Slope at end

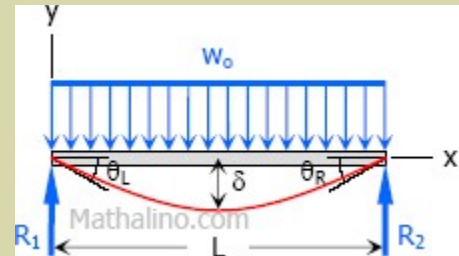
$$\theta_L = \theta_R = \frac{w_o L^3}{24EI}$$

Maximum deflection

$$\delta = \frac{5w_o L^4}{384EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{w_o x}{24} (L^3 - 2Lx^2 + x^3)$$



**Case 9: Triangle load with zero at one support and full at
the other support of simple beam**

Maximum Moment

$$M = \frac{w_o L^2}{9\sqrt{3}}$$

Slope at end

$$\theta_L = \frac{7w_o L^3}{360EI}$$

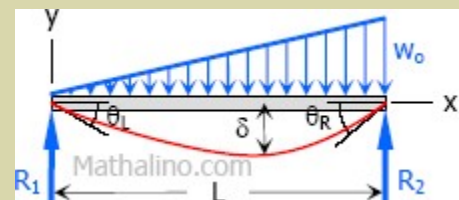
$$\theta_R = \frac{8w_o L^3}{360EI}$$

Maximum deflection

$$\delta = \frac{2.5w_o L^4}{384EI} \text{ at } x = 0.519L$$

Deflection Equation (y is positive downward)

$$EI y = \frac{w_o x}{360L} (7L^4 - 10L^2 x^2 + 3x^4)$$



Case 10: Triangular load with zero at each support and full at the midspan of simple beam

Maximum Moment

$$M = \frac{w_o L^2}{12}$$

Slope at end

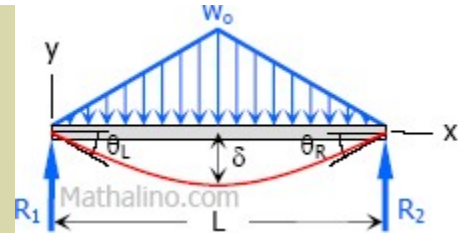
$$\theta_L = \theta_R = \frac{5w_o L^3}{192EI}$$

Maximum deflection

$$\delta = \frac{w_o L^4}{120EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{w_o x}{960L} (25L^4 - 40L^2 x^2 + 16x^4) \text{ for } 0 < x < \frac{L}{2}$$



Case 11: Moment load at the right support of simple beam

Maximum Moment

$$M = M$$

Slope at end

$$\theta_L = \frac{ML}{6EI}$$

$$\theta_R = \frac{ML}{3EI}$$

Maximum deflection

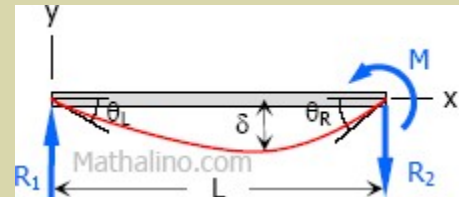
$$\delta = \frac{ML^2}{9\sqrt{3}EI} \text{ at } x = \frac{L}{\sqrt{3}}$$

Deflection at the center (not maximum)

$$\delta = \frac{ML^2}{16EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{MLx}{6} \left(1 - \frac{x^2}{L^2} \right)$$



Case 12: Moment load at the left support of simple beam

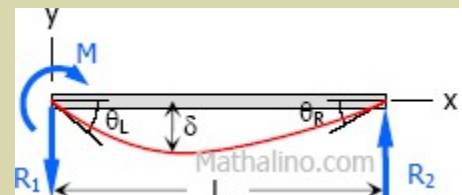
Maximum Moment

$$M = M$$

Slope at end

$$\theta_L = \frac{ML}{3EI}$$

$$\theta_R = \frac{ML}{6EI}$$



$$\delta = \frac{ML^2}{9\sqrt{3}EI} \text{ at } x = \left(L - \frac{L}{\sqrt{3}} \right)$$

Deflection at the center (not maximum)

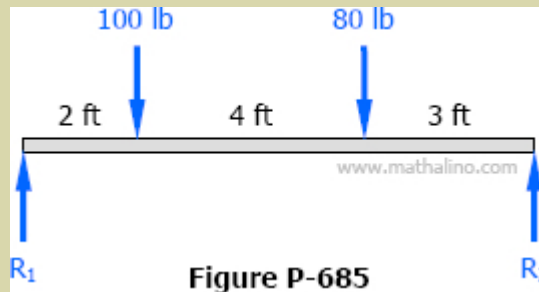
$$\delta = \frac{ML^2}{16EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Mx}{6L}(L-x)(2L-x)$$

Problem 685

Determine the midspan value of $EI\delta$ for the beam loaded as shown in Fig. P-685. Use the method of superposition.

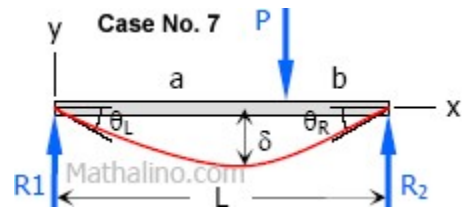


Solution 685

[Click here to show or hide the solution](#)

From [Case No. 7](#) of [Summary of Beam Loadings](#), deflection at the center is

$$\delta = \frac{Pb}{48EI}(3L^2 - 4b^2) \text{ when } a > b$$



Thus, for Fig. P-685

$EI \delta_{\text{midspan}} = EI \delta_{\text{midspan}} \text{ due to 100 lb force} + EI \delta_{\text{midspan}} \text{ due to 80 lb force}$

$$EI \delta_{\text{midspan}} = \sum \frac{Pb}{48}(3L^2 - 4b^2)$$

$$EI \delta_{midspan} = \frac{100(2)}{48} [3(9^2) - 4(2^2)] + \frac{80(3)}{48} [3(9^2) - 4(3^2)]$$

$$EI \delta_{midspan} = 945.83 + 1035$$

$$EI \delta_{midspan} = 1980.83 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

Problem 686

Determine the value of $EI\delta$ under each concentrated load in Fig. P-686.

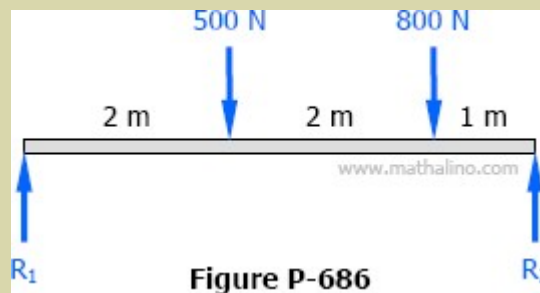


Figure P-686

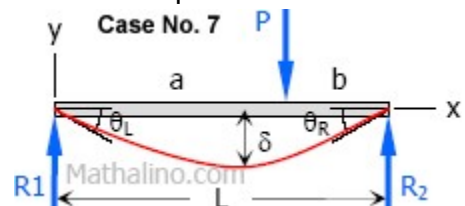
Solution 686

[HideClick here to show or hide the solution](#)

From [Case No. 7](#) of [Summary of Beam Loadings](#), the deflection equations are

$$EI y = \frac{Pbx}{6L} (L^2 - x^2 - b^2) \text{ for } 0 < x < a$$

$$EI y = \frac{Pb}{6L} \left[\frac{L}{b} (x - a)^3 + (L^2 - b^2)x - x^3 \right] \text{ for } a < x < L$$



The point under the load P is generally located at $x = a$ and at this point, both equations above will become

$$EI y = \frac{Pab}{6L} (L^2 - a^2 - b^2)$$

Deflection under the 500 N load

$EI\delta = EI\delta \text{ due to 500 N load} + EI\delta \text{ due to 800 N load}$

$$EI\delta = \frac{500(2)(3)}{6(5)}(5^2 - 2^2 - 3^2) + \frac{800(1)(2)}{6(5)}(5^2 - 2^2 - 1^2)$$

$$EI\delta = 1200 + 1066.67$$

$$EI\delta = 2266.67 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

Deflection under the 800 N load

$EI\delta = EI\delta \text{ due to 500 N load} + EI\delta \text{ due to 800 N load}$

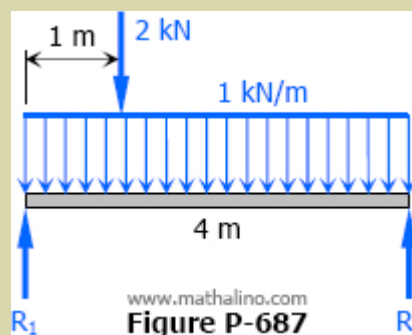
$$EI\delta = \frac{500(3)}{6(5)} \left[\frac{5}{3}(4-2)^3 + (5^2 - 3^2)(4) - 4^3 \right] + \frac{800(1)(4)}{6(5)}(5^2 - 4^2 - 1^2)$$

$$EI\delta = 666.67 + 853.33$$

$$EI\delta = 1520 \text{ N} \cdot \text{m}^3 \quad \text{answer}$$

Problem 687

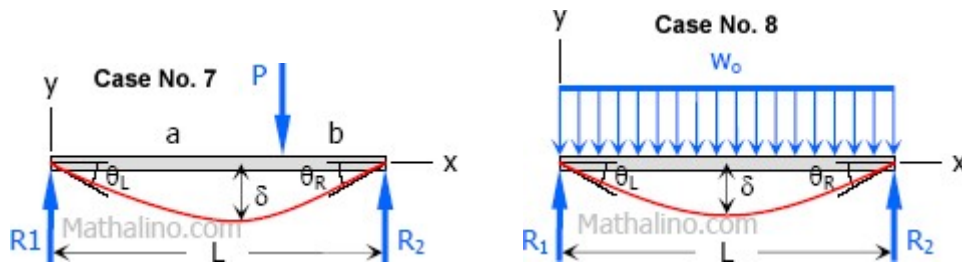
Determine the midspan deflection of the beam shown in Fig. P-687 if $E = 10 \text{ GPa}$ and $I = 20 \times 10^6 \text{ mm}^4$.



Solution 687

[HideClick here to show or hide the solution](#)

From Case No. 7, midspan deflection is $\delta = \frac{Pb}{48EI}(3L^2 - 4b^2)$ when $a > b$



From Case No. 8, midspan deflection is $\delta = \frac{5w_o L^4}{384EI}$

Midspan deflection of the given beam

$EI\delta = EI\delta$ due to 2 kN concentrated load + $EI\delta$ due to 1 kN/m uniform loading

$$EI\delta = \frac{Pb}{48}(3L^2 - 4b^2) + \frac{5w_o L^4}{384}$$

$$EI\delta = \frac{2(1)}{48}[3(4^2) - 4(1^2)] + \frac{5(1)(4^4)}{384}$$

$$EI\delta = \frac{11}{6} + \frac{10}{3}$$

$$EI\delta = \frac{31}{6} \text{ kN} \cdot \text{m}^3$$

$$\delta = \frac{\frac{31}{6}}{EI}$$

$$\delta = \frac{\frac{31}{6}(1000^4)}{10000(20 \times 10^6)}$$

$$\delta = 25.83 \text{ mm} \quad \text{answer}$$

Problem 688

Determine the midspan value of $EI\delta$ at the left end of the beam shown in Fig. P-688.

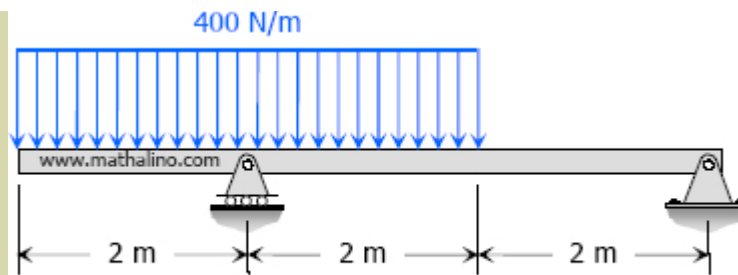


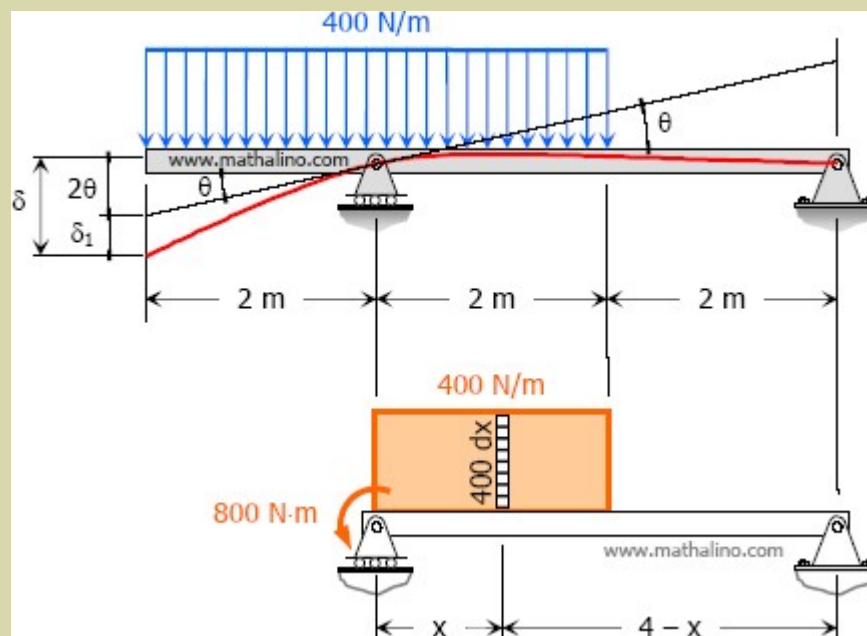
Figure P-688

Solution 688

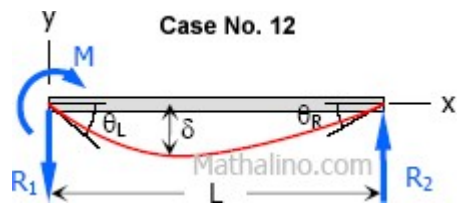
[HideClick here to show or hide the solution](#)

From the figure below, the total deformation at the end of overhang is

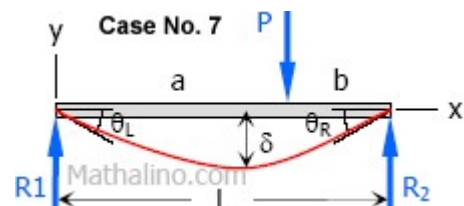
$$\delta = 2\theta + \delta_1$$



The rotation θ at the left support is combination of Case No. 12 and by integration of Case No. 7.



$$\theta_L = \frac{ML}{3EI}$$



$$\theta_L = \frac{Pb(L^2 - b^2)}{6EIL}$$

Solving for θ

$EI\theta = EI\theta$ due to 800 N·m moment at left support - $EI\theta$ due to 400 N/m uniform load

$$EI\theta = \frac{800(4)}{3} - \int_0^2 \frac{400dx(4-x)[4^2 - (4-x)^2]}{6(4)}$$

$$EI\theta = \frac{3200}{3} - \frac{50}{3} \int_0^2 [16(4-x) - (4-x)^3] dx$$

$$EI\theta = \frac{3200}{3} - \frac{50}{3} \left[-8(4-x)^2 + \frac{(4-x)^4}{4} \right]_0^2$$

$$EI\theta = \frac{3200}{3} - \frac{50}{3} \left[-8(2^2) + \frac{2^4}{4} \right] + \frac{50}{3} \left[-8(4^2) + \frac{4^4}{4} \right]$$

$$EI\theta = \frac{3200}{3} + \frac{1400}{3} - \frac{3200}{3}$$

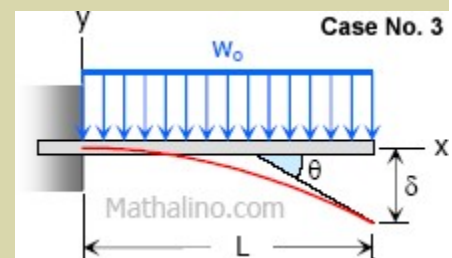
$$EI\theta = \frac{1400}{3} \text{ N} \cdot \text{m}^2$$

Apply Case No. 3 for solving δ_1 . From Case No. 3:

$$\delta = \frac{w_o L^4}{8EI}$$

Solving for δ_1 :

$$EI\delta_1 = \frac{400(2^4)}{8}$$



$$EI \delta_1 = 800 \text{ N} \cdot \text{m}^3$$

-

Total deflection at the free end

$$EI \delta = 2EI \theta + EI \delta_1$$

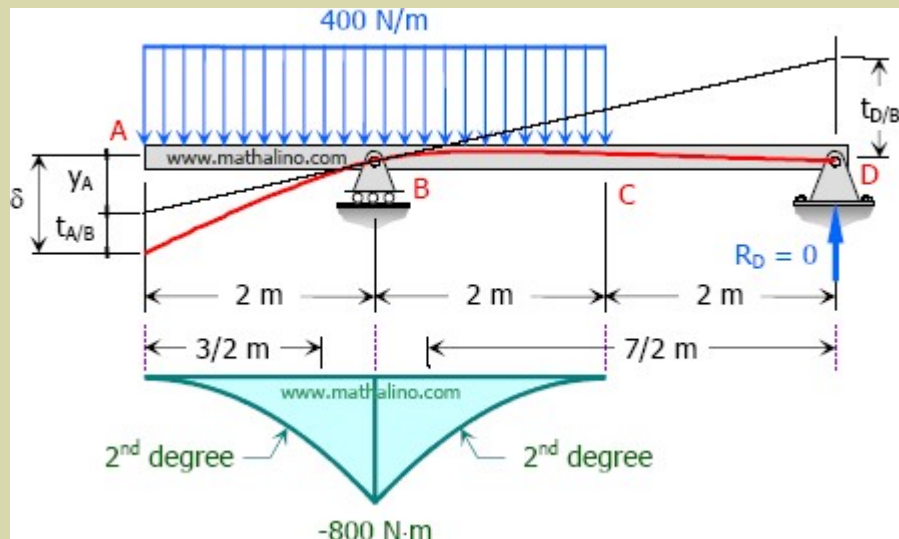
$$EI \delta = 2 \left(\frac{1400}{3} \right) + 800$$

$$EI \delta = \frac{5200}{3} \text{ N} \cdot \text{m}^3 \quad \underline{\text{answer}}$$

HideAnother Solution (Area-moment method)

This problem can be done with less effort by area-moment method.

-



-

$$EI t_{A/B} = (Area_{AB}) \bar{X}_A$$

$$EI t_{A/B} = -\frac{1}{3}(2)(800)\left(\frac{3}{2}\right)$$

$$EI t_{A/B} = -800 \text{ N} \cdot \text{m}^3$$

-

$$EI t_{D/B} = (Area_{AB}) \bar{X}_A$$

$$EI t_{D/B} = -\frac{1}{3}(2)(800)\left(\frac{7}{2}\right)$$

$$EI t_{D/B} = -\frac{5600}{3} \text{ N} \cdot \text{m}^3$$

-

The negative sign above indicates that the elastic curve is below the tangent line.

-

$$\frac{y_A}{2} = \frac{t_{D/B}}{4}$$

$$y_A = \frac{1}{2} t_{D/B}$$

$$EI y_A = \frac{1}{2} (EI t_{D/B})$$

$$EI y_A = \frac{1}{2} \left(\frac{5600}{3} \right)$$

$$EI y_A = \frac{2800}{3} \text{ N} \cdot \text{m}^3$$

-

$$EI \delta = EI y_A + EI t_{A/B}$$

$$EI \delta = \frac{2800}{3} + 800$$

$$EI \delta = \frac{5200}{3} \text{ N} \cdot \text{m}^3$$

$$EI \delta = \frac{5200}{3} \text{ N} \cdot \text{m}^3 \quad \text{--- (okay!)}$$

Problem 689

The beam shown in Fig. P-689 has a rectangular cross section 4 inches wide by 8 inches deep. Compute the value of P that will limit the midspan deflection to 0.5 inch. Use $E = 1.5 \times 10^6$ psi.

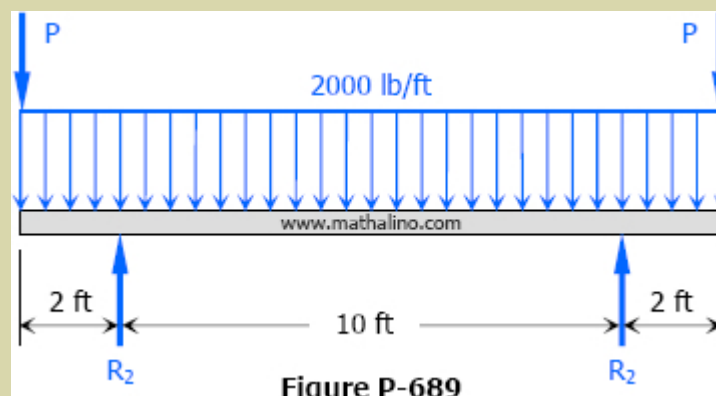


Figure P-689

Solution 689

[Hide](#)[Click here to show or hide the solution](#)

The overhang is resolved into simple beam with end moments. The magnitude of end moment is,

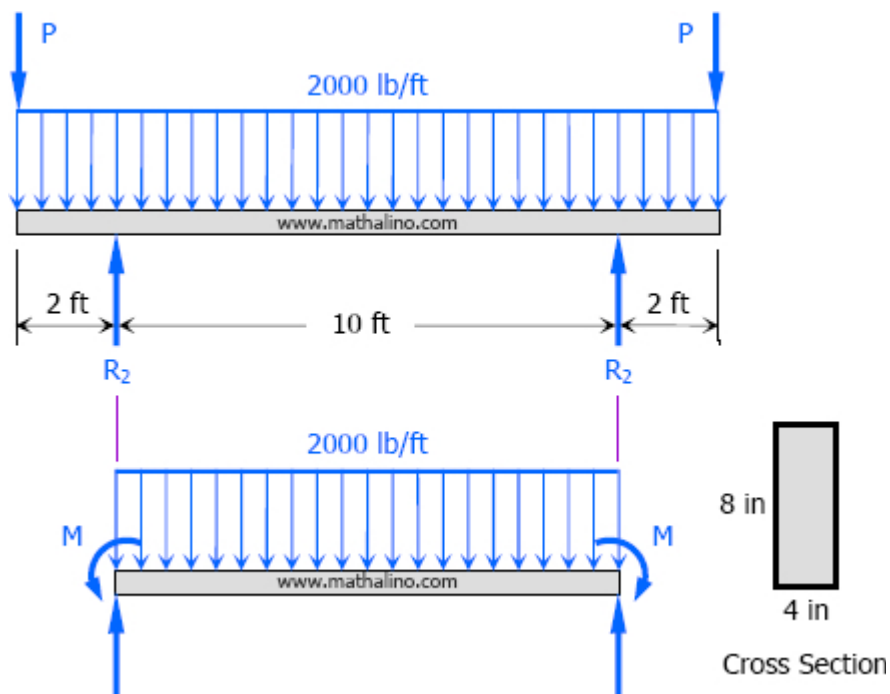
$$M = 2000(2)(1) + 2P$$

$$M = 4000 + 2P$$

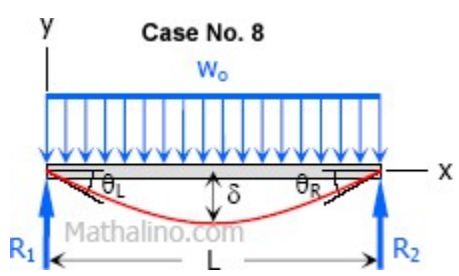
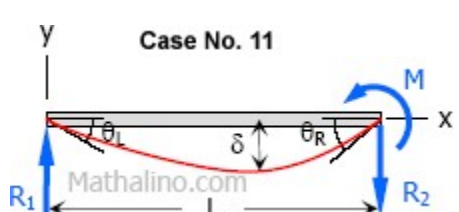
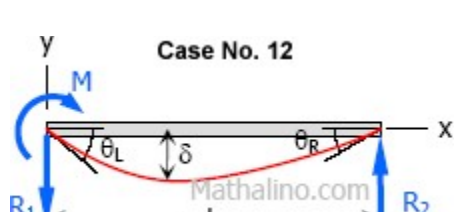
Moment of inertia of beam section

$$I = \frac{bd^3}{12} = \frac{4(8^3)}{12}$$

$$I = \frac{512}{3} \text{ in}^4$$



The midspan deflection is a combination of deflection due to uniform load and two end moments. Use Case No. 8 and Cases No. 8, 11, and 12 to solve for the midspan deflection.

Type of Loading	Midspan Deflection
Case No. 8 	$\delta = \frac{5w_o L^4}{384EI}$
Case No. 11 	$\delta = \frac{ML^2}{16EI}$
Case No. 12 	$\delta = \frac{ML^2}{16EI}$

$$\delta_{midspan} = \frac{5w_o L^4}{384EI} - 2 \left[\frac{ML^2}{16EI} \right]$$

$$0.5 = \frac{5(2000)(10^4)(12^3)}{384(1.5 \times 10^6)(\frac{512}{3})} - 2 \left[\frac{(4000 + 2P)(10^2)(12^3)}{16(1.5 \times 10^6)(\frac{512}{3})} \right]$$

$$0.5 = \frac{225}{128} - \frac{27(4000 + 2P)}{320\,000}$$

$$160\,000 = 562\,500 - 27(4000 + 2P)$$

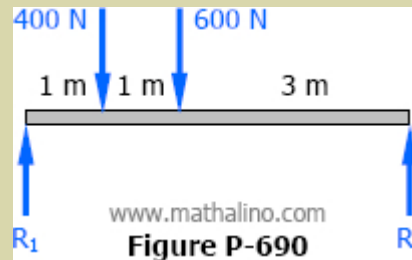
$$27(4000 + 2P) = 402\,500$$

$$2P = 10\,907.40$$

$$P = 5453.7 \text{ lb} \quad \text{answer}$$

Problem 690

The beam shown in Fig. P-690 has a rectangular cross section 50 mm wide. Determine the proper depth d of the beam if the midspan deflection of the beam is not to exceed 20 mm and the flexural stress is limited to 10 MPa. Use $E = 10 \text{ GPa}$.



Solution 690

[HideClick here to show or hide the solution](#)

$$\Sigma M_{R2} = 0$$

$$5R_1 = 4(400) + 3(600)$$

$$R_1 = 680 \text{ N}$$

$$\Sigma M_{R1} = 0$$

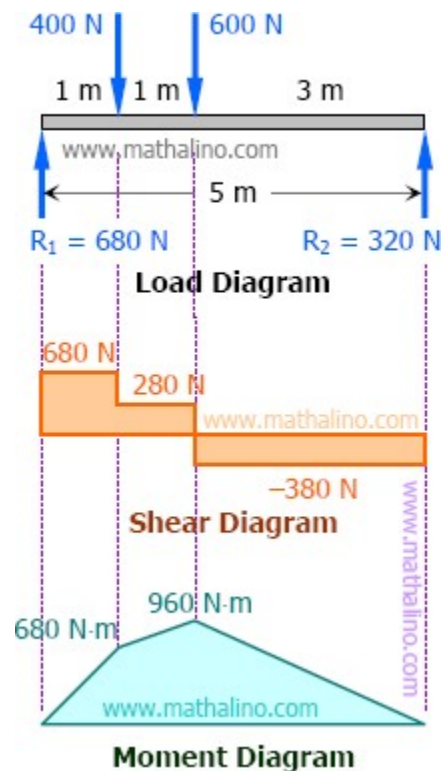
$$5R_2 = 1(400) + 2(600)$$

$$R_2 = 320 \text{ N}$$

Based on allowable flexural stress

$$(f_b)_{max} = \frac{6M_{max}}{bd^2}$$

$$10 = \frac{6(960)(1000)}{50d^2}$$

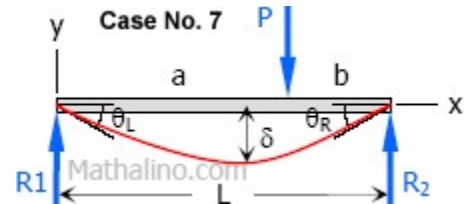


$$d^2 = 11\,520$$

$$d = 107.33 \text{ mm}$$

Based on allowable midspan deflection. Use Case No. 7, the midspan deflection of simple beam under concentrated load is given by

$$\delta = \frac{Pb}{48EI}(3L^2 - 4b^2) \text{ when } a > b$$



For the given beam, the midspan deflection is the sum of the midspan deflection of each load acting separately.

$$\delta = \sum \frac{Pb}{48EI}(3L^2 - 4b^2)$$

$$20 = \frac{400(1)(1000^3)}{48(10\,000I)}[3(5^2) - 4(1^2)] + \frac{600(2)(1000^3)}{48(10\,000I)}[3(5^2) - 4(2^2)]$$

$$\frac{20(10\,000I)}{1000^3} = \frac{400(1)}{48}[3(5^2) - 4(1^2)] + \frac{600(2)}{48}[3(5^2) - 4(2^2)]$$

$$\frac{I}{5000} = \frac{1775}{3} + 1475$$

$$\frac{I}{5000} = \frac{6200}{3}$$

$$I = 10\,333\,333.33$$

$$\frac{bd^3}{12} = 10\,333\,333.33$$

$$\frac{50d^3}{12} = 10\,333\,333.33$$

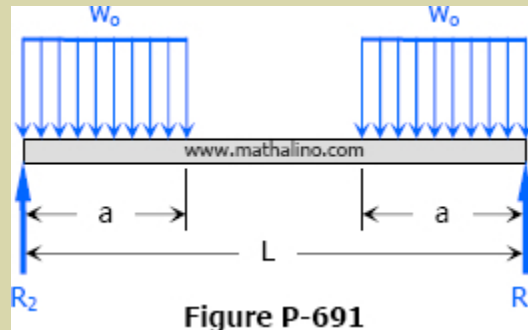
$$d^3 = 2\,480\,000$$

$$d = 135.36 \text{ mm}$$

Use **d = 135.36 mm** *answer*

Problem 691

Determine the midspan deflection for the beam shown in Fig. P-691. (Hint: Apply [Case No. 7](#) and integrate.)

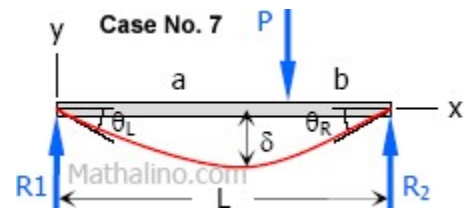


Solution 691

[Click here to show or hide the solution](#)

From Case No. 7, the midspan deflection is

$$\delta = \frac{Pb}{48EI} (3L^2 - 4b^2) \text{ when } a > b$$



For the given beam

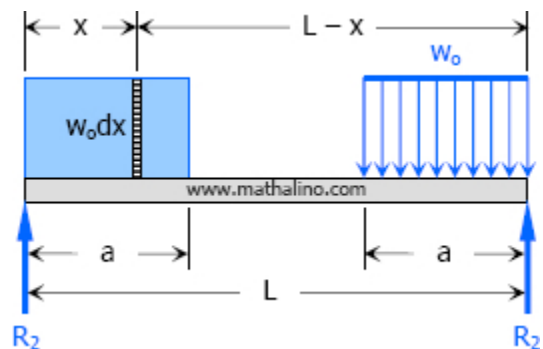
$$P = w_0 dx$$

$$b = x$$

$$\delta = 2 \int_0^a \frac{(w_0 dx)x}{48EI} (3L^2 - 4x^2)$$

$$\delta = \frac{w_0}{24EI} \int_0^a (3L^2 x - 4x^3) dx$$

$$\delta = \frac{w_0}{24EI} \left[\frac{3L^2 x^2}{2} - x^4 \right]_0^a$$



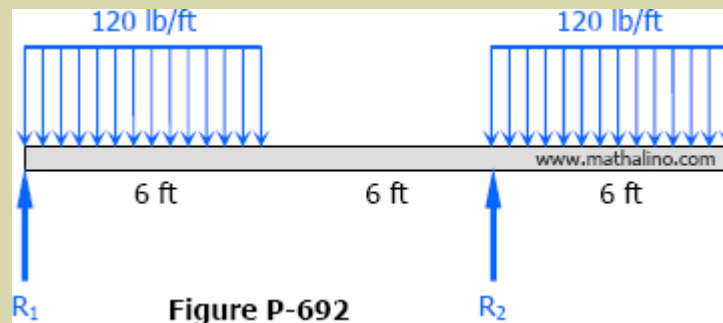
$$\delta = \frac{w_o}{24EI} \left[\frac{3L^2x^2 - 2x^4}{2} \right]_0^a$$

$$\delta = \frac{w_o}{48EI} [3L^2a^2 - 2a^4]$$

$$\delta = \frac{w_o a^2}{48EI} (3L^2 - 2a^2) \quad \text{answer}$$

Problem 692

Find the value of $EI\delta$ midway between the supports for the beam shown in Fig. P-692. (Hint: Combine Case No. 11 and one half of Case No. 8.)

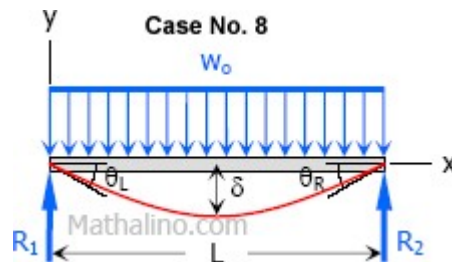


Solution 692

[Hide](#) [Click here to show or hide the solution](#)

The midspan deflection from [Case No. 8](#) and [Case No. 11](#) are respectively,

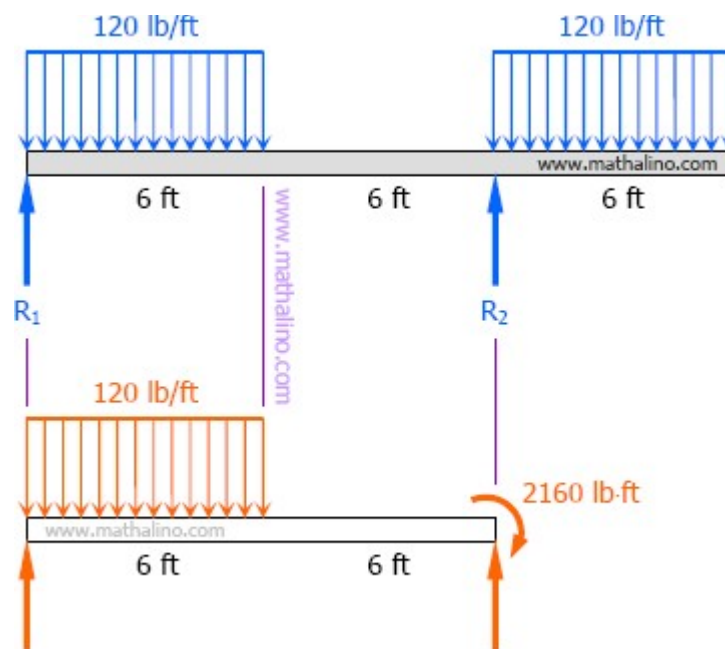
$$\delta = \frac{5w_o L^4}{384EI}$$



$$\delta = \frac{ML^2}{16EI}$$

Case No. 11

The given beam is transformed into a simple beam with end moment at the right support due to the load at the overhang as shown in the figure below.



$EI\delta = \frac{1}{2}$ of $EI\delta$ due to uniform load over the entire span - $EI\delta$ due to end moment

$$EI \delta_{midspan} = \frac{1}{2} \left[\frac{5(120)(12^4)}{384} \right] - \frac{2160(12^2)}{16}$$

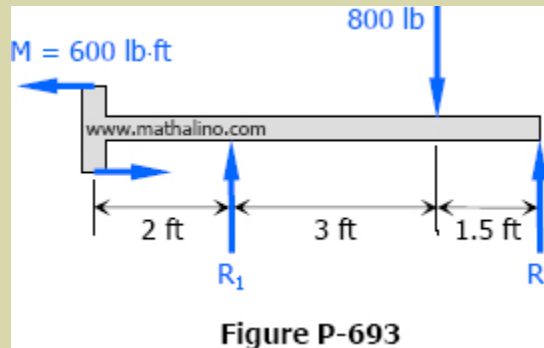
$$EI \delta_{midspan} = 16\,200 - 19\,440$$

$$EI \delta_{midspan} = -3\,240$$

$$EI \delta_{midspan} = 3240 \text{ lb} \cdot \text{ft}^3 \text{ upward} \quad \text{answer}$$

Problem 693

Determine the value of $EI\delta$ at the left end of the overhanging beam in Fig. P-693.



Solution 693

[Click here to show or hide the solution](#)

The rotation at the left support is the combination of [Case No. 7](#) and [Case No. 12](#).

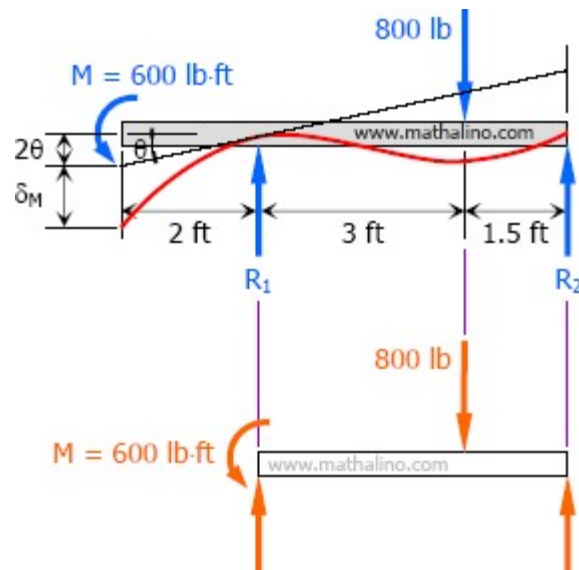
$$\theta_L = \frac{Pb(L^2 - b^2)}{6EIL}$$

Case No. 7

$$\theta_L = \frac{ML}{3EI}$$

Case No. 12

The overhang beam is transformed into a simple beam and the end moment at the free end of the overhang is carried to the left support of the transformed beam.



$$\theta = \frac{Pb(L^2 - b^2)}{6EIL} - \frac{ML}{3EI}$$

$$\theta = \frac{800(1.5)(4.5^2 - 1.5^2)}{6EI(4.5)} - \frac{600(4.5)}{3EI}$$

$$\theta = \frac{800}{EI} - \frac{900}{EI}$$

$$\theta = -\frac{100}{EI}$$

The negative sign indicates that the rotation at the left end contributed by the end moment (taken as negative) is greater than the rotation at the left end contributed by the concentrated load (taken as positive).

From [Case No. 5](#), the end deflection is

$$\delta = \frac{ML^2}{2EI}$$

The deflection at the overhang due to moment load alone is

$$\delta_M = \frac{600(2^2)}{2EI}$$

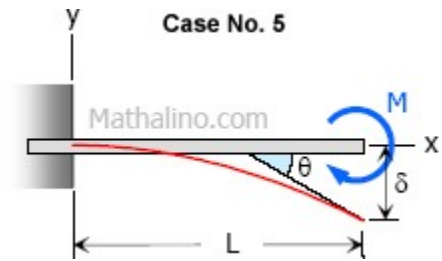
$$\delta_M = \frac{1200}{EI}$$

Total deflection at the left end of the given beam is

$$\delta = 2\theta + \delta_M$$

$$\delta = 2 \left(\frac{100}{EI} \right) + \frac{1200}{EI}$$

$$\delta = \frac{1400}{EI} \quad \text{answer}$$



Problem 694

The frame shown in [Fig. P-694](#) is of constant cross section and is perfectly restrained at its lower end. Compute the vertical deflection caused by the couple M.

