Contents

CHAPTER 1

An Introduction to Indeterminate Structures

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AN INTRODUCTION TO INDETERMINATE STRUCTURES

Types of Structural Systems

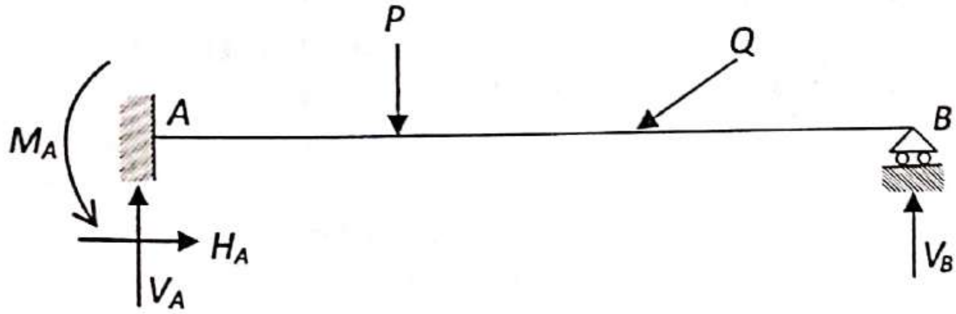
Structural systems and their components are broadly divided into beam, column, rigid frame and truss.

Determinate and Indeterminate Structures (Beam, Truss, Frame)

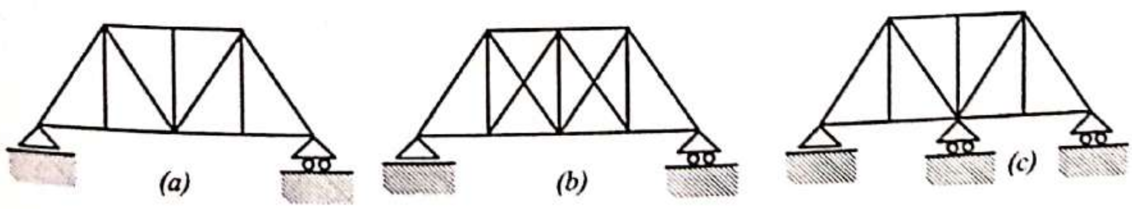
If the unknown forces, reactions and moments of a whole structure or it's components cannot be determined by the three equation of static equilibrium, then the structure is called indeterminate. These three equations of static equilibrium are:

$$\sum F_X = 0; \quad \sum F_Y = 0; \quad \sum M = 0$$

Beams are only externally indeterminate. If the support reactions are more than the equation of static equilibrium, it is indeterminate in the order of corresponding additional reaction more than three. The following beam has 4 unknown reactions, so it is indeterminate of (4-3) = 1^0



Trusses can be statically indeterminate by both reaction and bar force. If the number of bars present in a truss is more than the minimum requirement, the truss is indeterminate by bar force.



The determinacy of a truss by either bar forces or external reaction or both can be determined by the following equation:

$$b + r = 2j$$

Where b is the number of bars, r is the number of external reactions and j is the number of joints.

In the above Figure (*a*), the truss is determinate, because the minimum number of bar required to form a simple truss is, $b = 2j - 3$. So it is 13.

In Figure (*b*), the truss is indeterminate of 2° by bar force because there are two additional bars present in this truss.

In Figure (*c*), the truss is indeterminate of 1° by external reaction because there is one additional reaction presents in this truss than maximum three (equation of static equilibrium).

So, Degree of indeterminacy of a truss can be found by, *D.O.I* $= b + r - 2j$

if, $b+r = 2j$, the truss is statically determinate

$b+r > 2j$, the truss is statically indeterminate

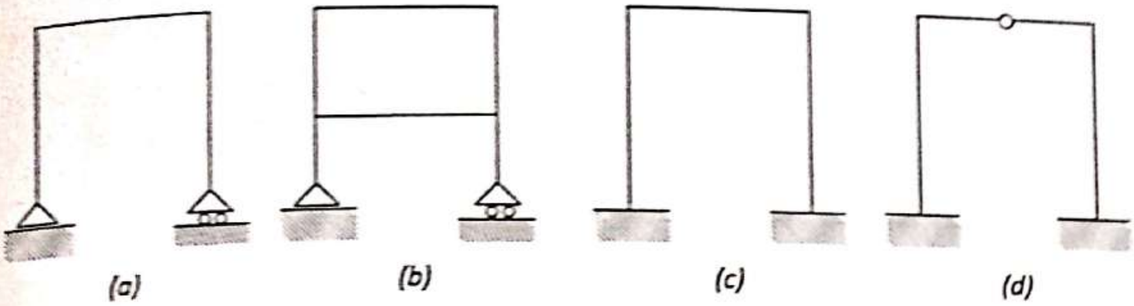
$b+r < 2j$, the truss is statically unstable

Rigid Frames can resist moment at their joints unlike truss, that is why they are called so. A rigid frame can be statically indeterminate by both internally and externally. The total number of unknown in a rigid frame is $3m+r$, where m is total number of members; r is the total number of external reactions. The three equations of static equilibrium can be applied each joint in the frame. So there are $3j$ number of simultaneous equations available to determine the unknowns. There can be some special condition presented in a frame, i.e. internal hinge, zero force member etc. Each special condition will produce additional conditional equation. So total number of equations is $3j + s$.

if, $3m+r = 3j+s$, the frame is statically determinate

$3m+r > 3j+s$, the frame is statically indeterminate

$3m+r < 3j+s$, the frame is unstable.



In the above Figure *(a)*, the frame has 3 members, 3 reactions and 4 joints. It gives,

3m+r = 3j+s; (3×3+3 =4×3+0). So it is statically determinate.

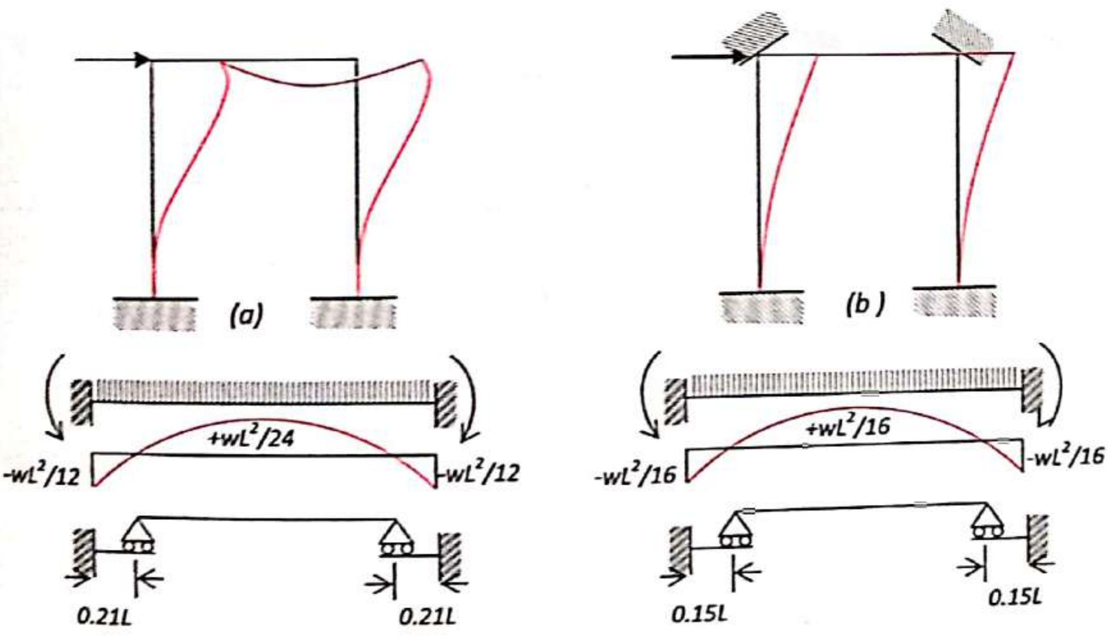
In Figure *(b)*, it has 4 members 3 reactions and 4 joints. It gives,

3m+r >3j+s; (4×3+3>3×4) So it is internally indeterminate of 3

Similarly, Figure *(c)* is externally indeterminate of 3° and Figure *(d)* is 2°

Rigid and Semi-Rigid Frame

In the rigid frame, though it is called so, the joints are not completely rigid from any rotation and translation. The joints are resisting moment with the joint rotation and side sway. In that sense it is actually semi-rigid frame. If it is completely rigid, it should look like Figure*(b)* below. For the semi-rigid condition the coefficient of end-moment will also be less than that of fully rigid condition as shown in Fig*(c)* and *(d)*.