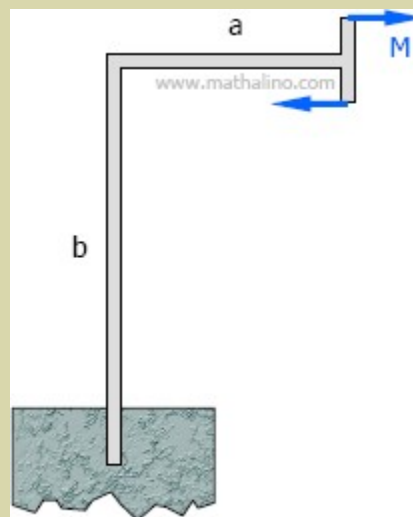
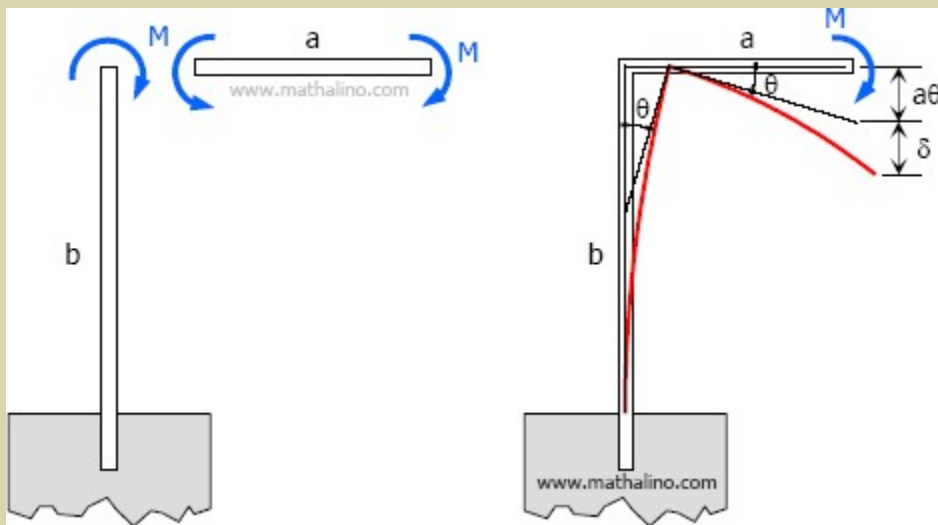


Figure P-694 and P-695

[HideClick here to read or hide Solution 694](#)



$$\delta_v = a\theta + \delta$$

$$\delta_v = a \left(\frac{Mb}{EI} \right) + \frac{Ma^2}{2EI}$$

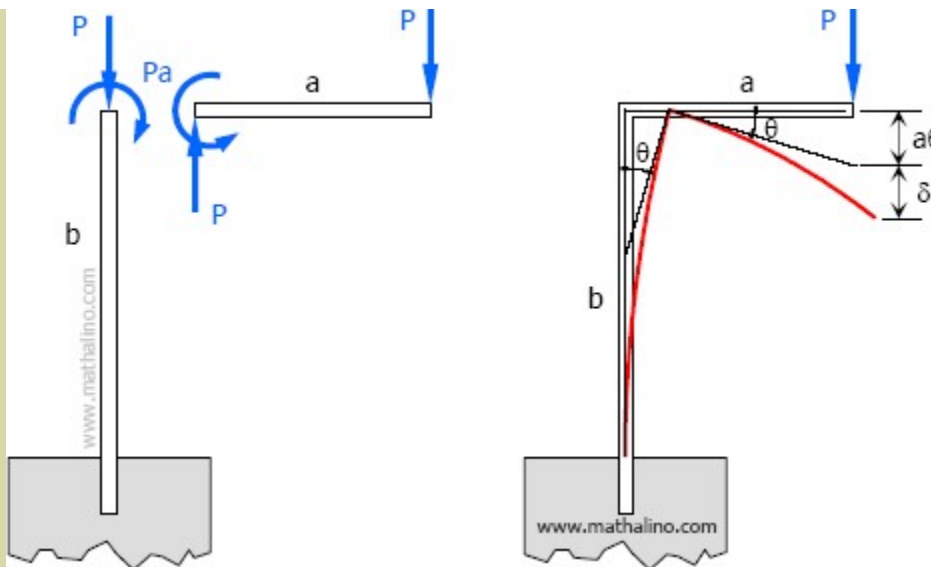
$$\delta_v = \frac{Mba}{EI} + \frac{Ma^2}{2EI}$$

$$\delta_v = \frac{Ma}{2EI} (2b + a) \quad \text{answer}$$

Problem 695

Solve Problem 694 if the couple is replaced by a downward load P.

[HideClick here to read or hide Solution 695](#)



$$\delta_v = a\theta + \delta$$

$$\delta_v = a \left(\frac{Pab}{EI} \right) + \frac{Pa^3}{3EI}$$

$$\delta_v = \frac{Pa^2b}{EI} + \frac{Pa^3}{3EI}$$

$$\delta_v = \frac{Pa^2}{3EI} (3b + a) \quad \text{answer}$$

Problem 696

In Fig. P-696, determine the value of P for which the deflection under P will be zero.

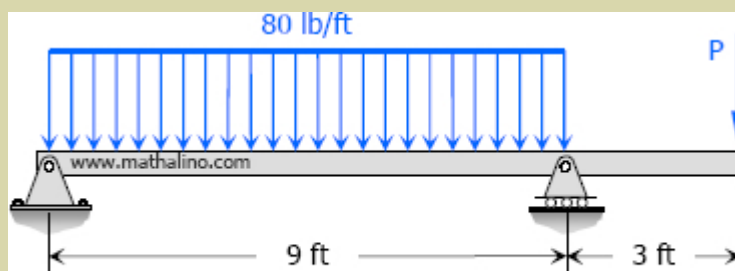
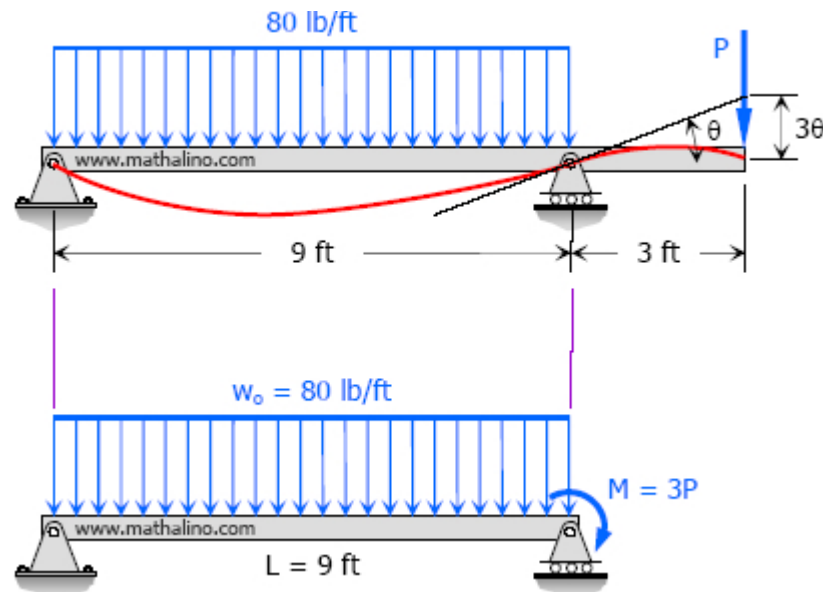


Figure P-696

[Hide](#) [Click here to read or hide Solution 696](#)

Apply [Case No. 8](#) and [Case No. 11](#) to find the slope at the right support.



$$\theta = \frac{w_o L^3}{24EI} - \frac{ML}{3EI}$$

$$\theta = \frac{80(9^3)}{24EI} - \frac{3P(9)}{3EI}$$

$$\theta = \frac{2430}{EI} - \frac{9P}{EI}$$

$$3\theta = \frac{7290}{EI} - \frac{27P}{EI}$$

Use [Case No. 1](#) for the deflection at the free end due to concentrated load P.

$$\delta = \frac{PL^3}{3EI}$$

$$\delta = \frac{P(3^3)}{3EI}$$

$$\delta = \frac{9P}{EI}$$

$$\delta = 3\theta$$

$$\frac{36P}{EI} = \frac{7290}{EI}$$

$$P = 202.5 \text{ lb} \quad \text{answer}$$

Problem 697

For the beam in Prob. 696, find the value of P for which the slope over the right support will be zero.

From Solution 696,

$$\theta = \frac{2430}{EI} - \frac{9P}{EI}$$

$$0 = \frac{2430}{EI} - \frac{9P}{EI}$$

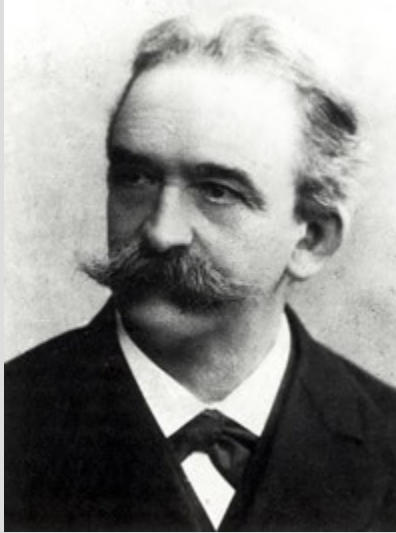
$$\frac{9P}{EI} = \frac{2430}{EI}$$

$$P = 270 \text{ lb} \quad \text{answer}$$

Conjugate Beam Method | Beam Deflection

Slope on real beam = Shear on conjugate beam
Deflection on real beam = Moment on conjugate beam

Properties of Conjugate Beam



Engr. Christian Otto Mohr

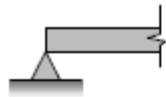
1. The length of a conjugate beam is always equal to the length of the actual beam.
2. The load on the conjugate beam is the M/EI diagram of the loads on the actual beam.
3. A simple support for the real beam remains simple support for the conjugate beam.
4. A fixed end for the real beam becomes free end for the conjugate beam.
5. The point of zero shear for the conjugate beam corresponds to a point of zero slope for the real beam.
6. The point of maximum moment for the conjugate beam corresponds to a point of maximum deflection for the real beam.

Supports of Conjugate Beam

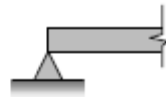
Knowing that the slope on the real beam is equal to the shear on conjugate beam and the deflection on real beam is equal to the moment on conjugate beam, the shear and bending moment at any point on the conjugate beam must be consistent with the slope and deflection at that point of the real beam. Take for example a real beam with fixed support; at the point of fixed support there is neither slope nor deflection, thus, the shear and moment of the corresponding conjugate beam at that point must be zero. Therefore, the conjugate of fixed support is free end.

Real Beam Support**Conjugate Beam Support**

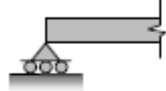
Hinged Support



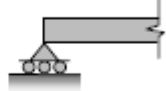
Hinged Support



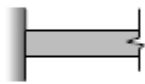
Roller Support



Roller Support



Fixed Support



Free End



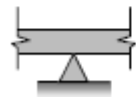
Free End



Fixed Support



Interior Support



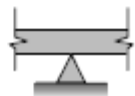
Internal Hinge



Internal Hinge



Interior Support

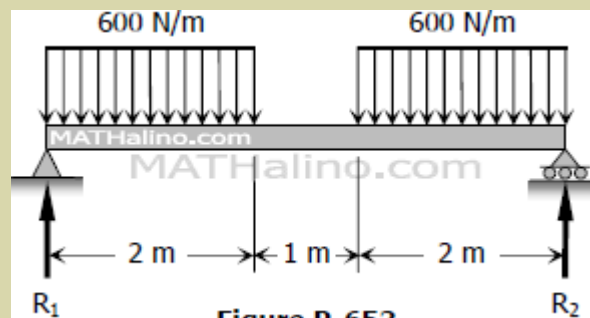
**Examples of Beam and its Conjugate**

The following are some examples of beams and its conjugate. Loadings are omitted.

Real Beam	Conjugate Beam

Problem 653

Compute the midspan value of $EI\delta$ for the beam shown in Fig. P-653. (Hint: Draw the M diagram by parts, starting from midspan toward the ends. Also take advantage of symmetry.



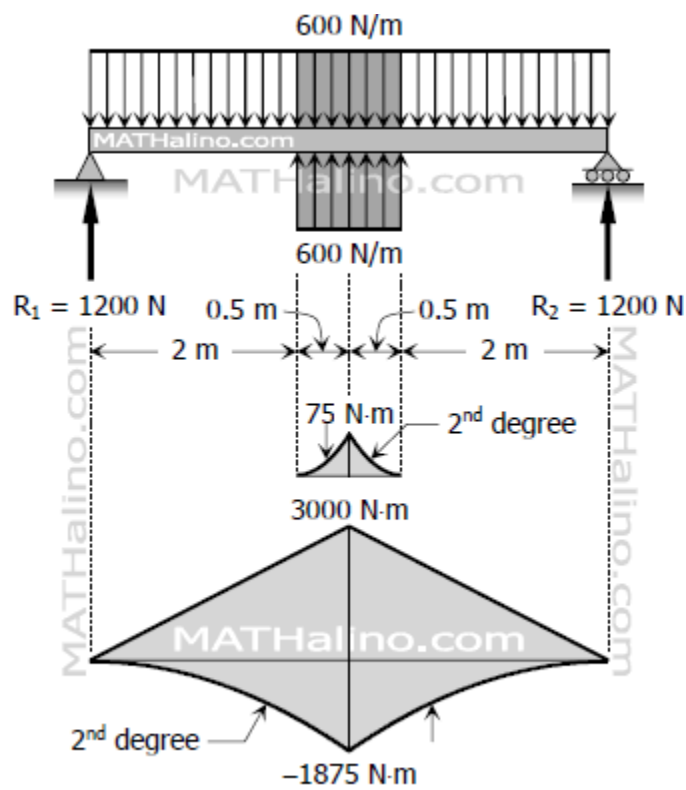
Solution 653 (Using Moment Diagram by Parts)

[Click here to show or hide the solution](#)

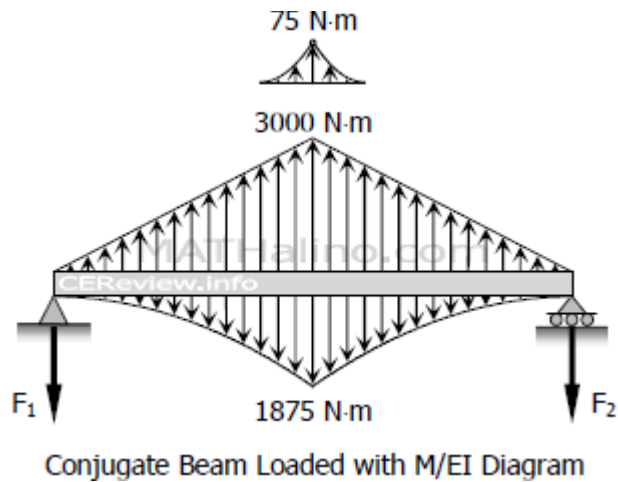
By symmetry,

$$R_1 = R_2 = 2(600)$$

$$R_1 = R_2 = 1200 \text{ N}$$



Moment Diagram by Parts

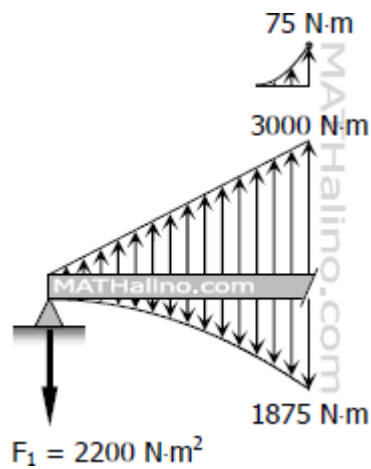


The loads of conjugate beam are symmetrical, thus,

$$F_1 = F_2 = \frac{1}{2} \left[\frac{1}{2}(5)(3000) + \frac{1}{3}(1)(75) - \frac{1}{3}(5)(1875) \right]$$

$$F_1 = F_2 = 2200 \text{ N} \cdot \text{m}^2$$

For this beam, the maximum deflection will occur at the midspan.



$$M_{midspan} = \frac{1}{2}(2.5)(3000) \left[\frac{1}{3}(2.5) \right] + \frac{1}{3}(0.5)(75) \left[\frac{1}{4}(0.5) \right] - \frac{1}{3}(2.5)(1875) \left[\frac{1}{4}(2.5) \right] - 2200(2.5)$$

$$M_{midspan} = -3350 \text{ N} \cdot \text{m}^3$$

Therefore, the maximum deflection is

$$EI \delta_{max} = M_{midspan}$$

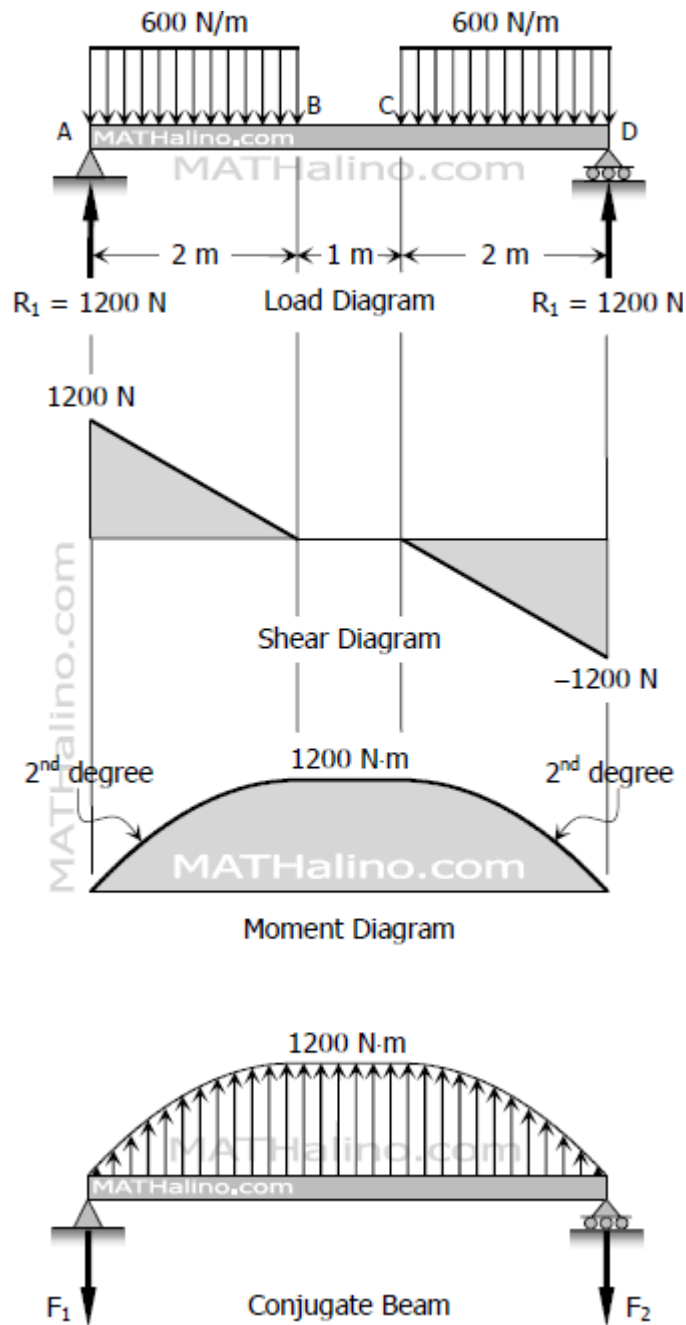
$$EI \delta_{max} = -3350 \text{ N} \cdot \text{m}^3$$

$$EI \delta_{max} = 3350 \text{ N} \cdot \text{m}^3 \quad \text{below the neutral axis} \quad \text{answer}$$

Another Solution (Using the Actual Moment Diagram)

[HideClick here to show or hide the solution](#)

(Conjugate beam method using the actual moment diagram)

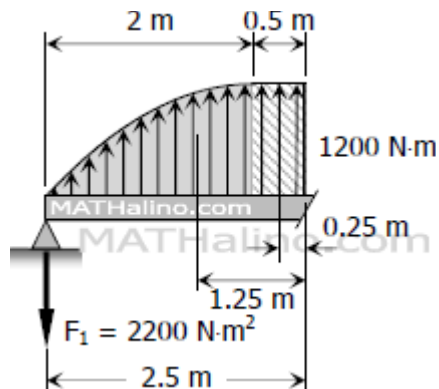


By symmetry of conjugate beam

$$F_1 = F_2 = \frac{1}{2} \left[\frac{2}{3}(2)(1200) + 1200(1) + \frac{2}{3}(2)(1200) \right]$$

$$F_1 = F_2 = 2200 \text{ N} \cdot \text{m}^2$$

The maximum deflection will occur at the midspan of this beam



$$M_{midspan} = \frac{2}{3}(2)(1200)(1.25) + 0.5(1200)(0.25) - 2.5(2200)$$

$$M_{midspan} = -3350 \text{ N} \cdot \text{m}^3$$

Therefore, the maximum deflection is

$$EI \delta_{max} = M_{midspan}$$

$$EI \delta_{max} = -3350 \text{ N} \cdot \text{m}^3$$

$$EI \delta_{max} = 3350 \text{ N} \cdot \text{m}^3 \text{ below the neutral axis} \quad \text{okay!}$$

Problem 654

For the beam in Fig. P-654, find the value of $EI\delta$ at 2 ft from R_2 .

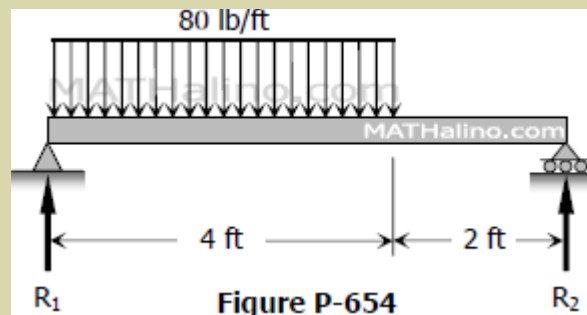


Figure P-654

Solution 654

[Hide](#)[Click here to show or hide the solution](#)

Solving for reactions

$$\Sigma M_{R2} = 0$$

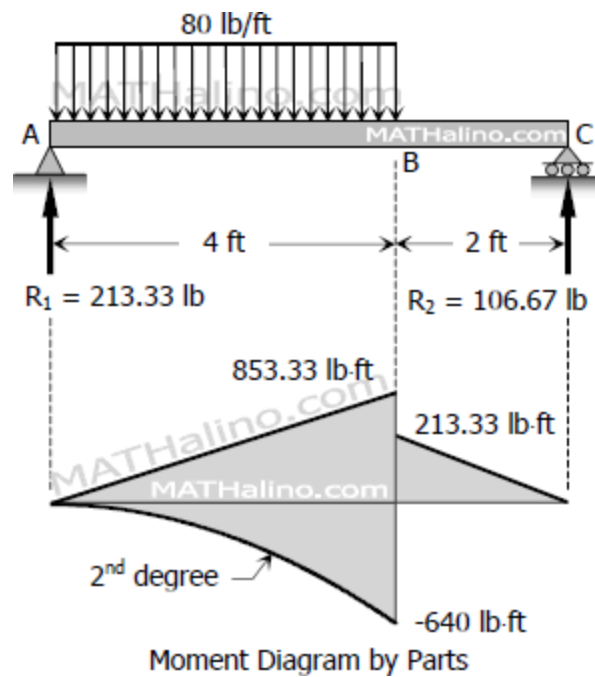
$$6R_1 = 80(4)(4)$$

$$R_1 = 213.33 \text{ lb}$$

$$\Sigma M_{R1} = 0$$

$$6R_2 = 80(4)(2)$$

$$R_2 = 106.67 \text{ lb}$$

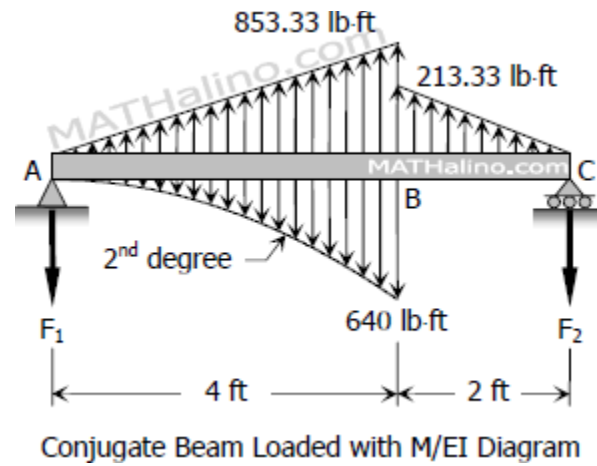


From the conjugate beam

$$\Sigma M_A = 0$$

$$6F_2 + \frac{1}{3}(4)(640)\left[\frac{3}{4}(4)\right] = \frac{1}{2}(4)(853.33)\left[\frac{2}{3}(4)\right] + \frac{1}{2}(2)(213.33)\left[4 + \frac{1}{3}(2)\right]$$

$$F_2 = 497.77 \text{ lb} \cdot \text{ft}^2$$



$$M_B = \frac{1}{2}(2)(213.33[\frac{1}{3}(2)]) - 2F_2$$

$$M_B = \frac{1}{2}(2)(213.33[\frac{1}{3}(2)]) - 2(497.77)$$

$$M_B = -853.32 \text{ lb} \cdot \text{ft}^3$$

Thus, the deflection at B is

$$EI \delta_B = M_B$$

$$EI \delta_B = -853.32 \text{ lb} \cdot \text{ft}^3$$

$$EI \delta_B = 853.32 \text{ lb} \cdot \text{ft}^3 \text{ downward} \quad \text{answer}$$

Problem 655

Find the value of $EI\delta$ under each concentrated load of the beam shown in Fig. P-655.

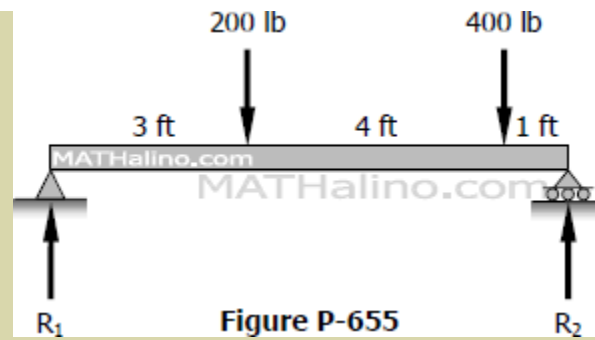


Figure P-655

Solution 655

[Hide](#)[Click here to show or hide the solution](#)

$$\Sigma M_D = 0$$

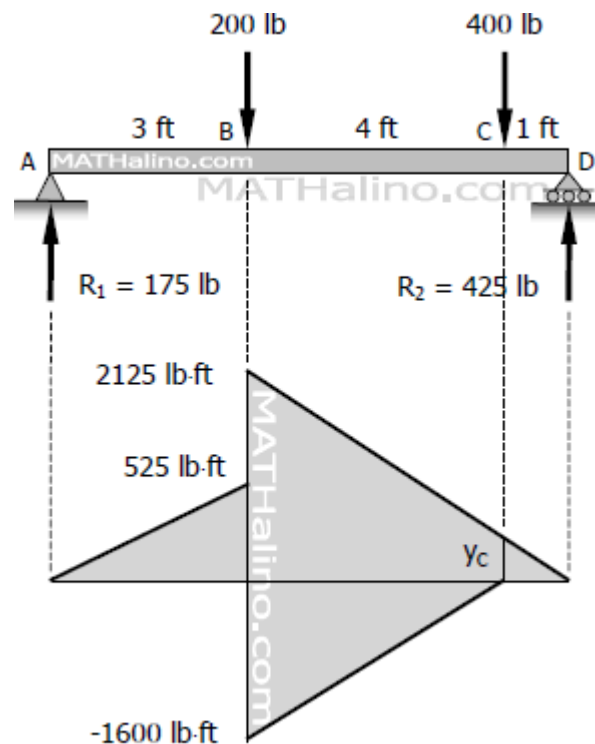
$$8R_1 = 200(5) + 400(1)$$

$$R_1 = 175 \text{ lb}$$

$$\Sigma M_A = 0$$

$$8R_2 = 200(3) + 400(7)$$

$$R_2 = 425 \text{ lb}$$



Moment Diagram by Parts

By ratio and proportion

$$\frac{y_C}{1} = \frac{2125}{5}$$

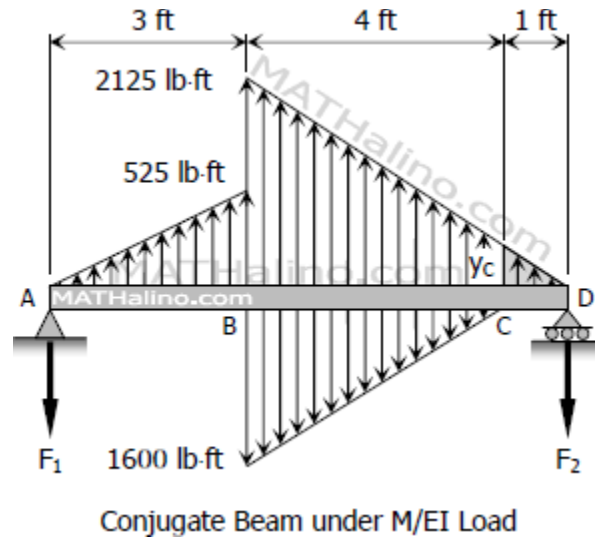
$$y_C = 425 \text{ lb} \cdot \text{ft}$$

From the conjugate beam

$$\Sigma M_D = 0$$

$$8F_1 + \frac{1}{2}(4)(1600)\left[1 + \frac{2}{3}(4)\right] = \frac{1}{2}(3)(525)\left[5 + \frac{1}{3}(3)\right] + \frac{1}{2}(5)(2125)\left[\frac{2}{3}(5)\right]$$

$$F_1 = 1337.5 \text{ lb} \cdot \text{ft}^2$$



$$\Sigma M_A = 0$$

$$8F_2 + \frac{1}{2}(4)(1600)\left[\frac{1}{3}(4)\right] = \frac{1}{2}(3)(525)\left[\frac{2}{3}(3)\right] + \frac{1}{2}(5)(2125)\left[3 + \frac{1}{3}(5)\right]$$

$$F_2 = 1562.5 \text{ lb} \cdot \text{ft}^2$$

Consider the section to the left of B in conjugate beam

$$M_B = \frac{1}{2}(3)(525)\left[\frac{1}{3}(3)\right] - 3F_1$$

$$MB = 787.5 - 3(1337.5)$$

$$M_B = -3225 \text{ lb} \cdot \text{ft}^3$$

Thus, the deflection at B is

$$EI \delta_B = M_B$$

$$EI \delta_B = 3225 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

Consider the section to the right of C in conjugate beam

$$M_C = \frac{1}{2}(1)(y_C)\left[\frac{1}{3}(1)\right] - 1F_2$$

$$M_C = \frac{1}{2}(1)(425)\left[\frac{1}{3}(1)\right] - 1(1562.5)$$

$$M_C = -1491.67 \text{ lb} \cdot \text{ft}^3$$

Thus, the deflection at C is

$$EI \delta_C = M_C$$

$$EI \delta_C = -1491.67 \text{ lb} \cdot \text{ft}^3$$

$$EI \delta_C = 1491.67 \text{ lb} \cdot \text{ft}^3 \text{ downward} \quad \text{answer}$$

Problem 656

Find the value of $EI\delta$ at the point of application of the 200 N·m couple in Fig. P-656.

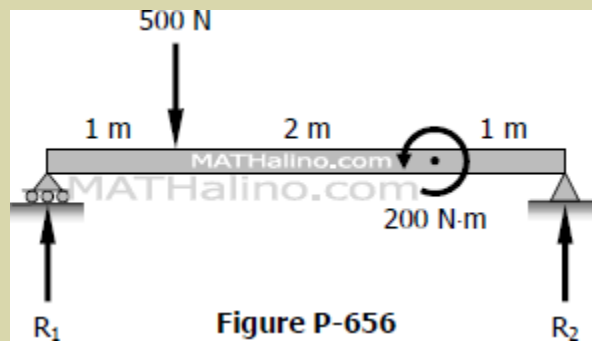


Figure P-656

Solution 656

[HideClick here to show or hide the solution](#)

From the real beam

$$\Sigma M_D = 0$$

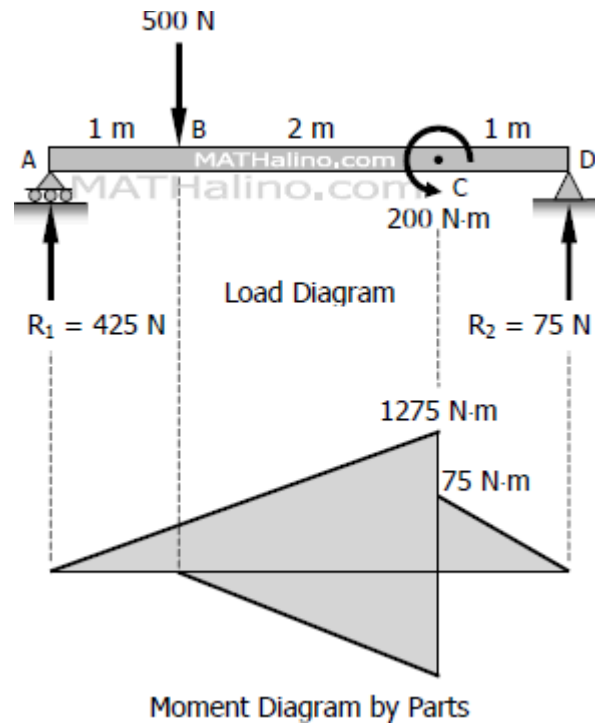
$$4R_1 = 3(500) + 200$$

$$R_1 = 425 \text{ N}$$

$$\Sigma M_A = 0$$

$$4R_2 + 200 = 1(500)$$

$$R_2 = 75 \text{ N}$$

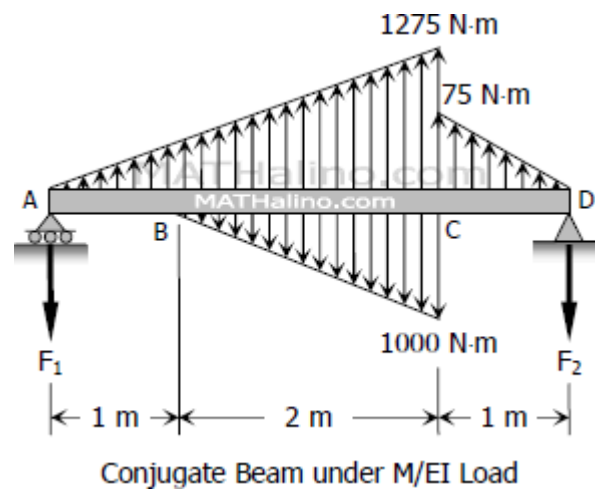


From the conjugate beam

$$\Sigma M_A = 0$$

$$4F_2 + \frac{1}{2}(2)(1000)\left[1 + \frac{2}{3}(2)\right] = \frac{1}{2}(3)(1275)\left[\frac{2}{3}(3)\right] + \frac{1}{2}(1)(75)\left[3 + \frac{1}{3}(1)\right]$$

$$F_2 = 404.17 \text{ N} \cdot \text{m}^3$$



$$M_C = \frac{1}{2}(1)(75)\left[\frac{1}{3}(1)\right] - 1(F_2)$$

$$M_C = 12.5 - 1(404.17)$$

$$M_C = -391.67 \text{ N} \cdot \text{m}^3$$

Therefore, the deflection at C is

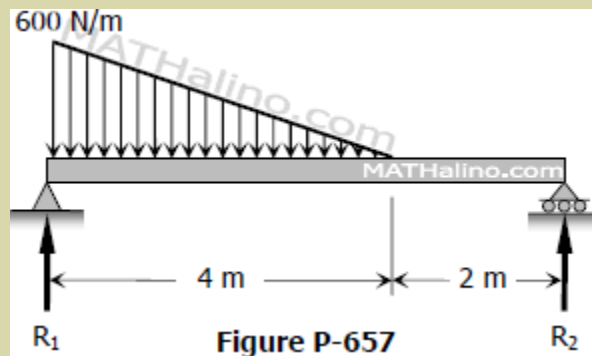
$$EI \delta_C = M_C$$

$$EI \delta_C = -391.67 \text{ N} \cdot \text{m}^3$$

$$EI \delta_C = 391.67 \text{ N} \cdot \text{m}^3 \text{ downward} \quad \text{answer}$$

Problem 657

Determine the midspan value of $EI\delta$ for the beam shown in Fig. P-657.



Solution 657

[HideClick here to show or hide the solution](#)

From the load diagram

$$\Sigma M_{R1} = 0$$

$$6R_2 = \frac{1}{2}(4)(600)\left[\frac{1}{3}(4)\right]$$

$$R_2 = 266.67 \text{ N}$$

$$\frac{y}{1} = \frac{600}{4}$$

$$y = 150 \text{ N/m}$$

From the moment diagram

$$a = 3R_2 = 3(266.67)$$

$$a = 800 \text{ N} \cdot \text{m}$$

$$b = -\frac{1}{2}(1)(y)\left[\frac{1}{3}(1)\right]$$

$$b = -\frac{1}{6}y = -\frac{1}{6}(150)$$

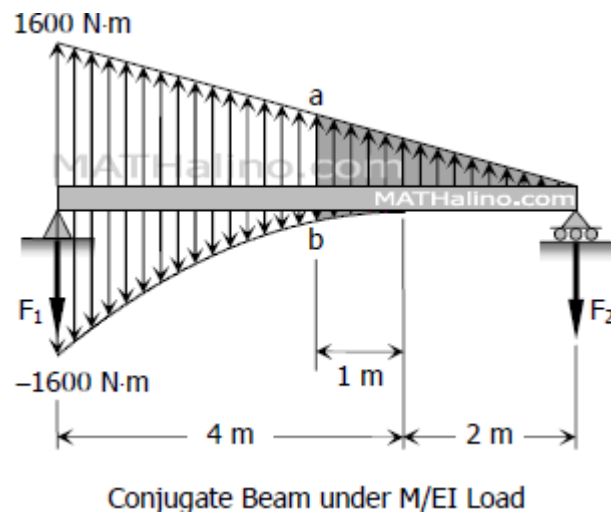
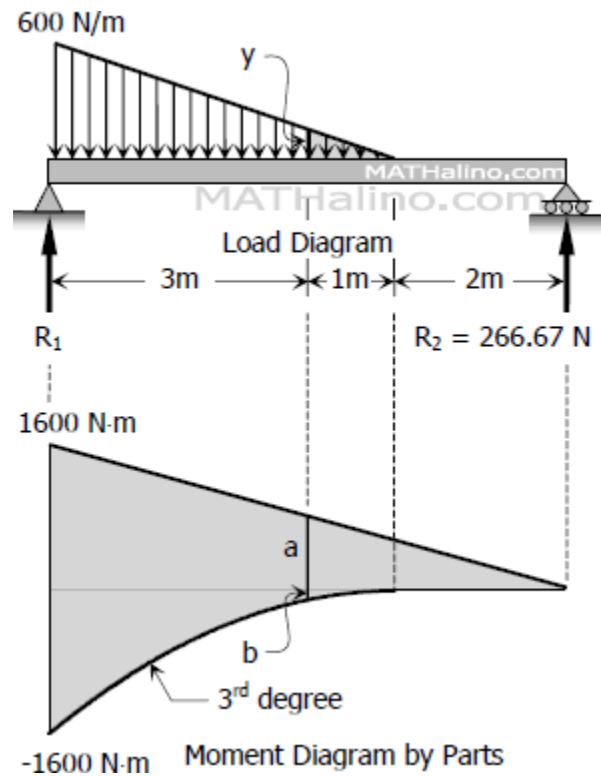
$$b = -25 \text{ N} \cdot \text{m}$$

From the conjugate beam

$$\Sigma M_{F_1} = 0$$

$$6F_2 + \frac{1}{4}(4)(1600)\left[\frac{1}{5}(4)\right] = 12(6)(1600)\left[\frac{1}{3}(6)\right]$$

$$F_2 = 1386.67 \text{ N} \cdot \text{m}^2$$



$$M_{midspan} = \frac{1}{2}(3a)\left[\frac{1}{3}(3)\right] - \frac{1}{4}(1b)\left[\frac{1}{5}(1)\right] - 3F_2$$

$$M_{midspan} = \frac{1}{2}(3)(800)\left[\frac{1}{3}(3)\right] - \frac{1}{4}(1)(25)\left[\frac{1}{5}(1)\right] - 3(1386.67)$$

$$M_{midspan} = -2961.25 \text{ N} \cdot \text{m}^3$$

Thus, the deflection at the midspan is

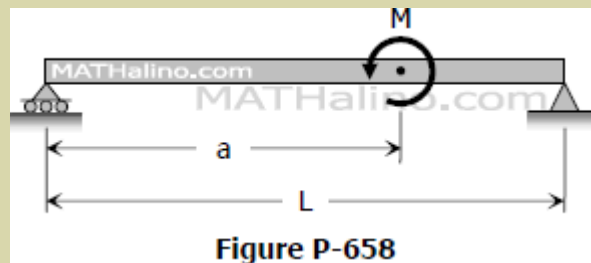
$$EI \delta_m = M_{midspan}$$

$$EI \delta_m = -2961.25 \text{ N} \cdot \text{m}^3$$

$$EI \delta_m = 2961.25 \text{ N} \cdot \text{m}^3 \text{ below the neutral axis} \quad \text{answer}$$

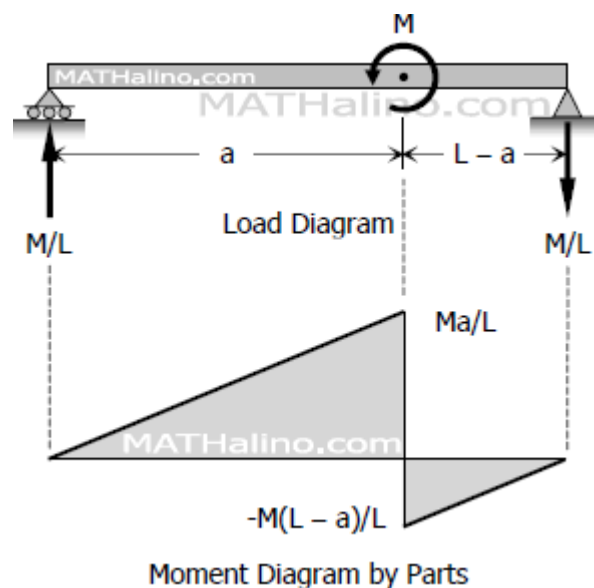
Problem 658

For the beam shown in Fig. P-658, find the value of $EI\delta$ at the point of application of the couple.



Solution 658

[Click here to show or hide the solution](#)



From the conjugate beam

$$\Sigma M_A = 0$$

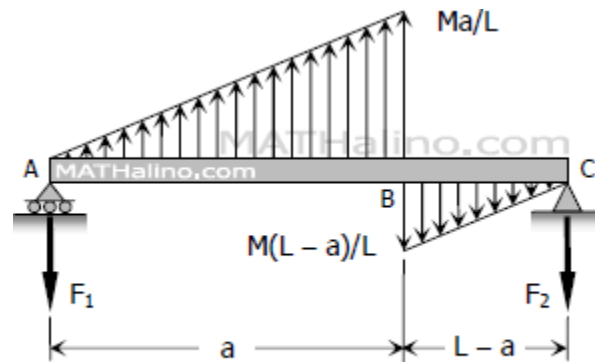
$$LF_2 + \frac{1}{2}(L-a)[M(L-a)/L][a + \frac{1}{3}(L-a)] = \frac{1}{2}a(Ma/L)[\frac{2}{3}a]$$

$$LF_2 + \frac{M(L-a)^2}{2L}[\frac{1}{3}L + \frac{2}{3}a] = \frac{Ma^3}{3L}$$

$$LF_2 + \frac{M(L-a)^2(L+2a)}{6L} = \frac{Ma^3}{3L}$$

$$LF_2 = \frac{Ma^3}{3L} - \frac{M(L-a)^2(L+2a)}{6L}$$

$$F_2 = \frac{Ma^3}{3L^2} - \frac{M(L-a)^2(L+2a)}{6L^2}$$



Conjugate Beam under M/EI Load

$$F_2 = \frac{M}{6L^2}[2a^3 - (L-a)^2(L+2a)]$$

$$F_2 = \frac{M}{6L^2}[2a^3 - (L^2 - 2aL + a^2)(L+2a)]$$

$$F_2 = \frac{M}{6L^2}[2a^3 - (L^3 - 2aL^2 + a^2L + 2aL^2 - 4a^2L + 2a^3)]$$

$$F_2 = \frac{M}{6L^2}[-L^3 + 3a^2L]$$

$$F_2 = \frac{M}{6L}(3a^2 - L^2)$$

$$M_B = -(L - a)F_2 - \frac{1}{2}(L - a)[M(L - a)/L][\frac{1}{3}(L - a)]$$

$$M_B = -(L - a)\frac{M}{6L}(3a^2 - L^2) - \frac{1}{2}(L - a)[M(L - a)/L][\frac{1}{3}(L - a)]$$

$$M_B = -\frac{M(L - a)}{6L}[(3a^2 - L^2) + (L - a)^2]$$

$$M_B = -\frac{M(L - a)}{6L}[3a^2 - L^2 + L^2 - 2aL + a^2]$$

$$M_B = -\frac{M(L - a)}{6L}(4a^2 - 2aL)$$

$$M_B = -\frac{M(L - a)}{6L}[-2a(L - 2a)]$$

$$M_B = \frac{Ma}{3L}(L - a)(L - 2a)$$

$$M_B = \frac{Ma}{3L}(L^2 - 3aL + 2a^2)$$

Thus,

$$EI \delta_B = \frac{Ma}{3L}(L^2 - 3aL + 2a^2) \quad \text{answer}$$

Strain Energy Method (Castigliano's Theorem) | Beam Deflection

Italian engineer Alberto Castigliano (1847 – 1884) developed a method of determining deflection of structures by strain energy method. His *Theorem of the Derivatives of Internal Work of Deformation* extended its application to the calculation of relative rotations and displacements between points in the structure and to the study of beams in flexure.

Energy of structure is its capacity of doing work and *strain energy* is the internal energy in the structure because of its deformation. By the principle of conservation of energy,

$$U = W_i$$

where U denotes the strain energy and W_i represents the work done by internal forces. The expression of strain energy depends therefore on the internal forces that can develop in the member due to applied external forces.

Castigliano's Theorem for Beam Deflection

For linearly elastic structures, the partial derivative of the strain energy with respect to an applied force (or couple) is equal to the displacement (or rotation) of the force (or couple) along its line of action.

$$\delta = \frac{\partial U}{\partial P} \quad \text{or} \quad \theta = \frac{\partial U}{\partial \bar{M}}$$

Where δ is the deflection at the point of application of force P in the direction of P , θ is the rotation at the point of application of the couple $\bar{M}$ in the direction of $\bar{M}$, and U is the strain energy.

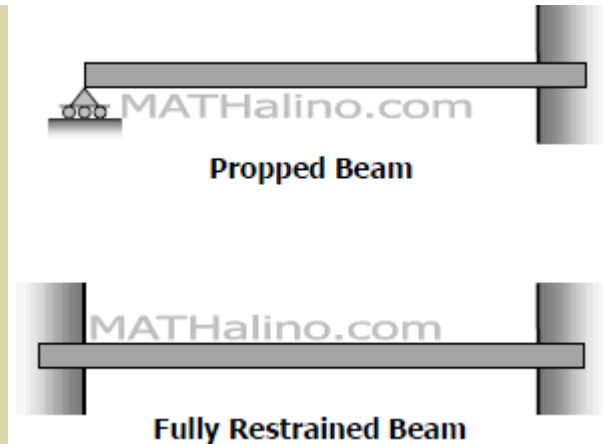
The strain energy of a beam was known to be $U = \int_0^L \frac{M^2}{2EI} dx$. Finding the partial derivative of this expression will give us the equations of Castigliano's deflection and rotation of beams. The equations are written below for convenience.

$$\delta = \int_0^L \left(\frac{\partial M}{\partial P} \right) \frac{M}{EI} dx \quad \text{and} \quad \theta = \int_0^L \left(\frac{\partial M}{\partial \bar{M}} \right) \frac{M}{EI} dx$$

Chapter 07 - Restrained Beams

Restrained Beams

In addition to the equations of static equilibrium, relations from the geometry of elastic curve are essential to the study of indeterminate beams. Such relations can be obtained from the study of deflection and rotation of beam. This section will focus on two types of indeterminate beams; the propped beams and the fully restrained beams.



A propped beam is fixed at one end and propped either at the other end or at any other point along its span. If the simple support is removed, propped beam will become cantilever beam. Fully restrained beam is fixed at both ends as shown in the figure above.

Deflection and Rotation of Propped Beam

Unless otherwise specified, the boundary conditions of propped beams are as follows.

- Deflection at both ends is zero.
- Rotation at fixed support is zero.

Deflection and Rotation of Fully Restrained Beam

Unless otherwise specified, the boundary conditions of propped beams are as follows.

- Deflection at both ends is zero.
- Rotation at both ends is zero.

Application of Double Integration and Superposition Methods to Restrained Beams

Superposition Method

There are 12 cases listed in the method of superposition for beam deflection.

- Cantilever beam with...
 1. concentrated load at the free end.
 2. concentrated load anywhere on the beam.
 3. uniform load over the entire span.

4. triangular load with zero at the free end

5. moment load at the free end.

- Simply supported beam with...

1. concentrated load at the midspan.

2. concentrated load anywhere on the beam span.

3. uniform load over the entire span.

4. triangular load which is zero at one end and full at the other end.

5. triangular load with zero at both ends and full at the midspan.

6. moment load at the right support.

7. moment load at the left support.

See [beam deflection by superposition method](#)

(The slope or deflection at any point on the beam is equal to the resultant of the slopes or deflections at that point caused by each of the load acting separately.)

Rotation and Deflection for Common Loadings

Case 1: Concentrated load at the free end of cantilever beam

Maximum Moment

$$M = -PL$$

Slope at end

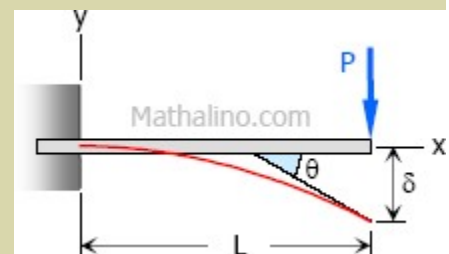
$$\theta = \frac{PL^2}{2EI}$$

Maximum deflection

$$\delta = \frac{PL^3}{3EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Px^2}{6}(3L - x)$$



Case 2: Concentrated load at any point on the span of cantilever beam

Maximum Moment

$$M = -Pa$$

Slope at end

$$\theta = \frac{Pa^2}{2EI}$$

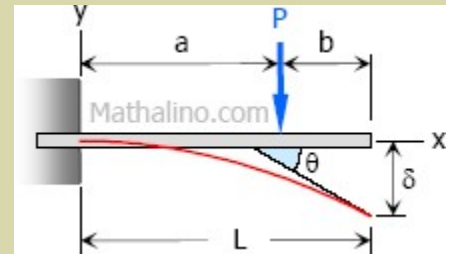
Maximum deflection

$$\delta = \frac{Pa^3}{6EI}(3L - a)$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Px^2}{6}(3a - x) \text{ for } 0 < x < a$$

$$EI y = \frac{Pa^2}{6}(3x - a) \text{ for } a < x < L$$



Case 3: Uniformly distributed load over the entire length of cantilever beam

Maximum Moment

$$M = -\frac{w_o L^2}{2}$$

Slope at end

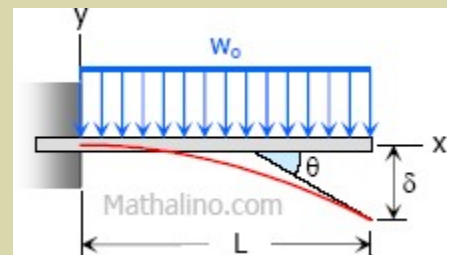
$$\theta = \frac{w_o L^3}{6EI}$$

Maximum deflection

$$\delta = \frac{w_o L^4}{8EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{w_o x^2}{120L}(6L^2 - 4Lx + x^2)$$



Case 4: Triangular load, full at the fixed end and zero at the free end, of cantilever beam

Maximum Moment

$$M = -\frac{w_o L^2}{6}$$

Slope at end

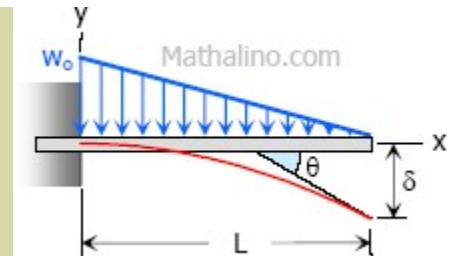
$$\theta = \frac{w_o L^3}{24EI}$$

Maximum deflection

$$\delta = \frac{w_o L^4}{30EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{w_o x^2}{120L} (10L^3 - 10L^2 x + 5Lx^2 - x^3)$$



Case 5: Moment load at the free end of cantilever beam

Maximum Moment

$$M = -M$$

Slope at end

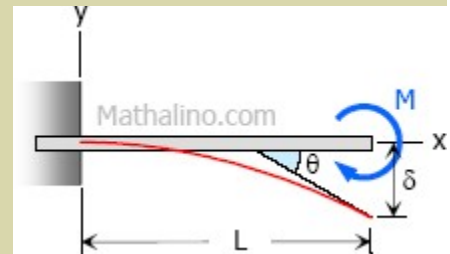
$$\theta = \frac{ML}{EI}$$

Maximum deflection

$$\delta = \frac{ML^2}{2EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Mx^2}{2}$$



Case 6: Concentrated load at the midspan of simple beam

Maximum Moment

$$M = \frac{PL}{4}$$

Slope at end

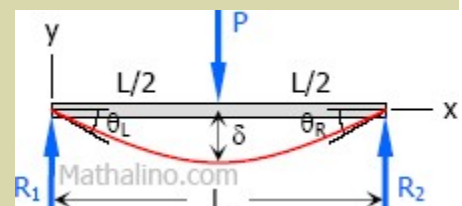
$$\theta_L = \theta_R = \frac{PL^2}{16EI}$$

Maximum deflection

$$\delta = \frac{PL^3}{48EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Px}{12} \left(\frac{3}{4}L^2 - x^2 \right) \text{ for } 0 < x < \frac{1}{2}L$$



Case 7: Concentrated load at any point on simple beam

Maximum Moment

$$M = \frac{Pab}{L} \text{ at } x = a$$

Slope at end

$$\theta_L = \frac{Pb(L^2 - b^2)}{6EIL}$$

$$\theta_R = \frac{Pa(L^2 - a^2)}{6EIL}$$

Maximum deflection

$$\delta = \frac{Pb(L^2 - b^2)^{3/2}}{9\sqrt{3}EIL} \text{ at } x = \sqrt{\frac{L^2 - b^2}{3}}$$

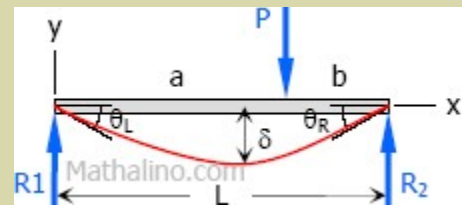
Deflection at the center (not maximum)

$$\delta = \frac{Pb}{48EI}(3L^2 - 4b^2) \text{ when } a > b$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Pbx}{6L}(L^2 - x^2 - b^2) \text{ for } 0 < x < a$$

$$EI y = \frac{Pb}{6L} \left[\frac{L}{b}(x - a)^3 + (L^2 - b^2)x - x^3 \right] \text{ for } a < x < L$$



Case 8: Uniformly distributed load over the entire span of simple beam

Maximum Moment

$$M = \frac{w_o L^2}{8}$$

Slope at end

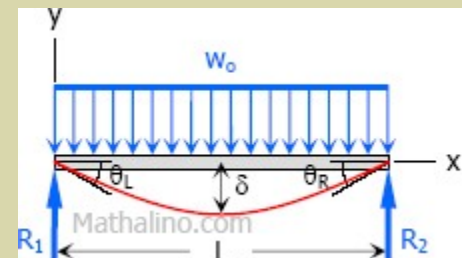
$$\theta_L = \theta_R = \frac{w_o L^3}{24EI}$$

Maximum deflection

$$\delta = \frac{5w_o L^4}{384EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{w_o x}{24}(L^3 - 2Lx^2 + x^3)$$



Case 9: Triangle load with zero at one support and full at the other support of simple beam

Maximum Moment

$$M = \frac{w_o L^2}{9\sqrt{3}}$$

Slope at end

$$\theta_L = \frac{7w_o L^3}{360EI}$$

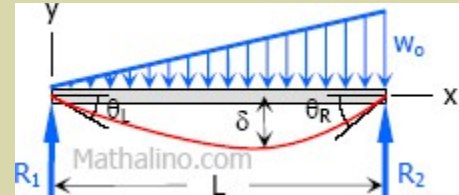
$$\theta_R = \frac{8w_o L^3}{360EI}$$

Maximum deflection

$$\delta = \frac{2.5w_o L^4}{384EI} \text{ at } x = 0.519L$$

Deflection Equation (y is positive downward)

$$EI y = \frac{w_o x}{360L} (7L^4 - 10L^2 x^2 + 3x^4)$$



Case 10: Triangular load with zero at each support and full at the midspan of simple beam

Maximum Moment

$$M = \frac{w_o L^2}{12}$$

Slope at end

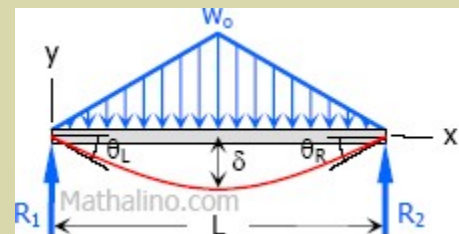
$$\theta_L = \theta_R = \frac{5w_o L^3}{192EI}$$

Maximum deflection

$$\delta = \frac{w_o L^4}{120EI}$$

Deflection Equation (y is positive downward)

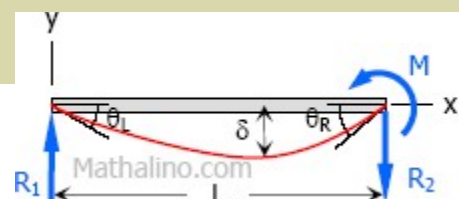
$$EI y = \frac{w_o x}{960L} (25L^4 - 40L^2 x^2 + 16x^4) \text{ for } 0 < x < \frac{L}{2}$$



Case 11: Moment load at the right support of simple beam

Maximum Moment

$$M = M$$



Slope at end

$$\theta_L = \frac{ML}{6EI}$$

$$\theta_R = \frac{ML}{3EI}$$

Maximum deflection

$$\delta = \frac{ML^2}{9\sqrt{3}EI} \text{ at } x = \frac{L}{\sqrt{3}}$$

Deflection at the center (not maximum)

$$\delta = \frac{ML^2}{16EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{MLx}{6} \left(1 - \frac{x^2}{L^2} \right)$$

Case 12: Moment load at the left support of simple beam

Maximum Moment

$$M = M$$

Slope at end

$$\theta_L = \frac{ML}{3EI}$$

$$\theta_R = \frac{ML}{6EI}$$

Maximum deflection

$$\delta = \frac{ML^2}{9\sqrt{3}EI} \text{ at } x = \left(L - \frac{L}{\sqrt{3}} \right)$$

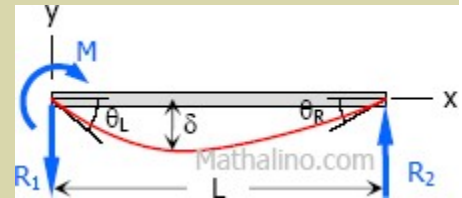
Deflection at the center (not maximum)

$$\delta = \frac{ML^2}{16EI}$$

Deflection Equation (y is positive downward)

$$EI y = \frac{Mx}{6L} (L - x)(2L - x)$$

)for details.



Double Integration Method

Moment at any exploratory section

$$EIy'' = M$$

Slope of the beam at any point

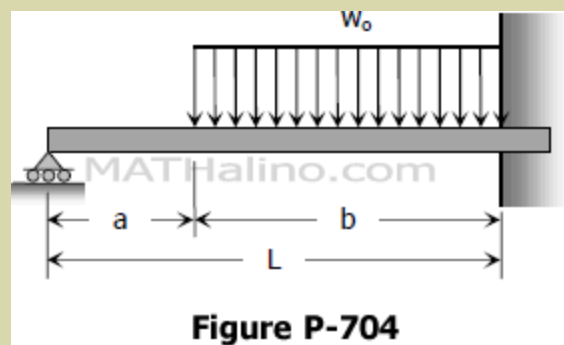
$$EIy' = \int M + C_1$$

Deflection of beam at any point

$$EIy = \int \int M + C_1x + C_2$$

Problem 704

Find the reactions at the supports and draw the shear and moment diagrams of the propped beam shown in Fig. P-704.



Solution by Double Integration Method

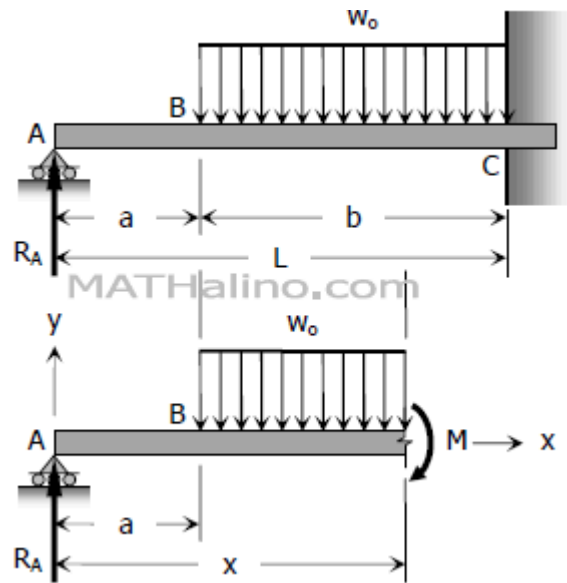
[HideClick here to show or hide the solution](#)

$$EI y'' = M$$

$$EI y'' = R_A x - \frac{1}{2} w_o \langle x-a \rangle^2$$

$$EI y' = \frac{1}{2} R_A x^2 - \frac{1}{6} w_o \langle x-a \rangle^3 + C_1$$

$$EI y = \frac{1}{6} R_A x^3 - \frac{1}{24} w_o \langle x-a \rangle^4 + C_1 x + C_2$$



Apply boundary conditions to solve for integration constants C_1 and C_2 :

At $x = 0$, $y = 0$, hence $C_2 = 0$

At $x = L$, $y = 0$, hence

$$0 = \frac{1}{6} R_A L^3 - \frac{1}{24} w_o (L-a)^4 + C_1 L + 0$$

$$C_1 L = \frac{1}{24} w_o (L-a)^4 - \frac{1}{6} R_A L^3$$

$$C_1 = \frac{w_o}{24L} (L-a)^4 - \frac{R_A L^2}{6}$$

At $x = L$, $y' = 0$, hence

$$0 = \frac{1}{2} R_A L^2 - \frac{1}{6} w_o (L-a)^3 + \frac{w_o}{24L} (L-a)^4 - \frac{R_A L^2}{6}$$

$$\frac{R_A L^2}{2} - \frac{w_o}{6} (L-a)^3 + \frac{w_o}{24L} (L-a)^4 - \frac{R_A L^2}{6} = 0$$

$$\frac{R_A L^2}{3} = \frac{w_o}{6}(L-a)^3 - \frac{w_o}{24L}(L-a)^4$$

$$R_A = \frac{w_o}{2L^2}(L-a)^3 - \frac{w_o}{8L^3}(L-a)^4$$

$$R_A = \frac{w_o b^3}{2L^2} - \frac{w_o b^4}{8L^3}$$

$$R_A = \frac{w_o b^3}{8L^3}(4L - b) \quad \text{answer}$$

Solution by Superposition Method

[Click here to show or hide the solution](#)

Deflection at A is zero. Thus, the deflection due to R_A denoted by δ_1 is equal to the sum of deflection at B denoted as δ_2 and the vertical deflection at A due to rotation of B which is denoted by θ .

$$\delta_1 = \delta_2 + a\theta$$

$$\frac{R_A L^3}{3EI} = \frac{w_o b^4}{8EI} + a \left(\frac{w_o b^3}{6EI} \right)$$

$$\frac{R_A L^3}{3} = \frac{w_o b^4}{8} + \frac{w_o a b^3}{6}$$

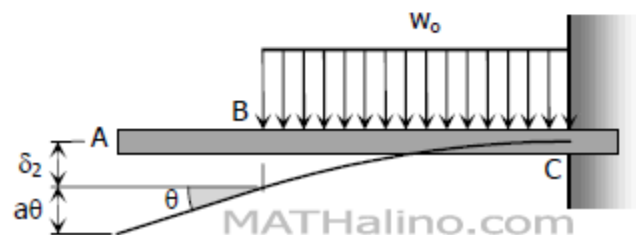
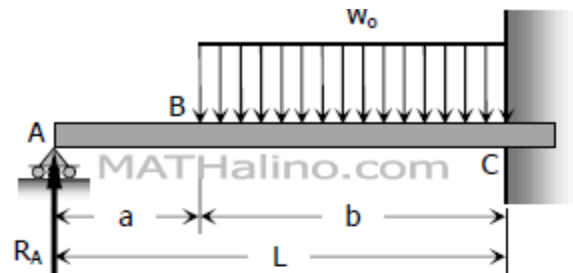
$$R_A = \frac{3w_o b^4}{8L^3} + \frac{w_o a b^3}{2L^3}$$

$$R_A = \frac{w_o b^3}{L^3} \left(\frac{3b}{8} + \frac{a}{2} \right)$$

$$R_A = \frac{w_o b^3}{L^3} \left(\frac{3b}{8} + \frac{L-b}{2} \right)$$

$$R_A = \frac{w_o b^3}{L^3} \left(\frac{3b + 4L - 4b}{8} \right)$$

$$R_A = \frac{w_o b^3}{8L^3}(4L - b) \quad \text{answer}$$



Shear and Moment Diagrams

[HideClick here to show or hide the solution](#)

The reaction at the simple support R_A was solved by two different methods above.

$$R_A = \frac{w_o b^3}{8L^3}(4L-b) \quad \text{answer}$$

With value of R_A known, it is now easy to solve for M_C and R_C .

Solving for fixed-end moment

$$M_C = R_A L - 0.5w_o b^2$$

$$M_C = \frac{w_o b^3}{8L^2}(4L-b) - \frac{w_o b^2}{2}$$

$$M_C = \frac{w_o b^2}{8L^2}[b(4L-b) - 4L^2]$$

$$M_C = \frac{w_o b^2}{8L^2}[4Lb - b^2 - 4L^2]$$

$$M_C = -\frac{w_o b^2}{8L^2}(4L^2 - 4Lb + b^2) \quad \text{answer}$$

Solving for fixed-end shear

$$\Sigma F_V = 0$$

$$R_C + R_A = w_o b$$

$$R_C + \frac{w_o b^3}{8L^3}(4L-b) = w_o b$$

$$R_C = -\frac{w_o b^3}{8L^3}(4L-b) + w_o b$$

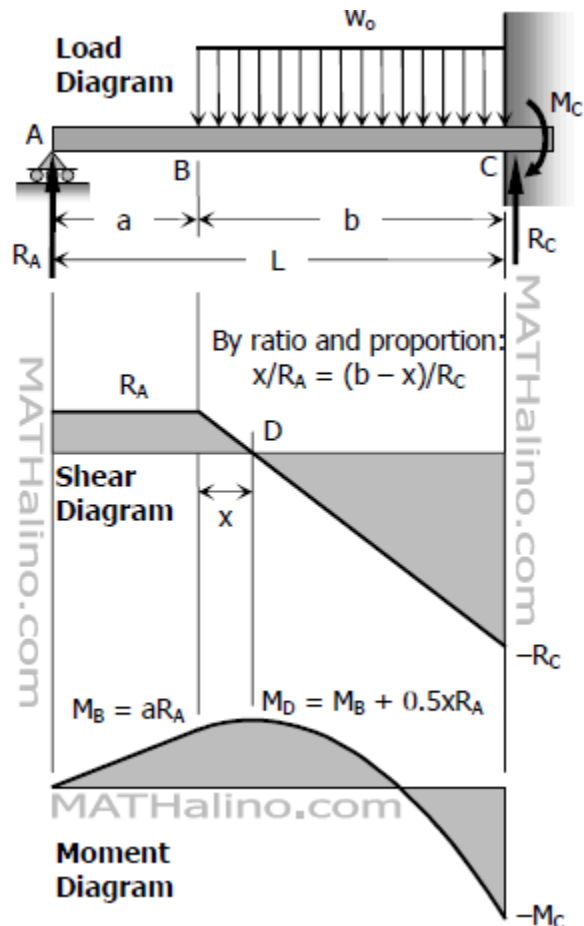
$$R_C = \frac{w_o b}{8L^3}[-b^2(4L-b) + 8L^3]$$

$$R_C = \frac{w_o b}{8L^3} [-4Lb^2 + b^3 + 8L^3]$$

$$R_C = \frac{w_o b}{8L^3} (8L^3 - 4Lb^2 + b^3) \quad \text{answer}$$

Shape of Shear Diagram

1. The shear at A is R_A
2. There is no load between segment BC, thus, the shear over BC is uniform and equal to R_A .
3. The load over segment CD is uniform and downward, thus, the shear on this segment is linear and decreasing from R_A to $-R_C$.
4. The shear diagram between BC is zero at D. The location of D can be found by ratio and proportion of the shear triangles of segments BD and DC.



Shape of Moment Diagram

1. The moment at A is zero.
2. The shear between AB is uniform and positive, thus the moment between AB is linearly increasing (straight line) from zero to aR_A .
3. The moment at B is aR_A which is equal to the area of the shear diagram of segment BC.
4. The shear between BC is linear with zero at point D, thus the moment diagram of segment BC is a second degree curve (parabola) with vertex at D. The parabola is open downward because the shear from B to C is decreasing.
5. The moment at D is equal to the sum of the moment at B and the area of the shear diagram of segment BD.

- Finally, the moment at C is equal to the moment at D minus the area of the shear diagram of segment DC. You can check the accuracy of the moment diagram by finding the moment at C in the load diagram.

In the event that you need to determine the location of zero moment (for construction joint most probably), compute for the area of the moment diagram from D to C and use the squared property of parabola to locate the zero moment. An easier way to find the point of zero moment is to write the moment equation (similar to what we did in double integration method above), equate the equation to zero and solve for x .

Problem 705

Find the reaction at the simple support of the propped beam shown in Fig. P-705 and sketch the shear and moment diagrams.

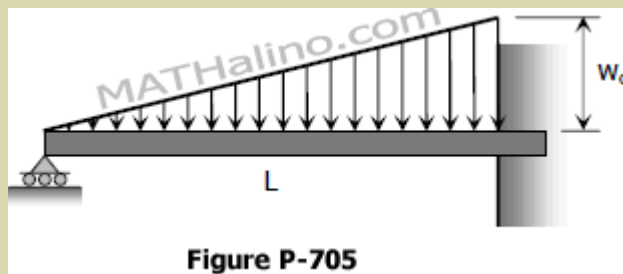


Figure P-705

Solution of Propped Reaction by Double Integration Method

[HideClick here to show or hide the solution](#)

$$\frac{y}{x} = \frac{w_o}{L}$$

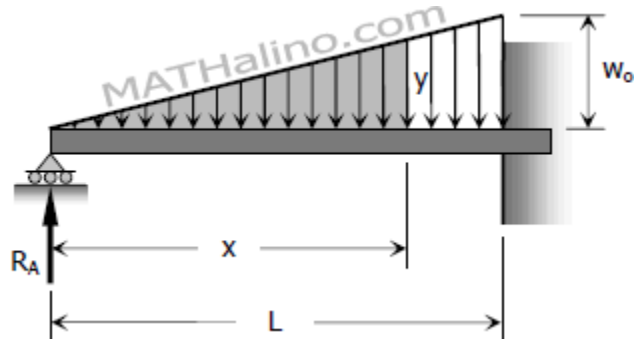
$$y = \frac{w_o x}{L}$$

Moment at x :

$$M = R_A x - \frac{1}{2} x y \left(\frac{1}{3} x \right)$$

$$M = R_A x - \frac{1}{6} x^2 y$$

$$M = R_A x - \frac{x^2}{6} \left(\frac{w_o x}{L} \right)$$



$$M = R_A x - \frac{w_o x^3}{6L}$$

Thus,

$$EI y'' = R_A x - \frac{w_o x^3}{6L}$$

$$EI y' = \frac{R_A x^2}{2} - \frac{w_o x^4}{24L} + C_1$$

$$EI y = \frac{R_A x^3}{6} - \frac{w_o x^5}{120L} + C_1 x + C_2$$

At $x = 0, y = 0$, thus $C_2 = 0$

At $x = L, y' = 0$

$$0 = \frac{R_A L^2}{2} - \frac{w_o L^4}{24L} + C_1$$

$$C_1 = \frac{w_o L^3}{24} - \frac{R_A L^2}{2}$$

Thus, the deflection equation is

$$EI y = \frac{R_A x^3}{6} - \frac{w_o x^5}{120L} + \left(\frac{w_o L^3}{24} - \frac{R_A L^2}{2} \right) x$$

At $x = L, y = 0$

$$0 = \frac{R_A L^3}{6} - \frac{w_o L^5}{120L} + \left(\frac{w_o L^3}{24} - \frac{R_A L^2}{2} \right) L$$

$$0 = \frac{R_A L^3}{6} - \frac{w_o L^4}{120} + \frac{w_o L^4}{24} - \frac{R_A L^3}{2}$$

$$0 = -\frac{R_A L^3}{3} + \frac{w_o L^4}{30}$$

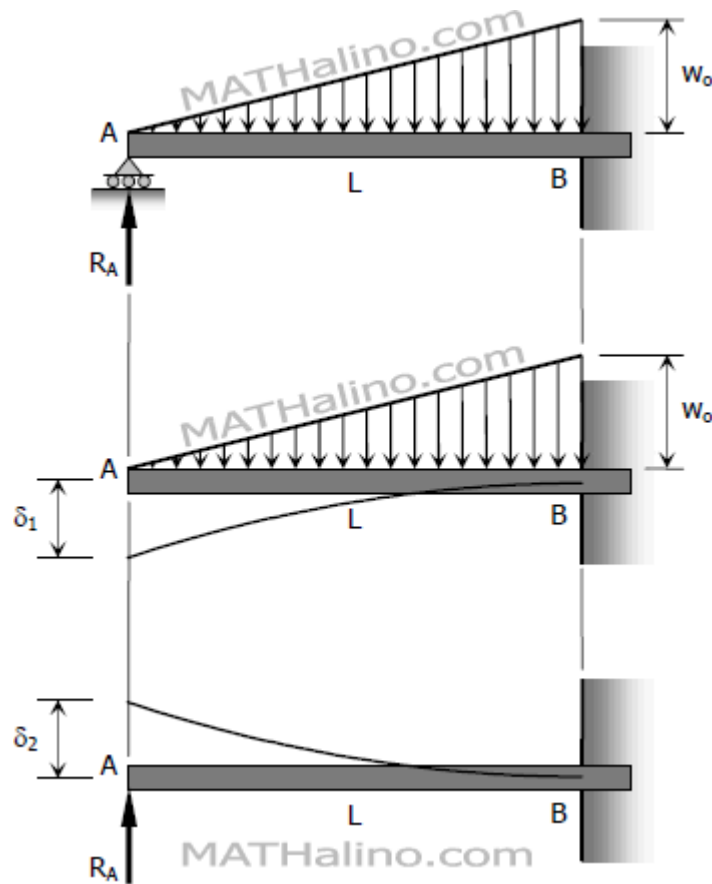
$$\frac{R_A L^3}{3} = \frac{w_o L^4}{30}$$

$$R_A = \frac{w_o L}{10} \quad \text{answer}$$

Solution of Propped Reaction by the Method of Superposition

[Click here to show or hide the solution](#)

Resolve the propped beam into two cantilever beams, one with uniformly varying load and the other with concentrated load as shown below. The concentrated load is the reaction at A.



The deflection at A is zero. Thus, by superposition method, the deflection due to triangular load is equal to the deflection due to concentrated load.

$$\delta_1 = \frac{w_o L^4}{30EI} \rightarrow \text{deflection due to triangular load}$$

$$\delta_2 = \frac{R_A L^3}{3EI} \rightarrow \text{deflection due to concentrated load}$$

$$\delta_1 = \delta_2$$

$$\frac{w_o L^4}{30EI} = \frac{R_A L^3}{3EI}$$

$$R_A = \frac{w_o L}{10} \quad \text{answer}$$

Sketching the Shear and Moments Diagrams

[HideClick here to show or hide the solution](#)

From the solutions above,

$$R_A = \frac{w_o L}{10}$$

Solving for reaction at B, R_B

$$\Sigma F_V = 0$$

$$R_A + R_B = \frac{1}{2} L w_o$$

$$\frac{w_o L}{10} + R_B = \frac{w_o L}{2}$$

$$R_B = \frac{2w_o L}{5} \quad \text{answer}$$

Solving for moment at B, M_B

$$M_B = R_A L - \frac{1}{2} L w_o \left(\frac{1}{3} L \right)$$

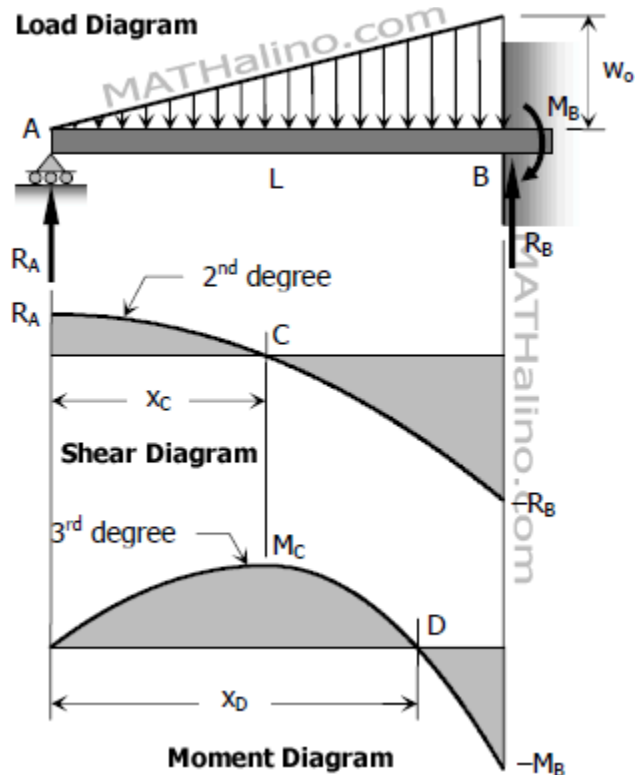
$$M_B = \left(\frac{w_o L}{10} \right) L - \frac{w_o L^2}{6}$$

$$M_B = \frac{w_o L^2}{10} - \frac{w_o L^2}{6}$$

$$M_B = -\frac{w_o L^2}{15} \quad \text{answer}$$

To Draw the Shear Diagram

1. The shear at A is equal to R_A
2. The load between AB is negatively increasing from zero at A to w_o at B, thus, the slope of the shear diagram between A and B is decreasing from zero at A to $-w_o$ at B.
3. The load between AB is linear, thus, the shear diagram between AB is a parabola (2nd degree curve) with vertex at A and open downward as stated with the decreasing slope in number 2.
4. The shear at B is equal to $-R_B$. See the magnitude of R_B in the solution above. To compute; shear at B = shear at A - load between AB.