(c) Fully rigid condition

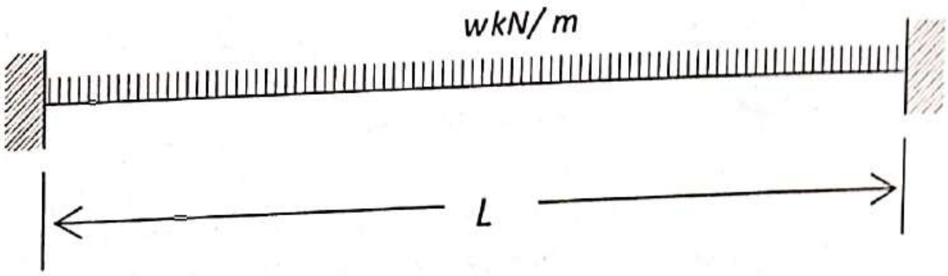
(d) Semi-rigid condition

Elastic and Inelastic Deflection

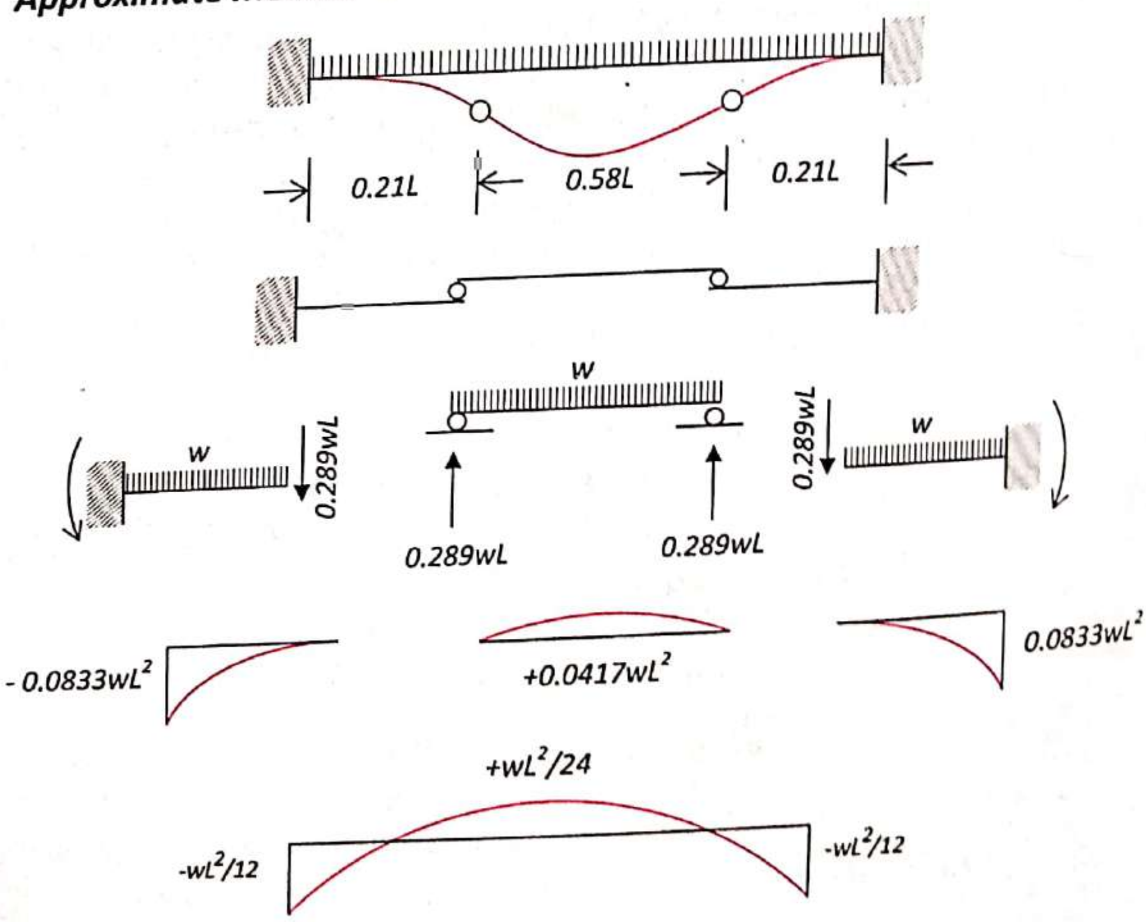
If a deflected beam returns to its original position as soon as the load is withdrawn, the deflection of this beam is called elastic deflection. After the removal of load if there is some remaining deflection, then it can be said that the beam underwent inelastic deflection. The remaining deflection when the applied load is zero, is termed as plastic deflection. In this book it is assumed that all the beams frames and trusses behave elastically and as soon as the load is removed the structure return to its original unloaded position.