5. The shear diagram between AB is zero at C as shown. Location of C, denoted by x_C can be found by squared property of parabola as follows.

$$\frac{x_C^2}{R_A} = \frac{L^2}{R_A + R_B}$$

$$x_C^2 = \frac{R_A L^2}{R_A + R_B}$$

$$x_C^2 = \frac{(\frac{1}{10} w_o L) L^2}{\frac{1}{10} w_o L + \frac{2}{5} w_o L}$$

$$x_C^2 = \frac{\frac{1}{10} w_o L^3}{\frac{1}{2} w_o L}$$

$$x_C^2 = \frac{1}{5}L^2$$

$$x_C = \frac{1}{\sqrt{5}}L \quad \text{from left support}$$

To Draw the Moment Diagram

1. The shear diagram from A to B is a 2nd degree curve, thus, the moment diagram between A and B is a third degree curve.
2. The moment at A is zero.
3. Moment at C is equal to the moment A plus the area of shear diagram between A and C.

$$M_C = \frac{2}{3}x_C R_A$$

$$M_C = \frac{2}{3}\left(\frac{1}{\sqrt{5}}L\right)\left(\frac{1}{10}w_oL\right)$$

$$M_C = \frac{w_oL^2}{15\sqrt{5}}$$

4. The moment at B is equal to $-M_B$, see the magnitude of M_B from the solution above. It is more easy to compute the moment at B by using the load diagram instead of shear diagram. In case, you need to solve the moment at B by the use of shear diagram; $M_B = M_C - \text{Area of shear between CB}$. You can follow the link for an [example of finding the area of shear diagram](#) of similar shape.
5. The moment is zero at point D. To locate this point, equate the moment equation developed in double integration method to zero.

$$M = R_Ax - \frac{w_o x^3}{6L} \quad \text{See double integration method above for finding M.}$$

$$M = \frac{w_o Lx}{10} - \frac{w_o x^3}{6L}$$

At D, $x = x_D$ and $M = 0$

$$0 = \frac{w_o Lx_D}{10} - \frac{w_o x_D^3}{6L}$$

$$\frac{w_o x_D^3}{6L} = \frac{w_o L x_D}{10}$$

$$\frac{x_D^2}{6L} = \frac{L}{10}$$

$$x_D^2 = \frac{3L^2}{5}$$

$$x_D = \frac{\sqrt{3}L}{\sqrt{5}} = \frac{\sqrt{3}L}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$x_D = \frac{\sqrt{15}}{5}L \text{ from left support}$$

Example 03

The propped beam shown in Fig. P -706 is loaded by decreasing triangular load varying from w_o from the simple end to zero at the fixed end. Find the support reactions and sketch the shear and moment diagrams

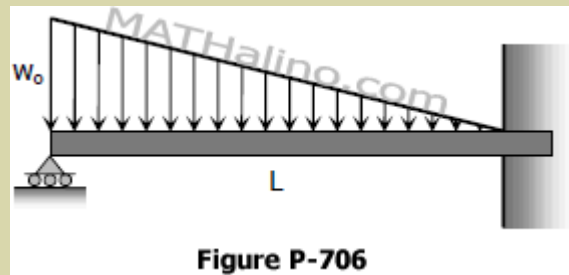


Figure P-706

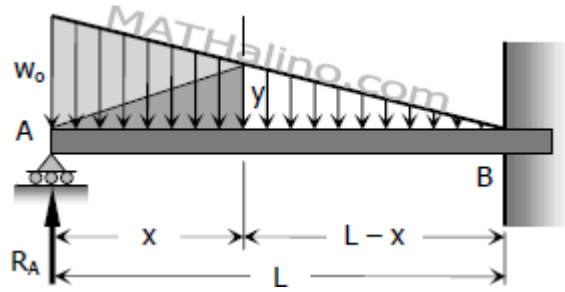
Solution by Double Integration Method

[HideClick here to show or hide the solution](#)

By ratio and proportion:

$$\frac{y}{L-x} = \frac{w_o}{L}$$

$$y = \frac{w_o}{L}(L-x)$$



Solving for moment equation

$$M = R_A x - \frac{1}{2}(x)(w_o)\left(\frac{2}{3}x\right) - \frac{1}{2}(x)(y)\left(\frac{1}{3}x\right)$$

$$M = R_A x - \frac{1}{3}w_o x^2 - \frac{1}{6}x^2 y$$

$$M = R_A x - \frac{1}{3}w_o x^2 - \frac{1}{6}x^2 \left[\frac{w_o}{L}(L - x) \right]$$

$$M = R_A x - \frac{w_o}{3}x^2 - \frac{w_o}{6L}(Lx^2 - x^3)$$

Doing the double integration

$$EI y'' = R_A x - \frac{w_o}{3}x^2 - \frac{w_o}{6L}(Lx^2 - x^3)$$

$$EI y' = \frac{R_A}{2}x^2 - \frac{w_o}{9}x^3 - \frac{w_o}{6L} \left(\frac{Lx^3}{3} - \frac{x^4}{4} \right) + C_1$$

$$EI y = \frac{R_A}{6}x^3 - \frac{w_o}{36}x^4 - \frac{w_o}{6L} \left(\frac{Lx^4}{12} - \frac{x^5}{20} \right) + C_1 x + C_2$$

Boundary conditions

At $x = 0$, $y = 0$, $C_2 = 0$

At $x = L$, $y = 0$

$$0 = \frac{R_A L^3}{6} - \frac{w_o L^4}{36} - \frac{w_o}{6L} \left(\frac{L^4}{12} - \frac{L^4}{20} \right) + C_1 L + 0$$

$$0 = \frac{R_A L^3}{6} - \frac{w_o L^4}{36} - \frac{w_o L^4}{180} + C_1 L$$

$$C_1 = \frac{w_o L^3}{30} - \frac{R_A L^2}{6}$$

At $x = L$, $y' = 0$

$$0 = \frac{R_A L^2}{2} - \frac{w_o L^3}{9} - \frac{w_o}{6L} \left(\frac{L^4}{3} - \frac{L^4}{4} \right) + \left(\frac{w_o L^3}{30} - \frac{R_A L^2}{6} \right)$$

$$0 = \frac{R_A L^2}{2} - \frac{w_o L^3}{9} - \frac{w_o L^3}{72} + \frac{w_o L^3}{30} - \frac{R_A L^2}{6}$$

$$\frac{R_A L^2}{3} = \frac{11 w_o L^3}{120}$$

$$R_A = \frac{11 w_o L}{40} \quad \text{answer}$$

Solution by Superposition Method

[HideClick here to show or hide the solution](#)

Decreasing triangular load is not listed in the [summary of beam loadings](#). It is therefore necessary to resolve this load into loads that are in the list. In this case, the load was resolved into uniformly distributed load and upward triangular load as shown. The sum of such loads is equal to the one that is given in the problem.

From the figure, it is clear that

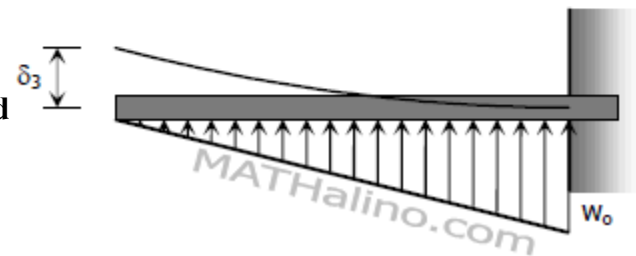
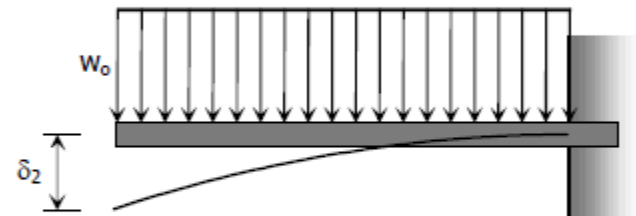
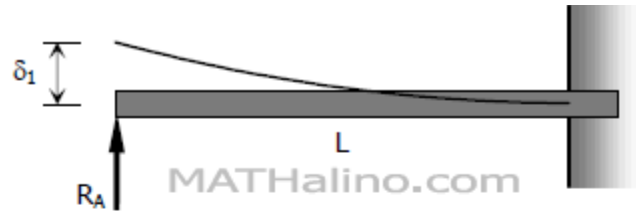
$$\delta_1 + \delta_3 = \delta_2$$

$$\frac{R_A L^3}{3EI} + \frac{w_o L^4}{30EI} = \frac{w_o L^4}{8EI}$$

$$\frac{R_A}{3} + \frac{w_o L}{30} = \frac{w_o L}{8}$$

$$\frac{R_A}{3} = \frac{11w_o L}{120}$$

$$R_A = \frac{11w_o L}{40} \quad \text{answer}$$



Superposition Method Using Point Load and Integration

[Hide](#) [Click here to show or hide the solution](#)

Another way of solving the reaction at A is by the use of derivative. From [Case No. 2](#), the deflection at the free end of cantilever beam is

$$\delta = \frac{Pa^2}{6EI} (3L - a)$$

For this problem $P = y \, dx$, $a = L - x$, and $\delta = d\delta_2$.

$$d\delta_2 = \frac{y \, dx (L - x)^2}{6EI} [3L - (L - x)]$$

From the figure,

$$\frac{y}{L - x} = \frac{w_o}{L}$$

$$y = \frac{w_o}{L}(L - x)$$

Thus,

$$d\delta_2 = \frac{\left[\frac{w_o}{L}(L - x)\right](L - x)^2}{6EI} [3L - (L - x)] dx$$

$$d\delta_2 = \frac{w_o(L - x)^3}{6LEI} [3L - (L - x)] dx$$

$$d\delta_2 = \frac{w_o}{2EI}(L - x)^3 dx - \frac{w_o}{6LEI}(L - x)^4 dx$$

$$\delta_2 = \frac{w_o}{2EI} \int_0^L (L - x)^3 dx - \frac{w_o}{6LEI} \int_0^L (L - x)^4 dx$$

δ_1 = end deflection due to R_A

$$\delta_1 = \frac{R_A L^3}{3EI}$$

$$\delta_1 = \delta_2$$

$$\frac{R_A L^3}{3EI} = \frac{w_o}{2EI} \int_0^L (L - x)^3 dx - \frac{w_o}{6LEI} \int_0^L (L - x)^4 dx$$

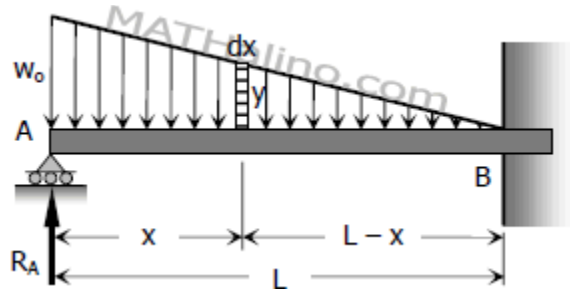
$$R_A = \frac{3w_o}{2L^3} \int_0^L (L - x)^3 dx - \frac{w_o}{2L^4} \int_0^L (L - x)^4 dx$$

$$R_A = \frac{3w_o}{2L^3} \left[-\frac{(L - x)^4}{4} \right] - \frac{w_o}{2L^4} \left[-\frac{(L - x)^5}{5} \right]$$

$$R_A = -\frac{3w_o}{8L^3} [0^4 - L^4] + \frac{w_o}{10L^4} [0^5 - L^5]$$

$$R_A = \frac{3w_o L}{8} - \frac{w_o L}{10}$$

$$R_A = \frac{11w_o L}{40} \quad \text{answer}$$



Shear and Moment Diagrams

[HideClick here to show or hide the solution](#)

The reaction at the simple support R_A was solved using two different methods above.

$$R_A = \frac{11w_oL}{40} \quad \text{answer}$$

Solving for vertical reaction at B
 $\Sigma F_V = 0$

$$R_A + R_B = \frac{1}{2}w_oL$$

$$\frac{11w_oL}{40} + R_B = \frac{w_oL}{2}$$

$$R_B = \frac{9w_oL}{40} \quad \text{answer}$$

Solving for moment reaction at B

$$M_B = R_AL - \frac{1}{2}w_oL\left(\frac{2}{3}L\right)$$

$$M_B = \left(\frac{9w_oL}{40}\right)L - \frac{w_oL^2}{3}$$

$$M_B = \frac{9w_oL^2}{40} - \frac{w_oL^2}{3}$$

$$M_B = -\frac{7w_oL^2}{120} \quad \text{answer}$$

To Draw the Shear Diagram

1. The shear at A is equal to R_A

- The load at AB is increasing from $-w_o$ at A to zero at B, thus, the slope of shear diagram from A to B is also increasing from $-w_o$ at A to zero at B.
- The load between AB is 1st degree (linear), thus, the shear diagram between AB is 2nd degree (parabolic) with vertex at B and open upward.
- The magnitude of shear at B is equal to $-R_B$. It is equal to the shear at A minus the triangular load between AB. See the magnitude of R_B above.
- The shear diagram from A to B will become zero somewhere along AB, the point is denoted by C in the figure. To locate point C, two solutions are presented below.

Location of C, the point of zero shear

Point C is the location of zero shear which may also be the location of maximum moment.

By squared property of parabola

$$\frac{(L - x_C)^2}{R_B} = \frac{L^2}{R_A + R_B}$$

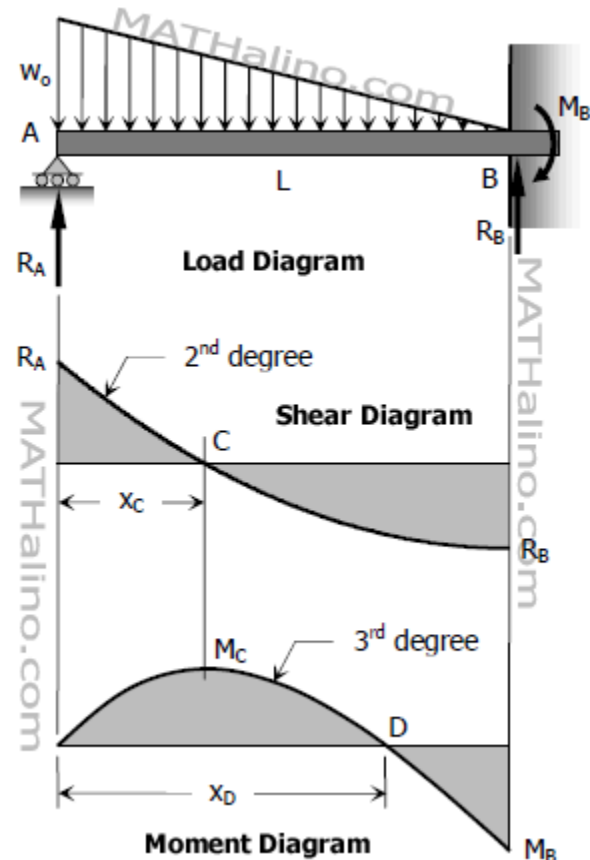
$$(L - x_C)^2 = \frac{R_B L^2}{R_A + R_B}$$

$$(L - x_C)^2 = \frac{\left(\frac{9}{40}w_o L\right)L^2}{\frac{11}{40}w_o L + \frac{9}{40}w_o L}$$

$$(L - x_C)^2 = \frac{\frac{9}{40}w_o L^3}{\frac{1}{2}w_o L}$$

$$(L - x_C)^2 = \frac{9}{20}L^2$$

$$L - x_C = \pm \sqrt{\frac{3}{20}}L$$



$$x_C = \left(1 \pm \frac{3}{\sqrt{20}}\right) L$$

$$x_C = 1.6708L \quad (\text{absurd})$$

$$x_C = 0.3292L \quad \text{answer}$$

By shear equation

Another method of solving for x_C is to pass an exploratory section anywhere on AB and sum up all the vertical forces to the left of the exploratory section. The location of x_C is where the sum of all vertical forces equate to zero.

Consider the figure shown to the right. Note that this figure is the same figure we used to find the reaction R_A by double integration method shown above.

The double integration method shows the relationship of x and y which

$$\text{is } y = \frac{w_o}{L}(L - x).$$

Sum of all vertical forces

$$\Sigma F_V = R_A - \frac{1}{2}w_o x - \frac{1}{2}xy$$

$$\Sigma F_V = \frac{11w_o L}{40} - \frac{w_o x}{2} - \frac{w_o x}{2L}(L - x)$$

$$\Sigma F_V = \frac{11w_o L}{40} - \frac{w_o x}{2} - \frac{w_o x}{2} + \frac{w_o x^2}{2L}$$

$$\Sigma F_V = \frac{11w_o L}{40} - w_o x + \frac{w_o x^2}{2L}$$

At point C, $\Sigma F_V = 0$ and $x = x_C$

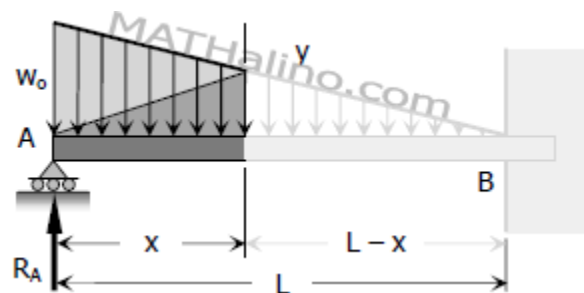
$$0 = \frac{11w_o L}{40} - w_o x_C + \frac{w_o x_C^2}{2L}$$

$$0 = \frac{11L}{40} - x_C + \frac{x_C^2}{2L}$$

$$0 = \frac{11}{20}L^2 - 2Lx_C + x_C^2$$

$$x_C^2 - 2Lx_C = -\frac{11}{20}L^2$$

$$x_C^2 - 2Lx_C + L^2 = -\frac{11}{20}L^2 + L^2$$



$$(x_C - L)^2 = \frac{9}{20} L^2$$

$$x_C - L = \pm \frac{3}{\sqrt{20}} L$$

$$x_C = \left(1 \pm \frac{3}{\sqrt{20}} \right) L$$

$$x_C = 1.6708L \quad (\text{absurd})$$

$$x_C = 0.3292L \quad \text{answer}$$

To Draw the Moment Diagram

1. The moment at simple support A is zero.
2. The shear diagram of AB is 2nd degree curve, thus, the moment diagram between AB is 3rd degree curve.
3. The moment at C can be computed in two ways; (a) by solving the area of shear diagram between A and C, and (b) by using the moment equation. For method (a), see the following links for similar situation of solving a partial area of parabolic spandrel.
 - [Simple beam with triangular load](#)
 - [Simple beam with rectangular and trapezoidal loads](#)

The solution below is using the approach mentioned in (b). From double integration method of solving R_A , the moment equation is given by

$$M = R_A x - \frac{w_o}{3} x^2 - \frac{w_o}{6L} (Lx^2 - x^3)$$

$$M = \frac{11w_o L}{40} x - \frac{w_o}{3} x^2 - \frac{w_o}{6L} (Lx^2 - x^3)$$

For $x = x_C = 0.3292L$, $M = M_C$

$$M_C = \frac{11w_o L}{40} (0.3292L) - \frac{w_o}{3} (0.3292L)^2 - \frac{w_o}{6L} [L(0.3292L)^2 - (0.3292L)^3]$$

$$M_C = 0.0423w_o L^2$$

4. In the same manner of solving for M_C , M_B can be found by using $x = L$. Thus,

$$M_B = \frac{11w_o L^2}{40} - \frac{w_o L^2}{3} - \frac{w_o}{6L}(L^3 - L^3)$$

$$M_B = -\frac{7w_o L^2}{120} \text{ which confirms the solution above for } M_B.$$

5. To locate the point of zero moment denoted by D in the figure, we will again use the moment equation; now with $M = 0$ and $x = x_D$.

$$0 = \frac{11w_o L}{40}x_D - \frac{w_o}{3}x_D^2 - \frac{w_o}{6L}(Lx_D^2 - x_D^3)$$

$$0 = \frac{11}{40}L - \frac{1}{3}x_D - \frac{x_D}{6L}(L - x_D)$$

$$0 = \frac{33}{20}L^2 - 2Lx_D - x_D(L - x_D)$$

$$0 = \frac{33}{20}L^2 - 3Lx_D + x_D^2$$

$$x_D^2 - 3Lx_D = -\frac{33}{20}L^2$$

$$x_D^2 - 3Lx_D + \frac{9}{4}L^2 = -\frac{33}{20}L^2 + \frac{9}{4}L^2$$

$$(x_D - \frac{3}{2}L)^2 = \frac{3}{5}L^2$$

$$x_D - \frac{3}{2}L = \pm \frac{\sqrt{3}}{\sqrt{5}}L$$

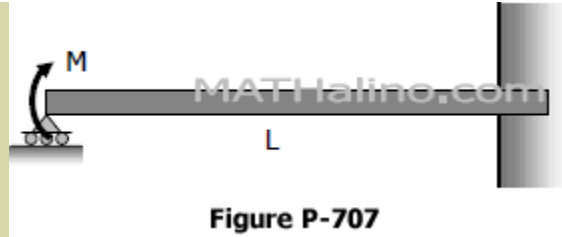
$$x_D = (\frac{3}{2} \pm \frac{\sqrt{3}}{\sqrt{5}})L$$

$$x_D = 2.2746L \quad (\text{absurd})$$

$$x_D = 0.7254L \quad \text{answer}$$

Problem 707

A couple M is applied at the propped end of the beam shown in Fig. P-707. Compute R at the propped end and also the wall restraining moment.



Solution 04

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The moment at any point on the beam which is at distance x from the left support is

$$M_x = M - Rx$$

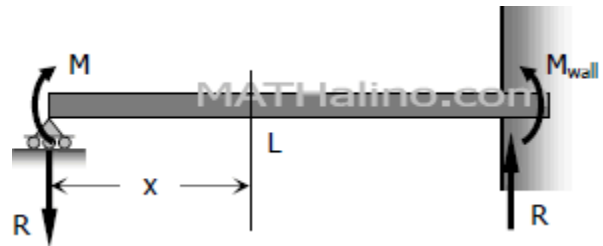
By double integration method

$$EI y'' = M_x$$

$$EI y'' = M - Rx$$

$$EI y' = Mx - \frac{1}{2}Rx^2 + C_1$$

$$EI y = \frac{1}{2}Mx^2 - \frac{1}{6}Rx^3 + C_1x + C_2$$



Boundary conditions

At $x = 0, y = 0; C_2 = 0$

At $x = L, y = 0;$

$$0 = \frac{1}{2}ML^2 - \frac{1}{6}RL^3 + C_1L$$

$$C_1 = \frac{1}{6}RL^2 - \frac{1}{2}ML$$

At $x = L, y' = 0;$

$$0 = ML - \frac{1}{2}RL^2 + \left(\frac{1}{6}RL^2 - \frac{1}{2}ML\right)$$

$$0 = M - \frac{1}{2}RL + \frac{1}{6}RL - \frac{1}{2}M$$

$$\frac{1}{3}RL = \frac{1}{2}M$$

$$R = \frac{3M}{2L} \quad \text{answer}$$

Problem 708

Two identical cantilever beams in contact at their ends support a distributed load over one of them as shown in Fig. P-708. Determine the restraining moment at each wall.

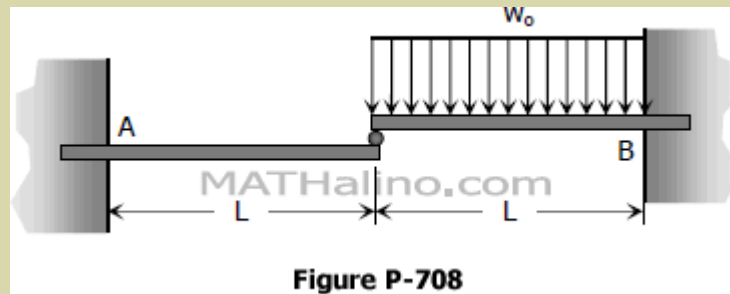


Figure P-708

Solution

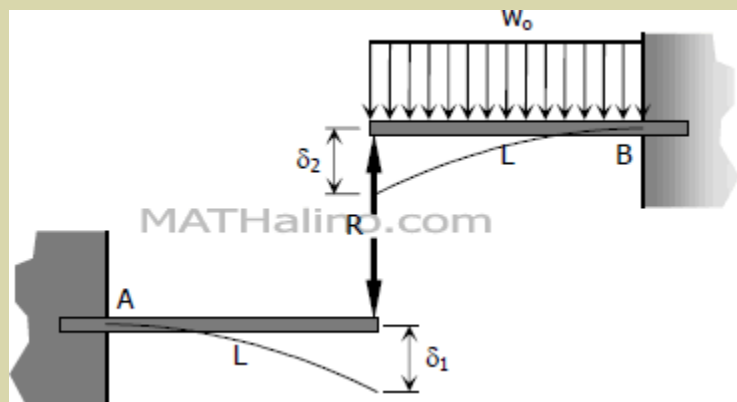
[HideClick here to show or hide the solution](#)

$$\delta_1 = \frac{RL^3}{3EI}$$

$$\delta_2 = \frac{w_o L^4}{8EI} - \frac{RL^3}{3EI}$$

$$\delta_1 = \delta_2$$

$$\frac{RL^3}{3EI} = \frac{w_o L^4}{8EI} - \frac{RL^3}{3EI}$$



$$\frac{2RL^3}{3EI} = \frac{w_o L^4}{8EI}$$

$$R = \frac{3w_o L}{16}$$

$$M_A = -RL$$

$$M_A = -\frac{3w_o L^2}{16} \quad \text{answer}$$

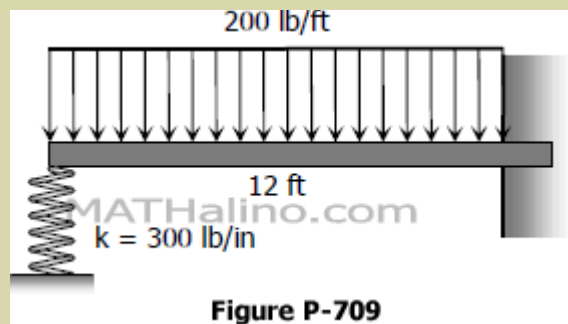
$$M_B = RL - w_o L\left(\frac{1}{2}L\right)$$

$$M_B = \frac{3w_o L^2}{16} - \frac{w_o L^2}{2}$$

$$M_B = -\frac{5w_o L^2}{16} \quad \text{answer}$$

Example 06

The beam in Figure PB-006 is supported at the left by a spring that deflects 1 inch for each 300 lb. For the beam $E = 1.5 \times 10^6$ psi and $I = 144$ in⁴. Compute the deflection of the spring.



Solution 06

[HideClick here to show or hide the solution](#)

Assume there is no spring support at the left end

$$\delta = \frac{w_o L^4}{8EI} = \frac{200(12^4)(12^3)}{8(1.5 \times 10^6)(144)}$$

$$\delta = 4.1472 \text{ in}$$

Considering the spring reaction

$$\delta - \delta_{spring} = \delta_R$$

$$4.1472 - \frac{R}{k} = \frac{RL^3}{3EI}$$

$$4.1472 - \frac{R}{300} = \frac{R(12^3)(12^3)}{3(1.5 \times 10^6)(144)}$$

$$4.1472 = 0.0079413R$$

$$R = 522.23 \text{ lb}$$

$$\delta_{spring} = \frac{R}{k} = \frac{522.23}{300}$$

$$\delta_{spring} = 1.74 \text{ in} \quad \text{answer}$$

Problem 710

Two timber beams are mounted at right angles and in contact with each other at their midpoints. The upper beam A is 2 in wide by 4 in deep and simply supported on an 8-ft span; the lower beam B is 3 in wide by 8 in deep and simply supported on a 10-ft span. At their cross-over point, they jointly support a load $P = 2000$ lb. Determine the contact force between the beams.

Solution

[HideClick here to show or hide the solution](#)

Let

R = contact force between beams A and B

Subscript (_A) = for upper beam

Subscript (_B) = for lower beam

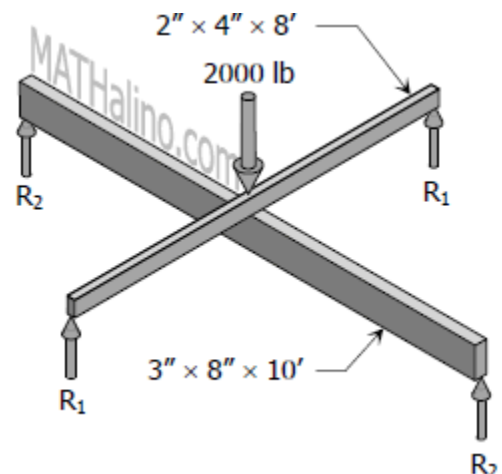
Moment of inertia

$$I_A = \frac{2(4^3)}{12} = \frac{32}{3} \text{ in}^4$$

$$I_B = \frac{3(8^3)}{12} = 128 \text{ in}^4$$

Note:

The midspan deflection of a simple beam loaded with



concentrated force at the midpoint is given by

$$\delta = \frac{PL^3}{48EI}$$

See Case No. 6 in the [Summary of Beam Loadings](#).

The midspan of upper beam A is under 2000 lb applied load and contact force R. R will act upward at beam A.

$$\delta_A = \frac{2000(8^3)(12^3)}{48E(\frac{32}{3})} - \frac{R(8^3)(12^3)}{48E(\frac{32}{3})}$$

$$\delta_A = \frac{3\,456\,000}{E} - \frac{1728R}{E}$$

The lower beam B is subjected by the contact force R at midspan.

$$\delta_B = \frac{R(10^3)(12^3)}{48E(128)}$$

$$\delta_B = \frac{1125R}{4E}$$

The deflections of upper beam A and lower beam B are obviously equal. R will act downward at beam B.

$$\delta_A = \delta_B$$

$$\frac{3\,456\,000}{E} - \frac{1728R}{E} = \frac{1125R}{4E}$$

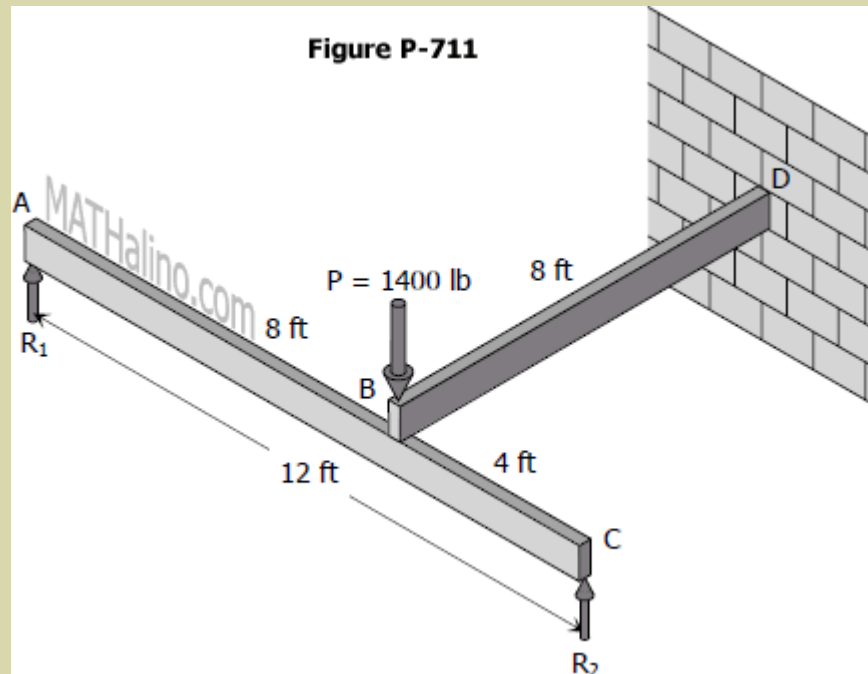
$$3\,456\,000 = \frac{8037R}{4}$$

$$R = 1720.04 \text{ lb} \quad \text{answer}$$

Problem 711

A cantilever beam BD rests on a simple beam AC as shown in Fig. P-711. Both beams are of the same material and are 3 in wide by 8 in deep. If they jointly carry a load $P = 1400$ lb, compute the

maximum flexural stress developed in the beams.



Solution

[Hide](#)[Click here to show or hide the solution](#)

For cantilever beam BD

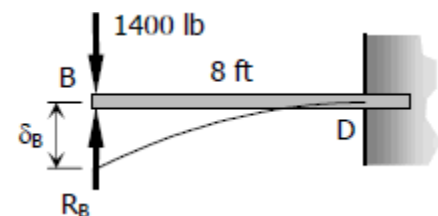
From Case No. 1 of [beam loading cases](#), the maximum deflection at the end of cantilever beam due to concentrated force at the free end is given by

$$\delta = \frac{PL^3}{3EI}$$

Thus,

$$\delta_B = \frac{1400(8^3)}{3EI} - \frac{R_B(8^3)}{3EI}$$

$$\delta_B = \frac{716\,800}{3EI} - \frac{512R_B}{3EI}$$



For the simple beam AC

The deflection at distance x from Case No. 7 of [different beam loadings](#) is

$$EI y = \frac{Pbx}{6L}(L^2 - x^2 - b^2) \quad \text{for } 0 \leq x \leq a$$

For $x = a$, the deflection equation will become

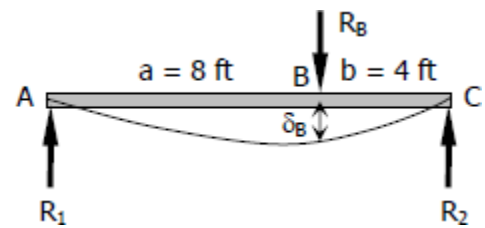
$$EI y = \frac{Pab}{6L}(L^2 - a^2 - b^2)$$

For beam AC; $P = R_B$, $a = 8$ ft, $b = 4$ ft, and $L = 12$ ft.

$$\delta_B = \frac{R_B(8)(4)}{6(12)EI}(12^2 - 8^2 - 4^2)$$

$$\delta_B = \frac{4R_B}{9EI}(64)$$

$$\delta_B = \frac{256R_B}{9EI}$$



Solving for the contact force, R_B

$$\delta_B = \delta_B$$

$$\frac{716800}{3EI} - \frac{512R_B}{3EI} = \frac{256R_B}{9EI}$$

$$\frac{716800}{3EI} = \frac{1972R_B}{9EI}$$

$$R = 1200 \text{ lb}$$

Determining the maximum moment

The maximum moment on cantilever beam will occur at D

$$M_D = 1200(8) - 1400(8)$$

$$M_D = -1600 \text{ lb} \cdot \text{ft}$$

The maximum moment on simple beam will occur at point B.

$$M_B = \frac{Pab}{L} = \frac{1200(8)(4)}{12}$$

$$M_B = 3200 \text{ lb} \cdot \text{ft}$$

Maximum moment is at point B

$$M_{max} = 3200 \text{ lb} \cdot \text{ft}$$

Solving for maximum flexural stress

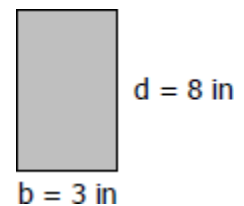
The bending stress of rectangular beam is given by

$$f_b = \frac{6M}{bd^2}$$

Thus,

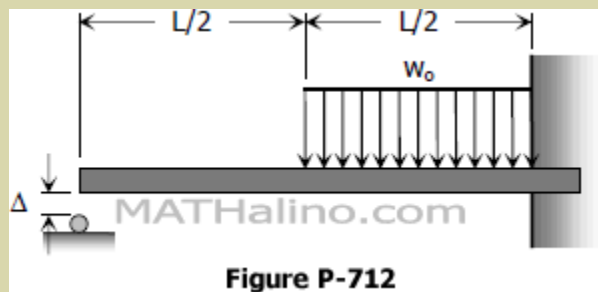
$$(f_b)_{max} = \frac{6(3200)(12)}{3(8^2)}$$

$$(f_b)_{max} = 1200 \text{ psi} \quad \text{answer}$$



Problem 712

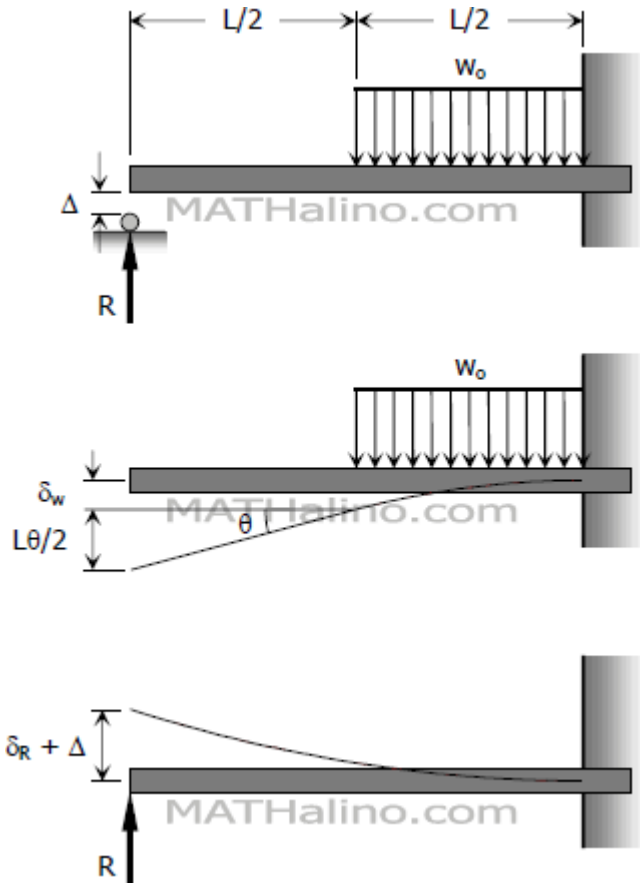
There is a small initial clearance D between the left end of the beam shown in Fig. P-712 and the roller support. Determine the reaction at the roller support after the uniformly distributed load is applied.



Solution

[Hide](#)[Click here to show or hide the solution](#)

$$\delta_R + \Delta = \delta_w + \frac{1}{2}L\theta$$



See Case 1 and Case 3 of [Superposition Method](#) for formulas:

$$\frac{RL^3}{3EI} + \Delta = \frac{w_o(\frac{1}{2}L)^4}{8EI} + \frac{L}{2} \left[\frac{w_o(\frac{1}{2}L)^3}{6EI} \right]$$

$$\frac{RL^3}{3EI} + \Delta = \frac{w_oL^4}{128EI} + \frac{w_oL^4}{96EI}$$

$$\frac{RL^3}{3EI} + \Delta = \frac{7w_oL^4}{384EI}$$

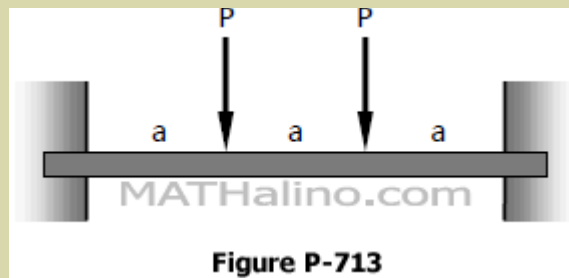
$$\frac{RL^3}{3EI} = \frac{7w_oL^4}{384EI} - \Delta$$

$$R = \frac{7w_o L}{128} - \frac{3EI\Delta}{L^3}$$

$$R = \frac{1}{128L^3}(7w_o L^4 - 384EI\Delta) \quad \text{answer}$$

Problem 713

Determine the end moment and midspan value of $EI\delta$ for the restrained beam shown in Fig. PB-010. (Hint: Because of symmetry, the end shears are equal and the slope is zero at midspan. Let the redundant be the moment at midspan.)



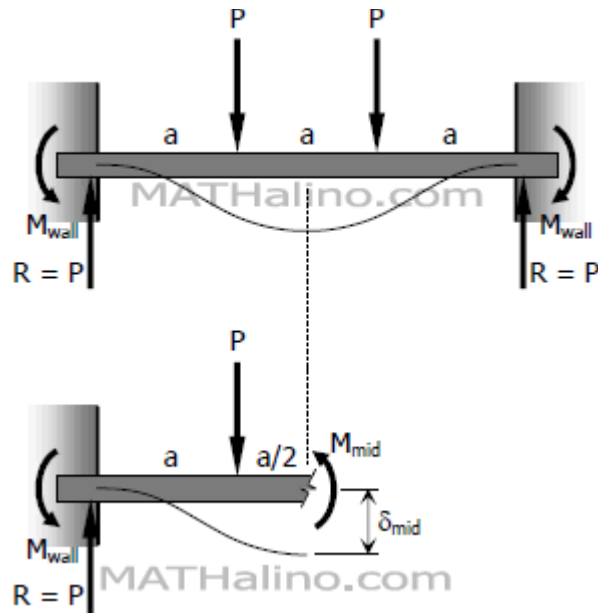
Solution 10

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$$\Sigma\theta_{mid} = 0$$

See Case 2 and Case 5 of [superposition method](#):

$$\frac{M_{mid}(\frac{3}{2}a)}{EI} - \frac{Pa^2}{2EI} = 0$$



$$\frac{3aM_{mid}}{2EI} = \frac{Pa^2}{2EI}$$

$$M_{mid} = \frac{1}{3}Pa$$

Thus,

$$M_{wall} = M_{mid} - Pa$$

$$M_{wall} = \frac{1}{3}Pa - Pa$$

$$M_{wall} = -\frac{2}{3}Pa \quad \text{answer}$$

Again, see Case 2 and Case 5 of [superposition method](#):

$$\delta_{mid} = \frac{Pa^2}{6EI} \left[3\left(\frac{3}{2}a\right) - a \right] - \frac{M_{mid}\left(\frac{3}{2}a\right)^2}{2EI}$$

$$\delta_{mid} = \frac{Pa^2}{6EI} \left(\frac{7}{2}a\right) - \frac{\frac{1}{3}Pa\left(\frac{9}{4}a^2\right)}{2EI}$$

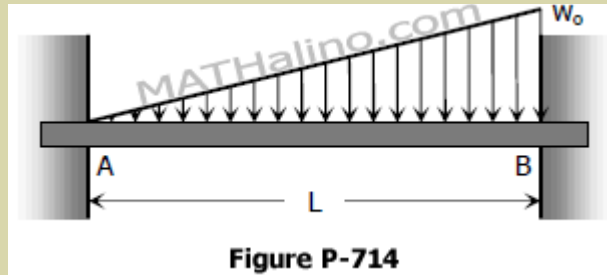
$$\delta_{mid} = \frac{7Pa^3}{12EI} - \frac{3Pa^3}{8EI}$$

$$\delta_{mid} = \frac{5Pa^3}{24EI}$$

$$EI\delta_{mid} = \frac{5}{24}Pa^3 \quad \text{answer}$$

Problem 714

Determine the end moments of the restrained beam shown in Fig. P-714.

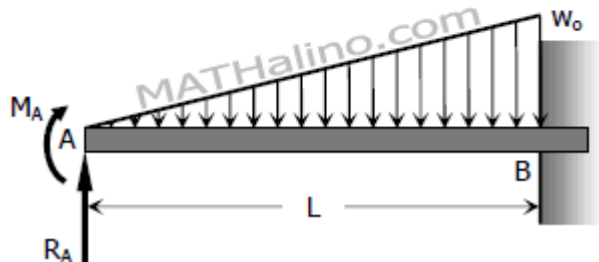
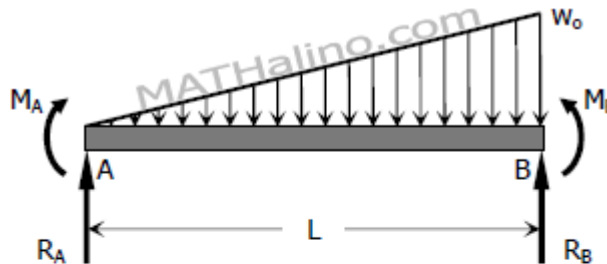


Solution

[HideClick here to show or hide the solution](#)

$$\delta_A = 0$$

$$\delta_{triangular\ load} - \delta_{fixed\ end\ moment} - \delta_{reaction\ at\ A} = 0$$



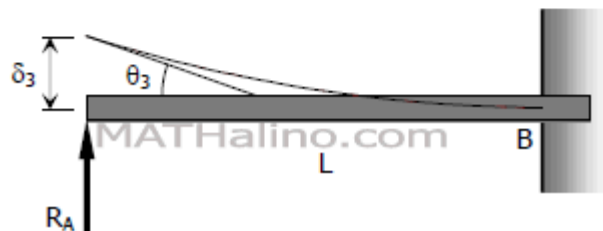
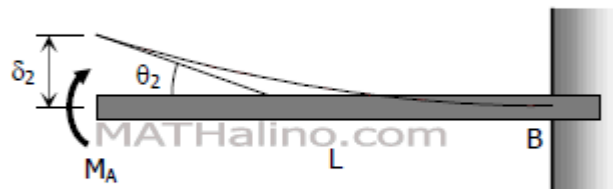
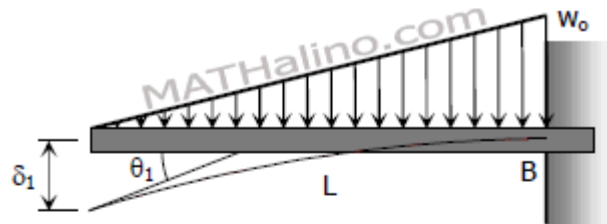
For formulas, see Case 1, Case 4, and Case 5 in [superposition method](#)
 $\delta_1 - \delta_2 - \delta_3 = 0$

$$\frac{w_o L^4}{30EI} - \frac{M_A L^2}{2EI} - \frac{R_A L^3}{3EI} = 0$$

$$\frac{w_o L^2}{30} - \frac{M_A}{2} - \frac{R_A L}{3} = 0$$

$$\frac{w_o L^2}{30} - \frac{M_A}{2} = \frac{R_A L}{3}$$

$$R_A = \frac{w_o L}{10} - \frac{3M_A}{2L}$$



$$\theta_A = 0$$

$$\theta_{triangular\ load} - \theta_{fixed\ end\ moment} - \theta_{reaction\ at\ A} = 0$$

$$\theta_1 - \theta_2 - \theta_3 = 0$$

$$\frac{w_o L^3}{24EI} - \frac{M_A L}{EI} - \frac{R_A L^2}{2EI} = 0$$

$$\frac{w_o L^2}{24} - M_A - \frac{R_A L}{2} = 0$$

$$w_o L^2 - 24M_A - 12R_A L = 0$$

$$w_o L^2 - 24M_A - 12 \left(\frac{w_o L}{10} - \frac{3M_A}{2L} \right) L = 0$$

$$w_o L^2 - 24M_A - \frac{6}{5} w_o L^2 + 18M_A = 0$$

$$-6M_A - \frac{1}{5} w_o L^2 = 0$$

$$6M_A = -\frac{1}{5} w_o L^2$$

$$M_A = -\frac{1}{30} w_o L^2 \quad \text{answer}$$

$$R_A = \frac{w_o L}{10} - \frac{3(-\frac{1}{30} w_o L^2)}{2L}$$

$$R_A = \frac{w_o L}{10} + \frac{w_o L}{20}$$

$$R_A = \frac{3}{20} w_o L$$

$$M_B = M_A + R_A L - \frac{1}{2} w_o L \left(\frac{1}{3} L \right)$$

$$M_B = -\frac{1}{30} w_o L^2 + \left(\frac{3}{20} w_o L \right) L - \frac{1}{2} w_o L \left(\frac{1}{3} L \right)$$

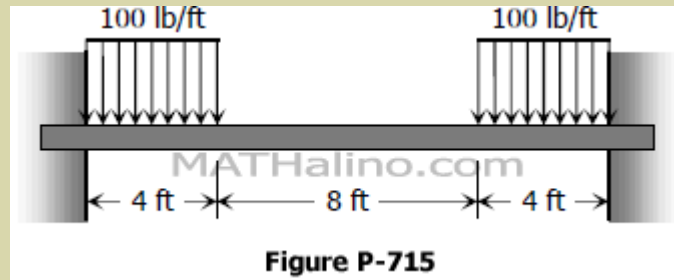
$$M_B = -\frac{1}{30} w_o L^2 + \frac{3}{20} w_o L^2 - \frac{1}{6} w_o L^2$$

$$M_B = -\frac{1}{20} w_o L^2 \quad \text{answer}$$

Problem 12

Determine the moment and maximum EIδ for the restrained beam shown in Fig. RB-012. (Hint: Let

the redundants be the shear and moment at the midspan. Also note that the midspan shear is zero.)



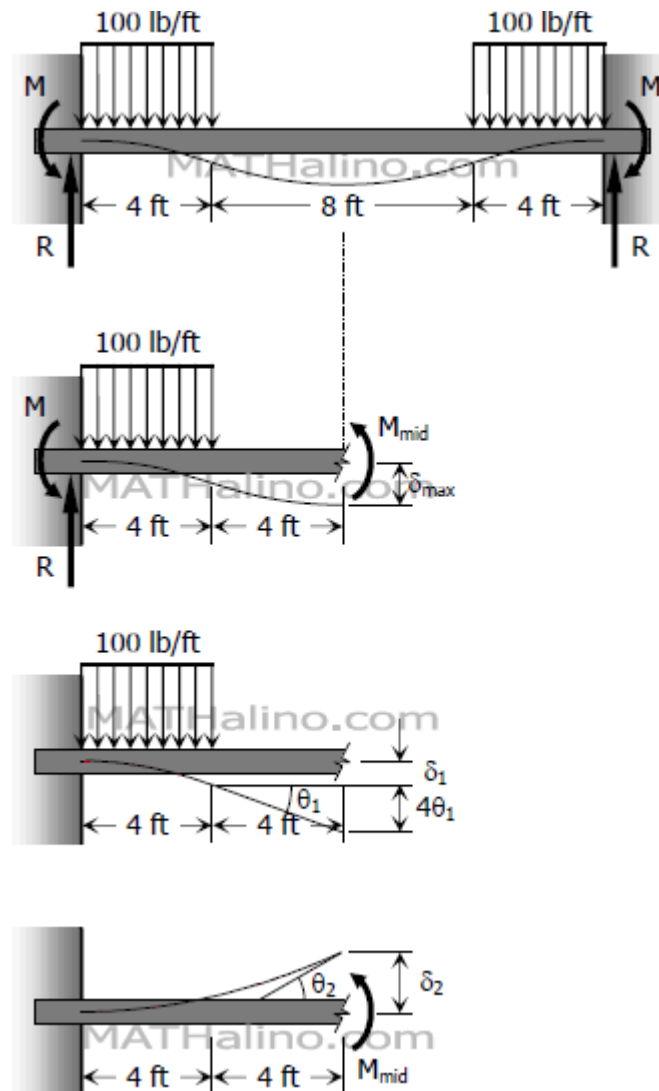
Solution 12

[Hide](#)[Click here to show or hide the solution](#)

$$\theta_{mid} = 0$$

$$\theta_{uniform\ load} - \theta_{moment\ at\ midspan} = 0$$

$$\theta_1 - \theta_2 = 0$$



See Case 3 and Case 5 of [superposition method](#)

$$\frac{100(4^3)}{6EI} - \frac{M_{mid}(8)}{EI} = 0$$

$$8M_{mid} = 1066.67$$

$$M_{mid} = 133.33 \text{ ft} \cdot \text{lb}$$

Moment at the fixed support (end moment):

$$M = M_{mid} - 100(4)(2)$$

$$M = 133.33 - 800$$

$$M = -666.67 \text{ ft} \cdot \text{lb} \quad \text{answer}$$

By symmetry, maximum deflection will occur at the midspan

$$\delta_{max} = \delta_1 + 4\theta_1 - \delta_2$$

$$\delta_{max} = \frac{100(4^4)}{8EI} + 4 \left[\frac{100(4^3)}{6EI} \right] - \frac{M_{mid}(8^2)}{2EI}$$

$$EI\delta_{max} = \frac{100(256)}{8} + \frac{4(100)(64)}{6} - \frac{133.33(64)}{2}$$

$$EI\delta_{max} = 3200 + 4266.67 - 4266.67$$

$$EI\delta_{max} = 3200 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

Note:

The fixed reactions are equal to $R = 100(4) = 400 \text{ lb}$. In the FBD of half left of the beam, R equates to the uniform load; this made the vertical shear at the midspan equal to zero.

Application of Area-Moment Method to Restrained Beams

See [deflection of beam by moment-area method](#) for details.

Rotation of beam from A to B

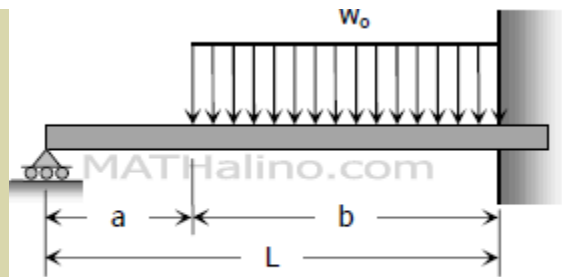
$$\theta_{AB} = \frac{1}{EI}(\text{Area}_{AB})$$

Deviation of B from a tangent line through A

$$t_{B/A} = \frac{1}{EI}(\text{Area}_{AB}) \bar{X}_B$$

Problem 704

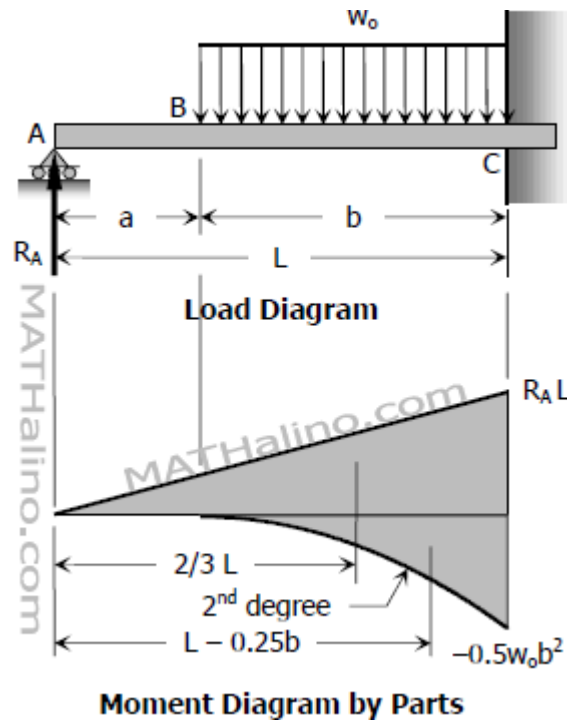
Find the reaction at the simple support of the propped beam shown in Figure PB-001 by using moment-area method.



Solution

[Click here to show or hide the solution](#)

The moment at C due to reaction R_A is R_AL and the moment at C due to uniform load w_0 is $-w_0b(0.5b) = -\frac{1}{2}w_0b^2$.



The deviation at A from tangent line through C is zero. Thus,

$$EI t_{A/C} = (Area_{AC}) \bar{X}_A = 0$$

$$\frac{1}{2}L(R_AL)(\frac{2}{3}L) - \frac{1}{3}b(\frac{1}{2}w_ob^2)(L - \frac{1}{4}b) = 0$$

$$\frac{1}{3}R_AL^3 - \frac{1}{6}w_ob^3(L - \frac{1}{4}b) = 0$$

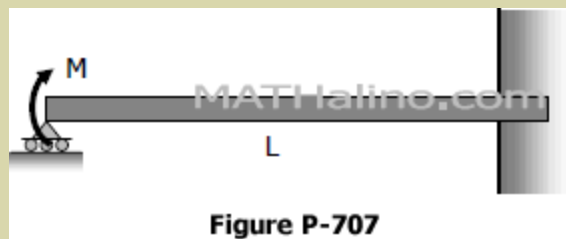
$$\frac{1}{3}R_AL^3 - \frac{1}{24}w_ob^3(4L - b) = 0$$

$$\frac{1}{3}R_AL^3 = \frac{1}{24}w_ob^3(4L - b)$$

$$R_A = \frac{w_ob^3}{8L^3}(4L - b) \quad \text{answer}$$

Problem 707

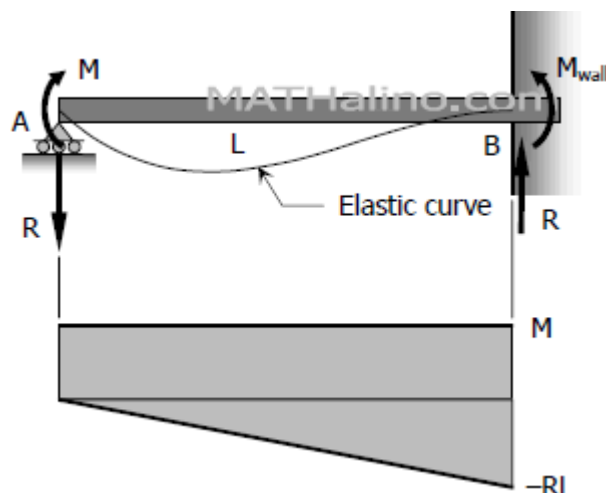
For the propped beam shown in Fig. P-707, solved for vertical reaction R at the simple support.



Solution

[HideClick here to show or hide the solution](#)

Taking the fixed support to be the moment center, the moment diagram by parts is shown to the right.



$$EI t_{A/B} = (Area_{AB}) \bar{X}_A = 0$$

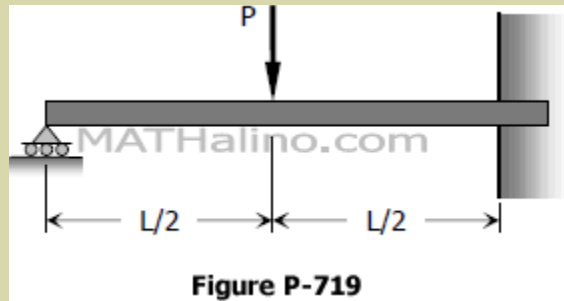
$$ML(\frac{1}{2}L) - \frac{1}{2}L(RL)(\frac{2}{3}L) = 0$$

$$\frac{1}{2}ML^2 - \frac{1}{3}RL^3 = 0$$

$$R = \frac{3M}{2L} \quad \text{answ}$$

Problem 719

For the propped beam shown in Fig. P-719, determine the propped reaction R and the midspan value of $EI\delta$.



Solution

[HideClick here to show or hide the solution](#)

Solving for the propped reaction R

$$EI t_{A/B} = 0$$

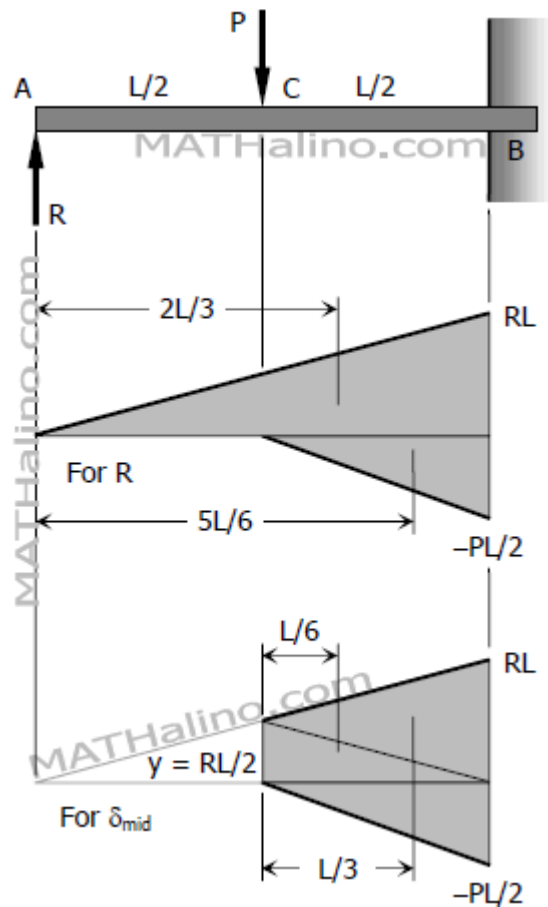
$$(Area_{AB}) \cdot \bar{X}_A = 0$$

$$\frac{1}{2}L(RL)(\frac{2}{3}L) - \frac{1}{2}(\frac{1}{2}L)(\frac{1}{2}PL)(\frac{5}{6}L) = 0$$

$$\frac{1}{3}RL^3 - \frac{5}{48}PL^3 = 0$$

$$\frac{1}{3}R - \frac{5}{48}P = 0$$

$$R = \frac{5}{16}P \quad \text{answer}$$



Solving for the midspan deflection δ_{mid}

$$\frac{y}{\frac{1}{2}L} = \frac{RL}{L}$$

$$y = \frac{1}{2}RL$$

$$EI t_{C/B} = (Area_{BC}) \cdot \bar{X}_A$$

$$EI t_{C/B} = \frac{1}{2}(\frac{1}{2}L)(RL)(\frac{1}{3}L) + \frac{1}{2}(\frac{1}{2}L)(\frac{1}{2}RL)(\frac{1}{6}L) - \frac{1}{2}(\frac{1}{2}L)(\frac{1}{2}PL)(\frac{1}{3}L)$$

$$EI t_{C/B} = \frac{1}{12}RL^3 + \frac{1}{48}RL^3 - \frac{1}{24}PL^3$$

$$EI t_{C/B} = \frac{5}{48}RL^3 - \frac{1}{24}PL^3$$

$$EI t_{C/B} = \frac{5}{48}(\frac{5}{16}P)L^3 - \frac{1}{24}PL^3$$

$$EI t_{C/B} = \frac{25}{768} PL^3 - \frac{1}{24} PL^3$$

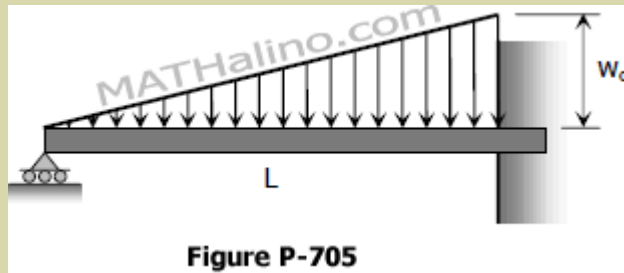
$$EI t_{C/B} = -\frac{7}{768} PL^3$$

Thus,

$$EI \delta_{mid} = \frac{7}{768} PL^3 \quad \text{answer}$$

Problem 720

Find the reaction at the simple support of the propped beam shown in Fig. P-705 by using moment-area method.

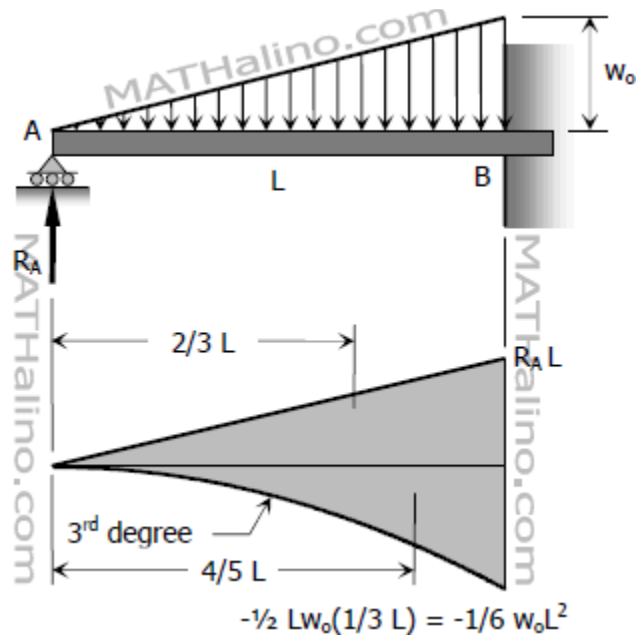


Solution

[HideClick here to show or hide the solution](#)

The moment at B due to R_A is $R_A L$ and the moment at B due to triangular load is $-\frac{1}{6} w_0 L^2$

Solution of R_A by Moment-Area Method



$$EI t_{A/B} = (\text{Area}_{AB}) \bar{X}_A = 0$$

$$\frac{1}{2} L (R_A L) \left(\frac{2}{3} L\right) - \frac{1}{4} L \left(\frac{1}{6} w_0 L^2\right) \left(\frac{4}{5} L\right) = 0$$

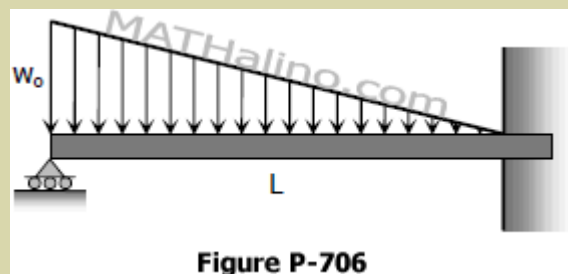
$$\frac{1}{3} R_A L^3 - \frac{1}{30} w_0 L^4 = 0$$

$$\frac{1}{3} R_A L^3 = \frac{1}{30} w_0 L^4$$

$$R_A = \frac{w_0 L}{10} \quad \text{answer}$$

Problem 721

By the use of moment-area method, determine the magnitude of the reaction force at the left support of the propped beam in Fig. P-706.

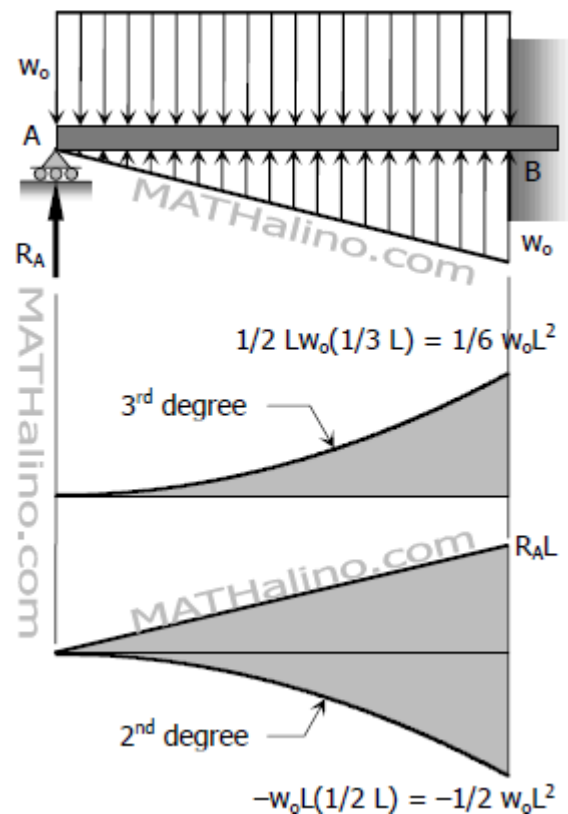


Solution

[Click here to show or hide the solution](#)

Transform the triangular load into a downward uniformly distributed load and upward increasing load. We do this so that we can easily draw the moment diagram by parts with moment center at the fixed support.

Solution of R_A by Moment-Area Method



$$EI t_{A/B} = (Area_{AB}) \bar{X}_A = 0$$

$$\frac{1}{4}(L)\left(\frac{1}{6}w_o L^2\right)\left(\frac{4}{5}L\right) + \frac{1}{2}(L)(R_A L)\left(\frac{2}{3}L\right) - \frac{1}{3}(L)\left(\frac{1}{2}w_o L^2\right)\left(\frac{3}{4}L\right) = 0$$

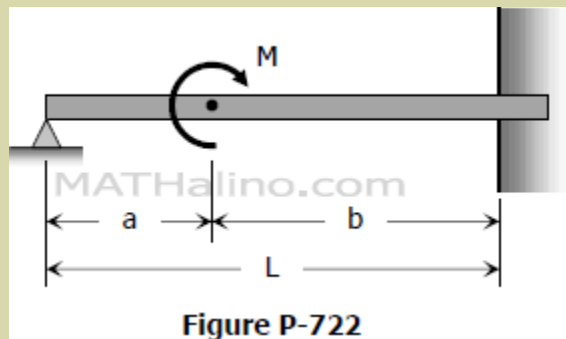
$$\frac{w_o L^4}{30} + \frac{R_A L^3}{3} - \frac{w_o L^4}{8} = 0$$

$$\frac{R_A L^3}{3} = \frac{11w_o L^4}{120}$$

$$R_A = \frac{11w_o L}{40} \quad \text{answer}$$

Problem 722

For the beam shown in Fig. P-722, compute the reaction R at the propped end and the moment at the wall. Check your results by letting $b = L$ and comparing with the results in [Problem 707](#).



Solution

[HideClick here to show or hide the solution](#)

$$EI \, t_{A/B} = 0$$

$$(\text{Area}_{AB}) \cdot \bar{X}_A = 0$$

$$Mb(L - \frac{1}{2}b) - \frac{1}{2}L(RL)(\frac{2}{3}L) = 0$$

$$MbL - \frac{1}{2}Mb^2 - \frac{1}{3}RL^3 = 0$$

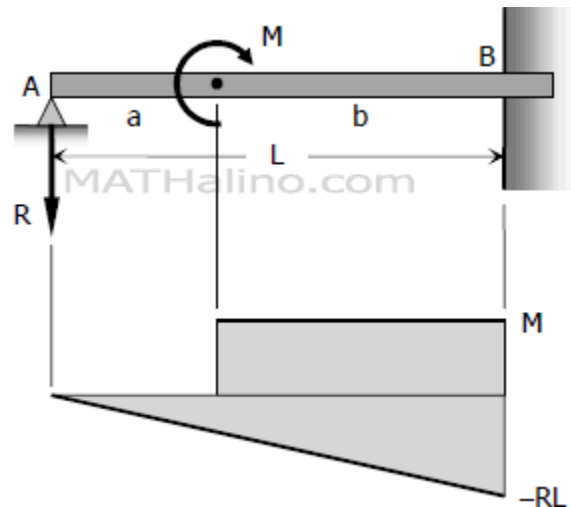
$$\frac{1}{3}RL^3 = MbL - \frac{1}{2}Mb^2$$

$$R = \frac{3Mb}{L^2} - \frac{3Mb^2}{2L^3}$$

$$R = \frac{3Mb}{2L^3}(2L - b) \quad \text{answer}$$

$$M_{\text{wall}} = M - RL$$

$$M_{\text{wall}} = M - \frac{3Mb}{2L^3}(2L - b)L$$



$$M_{wall} = M - \frac{3Mb}{2L^2}(2L - b) \quad \text{answer}$$

When $b = L$

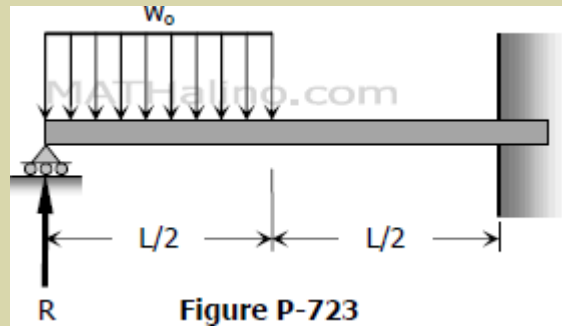
$$R = \frac{3ML}{2L^3}(2L - L) = \frac{3M}{2L}$$

$$M_{wall} = M - \frac{3ML}{2L^2}(2L - L) = M - \frac{3}{2}M = -\frac{3}{2}M$$

See Problem 707 for [propped beam with moment load at the simple support](#) for comparison.

Problem 723

Find the reaction R and the moment at the wall for the propped beam shown in Fig. P-723.



Solution

$$EI \, t_{A/B} = 0$$

$$\frac{1}{3}\left(\frac{1}{2}L\right)\left(\frac{1}{8}w_oL^2\right)\left(\frac{7}{8}L\right) + \frac{1}{2}(L)(RL)\left(\frac{2}{3}L\right) - \frac{1}{3}(L)\left(\frac{1}{2}w_oL^2\right)\left(\frac{3}{4}L\right) = 0$$

$$\frac{7}{384}w_oL^4 + \frac{1}{3}RL^3 - \frac{1}{8}w_oL^4 = 0$$

$$\frac{7}{384}w_o L + \frac{1}{3}R - \frac{1}{8}w_o L = 0$$

$$\frac{1}{3}R = \frac{1}{8}w_o L - \frac{7}{384}w_o L$$

$$\frac{1}{3}R = \frac{41}{384}w_o L$$

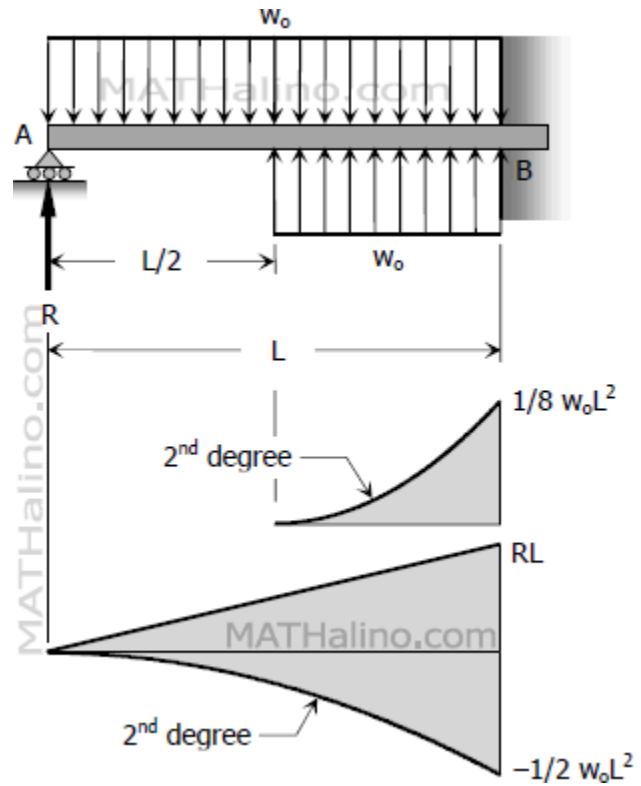
$$R = \frac{41}{128}w_o L \quad \text{answer}$$

$$M_{wall} = RL + \frac{1}{8}w_o L^2 - \frac{1}{2}w_o L^2$$

$$M_{wall} = \left(\frac{41}{128}w_o L\right)L + \frac{1}{8}w_o L^2 - \frac{1}{2}w_o L^2$$

$$M_{wall} = \frac{41}{128}w_o L^2 + \frac{1}{8}w_o L^2 - \frac{1}{2}w_o L^2$$

$$M_{wall} = -\frac{7}{128}w_o L^2 \quad \text{answer}$$



Problem 724

The beam shown in Fig. P-724 is only partially restrained at the wall so that, after the uniformly distributed load is applied, the slope at the wall is $w_o L^3 / 48EI$ upward to the right. If the supports remain at the same level, determine R .

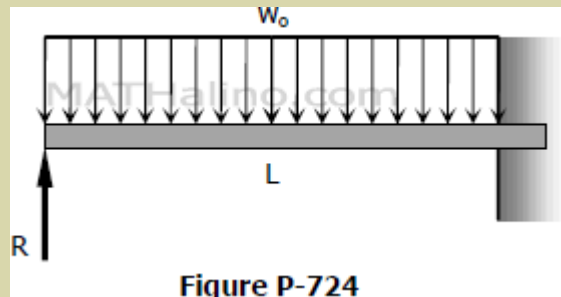


Figure P-724

Solution

$$\theta = \frac{w_o L^3}{48EI}$$

$$L\theta = \frac{w_o L^4}{48EI}$$

$$t_{A/B} = L\theta$$

$$t_{A/B} = \frac{w_o L^4}{48EI}$$

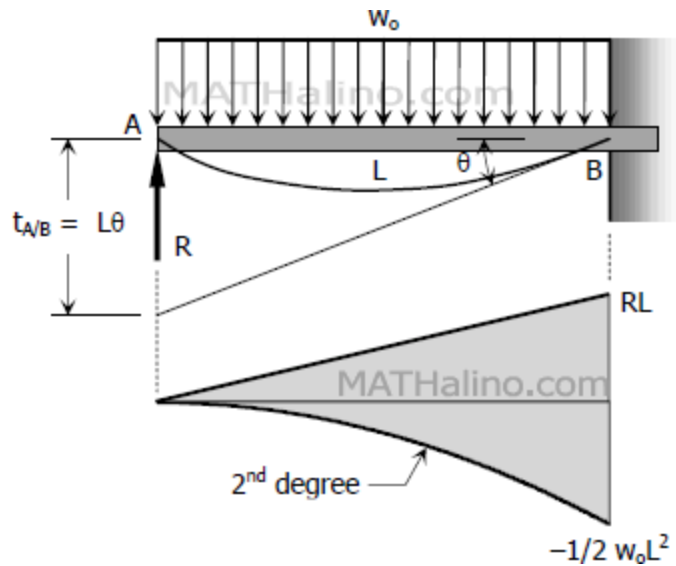
$$EI t_{A/B} = \frac{1}{48} w_o L^4$$

$$\frac{1}{2}(L)(RL)\left(\frac{2}{3}L\right) - \frac{1}{3}(L)\left(\frac{1}{2}w_o L^2\right)\left(\frac{3}{4}L\right) = \frac{1}{48} w_o L^4$$

$$\frac{1}{3}RL^3 - \frac{1}{8}w_o L^4 = \frac{1}{48} w_o L^4$$

$$\frac{1}{3}RL^3 = \frac{7}{48} w_o L^4$$

$$R = \frac{7}{16} w_o L^4 \quad \text{answer}$$



Problem 725

If the support under the propped beam in [Problem 724](#) settles an amount δ , show that the propped reaction decreases by $3EI\delta/L^3$.

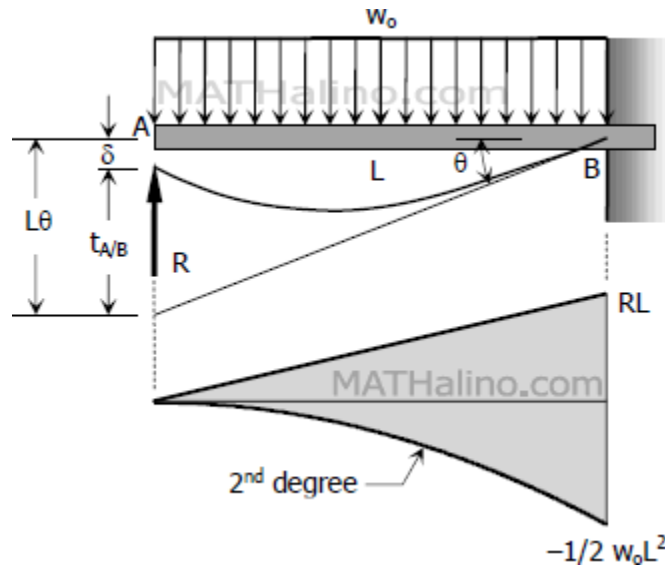
Solution

[HideClick here to show or hide the solution](#)

$$\theta = \frac{w_o L^3}{48EI}$$

$$EI \theta = \frac{1}{48} w_o L^3$$

$$L(EI \theta) = \frac{1}{48} w_o L^4$$



$$t_{A/B} = L \theta - \delta$$

$$EI t_{A/B} = L(EI \theta) - EI \delta$$

$$\frac{1}{3}RL^3 = \frac{7}{48}w_o L^4 - EI \delta$$

$$R = \frac{7}{16}w_o L - \frac{3EI \delta}{L^3}$$

The quantity $\frac{7}{16}w_o L$ is the simple reaction when there is no settlement δ at the propped support, thus the reaction R decreased by $\frac{3EI\delta}{L^3}$.

Problem 726

A beam of length L , perfectly restrained at both ends, supports a concentrated load P at midspan. Determine the end moment and maximum deflection.

Solution

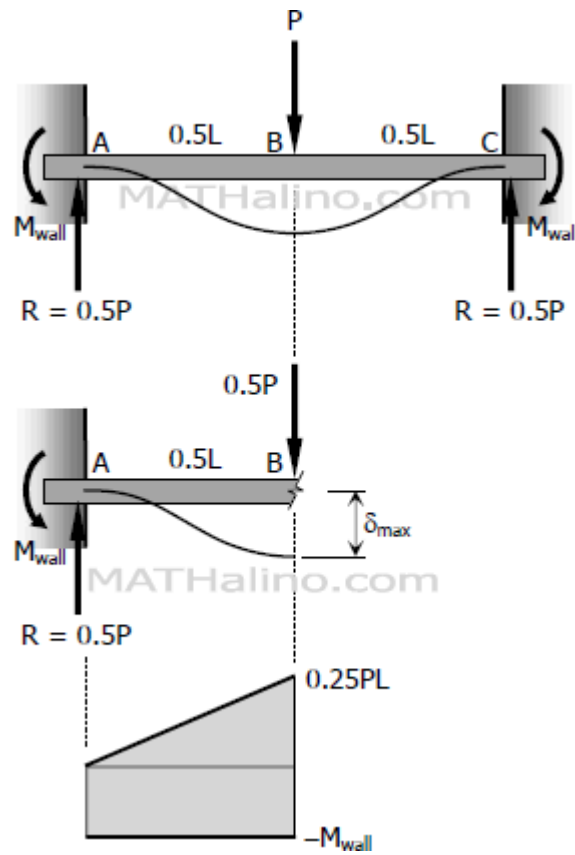
[HideClick here to show or hide the solution](#)

$$EI \theta_{AB} = 0$$

$$\frac{1}{2}(0.5L)(0.25PL) - 0.5L(M_{wall}) = 0$$

$$\frac{1}{2}M_{wall}L = \frac{1}{16}PL^2$$

$$M_{wall} = \frac{1}{8}PL \quad \text{answer}$$



$$\delta_{max} = t_{A/B}$$

$$EI \delta_{max} = EI t_{A/B}$$

$$EI \delta_{max} = (Area_{AB}) \cdot \bar{X}_A$$

$$EI \delta_{max} = \frac{1}{2}(0.5L)(0.25PL) \left[\frac{2}{3}(0.5L) \right] - 0.5L(M_{wall}) \left[\frac{1}{2}(0.5L) \right]$$

$$EI \delta_{max} = \frac{1}{48}PL^3 - \frac{1}{8}M_{wall}L^2$$

$$EI \delta_{max} = \frac{1}{48}PL^3 - \frac{1}{8} \left(\frac{1}{8}PL \right) L^2$$

$$EI \delta_{max} = \frac{1}{48}PL^3 - \frac{1}{64}PL^3$$

$$EI \delta_{max} = \frac{1}{192}PL^3 \quad \text{answer}$$

Problem 727

Repeat [Problem 726](#) assuming that the concentrated load is replaced by a uniformly distributed load of intensity w_o over the entire length.