Difference in Solution by Approximate method and Flexibility method

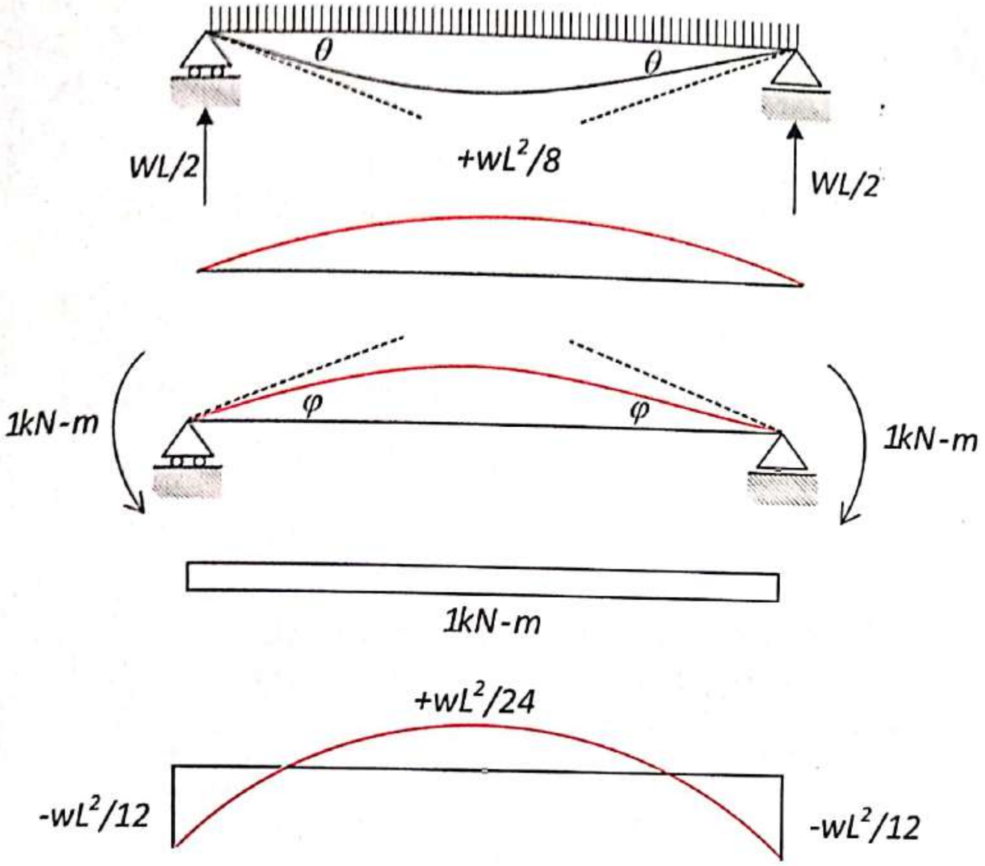
Consider the following both ends fixed beam. Find the unknown moments and reactions of this beam by different methods:



Approximate Method



Flexibility Method



By the conjugate beam method,

$$\theta + M\varphi = 0 \qquad \Rightarrow M = \frac{\theta}{\varphi}$$

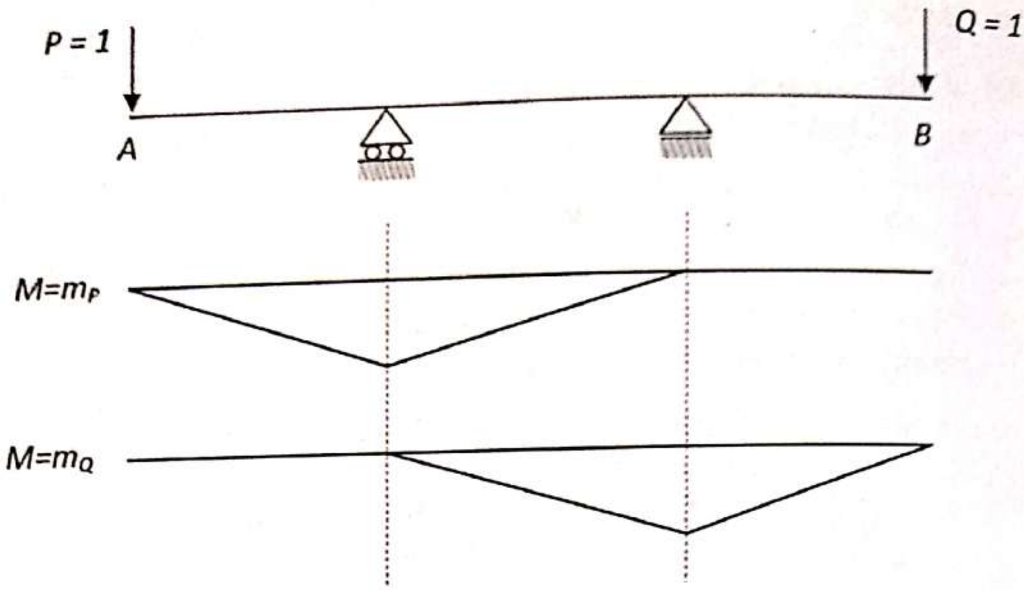
$$EI\theta = \frac{1}{2}\frac{2}{3}\frac{wL^2}{8}L = \frac{wL^3}{24}$$

$$EI\varphi = \frac{1}{2}.L = \frac{L}{2}$$

$$M = \frac{EI\theta}{EI\varphi} = \frac{wL^2}{12}$$

The Law of Reciprocal Deflection

It comes *from the general reciprocal virtual work theorem* which states that, *"The virtual work done by a P-force system is going through a deflection of a Q-force system is equal to the virtual work done by the Q-force system is going through the deformation of the P-force system"*



δ_{BA} = Deflection at B due to unit load at A
δ_{AB} = Deflection at A due to unit load at B

Now, by unit load method, $\delta_{BA} = \int_0^L \frac{Mm}{EI} dx = \int_0^L \frac{m_P m_Q}{EI} dx$

and, $\delta_{AB} = \int_0^L \frac{Mm}{EI} dx = \int_0^L \frac{m_Q m_P}{EI} dx$

$$\therefore \delta_{AB} = \delta_{BA}$$

This law of reciprocal deflection can also be true for truss as long as the deflection remain elastic.

CHAPTER 2

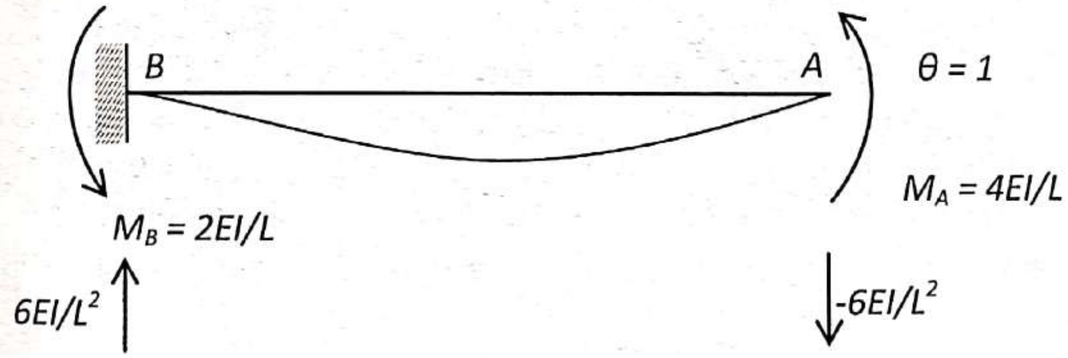
Stiffness Method of Structure Analysis

CHAPTER 2

STIFFNESS METHOD OF STRUCTURE ANALYSIS

What is Stiffness?