Solution

[Click here to show or hide the solution](#)

$$EI \theta_{AB} = 0$$

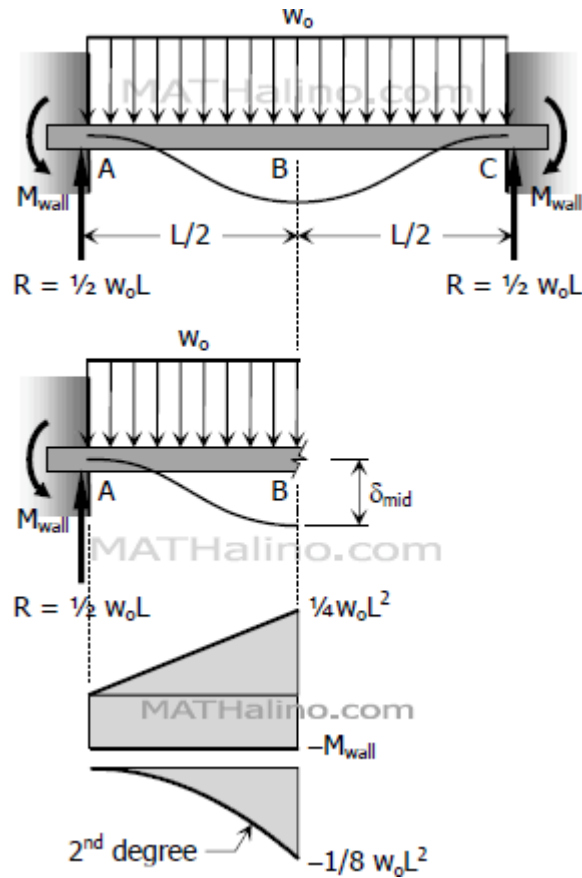
$$\frac{1}{2}(\frac{1}{2}L)(\frac{1}{4}w_o L^2) - \frac{1}{2}L(M_{wall}) - \frac{1}{3}(\frac{1}{2}L)(\frac{1}{8}w_o L^2) = 0$$

$$\frac{1}{16}w_o L^3 - \frac{1}{2}M_{wall}L - \frac{1}{48}w_o L^3 = 0$$

$$\frac{1}{2}M_{wall}L = \frac{1}{24}w_o L^3$$

$$M_{wall} = \frac{1}{12}w_o L^2 \quad \text{answer}$$

Note that the actual bending moment is a negative moment (bending the beam downward) as shown in the figure. The answer above is positive which indicates that our assumption of downward arrows is correct. Had we assumed positive M_{wall} (upward arrows), the answer would be negative pointing out that the actual moment is negative (or downward arrows).



$$\delta_{max} = t_{A/B}$$

$$EI \delta_{max} = EI t_{A/B}$$

$$EI \delta_{max} = (Area_{AB}) \cdot \bar{X}_A$$

$$EI \delta_{max} = \frac{1}{2} \left(\frac{1}{2} L \right) \left(\frac{1}{4} w_o L^2 \right) \left[\frac{2}{3} \left(\frac{1}{2} L \right) \right] - \frac{1}{2} L (M_{wall}) \left[\frac{1}{2} \left(\frac{1}{2} L \right) \right] - \frac{1}{3} \left(\frac{1}{2} L \right) \left(\frac{1}{8} w_o L^2 \right) \left[\frac{3}{4} \left(\frac{1}{2} L \right) \right]$$

$$EI \delta_{max} = \frac{1}{16} w_o L^3 \left(\frac{1}{3} L \right) - \frac{1}{2} M_{wall} L \left(\frac{1}{4} L \right) - \frac{1}{48} w_o L^3 \left(\frac{3}{8} L \right)$$

$$EI \delta_{max} = \frac{1}{48} w_o L^4 - \frac{1}{8} M_{wall} L^2 - \frac{1}{128} w_o L^4$$

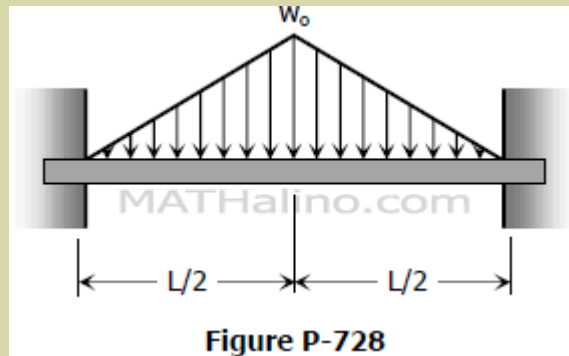
$$EI \delta_{max} = \frac{1}{48} w_o L^4 - \frac{1}{8} \left(\frac{1}{12} w_o L^2 \right) L^2 - \frac{1}{128} w_o L^4$$

$$EI \delta_{max} = \frac{1}{48} w_o L^4 - \frac{1}{96} w_o L^4 - \frac{1}{128} w_o L^4$$

$$EI \delta_{max} = \frac{1}{384} w_o L^4 \quad \text{answer}$$

Problem 728

Determine the end moment and maximum deflection of a perfectly restrained beam loaded as shown in Fig. P-728.

**Solution**

[HideClick here to show or hide the solution](#)

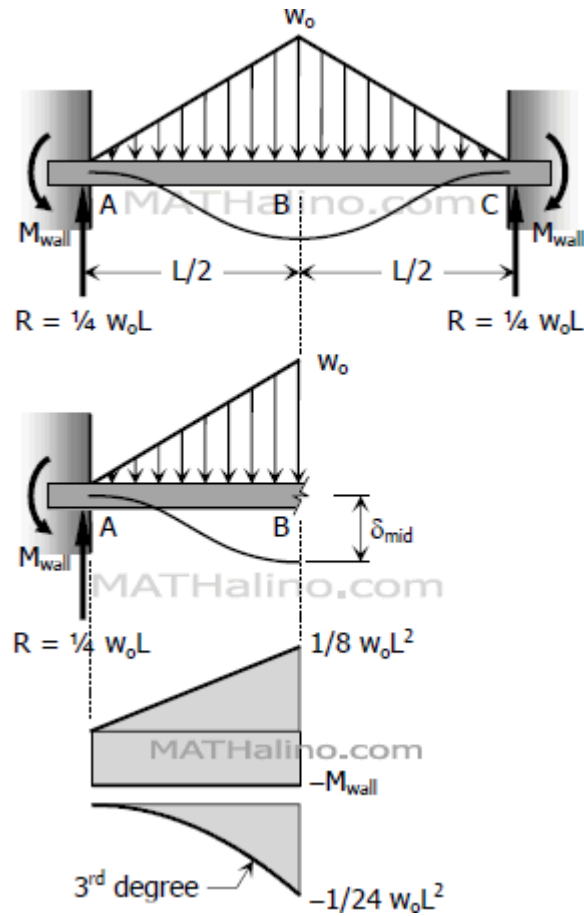
$$EI \theta_{AB} = 0$$

$$\frac{1}{2} \left(\frac{1}{2} L \right) \left(\frac{1}{8} w_o L^2 \right) - \frac{1}{2} L (M_{wall}) - \frac{1}{4} \left(\frac{1}{2} L \right) \left(\frac{1}{24} w_o L^2 \right) = 0$$

$$\frac{1}{32} w_o L^3 - \frac{1}{2} M_{wall} L - \frac{1}{192} w_o L^3 = 0$$

$$\frac{1}{2} M_{wall} L = \frac{5}{192} w_o L^3$$

$$M_{wall} = \frac{5}{96} w_o L^2$$



$$\delta_{max} = t_{A/B}$$

$$EI \delta_{max} = EI t_{A/B}$$

$$EI \delta_{max} = (Area_{AB}) \cdot \bar{X}_A$$

$$EI \delta_{max} = \frac{1}{2} \left(\frac{1}{2} L \right) \left(\frac{1}{8} w_o L^2 \right) \left[\frac{2}{3} \left(\frac{1}{2} L \right) \right] - \frac{1}{2} L (M_{wall}) \left[\frac{1}{2} \left(\frac{1}{2} L \right) \right] - \frac{1}{4} \left(\frac{1}{2} L \right) \left(\frac{1}{24} w_o L^2 \right) \left[\frac{4}{5} \left(\frac{1}{2} L \right) \right]$$

$$EI \delta_{max} = \frac{1}{32} w_o L^3 \left(\frac{1}{3} L \right) - \frac{1}{2} M_{wall} L \left(\frac{1}{4} L \right) - \frac{1}{192} w_o L^3 \left(\frac{2}{5} L \right)$$

$$EI \delta_{max} = \frac{1}{96} w_o L^4 - \frac{1}{8} M_{wall} L^2 - \frac{1}{480} w_o L^4$$

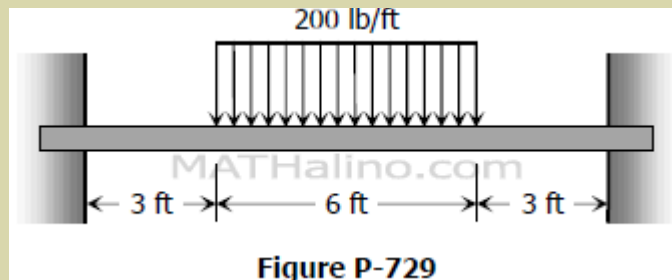
$$EI \delta_{max} = \frac{1}{96} w_o L^4 - \frac{1}{8} \left(\frac{5}{96} w_o L^2 \right) L^2 - \frac{1}{480} w_o L^4$$

$$EI \delta_{max} = \frac{1}{96} w_o L^4 - \frac{5}{768} w_o L^4 - \frac{1}{480} w_o L^4$$

$$EI \delta_{max} = \frac{7}{3840} w_o L^4 \quad \text{answer}$$

Problem 729

For the restrained beam shown in Fig. P-729, compute the end moment and maximum $EI\delta$.



Solution

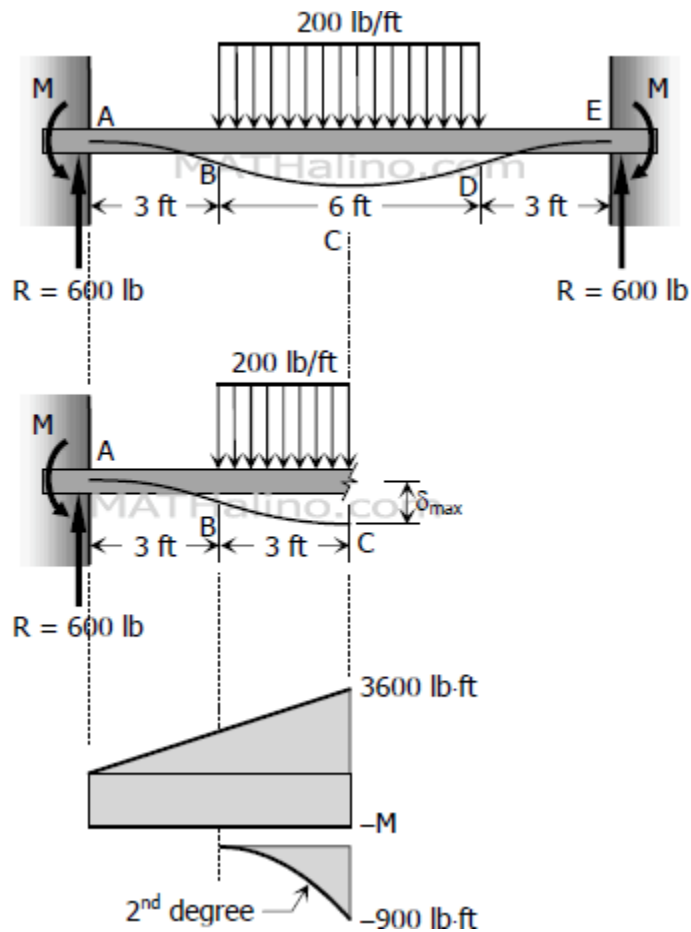
[Hide](#) [Click here to show or hide the solution](#)

$$EI \theta_{AC} = 0$$

$$\frac{1}{2}(6)(3600) - 6M - \frac{1}{3}(3)(900) = 0$$

$$10800 - 6M - 900 = 0$$

$$M = 1650 \text{ lb} \cdot \text{ft} \quad \text{answer}$$



$$\delta_{max} = t_{A/C}$$

$$EI \delta_{max} = EI t_{A/C}$$

$$EI \delta_{max} = (Area_{AC}) \cdot \bar{X}_A$$

$$EI \delta_{max} = \frac{1}{2}(6)(3600)\left[\frac{2}{3}(6)\right] - 6M(3) - \frac{1}{3}(3)(900)\left[3 + \frac{3}{4}(3)\right]$$

$$EI \delta_{max} = 10\,800(4) - 18M - 900(5.25)$$

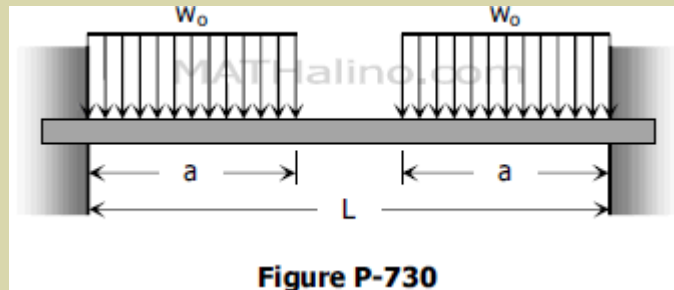
$$EI \delta_{max} = 10\,800(4) - 18(1650) - 900(5.25)$$

$$EI \delta_{max} = 8775 \text{ lb} \cdot \text{ft}^3 \quad \text{answer}$$

Problem 703

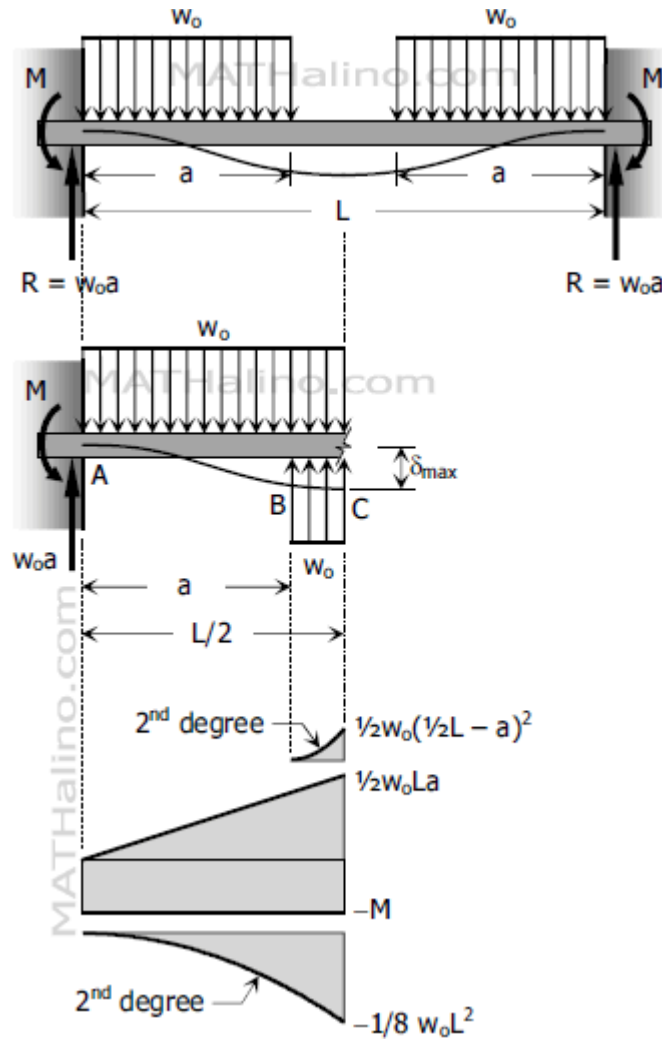
Determine the end moment and maximum deflection for a perfectly restrained beam loaded as

shown in Fig. P-730.



Solution

[HideClick here to show or hide the solution](#)



$$EI \theta_{AC} = 0$$

$$\frac{1}{3} \left(\frac{1}{2} L - a \right) \left[\frac{1}{2} w_o \left(\frac{1}{2} L - a \right)^2 \right] + \frac{1}{2} \left(\frac{1}{2} L \right) \left(\frac{1}{2} w_o L a \right) - \frac{1}{2} L M - \frac{1}{3} \left(\frac{1}{2} L \right) \left(\frac{1}{8} w_o L^2 \right) = 0$$

$$\frac{1}{6} w_o \left(\frac{1}{2} L - a \right)^3 + \frac{1}{8} w_o L^2 a - \frac{1}{2} L M - \frac{1}{48} w_o L^3 = 0$$

$$\frac{1}{2} L M = \frac{1}{6} w_o \left(\frac{1}{2} L - a \right)^3 - \frac{1}{48} w_o L^3 + \frac{1}{8} w_o L^2 a$$

$$\frac{1}{2} L M = \frac{1}{48} w_o (L - 2a)^3 - \frac{1}{48} w_o L^2 (L - 6a)$$

$$\frac{1}{2} L M = \frac{1}{48} w_o [(L - 2a)^3 - L^2 (L - 6a)]$$

$$M = \frac{w_o}{24L} [L^3 - 3L^2(2a) + 3L(2a)^2 - (2a)^3 - L^3 + 6L^2 a]$$

$$M = \frac{w_o}{24L} [L^3 - 6L^2a + 12La^2 - 8a^3 - L^3 + 6L^2a]$$

$$M = \frac{w_o}{24L} [12La^2 - 8a^3]$$

$$M = \frac{w_o}{24L} [4a^2(3L - 2a)]$$

$$M = \frac{w_o a^2}{6L} (3L - 2a) \quad \text{answer}$$

$$\delta_{max} = t_{A/C}$$

$$EI \delta_{max} = EI t_{A/C}$$

$$EI \delta_{max} = (Area_{AC}) \cdot \bar{X}_A$$

$$\begin{aligned} EI \delta_{max} &= \frac{1}{3}(\frac{1}{2}L - a) \left[\frac{1}{2}w_o(\frac{1}{2}L - a)^2 \right] \left[a + \frac{3}{4}(\frac{1}{2}L - a) \right] \\ &+ \frac{1}{2}(\frac{1}{2}L)(\frac{1}{2}w_oLa) \left[\frac{2}{3}(\frac{1}{2}L) \right] - \frac{1}{2}LM \left[\frac{1}{2}(\frac{1}{2}L) \right] \\ &- \frac{1}{3}(\frac{1}{2}L)(\frac{1}{8}w_oL^2) \left[\frac{3}{4}(\frac{1}{2}L) \right] \end{aligned}$$

$$\begin{aligned} EI \delta_{max} &= \frac{1}{6}w_o(\frac{1}{2}L - a)^3(\frac{1}{4}a + \frac{3}{8}L) + \frac{1}{8}w_oL^2a(\frac{1}{3}L) \\ &- \frac{1}{2}LM(\frac{1}{4}L) - \frac{1}{48}w_oL^3(\frac{3}{8}L) \end{aligned}$$

$$EI \delta_{max} = \frac{1}{384}w_o(L - 2a)^3(2a + 3L) + \frac{1}{24}w_oL^3a - \frac{1}{8}L^2M - \frac{1}{128}w_oL^4$$

$$\begin{aligned} EI \delta_{max} &= \frac{1}{384}w_o [L^3 - 3L^2(2a) + 3L(2a)^2 - (2a)^3](2a + 3L) \\ &+ \frac{1}{24}w_oL^3a - \frac{1}{8}L^2M - \frac{1}{128}w_oL^4 \end{aligned}$$

$$\begin{aligned} EI \delta_{max} &= \frac{1}{384}w_o(L^3 - 6L^2a + 12La^2 - 8a^3)(2a + 3L) \\ &+ \frac{1}{24}w_oL^3a - \frac{1}{8}L^2M - \frac{1}{128}w_oL^4 \end{aligned}$$

$$\begin{aligned} EI \delta_{max} &= \frac{1}{384}w_o(2L^3a - 12L^2a^2 + 24La^3 - 16a^4 + 3L^4 - 18L^3a + 36L^2a^2 \\ &- 24La^3) + \frac{1}{24}w_oL^3a - \frac{1}{8}L^2M - \frac{1}{128}w_oL^4 \end{aligned}$$

$$EI \delta_{max} = \frac{1}{384}w_o(24L^2a^2 - 16a^4 + 3L^4 - 16L^3a) + \frac{1}{24}w_oL^3a - \frac{1}{8}L^2M - \frac{1}{128}w_oL^4$$

$$\begin{aligned} EI \delta_{max} &= (\frac{1}{16}w_oL^2a^2 - \frac{1}{24}w_oa^4 + \frac{1}{128}w_oL^4 - \frac{1}{24}w_oL^3a) + \frac{1}{24}w_oL^3a \\ &- \frac{1}{8}L^2M - \frac{1}{128}w_oL^4 \end{aligned}$$

$$EI \delta_{max} = \frac{1}{16}w_oL^2a^2 - \frac{1}{24}w_oa^4 - \frac{1}{8}L^2M$$

$$EI \delta_{max} = \frac{1}{16} w_o L^2 a^2 - \frac{1}{24} w_o a^4 - \frac{1}{8} L^2 \left[\frac{w_o a^2}{6L} (3L - 2a) \right]$$

$$EI \delta_{max} = \frac{1}{16} w_o L^2 a^2 - \frac{1}{24} w_o a^4 - \frac{1}{48} w_o L a^2 (3L - 2a)$$

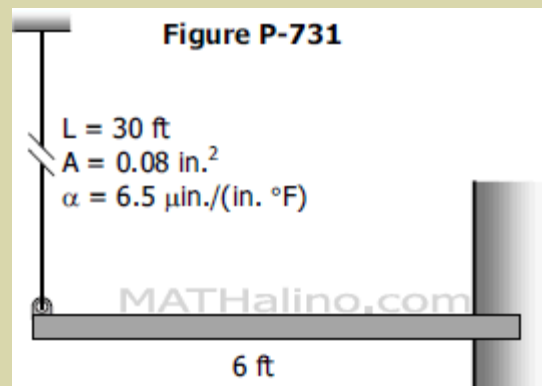
$$EI \delta_{max} = \frac{1}{16} w_o L^2 a^2 - \frac{1}{24} w_o a^4 - \frac{1}{16} w_o L^2 a^2 + \frac{1}{24} w_o a^3$$

$$EI \delta_{max} = -\frac{1}{24} w_o a^4 + \frac{1}{24} w_o L a^3$$

$$EI \delta_{max} = \frac{1}{24} w_o a^3 (L - a) \quad \text{answer}$$

Problem 731

The beam shown in Fig. P-731 is connected to a vertical rod. If the beam is horizontal at a certain temperature, determine the increase in stress in the rod if the temperature of the rod drops 90°F. Both the beam and the rod are made of steel with $E = 29 \times 10^6$ psi. For the beam, use $I = 154$ in.⁴



Solution

[HideClick here to show or hide the solution](#)

Assuming complete freedom for the rod, the deformation due to drop of temperature is...

$$\delta_T = \alpha L \Delta T$$

$$\delta_T = 0.0000065(30)(90)$$

$$\delta_T = 0.01755 \text{ ft} = 0.2106 \text{ in.}$$

subscript b (_b) = refers to the beam

subscript r (_r) = refers to the rod

$$\delta_b + \delta_r = \delta_T$$

$$\left(\frac{PL^3}{3EI} \right)_b + \left(\frac{PL}{AE} \right)_r = 0.2106$$

$$\left(\frac{PL^3}{3I} \right)_b + \left(\frac{PL}{A} \right)_r = 0.2106E$$

$$\frac{P(6^3)(12^3)}{3(154)} + \frac{P(30)(12)}{0.08} = 0.2106(29 \times 10^6)$$

$$807.90P + 4500P = 6107400$$

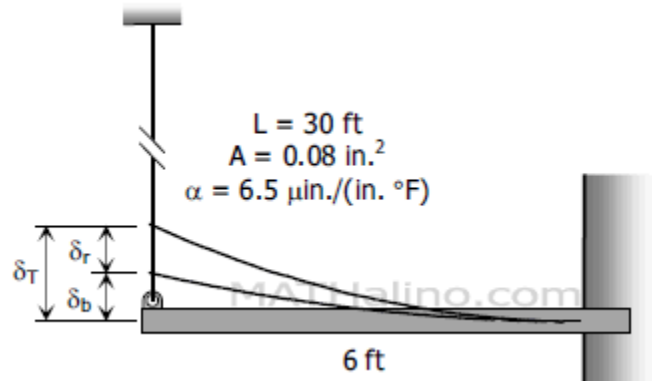
$$5307.9P = 6107400$$

$$P = 1150.62 \text{ lb} = 1.151 \text{ kips}$$

Stress increase on the rod

$$\sigma = \frac{P}{A} = \frac{1.151}{0.08}$$

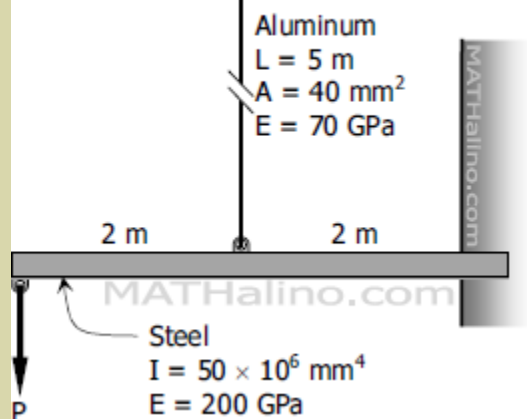
$$\sigma = 14.39 \text{ ksi} \quad \text{answer}$$



Problem 732

The midpoint of the steel in Fig. P-732 is connected to the vertical aluminum rod. Determine the maximum value of P if the stress in the rod is not to exceed 120 MPa.

Figure P-732



Solution 732

[Click here to show or hide the solution](#)

For the aluminum rod

$$P_{al} = (\sigma A)_{al}$$

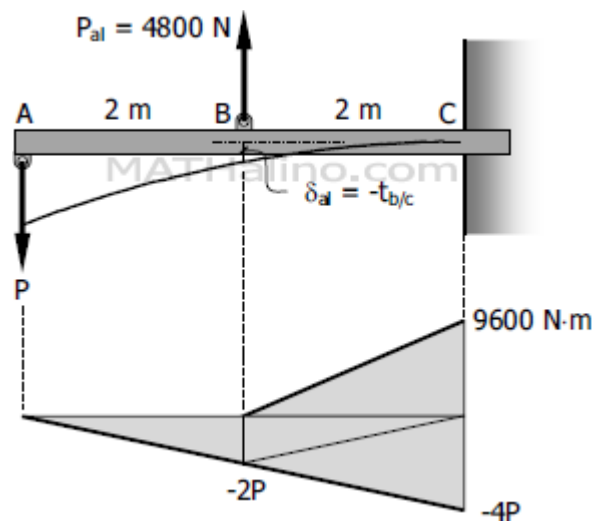
$$P_{al} = 120(40)$$

$$P_{al} = 4800 \text{ N}$$

$$\delta_{al} = \left(\frac{\sigma L}{E} \right)_{al}$$

$$\delta_{al} = \frac{120(5)(1000)}{70000}$$

$$\delta_{al} = \frac{60}{7} \text{ mm}$$



For the steel beam

$$t_{B/C} = \frac{1}{EI} (\text{Area}_{BC}) \cdot \bar{X}_B$$

$$t_{B/C} = \frac{1}{200\,000(50 \times 10^6)} \left[\frac{1}{2}(2)(9600)\left(\frac{4}{3}\right) - \frac{1}{2}(2)(2P)\left(\frac{2}{3}\right) - \frac{1}{2}(2)(4P)\left(\frac{4}{3}\right) \right] (1000^3)$$

$$t_{B/C} = \frac{1}{10\,000} (25\,600 - \frac{2}{3}P)$$

$$t_{B/C} = \frac{64}{25} - \frac{1}{1500}P$$

From the figure

$$\delta_{al} = -t_{B/C}$$

$$\frac{60}{7} = -(\frac{64}{25} - \frac{1}{1500}P)$$

$$\frac{1}{1500}P = \frac{1948}{175}$$

$$P = 16\,697.14 \text{ N}$$

$$P = 16.7 \text{ kN} \quad \text{answer}$$

Problem 733

The load P in [Prob. 732](#) is replaced by a counterclockwise couple M . Determine the maximum value of M if the stress in the vertical rod is not to exceed 150 MPa.

Solution 733

[Hide](#) [Click here to show or hide the solution](#)

For the aluminum rod

$$P_{al} = (\sigma A)_{al}$$

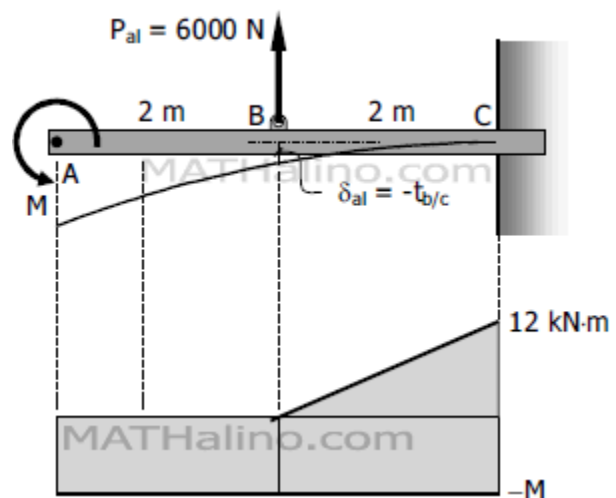
$$P_{al} = 150(40)$$

$$P_{al} = 6000 \text{ N}$$

$$\delta_{al} = \left(\frac{\sigma L}{E} \right)_{al}$$

$$\delta_{al} = \frac{150(5)(1000)}{70\,000}$$

$$\delta_{al} = \frac{75}{7} \text{ mm}$$



For the steel beam

$$t_{B/C} = \frac{1}{(EI)_{st}} (\text{Area}_{BC}) \cdot \bar{X}_B$$

$$t_{B/C} = \frac{1}{200\,000(50 \times 10^6)} \left[\frac{1}{2}(2)(12\,000)\left(\frac{4}{3} - 2M\right) \right] (1000^3)$$

$$t_{B/C} = \frac{1}{10\,000} (16\,000 - 2M)$$

$$t_{B/C} = \frac{8}{5} - \frac{1}{5000} M$$

From the figure

$$\delta_{at} = -t_{B/C}$$

$$\frac{75}{7} = -\left(\frac{8}{5} - \frac{1}{5000} M\right)$$

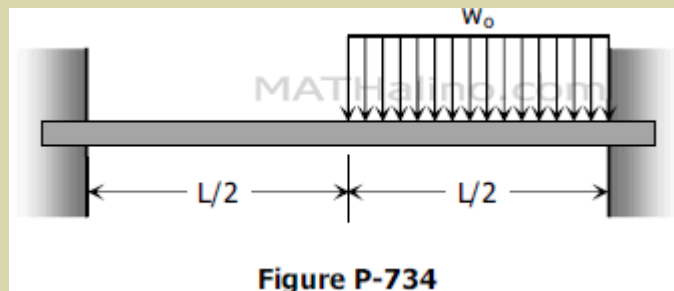
$$\frac{1}{5000} M = \frac{431}{35}$$

$$P = 61\,571.43 \text{ N}$$

$$P = 61.57 \text{ kN} \quad \text{answer}$$

Problem 734

Determine the end moments for the restrained beams shown in Fig. P-734.



Solution 734

[HideClick here to show or hide the solution](#)

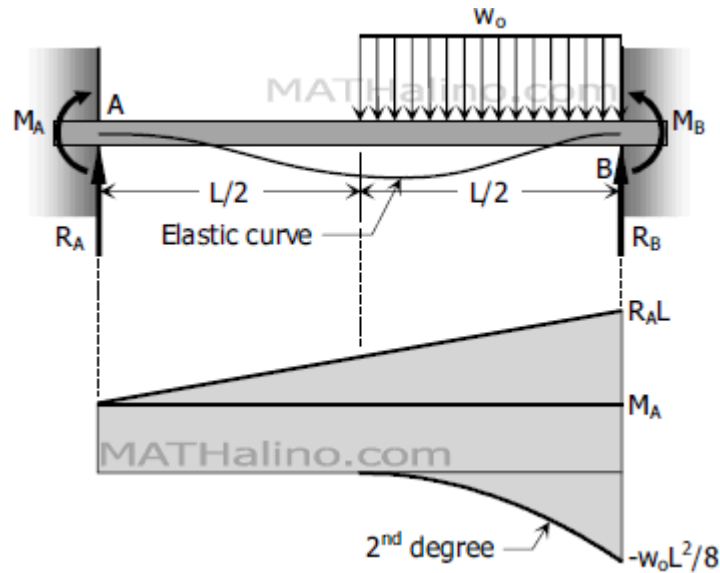
The angle between tangents through A and B is zero

$$EI \theta_{AB} = (\text{Area}_{AB}) = 0$$

$$\frac{1}{2} L (R_A L) + M A L - \frac{1}{3} \left(\frac{1}{2} L \right) \left(\frac{1}{8} w_o L^2 \right) = 0$$

$$\frac{1}{2}R_A L^2 + M_A L - \frac{1}{48}w_o L^3 = 0$$

$$M_A = \frac{1}{48}w_o L^2 - \frac{1}{2}R_A L \rightarrow \text{equation (1)}$$



The deviation of B from a tangent line through A is zero

$$EI t_{B/A} = (\text{Area}_{AB}) \cdot \bar{X}_B = 0$$

$$\frac{1}{2}L(R_A L)\left(\frac{1}{3}L\right) + M_A L\left(\frac{1}{2}L\right) - \frac{1}{3}\left(\frac{1}{2}L\right)\left(\frac{1}{8}w_o L^2\right)\left[\frac{1}{4}\left(\frac{1}{2}L\right)\right] = 0$$

$$\frac{1}{6}R_A L^3 + \frac{1}{2}M_A L^2 - \frac{1}{384}w_o L^4 = 0$$

$$64R_A L + 192M_A = w_o L^2 \rightarrow \text{equation (2)}$$

Substitute \$M_A\$ of equation (1) to equation (2) above

$$64R_A L + 192\left(\frac{1}{48}w_o L^2 - \frac{1}{2}R_A L\right) = w_o L^2$$

$$64R_A L + 4w_o L^2 - 96R_A L = w_o L^2$$

$$32R_A L = 3w_o L^2$$

$$R_A = \frac{3}{32}w_o L$$

Substitute R_A to equation (1)

$$M_A = \frac{1}{48}w_o L^2 - \frac{1}{2}\left(\frac{3}{32}w_o L\right)L$$

$$M_A = \frac{1}{48}w_o L^2 - \frac{3}{64}w_o L^2$$

$$M_A = -\frac{5}{192}w_o L^2 \quad \text{answer}$$

Sum up moments at right support B

$$M_B = M_A + R_A L - \frac{1}{8}w_o L^2 \rightarrow \text{see the right end of moment diagram by parts}$$

$$M_B = -\frac{5}{192}w_o L^2 + \left(\frac{3}{32}w_o L\right)L - \frac{1}{8}w_o L^2$$

$$M_B = -\frac{5}{192}w_o L^2 + \frac{3}{32}w_o L^2 - \frac{1}{8}w_o L^2$$

$$M_B = -\frac{11}{192}w_o L^2 \quad \text{answer}$$

Problem 735

The beam shown in Fig. P-735 is perfectly restrained at A but only partially restrained at B, where the slope is $w_o L^3/48EI$ directed up to the right. Solve for the end moments.

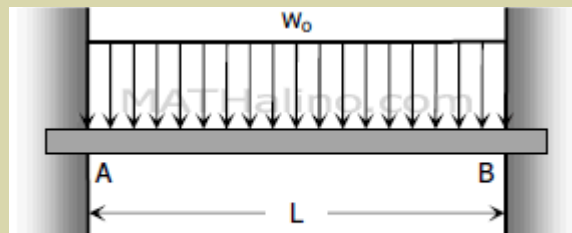


Figure P-735

Solution 735

[HideClick here to show or hide the solution](#)

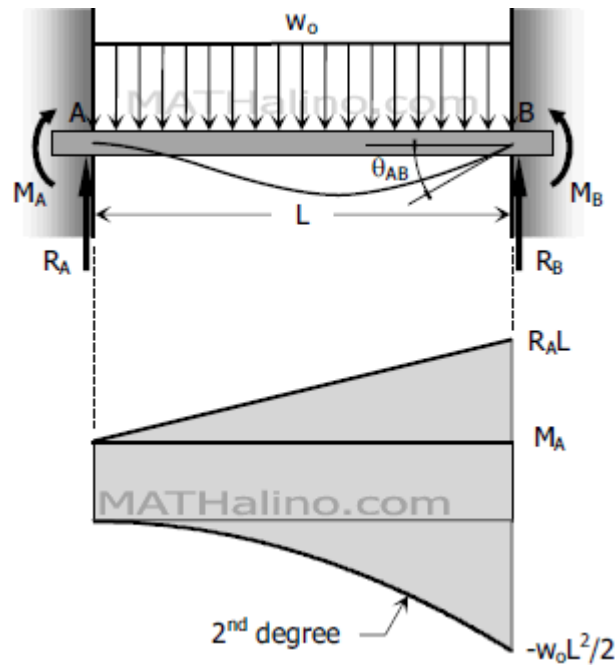
$$EI \theta_{AB} = (Area_{AB})$$

$$EI \left(\frac{w_o L^3}{48EI} \right) = \frac{1}{2}(R_A L)L + M_A L - \frac{1}{3}\left(\frac{1}{2}w_o L^2\right)L$$

$$\frac{1}{48}w_o L^3 = \frac{1}{2}R_A L^2 + M_A L - \frac{1}{6}w_o L^3$$

$$M_A L = \frac{3}{16} w_o L^3 - \frac{1}{2} R_A L^2$$

$$M_A = \frac{3}{16} w_o L^2 - \frac{1}{2} R_A L \rightarrow \text{Equation (1)}$$



$$EI t_{B/A} = (\text{Area}_{AB}) \cdot \bar{X}_B = 0$$

$$\frac{1}{2} (R_A L) L \left(\frac{1}{3} L \right) + M_A L \left(\frac{1}{2} L \right) - \frac{1}{3} \left(\frac{1}{2} w_o L^2 \right) L \left(\frac{1}{4} L \right) = 0$$

$$\frac{1}{6} R_A L^3 + \frac{1}{2} M_A L^2 - \frac{1}{24} w_o L^4 = 0$$

$$\frac{1}{6} R_A L + \frac{1}{2} M_A - \frac{1}{24} w_o L^2 = 0$$

Substitute M_A defined in equation (1)

$$\frac{1}{6} R_A L + \frac{1}{2} \left(\frac{3}{16} w_o L^2 - \frac{1}{2} R_A L \right) - \frac{1}{24} w_o L^2 = 0$$

$$\frac{1}{6} R_A L + \frac{3}{32} w_o L^2 - \frac{1}{4} R_A L - \frac{1}{24} w_o L^2 = 0$$

$$\frac{1}{12} R_A L = \frac{5}{96} w_o L^2$$

$$R_A = \frac{5}{8} w_o L$$

From equation (1)

$$M_A = \frac{3}{16}w_o L^2 - \frac{1}{2}\left(\frac{5}{8}w_o L\right)L$$

$$M_A = \frac{3}{16}w_o L^2 - \frac{5}{16}w_o L^2$$

$$M_A = -\frac{1}{8}w_o L^2 \quad \text{answer}$$

From moment diagram by parts

$$M_B = R_A L + M_A - \frac{1}{2}w_o L^2$$

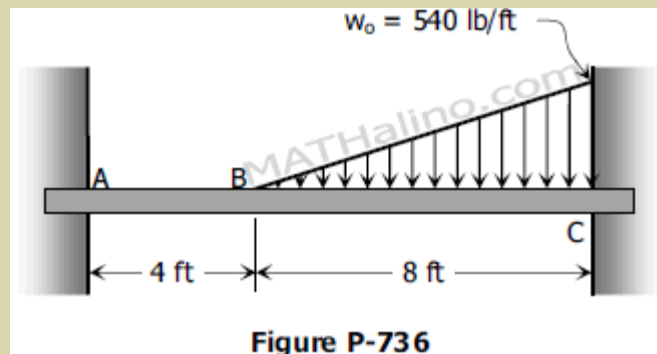
$$M_B = \left(\frac{5}{8}w_o L\right)L - \frac{1}{8}w_o L^2 - \frac{1}{2}w_o L^2$$

$$M_B = \frac{5}{8}w_o L^2 - \frac{1}{8}w_o L^2 - \frac{1}{2}w_o L^2$$

$$M_B = 0 \quad \text{answer}$$

Problem 736

Determine the end shears and end moments for the restrained beam shown in Fig. P-736 and sketch the shear and moment diagrams.



Solution 736

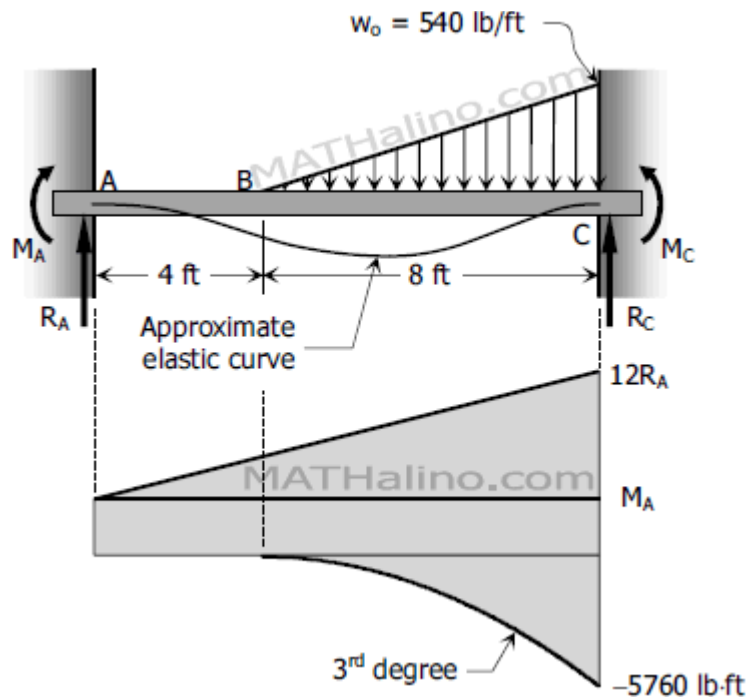
[HideClick here to show or hide the solution](#)

$$EI \theta_{AC} = (\text{Area}_{AC}) = 0$$

$$\frac{1}{2}(12)(12R_A) + 12M_A - \frac{1}{4}(8)(5760) = 0$$

$$72R_A + 12M_A - 11520 = 0$$

$$6R_A + M_A = 960 \rightarrow \text{Equation (1)}$$



$$EI t_{C/A} = (\text{Area}_{AC}) \cdot \bar{X}_C = 0$$

$$\frac{1}{2}(12)(12R_A)\left[\frac{1}{3}(12)\right] + 12M_A\left[\frac{1}{2}(12)\right] - \frac{1}{4}(8)(5760)\left[\frac{1}{5}(8)\right] = 0$$

$$288R_A + 72M_A - 18432 = 0$$

$$4R_A + M_A = 256 \rightarrow \text{Equation (2)}$$

Solving Equations (1) and (2)

$$R_A = 352 \text{ lb} \quad \text{answer}$$

$$M_A = -1152 \text{ lb}\cdot\text{ft} \quad \text{answer}$$

Checking

$$EI t_{A/C} = (\text{Area}_{AC}) \cdot \bar{X}_A = 0$$

$$\frac{1}{2}(12)(12R_A)\left[\frac{2}{3}(12)\right] + 12M_A\left[\frac{1}{2}(12)\right] - \frac{1}{4}(8)(5760)\left[4 + \frac{4}{5}(8)\right] = 0$$

$$576R_A + 72M_A - 119\,808 = 0$$

$$576(352) + 72(-1152) - 119\,808 = 0$$

$$0 = 0 \quad \text{okay!}$$

$$M_C = 12R_A + M_A - 5760$$

$$M_C = 12(352) - 1152 - 5760$$

$$M_C = -2688 \text{ lb} \cdot \text{ft} \quad \text{answer}$$

$$\Sigma F_V = 0$$

$$R_A + R_C = \frac{1}{2}(8)(540)$$

$$352 + R_C = 2160$$

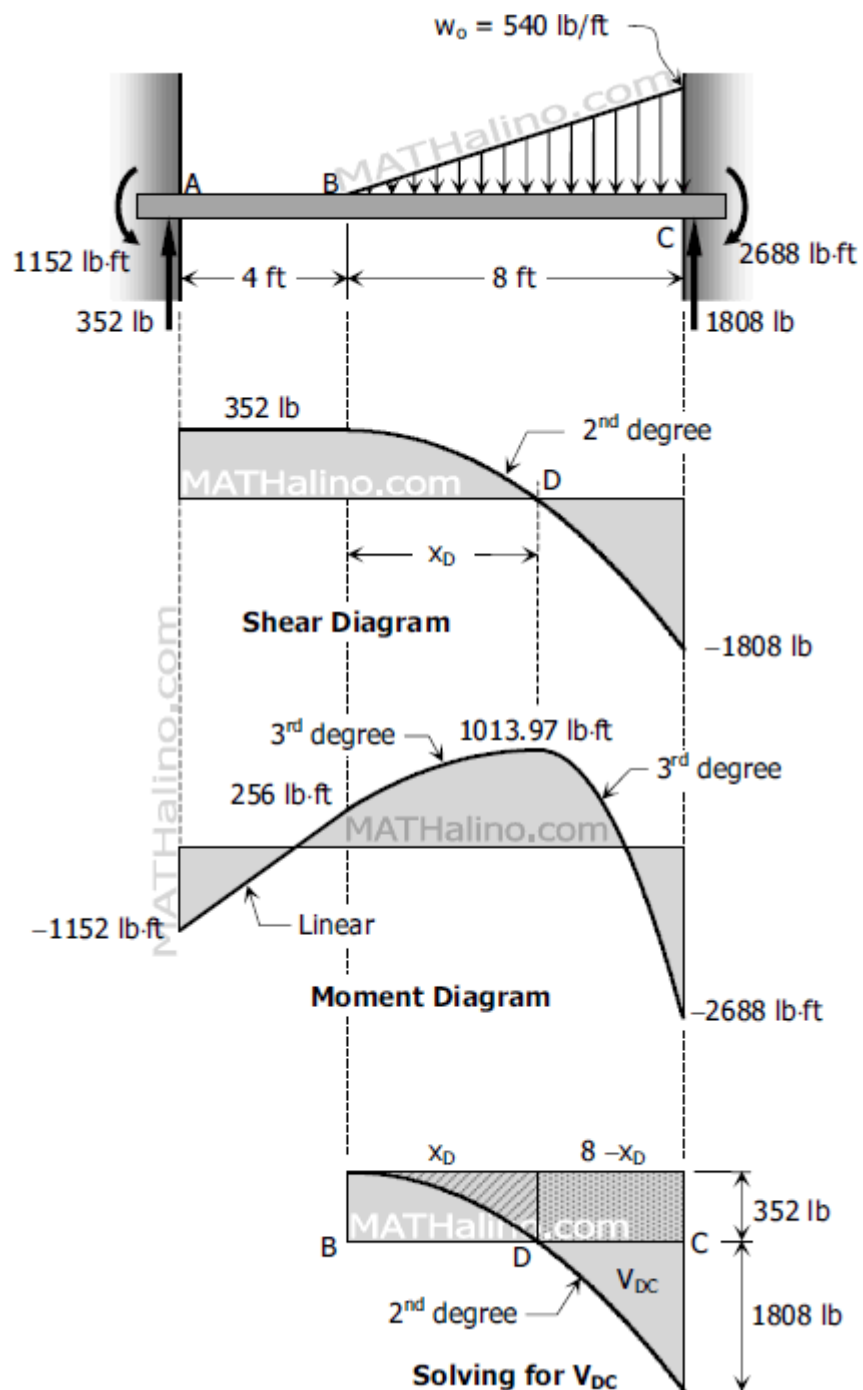
$$R_C = 1\,808 \text{ lb} \quad \text{answer}$$

To draw the shear diagram

1. $V_A = 352 \text{ lb}$
2. $V_B = V_A + \text{Load}_{AB}$
 $V_B = 352 + 0$
 $V_B = 352 \text{ lb}$
3. There is no load between AB, thus, shear in AB is constant.
4. $V_C = V_B + \text{Load}_{BC}$
 $V_C = 352 - (1/2)(8)(540)$
 $V_C = -1808 \text{ lb}$
5. Load between B and C is linearly decreasing from zero to -540 lb/ft, thus, shear in segment BC is a concave downward second degree curve (parabola) with vertex at B.
6. Location of point D by squared property of parabola:

$$\frac{x_D^2}{352} = \frac{8^2}{3521808}$$

$x_D = 3.23 \text{ ft}$ to the right of B



To draw the moment diagram

1. $M_A = -1152 \text{ lb}\cdot\text{ft}$
2. $M_B = M_A + V_{AB}$
 $M_B = -1152 + 352(4)$
 $M_B = 256 \text{ lb}\cdot\text{ft}$
3. The shear between A and B is uniform and positive, thus, the moment in AB is linear and increasing.
4. $M_D = M_B + V_{BD}$
 $M_D = 256 + (2/3)(x_D)(352)$
 $M_D = 256 + (2/3)(3.23)(352)$
 $M_D = 1013.97 \text{ lb}\cdot\text{ft}$
5. $M_C = M_D + V_{DC}$

Solving for VDC

$$V_{DC} = (-1/3)(8)(352 + 1808) + (1/3)(x_D)(352) + 352(8 - x_D)$$

$$V_{DC} = -5760 + (1/3)(3.23)(352) + 352(8 - 3.23)$$

$$V_{DC} = -3701.97 \text{ lb}$$

Thus,

$$M_C = 1013.97 - 3701.97$$

$$M_C = -2688 \text{ lb}\cdot\text{ft}$$

6. The shear diagram from B to C is a parabola, thus, the moment diagram of segment BC is a third degree curve. The value of shear from B to C decreases, thus, the slope of moment diagram between B and C also decreases making the cubic curve concave downward.

7. Problem 737

In the perfectly restrained beam shown in Fig. P-737, support B has settled a distance Δ below support A. Show that $M_B = -M_A = 6EI\Delta/L^2$.

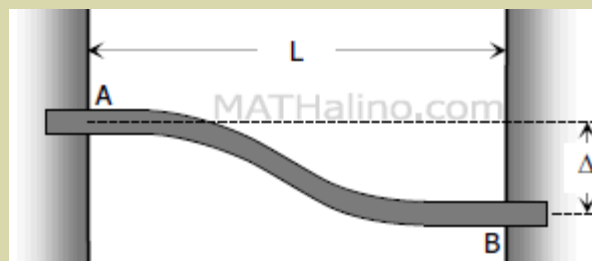


Figure P-737

8.

10. Solution 737

11. [Click here to show or hide the solution](#)

12. Rotation between A and B

$$\theta_{AB} = \frac{1}{EI}(\text{Area}_{AB}) = 0$$

$$13. \frac{1}{2}L(R_AL) + M_AL = 0$$

$$14. \frac{1}{2}R_AL^2 + M_AL = 0$$

$$15. R_AL + 2M_A = 0$$

$$16. R_A = -\frac{2M_A}{L} \rightarrow \text{equation (1)}$$

17. Deviation of B from tangent through A

$$t_{B/A} = \frac{1}{EI}(\text{Area}_{AB}) \cdot \bar{X}_B = -\Delta$$

$$18. \frac{1}{2}L(R_AL)\left(\frac{1}{3}L\right) + M_AL\left(\frac{1}{2}L\right) = -EI\Delta$$

$$19. \frac{1}{6}R_AL^3 + \frac{1}{2}M_AL^2 = -EI\Delta$$

$$20. R_AL^3 + 3M_AL^2 = -6EI\Delta$$

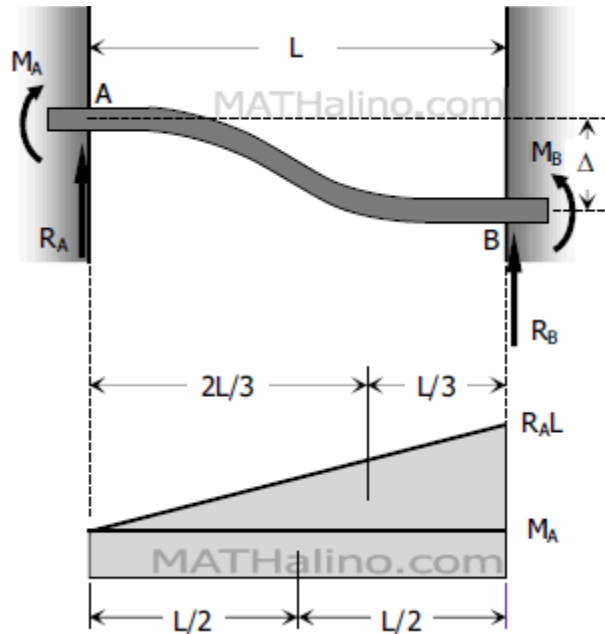
21. Substitute R_A defined in equation (1)

$$\left(-\frac{2M_A}{L}\right)L^3 + 3M_AL^2 = -6EI\Delta$$

$$22. -2M_AL^2 + 3M_AL^2 = -6EI\Delta$$

$$23. M_AL^2 = -6EI\Delta$$

$$24. M_A = -6EI\Delta/L^2$$



25. Substitute M_A to equation (1)

$$R_A = -\frac{2(-6EI\Delta/L^2)}{L}$$

26. $R_A = \frac{12EI\Delta}{L^3}$

27. Moment at B (See moment diagram by parts)

$$M_B = M_A + R_AL$$

28. $M_B = -6EI\Delta/L^2 + (12EI\Delta/L^3)L$

29. $M_B = -6EI\Delta/L^2 + 12EI\Delta/L^2$

30. $M_B = 6EI\Delta/L^2$

31. Thus,

$$M_B = -M_A = 6EI\Delta/L^2 \quad \text{okay!}$$

32. Problem 738

A perfectly restrained beam is loaded by a couple M applied where shown in Fig. P-738. Determine the end moments.

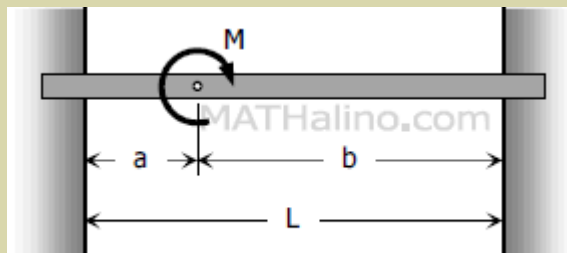


Figure P-738

33.

34.

35. Solution 738

36. [HideClick here to show or hide the solution](#)

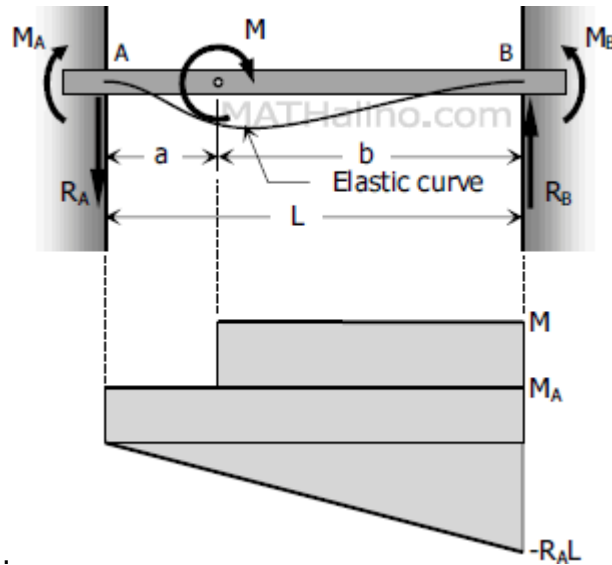
37. Rotation of AB is zero

$$EI\theta = (\text{Area}_{AB}) = 0$$

38. $Mb + M_AL - \frac{1}{2}L(R_AL) = 0$

$$39. Mb + M_A L = \frac{1}{2} R_A L^2$$

$$40. R_A = \frac{2(Mb + M_A L)}{L^2} \rightarrow \text{equation (1)}$$



41.

42.

43. Deviation of B from a tangent line through A is zero

$$EI t_{B/A} = (\text{Area}_{AB}) \bar{X}_B = 0$$

$$44. Mb(\frac{1}{2}b) + M_A L(\frac{1}{2}L) - \frac{1}{2}L(R_A L)(\frac{1}{3}L) = 0$$

$$45. \frac{1}{2}Mb^2 + \frac{1}{2}M_A L^2 - \frac{1}{6}R_A L^3 = 0$$

$$46. 3Mb^2 + 3M_A L^2 - R_A L^3 = 0$$

47. Substitute R_A of equation (1) to the equation above

$$3Mb^2 + 3M_A L^2 - \left[\frac{2(Mb + M_A L)}{L^2} \right] L^3 = 0$$

$$48. 3Mb^2 + 3M_A L^2 - 2MbL - 2M_A L^2 = 0$$

$$49. M_A L^2 = 2MbL - 3Mb^2$$

$$50. M_A L^2 = Mb(2L - 3b)$$

$$51. M_A L^2 = Mb[2L - 3(L - a)]$$

$$52. M_A L^2 = Mb(3a - L)$$

$$53. M_A = \frac{Mb}{L^2}(3a - L)$$

$$54. M_A = \frac{Mb}{L} \left(\frac{3a}{L} - 1 \right) \quad \text{answer}$$

55. From equation (1)

$$R_A = \frac{2Mb}{L^2} + \frac{2M_A}{L}$$

$$56. R_A = \frac{2Mb}{L^2} + \frac{2}{L} \left[\frac{Mb}{L} \left(\frac{3a}{L} - 1 \right) \right]$$

$$57. R_A = \frac{2Mb}{L^2} + \frac{6Mab}{L^3} - \frac{2Mb}{L^2}$$

$$58. R_A = \frac{6Mab}{L^3}$$

59. From moment diagram by parts

$$M_B = M + M_A - R_A L$$

$$60. M_B = M + \frac{Mb}{L} \left(\frac{3a}{L} - 1 \right) - \frac{6Mab}{L^3}(L)$$

$$61. M_B = M + \frac{3Mab}{L^2} - \frac{Mb}{L} - \frac{6Mab}{L^2}$$

$$62. M_B = M - \frac{Mb}{L} - \frac{3Mab}{L^2}$$

$$63. M_B = M - \frac{M(L - a)}{L} - \frac{3Mab}{L^2}$$

$$64. \quad M_B = M - M + \frac{Ma}{L} - \frac{3Mab}{L^2}$$

$$65. \quad M_B = \frac{Ma}{L} - \frac{3Mab}{L^2}$$

$$66. \quad M_B = -\frac{Ma}{L} \left(\frac{3b}{L} - 1 \right) \quad \text{answer}$$

Fixed-end moments of fully restrained beam

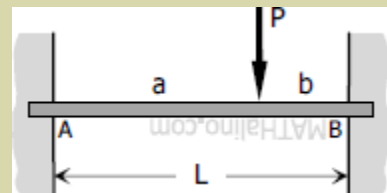
Summary for the value of end moments and deflection of perfectly restrained beam carrying various loadings. Note that for values of EIy , y is positive downward.

Case 1: Concentrated load anywhere on the span of fully restrained beam

End moments

$$M_A = -\frac{Pab^2}{L^2}$$

$$M_B = -\frac{Pa^2b}{L^2}$$



Value of EIy

$$\text{Midspan } EIy = \frac{Pb^2}{48} (3L - 4b)$$

Note: only for $b > a$

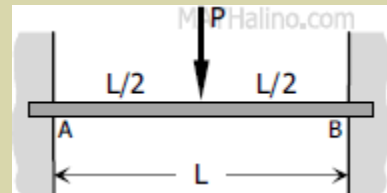
Case 2: Concentrated load on the midspan of fully restrained beam

End moments

$$M_A = M_B = -\frac{PL}{8}$$

Value of EIy

$$\text{Maximum } EIy = \frac{PL^3}{192}$$

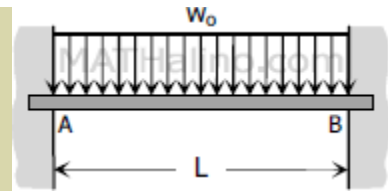


Case 3: Uniformly distributed load over the entire span of fully restrained beam

End moments

$$M_A = M_B = -\frac{w_o L^2}{12} = -\frac{WL}{12}$$

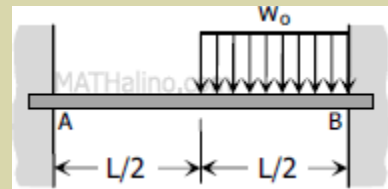
$$\text{Maximum } EI y = \frac{w_o L^4}{384} = \frac{WL^3}{384}$$



Case 4: Uniformly distributed load over half the span of fully restrained beam

$$M_A = -\frac{5w_o L^2}{192} = -\frac{5WL}{96}$$

$$M_B = -\frac{11w_o L^2}{192} = -\frac{11WL}{96}$$

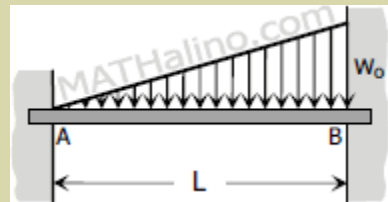


$$\text{Midspan } EI y = \frac{w_o L^4}{384} = \frac{WL^3}{384}$$

Case 5: Triangular load over the entire span of fully restrained beam

$$M_A = -\frac{w_o L^2}{30} = -\frac{WL}{15}$$

$$M_B = -\frac{w_o L^2}{20} = -\frac{WL}{10}$$

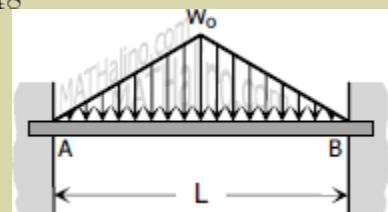


$$\text{Midspan } EI y = \frac{w_o L^4}{768} = \frac{WL^3}{384}$$

Case 6: Isosceles triangle loadings over the entire span of fully restrained beam

$$M_A = M_B = -\frac{5w_o L^2}{96} = -\frac{5WL}{48}$$

$$\text{Maximum } EI y = \frac{7w_o L^4}{3840} = \frac{7WL^3}{1920}$$



Case 7: Moment load anywhere on the span of fully restrained beam

$$M_A = \frac{Mb}{L} \left(\frac{3a}{L} - 1 \right)$$

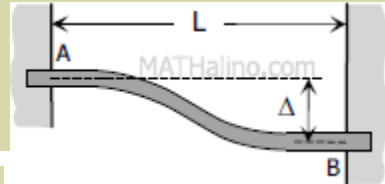
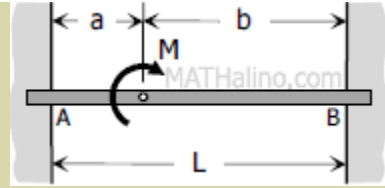
$$M_B = -\frac{Ma}{L} \left(\frac{3b}{L} - 1 \right)$$

Case 8: Fully restrained beam with one support settling

End moments

$$M_A = -\frac{6EI\Delta}{L^2}$$

$$M_B = \frac{6EI\Delta}{L^2}$$



Chapter 08 - Continuous Beams

Continuous beams are those that rest over three or more supports, thereby having one or more redundant support reactions.

These section includes

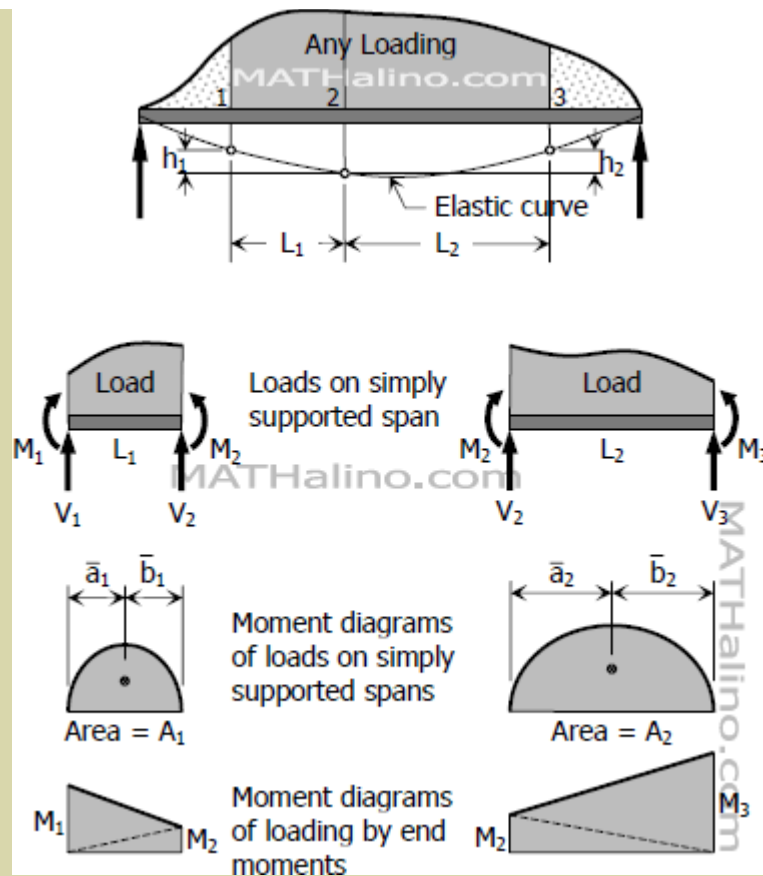
1. Generalized form of three-moment equation
2. Factors for three-moment equation
3. Application of the three-moment equation
4. Reactions of continuous beams
5. Shear and moment diagrams of continuous beams
6. Continuous beams with fixed ends
7. Deflection determined by three-moment equation
8. Moment distribution method

Explore the links under this page for available topics. Any topic not on the link only means that it is not yet available in this site.

The Three-Moment Equation

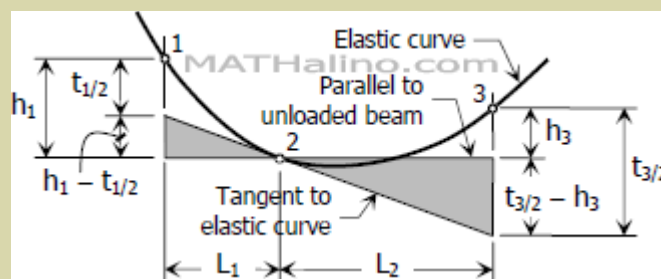
The three-moment equation gives us the relation between the moments between any three points in a beam and their relative vertical distances or deviations. This method is widely used in finding the reactions in a continuous beam.

Consider three points on the beam loaded as shown.



From proportions between similar triangles:

$$\frac{h_1 - t_{1/2}}{L_1} = \frac{t_{3/2} - h_3}{L_2}$$



$$\frac{h_1}{L_1} - \frac{t_{1/2}}{L_1} = \frac{t_{3/2}}{L_2} - \frac{h_3}{L_2}$$

$$\frac{t_{1/2}}{L_1} + \frac{t_{3/2}}{L_2} = \frac{h_1}{L_1} + \frac{h_3}{L_2} \rightarrow \text{equation (1)}$$

Values of $t_{1/2}$ and $t_{3/2}$

$$t_{1/2} = \frac{1}{E_1 I_1} (\text{Area}_{1-2}) \cdot \bar{X}_1$$

$$t_{1/2} = \frac{1}{E_1 I_1} \left[A_1 \bar{a}_1 + \left(\frac{1}{2} M_1 L_1 \right) \left(\frac{1}{3} L_1 \right) + \left(\frac{1}{2} M_2 L_1 \right) \left(\frac{2}{3} L_1 \right) \right]$$

$$t_{1/2} = \frac{1}{6 E_1 I_1} (6 A_1 \bar{a}_1 + M_1 L_1^2 + 2 M_2 L_1^2)$$

$$t_{3/2} = \frac{1}{E_2 I_2} (\text{Area}_{2-3}) \cdot \bar{X}_3$$

$$t_{3/2} = \frac{1}{E_2 I_2} \left[A_2 \bar{b}_2 + \left(\frac{1}{2} M_2 L_2 \right) \left(\frac{2}{3} L_2 \right) + \left(\frac{1}{2} M_3 L_2 \right) \left(\frac{1}{3} L_2 \right) \right]$$

$$t_{3/2} = \frac{1}{6 E_2 I_2} (6 A_2 \bar{b}_2 + 2 M_2 L_2^2 + M_3 L_2^2)$$

Substitute $t_{1/2}$ and $t_{3/2}$ to equation (1)

$$\frac{1}{6 E_1 I_1} \left(\frac{6 A_1 \bar{a}_1}{L_1} + M_1 L_1 + 2 M_2 L_1 \right) + \frac{1}{6 E_2 I_2} \left(\frac{6 A_2 \bar{b}_2}{L_2} + 2 M_2 L_2 + M_3 L_2 \right) = \frac{h_1}{L_1} + \frac{h_3}{L_2}$$

Multiply both sides by 6

$$\frac{1}{E_1 I_1} \left(\frac{6 A_1 \bar{a}_1}{L_1} + M_1 L_1 + 2 M_2 L_1 \right) + \frac{1}{E_2 I_2} \left(\frac{6 A_2 \bar{b}_2}{L_2} + 2 M_2 L_2 + M_3 L_2 \right) = 6 \left(\frac{h_1}{L_1} + \frac{h_3}{L_2} \right)$$

Distribute $1/EI$

$$\frac{6 A_1 \bar{a}_1}{E_1 I_1 L_1} + \frac{M_1 L_1}{E_1 I_1} + \frac{2 M_2 L_1}{E_1 I_1} + \frac{6 A_2 \bar{b}_2}{E_2 I_2 L_2} + \frac{2 M_2 L_2}{E_2 I_2} + \frac{M_3 L_2}{E_2 I_2} = 6 \left(\frac{h_1}{L_1} + \frac{h_3}{L_2} \right)$$

Combine similar terms and rearrange

$$\frac{M_1 L_1}{E_1 I_1} + 2 M_2 \left(\frac{L_1}{E_1 I_1} + \frac{L_2}{E_2 I_2} \right) + \frac{M_3 L_2}{E_2 I_2} + \frac{6 A_1 \bar{a}_1}{E_1 I_1 L_1} + \frac{6 A_2 \bar{b}_2}{E_2 I_2 L_2} = 6 \left(\frac{h_1}{L_1} + \frac{h_3}{L_2} \right)$$

If E is constant this equation becomes,

$$\frac{M_1 L_1}{I_1} + 2 M_2 \left(\frac{L_1}{I_1} + \frac{L_2}{I_2} \right) + \frac{M_3 L_2}{I_2} + \frac{6 A_1 \bar{a}_1}{I_1 L_1} + \frac{6 A_2 \bar{b}_2}{I_2 L_2} = 6 E \left(\frac{h_1}{L_1} + \frac{h_3}{L_2} \right)$$

If E and I are constant then,

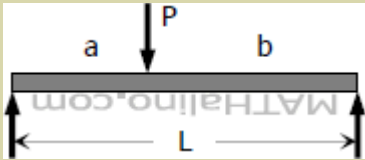
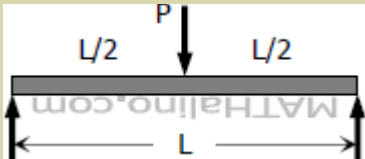
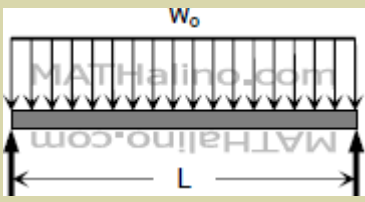
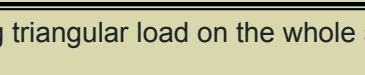
$$M_1 L_1 + 2M_2(L_1 + L_2) + M_3 L_2 + \frac{6A_1 \bar{a}_1}{L_1} + \frac{6A_2 \bar{b}_2}{L_2} = 6EI \left(\frac{h_1}{L_1} + \frac{h_3}{L_2} \right)$$

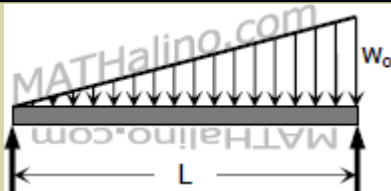
For the application of three-moment equation to continuous beam, points 1, 2, and 3 are usually unsetting supports, thus h_1 and h_3 are zero. With E and I constants, the equation will reduce to

$$M_1 L_1 + 2M_2(L_1 + L_2) + M_3 L_2 + \frac{6A_1 \bar{a}_1}{L_1} + \frac{6A_2 \bar{b}_2}{L_2} = 0$$

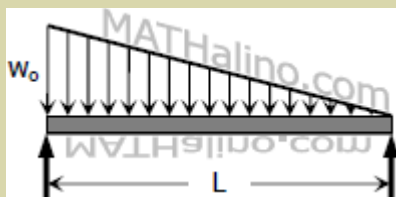
Factors for the three-moment equation

The table below list the value of $6A\bar{a}/L$ and $6A\bar{b}/L$ for different types of loading.

Type of Loading	$\frac{6A\bar{a}}{L}$	$\frac{6A\bar{b}}{L}$
Concentrated load anywhere on the span. 	$\frac{Pa}{L}(L^2 - a^2)$	$\frac{Pb}{L}(L^2 - b^2)$
Concentrated load at the midspan. 	$\frac{3PL^2}{8}$	$\frac{3PL^2}{8}$
Uniform load over the entire span. 	$\frac{w_o L^3}{4}$	$\frac{w_o L^3}{4}$
Increasing triangular load on the whole span. 	$\frac{8w_o L^3}{60}$	$\frac{7w_o L^3}{60}$



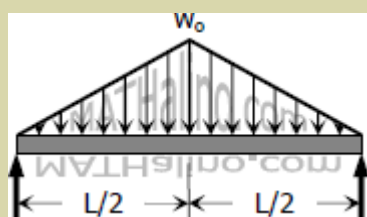
Decreasing triangular load on the whole span.



$$\frac{7w_oL^3}{60}$$

$$\frac{8w_oL^3}{60}$$

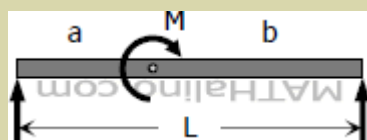
Isosceles triangular load over the entire span.



$$\frac{5w_oL^3}{32}$$

$$\frac{5w_oL^3}{32}$$

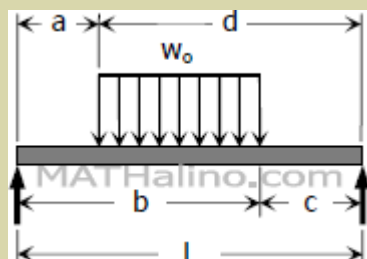
Moment load at any point on the span.



$$-\frac{M}{L}(3a^2 - L^2)$$

$$+\frac{M}{L}(3b^2 - L^2)$$

General uniform loading.

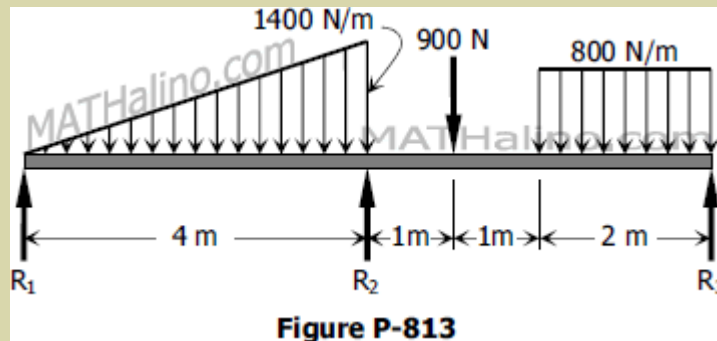


$$\frac{6A\bar{a}}{L} = \frac{w_o}{4L} [b^2(2L^2 - b^2) - a^2(2L^2 - a^2)]$$

$$\frac{6A\bar{b}}{L} = \frac{w_o}{4L} [d^2(2L^2 - d^2) - c^2(2L^2 - c^2)]$$

Problem 813

Determine the moment over the support R_2 of the beam shown in Fig. P-813.



Solution 813

[Click here to show or hide the solution](#)

$$M_1 L_1 + 2M_2(L_1 + L_2) + M_3 L_2 + \frac{6A_1 \bar{a}_1}{L_1} + \frac{6A_2 \bar{b}_2}{L_2} = 0$$

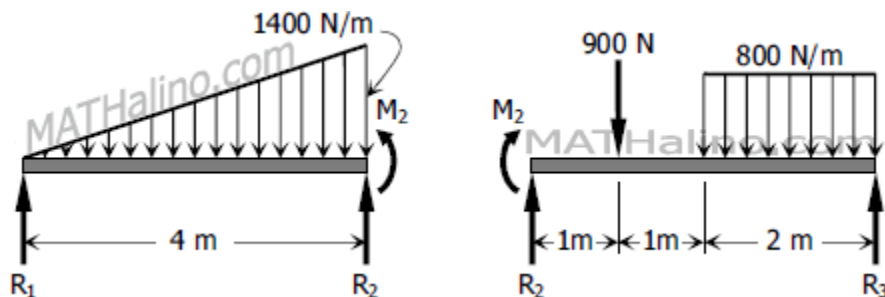
Where

$$M_1 = M_3 = 0$$

$$L_1 = L_2 = 4 \text{ m}$$

$$\frac{6A_1 \bar{a}_1}{L_1} = \frac{8w_o L^3}{60} = \frac{8(1400)(4^3)}{60}$$

$$\frac{6A_1 \bar{a}_1}{L_1} = 11\,946.67 \text{ N} \cdot \text{m}^2$$



$$\frac{6A_2\bar{b}_2}{L_2} = \frac{Pb}{L}(L^2 - b^2) + \frac{w_0d^2}{4L}(2L^2 - d^2)$$

$$\frac{6A_2\bar{b}_2}{L_2} = \frac{900(3)}{4}(4^2 - 3^2) + \frac{800(2^2)}{4(4)}[2(4^2) - 2^2]$$

$$\frac{6A_2\bar{b}_2}{L_2} = 10\,325 \text{ N} \cdot \text{m}^2$$

Thus,

$$0 + 2M_2(4 + 4) + 0 + 11\,946.67 + 10\,325$$

$$M_2 = -1391.98 \text{ N} \cdot \text{m} \quad \text{answer}$$

Problem 814

Find the moment at R_2 of the continuous beam shown in Fig. P-814.