Stiffness can be defined as the force required to produce unit deformation. Here the deformation can be rotation or translation and the force can be axial, shear or moment. In the following figure, one unit rotation *($\theta = 1$)* is applied at the near end *A* where the far end *B* is fixed. To produce this one unit rotation, the required moment is *4EI/L* at end *A* and *2EI/L* at end *B*. So, the rotational stiffness of end *A* is *4EI/L* and that of *B* is *2EI/L*.



Degree of Freedom (DOF) and Degree of Kinematic Indeterminacy (d.o.k.i)

A support in 2D (henceforth will be called as node/joint) can undergo three deformations/displacements; one rotation and two translations. So each node can have three kinematic degree of freedom. The total number of degree of freedom of a structure is termed as degree of kinematic indeterminacy *(d.o.k.i)*. If the total degree of kinematic indeterminacy in a structure is appeared to be zero, the structure is called kinematically determinate structure. In other word fully restrained structure (FRS).

Procedure for Method of Stiffness Analysis

The steps to be followed in performing a stiffness analysis:

1) Determine the number of unknowns displacement *(d.o.k.i)* at the nodes/joints and label them $u_1, u_2, \ldots\ldots, u_n$ in sequence where *n* is the number of unknowns displacement or degrees of freedom (DOF).

2) Locked the structure against all of the displacements *(d.o.k.i)* such that it is kinematically determinate or fully restrained, i.e., the all displacements in step 1 is set equal to zero.

3) Calculate the member fixed-end forces (moments) (FEF, FEM) in this fully restrained state at the nodes/joints of the structure due to the external applied loads on the member. The member-end forces (moments) are vectorially added (anticlockwise is assumed +ve) at the nodes/joints to produce the equivalent fixed-end structure forces (moments), which are labeled as P_{fi} for $i = 1, 2, \ldots\ldots, n$.

4) Introduce a unit rotation (or unit translation) at each degree of freedom identified in step 1, one at a time while all other DOF will remain equal to zero and without any loading on the structure, i.e., $u_1 = 1$ while u_2, ..., $u_n = 0$. Sketch the displaced structure for each of these cases. Determine the member-end forces (moments) introduced as a result of each unit displacement for this fully restrained structure. These member-end forces (moments) define the member-end stiffness coefficients, i.e., forces (moments) per unit displacement. The member-end stiffness coefficients are vectorially added at the nodes/joints to produce the structure stiffness coefficients, which are labeled k_{ij} for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. [e.g. k_{12} is read as the stiffness towards DOF 1 (u_1) for unit ration/translation at DOF 2 (u_2)]

5) Write the equilibrium stiffness equation or form stiffness matrix for *d.o.k.i > 2*

$$[K][u]+[P_f]=0 \quad \text{i.e, Stiffness} \times \text{displacement} = \text{Force}$$

6) Solve the equation (matrix) for unknown nodal/ joint displacements.

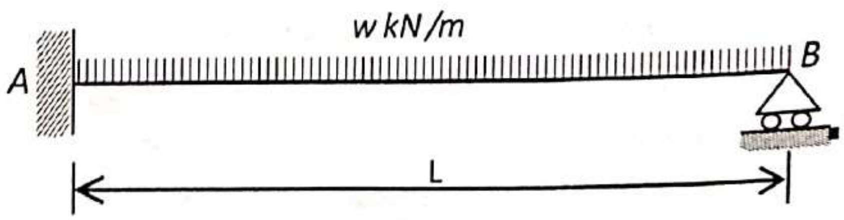
7) Calculate the member-end forces by member stiffness equilibrium equation,

$$[F]_m = [P_f]_m + [K]_m [u]_m$$

Worked out Examples

Example 1

For the following prop cantilever beam, find the unknown force in the beam.



Solution

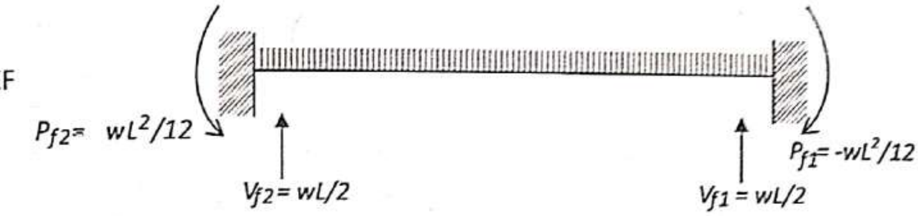
Step 1 *d.o.k.i = 1*

Step 2

Fully restrained structure, $u_1 = 0$

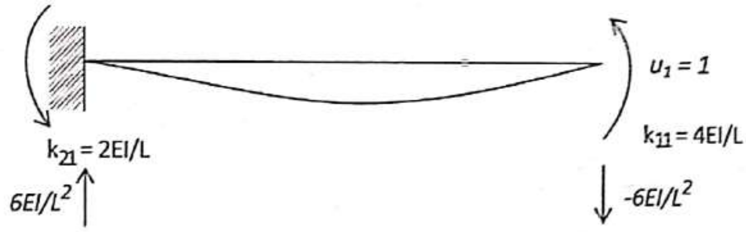
Step 3

Calculate FEM/FEF



Step 4

Apply $u_1 = 1$, and calculate joint stiffness k_{ij} and force due to stiffness



Step 5 *Write the stiffness equation and solve for unknown joint displacement.*

$$u_1 k_{11} + P_{f1} = 0$$

$$\Rightarrow \; u_1 \frac{4EI}{L} - \frac{wL^2}{12} = 0$$

$$\Rightarrow \qquad u_1 = \frac{wL^3}{48EI}$$

Step 6 *Calculate the member-end forces,*

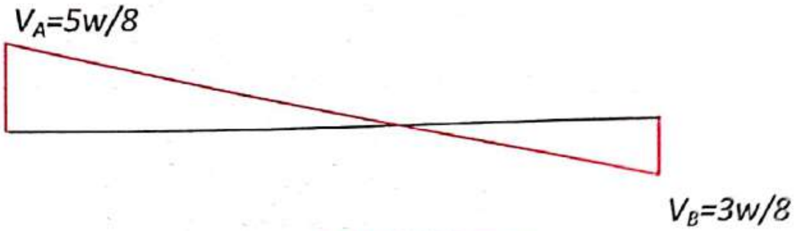
$$M_A = \frac{wL^2}{12} + k_{21} u_1 = \frac{wL^2}{12} + \frac{2EI}{L}\frac{wL^3}{48EI} = \frac{wL^2}{8}$$

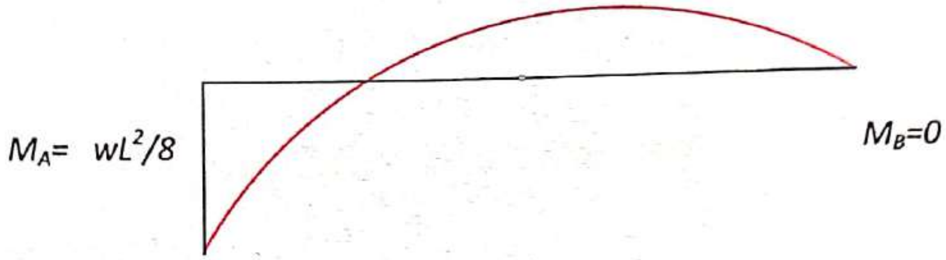
$$M_B = -\frac{wL^2}{12} + k_{11} u_1 = -\frac{wL^2}{12} + \frac{4EI}{L}\frac{wL^3}{48EI} = 0$$

$$V_A = \frac{wL}{2} + \frac{6EI}{L^2} u_1 = \frac{wL}{2} + \frac{6EI}{L^2}\frac{wL^3}{48EI} = \frac{5wL}{8}$$

$$V_B = \frac{wL}{2} - \frac{6EI}{L^2} u_1 = \frac{wL}{2} - \frac{6EI}{L^2}\frac{wL^3}{48EI} = \frac{3wL}{8}$$

Draw the final S.F.D and B.M.D





More Example of Beam by Stiffness Method

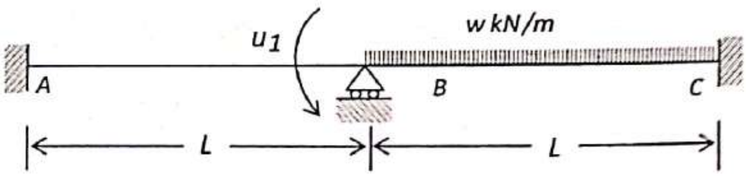
Classical or Conventional method

In the classical analysis, degree of freedoms *(d.o.k.i)* are taken only where they are necessary.