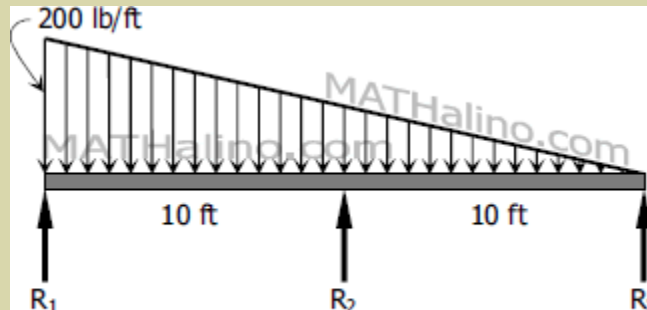


Figure P-814

Solution 814

[HideClick here to show or hide the solution](#)

$$M_1L_1 + 2M_2(L_1 + L_2) + M_3L_2 + \frac{6A_1\bar{a}_1}{L_1} + \frac{6A_2\bar{b}_2}{L_2} = 0$$

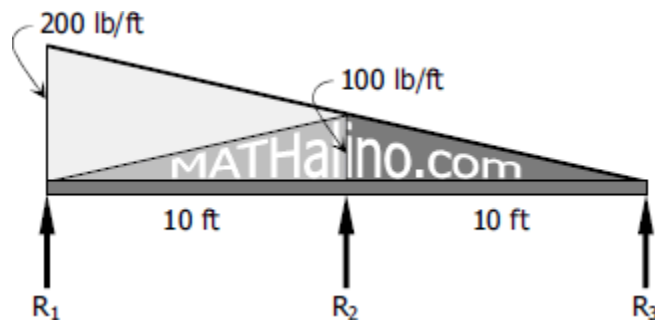
Where

$$M_1 = M_3 = 0$$

$$L_1 = L_2 = 10 \text{ ft}$$

$$\frac{6A_1\bar{a}_1}{L_1} = \frac{7w_oL^3}{60} + \frac{8w_oL^3}{60} = \frac{7(200)(10^3)}{60} + \frac{8(100)(10^3)}{60}$$

$$\frac{6A_1\bar{a}_1}{L_1} = 36\,666.67 \text{ lb} \cdot \text{ft}^2$$



$$\frac{6A_2\bar{b}_2}{L_2} = \frac{8w_oL^3}{60} = \frac{8(100)(10^3)}{60}$$

$$\frac{6A_2\bar{b}_2}{L_2} = 13\,333.33 \text{ lb} \cdot \text{ft}^2$$

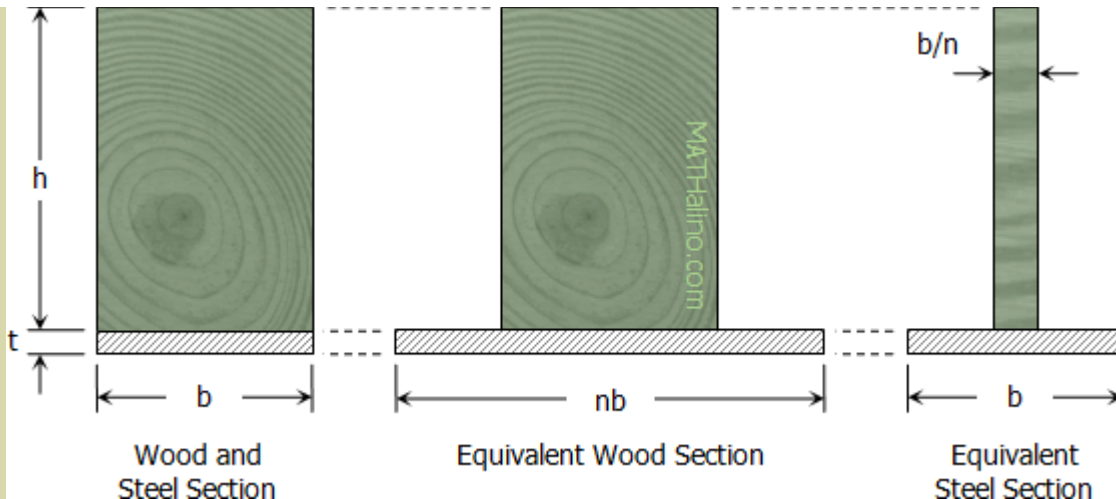
Thus,

$$0 + 2M_2(10 + 10) + 0 + 36\,666.67 + 13\,333.33 = 0$$

$$M_2 = -1250 \text{ lb} \cdot \text{ft} \quad \text{answer}$$

Chapter 10 - Reinforced Beams

Flexure formula do not apply directly to composite beams because it was based on the assumption that the beam was homogeneous. It is therefore necessary to transform the composite material into equivalent homogeneous section. To do this, consider a steel and wood section to be firmly bolted together so that they can act as one unit. Shown below are the composite wood and steel section and the corresponding equivalent in wood and steel sections.



The quantity n is the ratio of the moduli of elasticity of stronger material to the weaker material. In the above case, $n = E_s / E_w$.

The use of the above concept is governed by the following assumptions:

1. The composite materials are firmly bonded together and act as a unit.
2. The strain and load capacities of each material remain unchanged.
3. The strains of any two adjacent materials at their junction point are equal.
4. The loads carried by equivalent fibers are equal.
5. The equivalent fibers must be at the same distance from the neutral axis as the original fibers.
6. The flexure formula can be applied only to the equivalent section.

Beams with Different Materials

From assumption no. (3) in the previous page: The strains of any two adjacent materials at their junction point are equal.

$$\epsilon_s = \epsilon_w$$

$$\frac{f_{bs}}{E_s} = \frac{f_{bw}}{E_w}$$

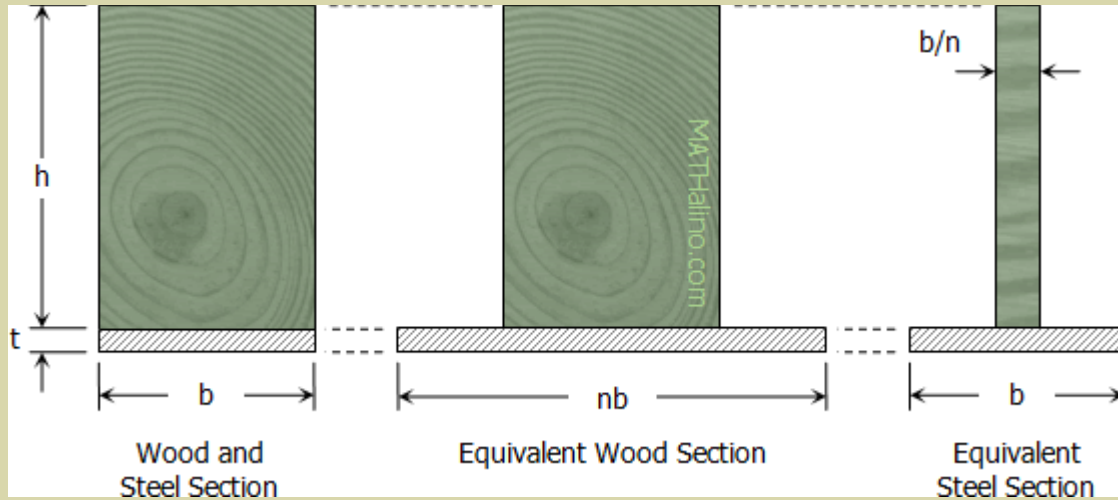
$$\frac{f_{bs}}{f_{bw}} = \frac{E_s}{E_w}$$

We let the moduli ratio be equal to n

$$n = \frac{E_s}{E_w}$$

It will follow also that

$$f_{bw} = \frac{f_{bs}}{n}$$



From assumption no. (4): The loads carried by equivalent fibers are equal.

$$P_w = P_s$$

$$f_{bw} A_w = f_{bs} A_s$$

$$A_w = \frac{f_{bs}}{f_{bw}} \cdot A_s$$

$$A_w = \frac{E_s}{E_w} \cdot A_s$$

The equivalent area of steel in wood is

$$A_w = n A_s$$

Problem 1002

A timber beam is reinforced with steel plates rigidly attached at the top and bottom as shown in Fig. P-1002. By what amount is moment increased by the reinforcement if $n = 15$ and the allowable stresses in the wood and steel are 8 MPa and 120 MPa, respectively?

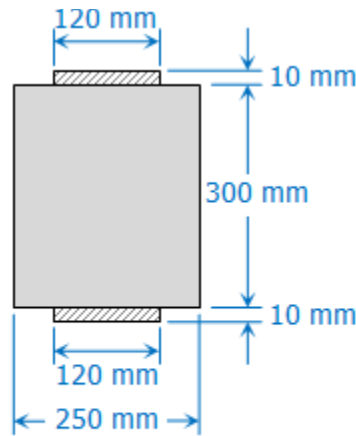


**Figure P-1002,
P-1003, & P-1004**

Solution 1002

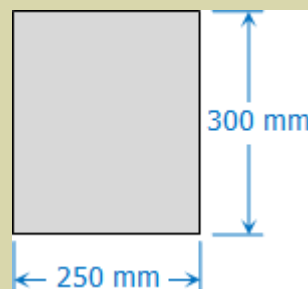
[HideClick here to show or hide the solution](#)

From the flexure formula

$$f_b = \frac{Mc}{I}$$

$$M = \frac{f_b I}{c}$$

Without the steel plate reinforcement



$$I = \frac{250(300^3)}{12} = 562\,500\,000 \text{ mm}^4$$

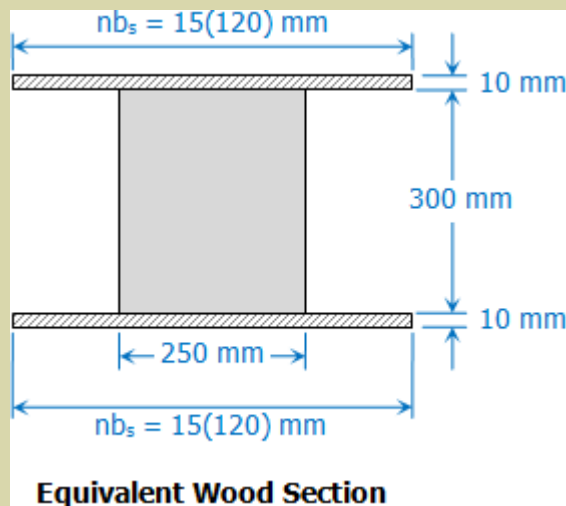
$$c = \frac{300}{2} = 150 \text{ mm}$$

$$M_1 = \frac{8(562\,500\,000)}{150}$$

$$M_1 = 30\,000\,000 \text{ N} \cdot \text{mm}$$

$$M_1 = 30 \text{ kN} \cdot \text{m}$$

With the steel plate reinforcement



$$I = \frac{15(120)(300 + 10 + 10)^3}{12} - \frac{[15(120) - 250](300^3)}{12} = 1\,427\,700\,000 \text{ mm}^4$$

$$c = \frac{300 + 10 + 10}{2} = 160 \text{ mm}$$

Convert steel to wood, use $f_{bw} = f_{bs} / n$

$$M_2 = \frac{\frac{120}{15}(1\,427\,700\,000)}{160}$$

$$M_2 = 71\,385\,000 \text{ N} \cdot \text{mm}$$

$$M_2 = 71.4 \text{ kN} \cdot \text{m}$$

Increase in moment capacity
 $\Delta M = M_2 - M_1 = 71.4 - 30$

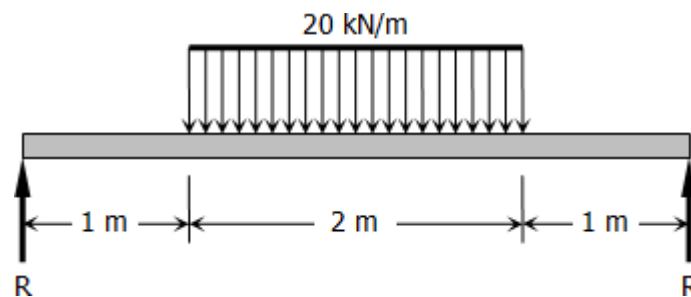
$\Delta M = 41.4 \text{ kN} \cdot \text{m}$ *answer*

Problem 1003

A simply supported beam 4 m long has the cross section shown in Fig. P-1002. It carries a uniformly distributed load of 20 kN/m over the middle half of the span. If $n = 15$, compute the maximum stresses in the wood and the steel.

Solution 1003

[Click here to show or hide the solution](#)



$$R = \frac{1}{2}(20)(2) = 20 \text{ kN}$$

Maximum moment will occur at the midspan:

$$M_{max} = 2R - 20(1)(0.5) = 2(20) - 10$$

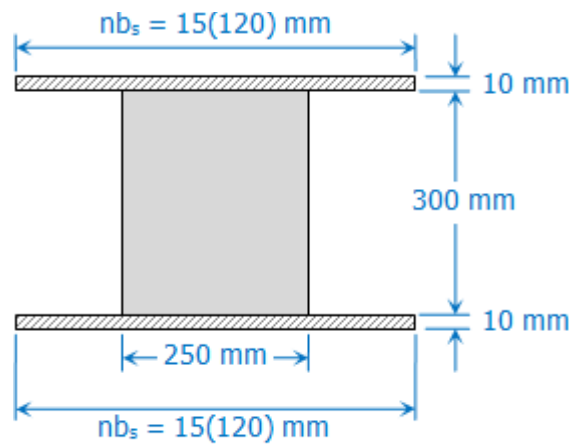
$$M_{max} = 30 \text{ kN} \cdot \text{m}$$

Flexural stress:

$$f_b = \frac{My}{I}$$

Where:

$$M = M_{max} = 30 \text{ kN} \cdot \text{m}$$



Equivalent Wood Section

$$I = \frac{15(120)(300 + 10 + 10)^3}{12} - \frac{[15(120) - 250](300^3)}{12}$$

$$I = 1\,427\,700\,000 \text{ mm}^4$$

Maximum stress in the wood ($y = 150 \text{ mm}$)

$$f_{bw} = \frac{30(150)(1000^2)}{1\,427\,700\,000}$$

$$f_{bw} = 3.152 \text{ MPa}$$

Maximum stress in the steel ($y = 160 \text{ mm}$)

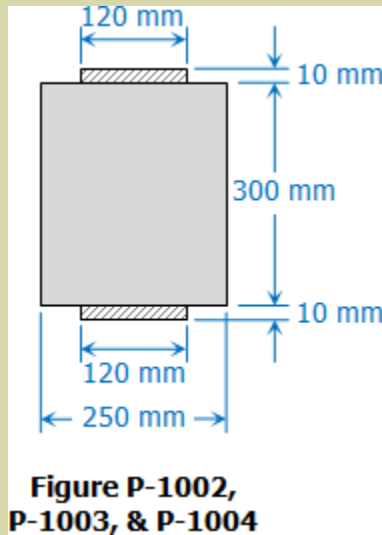
$$\frac{f_{bs}}{15} = \frac{30(160)(1000^2)}{1\,427\,700\,000}$$

$$f_{bs} = 47.279 \text{ MPa}$$

Problem 1004

Repeat [Problem 1002](#) assuming that the reinforcement consists of aluminum plates for which the

allowable stress is 80 MPa. Use $n = 5$.



Solution 1004

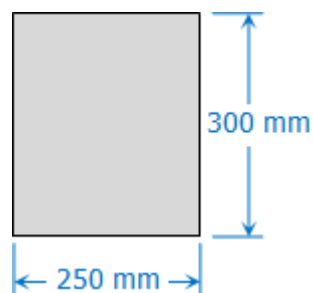
[HideClick here to show or hide the solution](#)

From the flexure formula

$$f_b = \frac{Mc}{I}$$

$$M = \frac{f_b I}{c}$$

Without the aluminum plate reinforcement



$$I = \frac{250(300^3)}{12} = 562\,500\,000 \text{ mm}^4$$

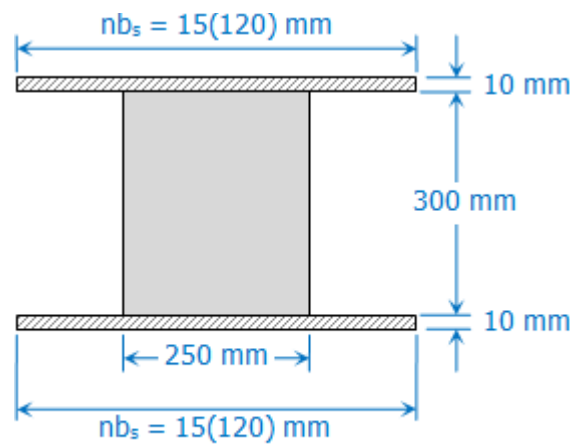
$$c = \frac{300}{2} = 150 \text{ mm}$$

$$M_1 = \frac{8(562\,500\,000)}{150}$$

$$M_1 = 30\,000\,000 \text{ N} \cdot \text{mm}$$

$$M_1 = 30 \text{ kN} \cdot \text{m}$$

With the aluminum plate reinforcement



Equivalent Wood Section

$$I = \frac{5(120)(300 + 10 + 10)^3}{12} - \frac{[5(120) - 250](300^3)}{12} = 850\,900\,000 \text{ mm}^4$$

$$c = \frac{300 + 10 + 10}{2} = 160 \text{ mm}$$

Convert aluminum to wood, use $f_{bw} = f_{ba} / n$

$$M_2 = \frac{\frac{80}{5}(850\,900\,000)}{160}$$

$$M_2 = 85\,090\,000 \text{ N} \cdot \text{mm}$$

$$M_2 = 85.09 \text{ kN} \cdot \text{m}$$

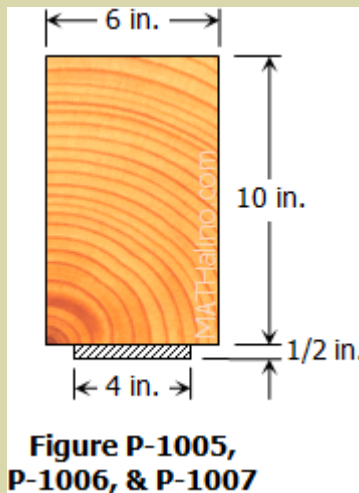
Increase in moment capacity

$$\Delta M = M_2 - M_1 = 85.09 - 30$$

$$\Delta M = 55.09 \text{ kN} \cdot \text{m} \quad \text{answer}$$

Problem 1005

A timber beam 6 in. by 10 in. is reinforced only at the bottom by a steel plate as shown in Fig. P-1005. Determine the concentrated load that can be applied at the center of a simply supported span 18 ft long if $n = 20$, $f_s \leq 18 \text{ ksi}$ and $f_w \leq 1200 \text{ psi}$. Show that the neutral axis is 7.1 in. below the top and that $I_{NA} = 1160 \text{ in.}^4$.



Solution 1005

[HideClick here to show or hide the solution](#)

From the equivalent wood section shown

$$A_w = 6(10) = 60 \text{ in}^2$$

$$A_s = 80(0.5) = 40 \text{ in}^2$$

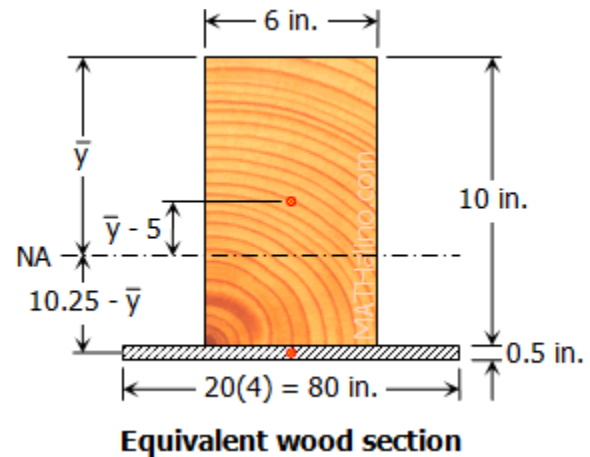
$$A = A_w + A_s = 100 \text{ in}^2$$

Location of NA from the top of the beam

$$A\bar{y} = A_w y_w + A_s y_s$$

$$100\bar{y} = 60(5) + 40(10.25)$$

$$\bar{y} = 7.1 \text{ in}$$



Moment of inertia of each section

$$I_g = \frac{bd^3}{12}$$

$$I_{gw} = \frac{6(10^3)}{12} = 500 \text{ in}^4$$

$$I_{gs} = \frac{80(0.5^3)}{12} = \frac{5}{6} \text{ in}^4$$

Moment of inertia about NA

$$I = \Sigma(I_g + Ad^2)$$

$$I_{NA} = [I_{gw} + A_w(\bar{y} - 5)^2] + [I_{gs} + A_s(10.25 - \bar{y})^2]$$

$$I_{NA} = [500 + 60(7.1 - 5)^2] + [\frac{5}{6} + 40(10.25 - 7.1)^2]$$

$$I_{NA} = \frac{3487}{3} \text{ in}^4 = 1162.33 \text{ in}^4$$

Maximum moment that the beam can carry

$$f_b = \frac{My}{I}$$

$$M = \frac{f_b I}{y}$$

Based on strength of wood ($y = \bar{y} = 7.1$ in)

$$M_w = \frac{1200\left(\frac{3487}{3}\right)}{7.1}$$

$$M_w = 196\,450.70 \text{ lb} \cdot \text{in}$$

$$M_w = 16\,370.89 \text{ lb} \cdot \text{ft}$$

Based on strength of wood ($y = 10.5 - \bar{y} = 3.4$ in)

$$M_s = \frac{\frac{18\,000}{20} \times \frac{3487}{3}}{3.4}$$

$$M_s = 307\,676.47 \text{ lb} \cdot \text{in}$$

$$M_s = 25\,639.70 \text{ lb} \cdot \text{ft}$$

For safe value of M , use the smaller of the two

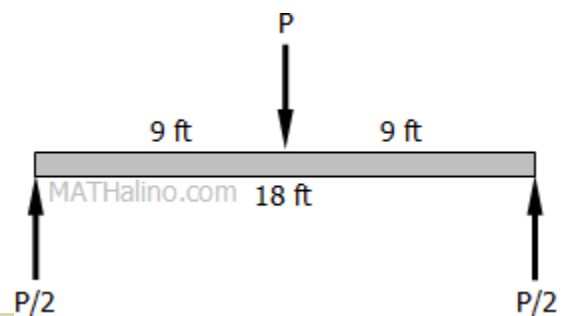
$$M = 16\,370.89 \text{ lb} \cdot \text{ft}$$

Safe value of P at the midspan that beam can carry

$$M = \frac{PL}{4}$$

$$16\,370.89 = \frac{P(18)}{4}$$

$$P = 3637.98 \text{ lb} \quad \text{answer}$$



Problem 1006

Determine the width b of the 1/2-in. steel plate fastened to the bottom of the beam in [Problem 1005](#) that will simultaneously stress the wood and the steel to their permissible limits of 1200 psi and 18 ksi, respectively.

Solution 1006

[HideClick here to show or hide the solution](#)

$$f_b = \frac{My}{I}$$

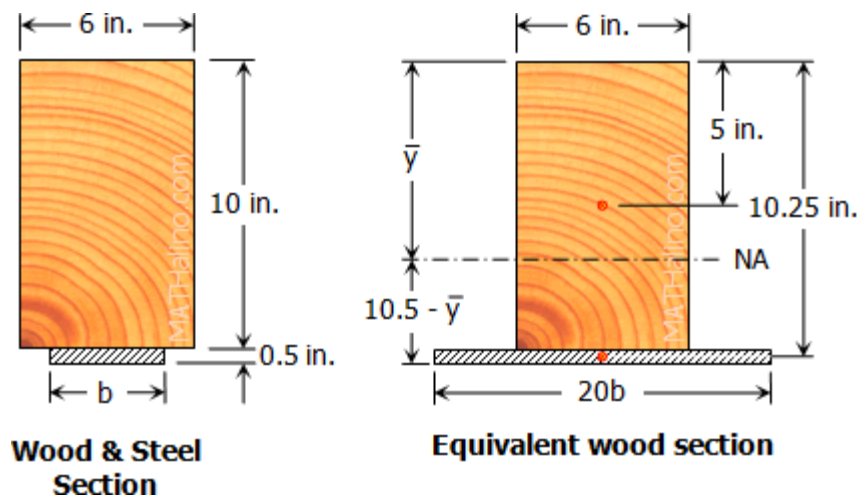
$$\frac{M}{I} = \frac{f_b}{y}$$

At the extreme wood fiber

$$\left(\frac{M}{I}\right)_{\text{wood}} = \frac{1200}{\bar{y}}$$

At the extreme steel fiber

$$\left(\frac{M}{I}\right)_{\text{steel}} = \frac{18\,000/20}{10.5 - \bar{y}} = \frac{900}{10.5 - \bar{y}}$$



$$\left(\frac{M}{I}\right)_{\text{wood}} = \left(\frac{M}{I}\right)_{\text{steel}}$$

$$\frac{1200}{\bar{y}} = \frac{900}{10.5 - \bar{y}}$$

$$1200(10.5 - \bar{y}) = 900\bar{y}$$

$$12\,600 - 1200\bar{y} = 900\bar{y}$$

$$2100\bar{y} = 12\,600$$

$$\bar{y} = 6 \text{ in}$$

You can also find $\bar{y}$ with the aid of stress diagram as follows:

[ShowClick here to show or hide the solution](#)

$$A_w = 6(10) = 60 \text{ in}^2$$

$$A_s = 0.5(20b) = 10b$$

$$A = A_w + A_s = 60 + 10b$$

$$A\bar{y} = A_w y_w + A_s y_s$$

$$(60 + 10b)(6) = 60(5) + 10b(10.25)$$

$$360 + 60b = 300 + 102.5b$$

$$42.5b = 60$$

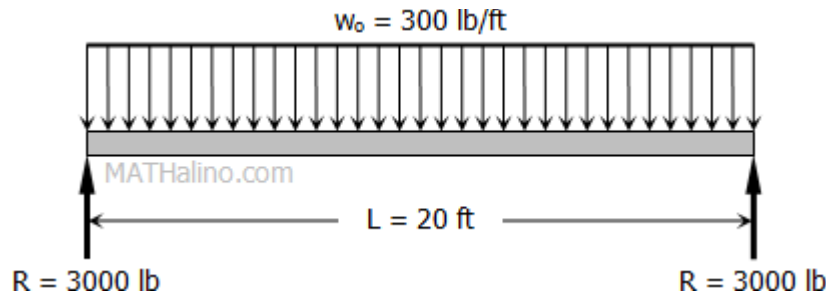
$$b = \frac{24}{17} \text{ in} = 1.41 \text{ in} \quad \text{answer}$$

Problem 1007

A uniformly distributed load of 300 lb/ft (including the weight of the beam) is simply supported on a 20-ft span. The cross section of the beam is described in [Problem 1005](#). If $n = 20$, determine the maximum stresses produced in the wood and the steel.

Solution 1007

[HideClick here to show or hide the solution](#)



Maximum moment will occur at the midspan

$$M_{max} = \frac{w_o L^2}{8} = \frac{300(20^2)}{8}$$

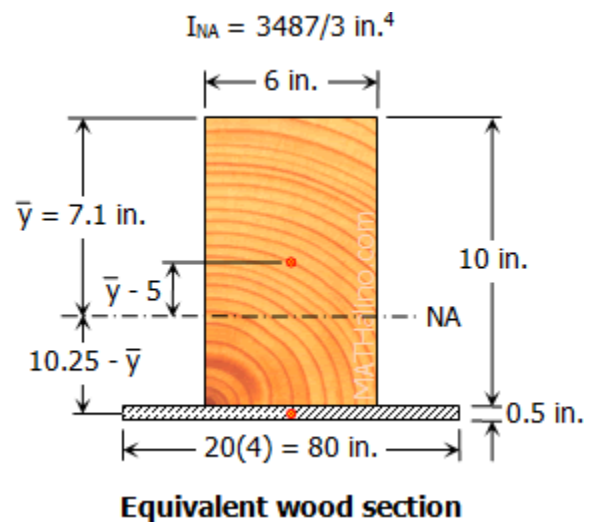
$$M_{max} = 15\,000 \text{ lb} \cdot \text{ft}$$

From [Solution 105](#):

$$\bar{y} = 7.1 \text{ in}$$

$$I_{NA} = \frac{3487}{3} \text{ in}^4$$

$$f_b = \frac{Mc}{I}$$



At the extreme wood fiber ($c = \bar{y}$)

$$f_{bw} = \frac{M_{max} \bar{y}}{I_{NA}}$$

$$f_{bw} = \frac{15\,000(7.1)(12)}{\frac{3487}{3}}$$

$$f_{bw} = 1099.51 \text{ psi} \quad \text{answer}$$

At the extreme steel fiber ($c = 10.5 - \bar{y}$)

$$\frac{f_{bs}}{n} = \frac{M_{max}(10.5 - \bar{y})}{I_{NA}}$$

$$\frac{f_{bs}}{20} = \frac{15\,000(10.5 - 7.1)(12)}{\frac{3487}{3}}$$

$$f_{bs} = 10\,530.54 \text{ psi} \quad \text{answer}$$

Problem 1008

A timber beam 150 mm wide by 250 mm deep is to be reinforced at the top and bottom by steel plates 10 mm thick. How wide should the steel plates be if the beam is to resist a moment of 40 kN·m? Assume that $n = 15$ and the allowable stresses in the wood and steel are 10 MPa and 120 MPa, respectively.

Solution 1008

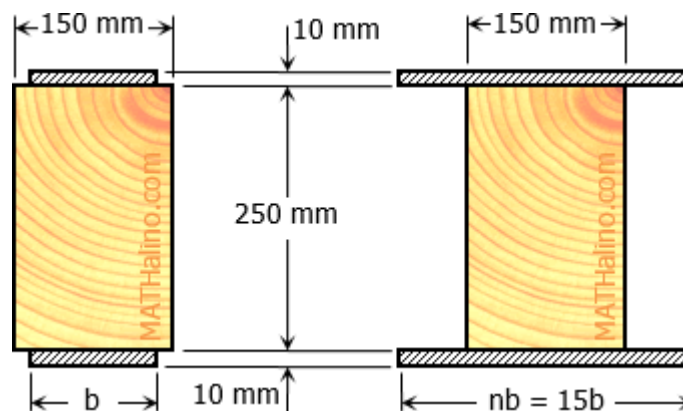
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$$M = 40 \text{ kN} \cdot \text{m}$$

$$I = \frac{15b(270^3)}{12} - \frac{(15b - 150)(250^3)}{12}$$

$$I = 24\,603\,750b - 19\,531\,250b + 195\,312\,500$$

$$I = 5\,072\,500b + 195\,312\,500$$



$$f_b = \frac{Mc}{I}$$

Based on allowable flexural stress of steel:

$$f_b = 120/n$$

$$c = \frac{1}{2}(250) + 10 = 135 \text{ mm}$$

Thus,

$$\frac{120}{15} = \frac{40(1000^2)(135)}{5\,072\,500b + 195\,312\,500}$$

$$b = 94.57 \text{ mm}$$

Based on allowable flexural stress of wood:

$$f_b = 10 \text{ MPa}$$

$$c = \frac{1}{2}(250) = 125 \text{ mm}$$

Thus,

$$10 = \frac{40(1000^2)(125)}{5\,072\,500b + 195\,312\,500}$$

$$b = 60.01 \text{ mm}$$

For stronger section, use $b = 94.6 \text{ mm}$ *answer*

Problem 1009

A timber beam 150 mm wide by 200 mm deep is to be reinforced at the top and bottom by aluminum plates 6 mm thick. Determine the width of the aluminum plates if the beam is to resist a moment of 14 kN·m. Assume $n = 5$ and take the allowable stresses as 10 MPa and 80 MPa in the wood and aluminum, respectively.

Solution 1009

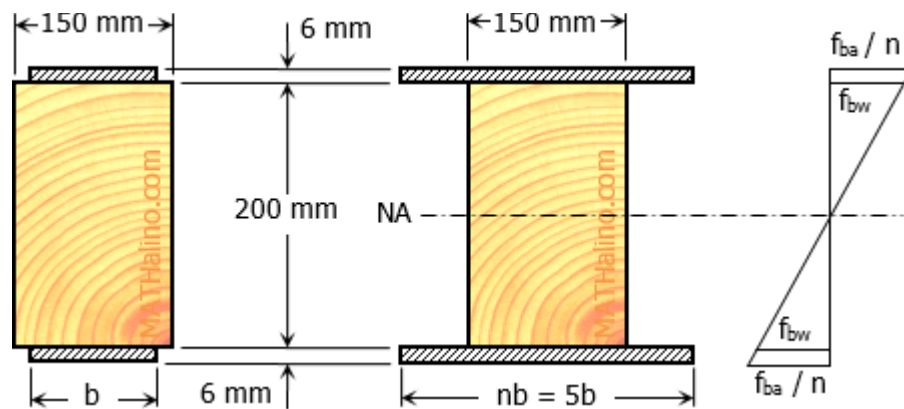
[HideClick here to show or hide the solution](#)

$$M = 14 \text{ kN} \cdot \text{m}$$

$$I = \frac{5b(212^3)}{12} - \frac{(5b - 150)(200^3)}{12}$$

$$I = 3\,970\,053.33b - 3\,333\,333.33b + 100\,000\,000$$

$$I = 636\,720b + 100\,000\,000$$



$$f_b = \frac{Mc}{I}$$

Based on allowable flexural stress of aluminum:

$$f_b = 80/n$$

$$c = \frac{1}{2}(200) + 6 = 106 \text{ mm}$$

Thus,

$$\frac{80}{n} = \frac{14(1000^2)(106)}{636\,720b + 100\,000\,000}$$

$$I = -11.38 \text{ mm}$$

Based on allowable flexural stress of wood:

$$f_b = 10 \text{ MPa}$$

$$c = \frac{1}{2}(100) = 100 \text{ mm}$$

Thus,

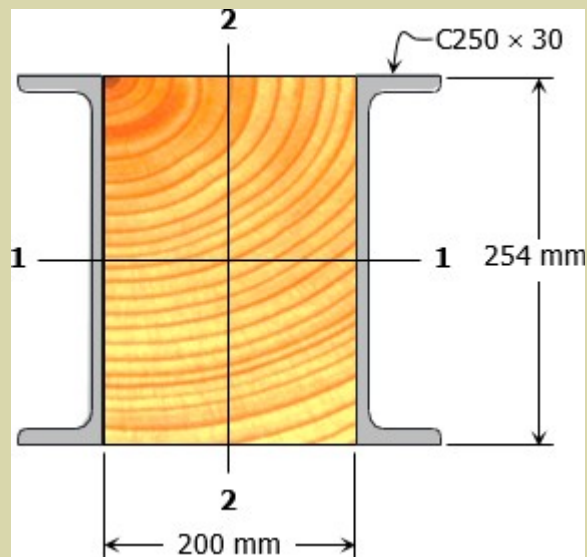
$$10 = \frac{14(1000^2)(100)}{636\,720b + 100\,000\,000}$$

$$b = 62.82 \text{ mm}$$

For stronger section, use $b = 62.82 \text{ mm}$ *answer*

Problem 1010

A pair of C250 × 30 steel channels are securely bolted to wood beam 200 mm by 254 mm, as shown in Fig. P-1010. From Table B-2 in Appendix B, the depth of the channel is also 254 mm.) If bending occurs about the axis 1-1, determine the safe resisting moment if the allowable stresses $\sigma_s = 120 \text{ MPa}$ and $\sigma_w = 8 \text{ MPa}$. Assume $n = 20$.



Problem 1011

In Problem 1010, determine the safe resisting moment if bending occurs about axis 2-2.

Solution

[HideClick here to show or hide the solution](#)

Relevant data for C250 × 30 steel channel

$$I_x = 32.7 \times 10^6 \text{ mm}^4$$

$$I_y = 1.16 \times 10^6 \text{ mm}^4$$

$$x = 15.3 \text{ mm}$$

$$\text{Area} = 3780 \text{ mm}^2$$

$$\text{Depth} = 254 \text{ mm}$$

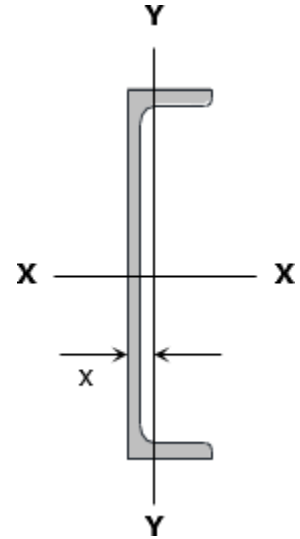
$$\text{Flange width} = 69 \text{ mm}$$

Equivalent wood area

$$A_w = 200(254) = 50\,800 \text{ mm}^2$$

$$A_s = 2(n \times \text{Area}) = 2(20 \times 3780) = 151\,200 \text{ mm}^2$$

$$A = A_w + A_s = 50\,800 + 151\,200 = 202\,000 \text{ mm}^2$$

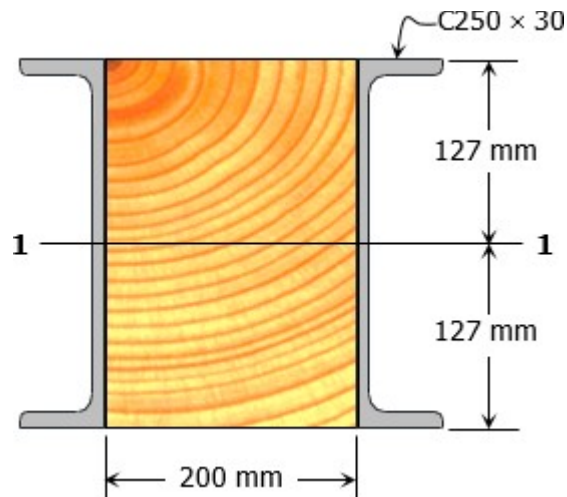


Solution 1010: Bending occurs at section 1-1

[HideClick here to show or hide the solution](#)

$$I_{1-1} = I_w + 2nI_x = \frac{200(254^3)}{12} + 2(20)(32.7 \times 10^6)$$

$$I_{1-1} = 1\,581\,117\,733 \text{ mm}^4$$



Measured from Section 1-1, the distance of extreme wood fiber and extreme steel fiber are equal. In this case, no need to investigate both materials.

$$\frac{f_{bs}}{n} = \frac{120}{20} = 6 \text{ MPa}$$

$$f_{bw} = 8 \text{ MPa}$$

Since $f_{bs} / n < f_{bw}$, steel is critical.

$$\frac{f_{bs}}{n} = \frac{Mc}{I_{1-1}}$$

$$6 = \frac{M(127)}{1\,581\,117\,733}$$

$$M = 74\,698\,475.57 \text{ N} \cdot \text{mm}$$

$$M = 74.7 \text{ kN} \cdot \text{m} \quad \text{answer}$$

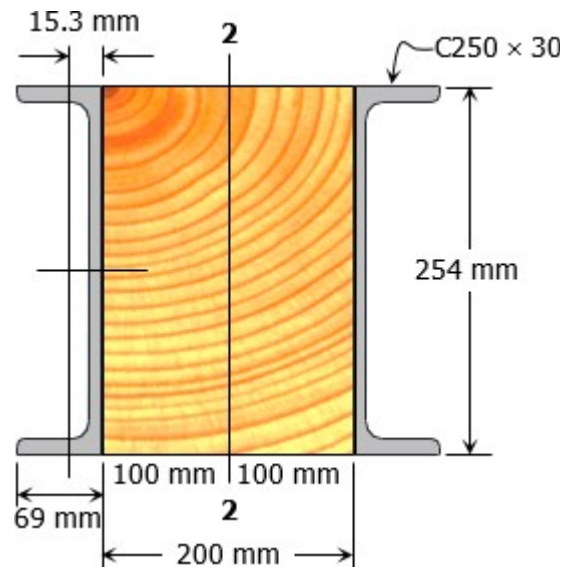
Solution 1011: Bending occurs at section 2-2

[HideClick here to show or hide the solution](#)

$$I_{2-2} = I_w + 2n[I_y + \text{Area}(100 + x)^2]$$

$$I_{2-2} = \frac{254(200^3)}{12} + 2(20)[(1.16 \times 10^6) + 3780(100 + 15.3)^2]$$

$$I_{2-2} = 2\,225\,799\,741 \text{ mm}^4$$



Since $\left(\frac{f_{bs}}{n} = 6 \text{ MPa}\right) < (f_{bw} = 8 \text{ MPa})$ and the extreme steel fiber is farther than extreme wood fiber from Section 2-2, steel is critical. In this case, no need to investigate the wood.

$$\frac{f_{bs}}{n} = \frac{Mc}{I_{2-2}}$$

$$6 = \frac{M(100 + 69)}{2\,225\,799\,741}$$

$$M = 79\,022\,476.01 \text{ N} \cdot \text{mm}$$

$$M = 79.02 \text{ kN} \cdot \text{m} \quad \text{answer}$$

From the above results, this section is stronger in bending at Section 2-2.