Example 2

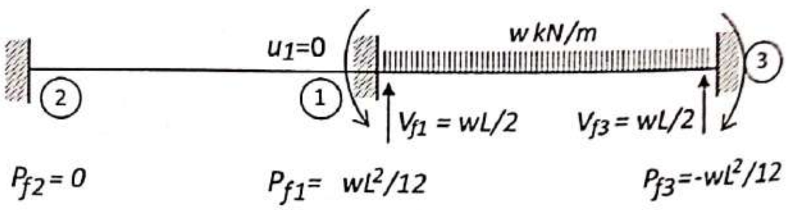
Find the unknown moments and reaction of the following continuous beam with a UDL of *w kN/m.*

d.o.k.i =1

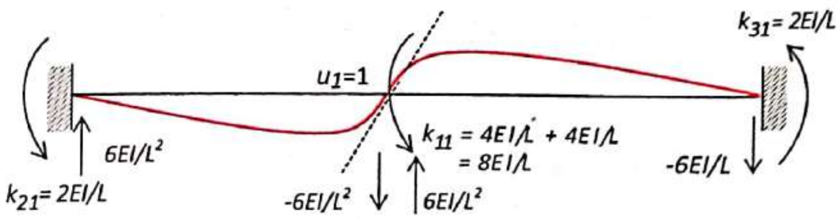


Solution

For fully restrained structure, calculate fixed end force (moment) FEF/FEM



Apply unit rotation at u_1=1 and calculate the stiffness coefficient



Solve the stiffness equilibrium equation, $$u_1 k_{11} + P_{f1} = 0$$

$$\Rightarrow \ u_1 \frac{8EI}{L} + \frac{wL^2}{12} = 0$$

$$\Rightarrow \quad u_1 = -\frac{wL^3}{96EI}$$

Calculate the unknown reactions and moments from member stiffness equilibrium equations,

$$M_A = 0 + k_{21}u_1 = -\frac{2EI}{L}\frac{wL^3}{96EI} = -\frac{wL^2}{48}$$

$$M_{BL} = 0 + \frac{4EI}{L}u_1 = -\frac{4EI}{L}\frac{wL^3}{96EI} = -\frac{wL^2}{24}$$

$$M_{BR} = \frac{wL^2}{12} + \frac{4EI}{L}u_1 = \frac{wL^2}{12} - \frac{4EI}{L}\frac{wL^3}{96EI} = \frac{wL^2}{24}$$

$$M_C = -\frac{wL^2}{12} + k_{31}u_1 = -\frac{wL^2}{12} - \frac{2EI}{L}\frac{wL^3}{96EI} = -\frac{5wL^2}{48}$$

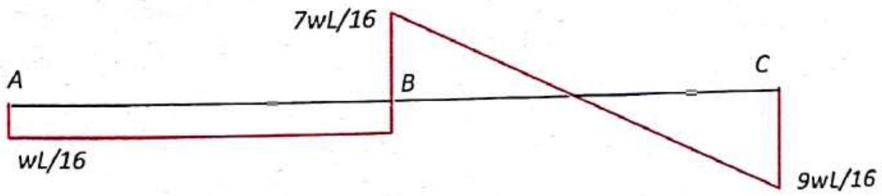
$$V_A = 0 + \frac{6EI}{L^2}u_1 = -\frac{6EI}{L^2}\frac{wL^3}{96EI} = -\frac{wL}{16}$$

$$V_{BL} = 0 + \frac{6EI}{L^2}u_1 = \frac{6EI}{L^2}\frac{wL^3}{96EI} = \frac{wL}{16}$$

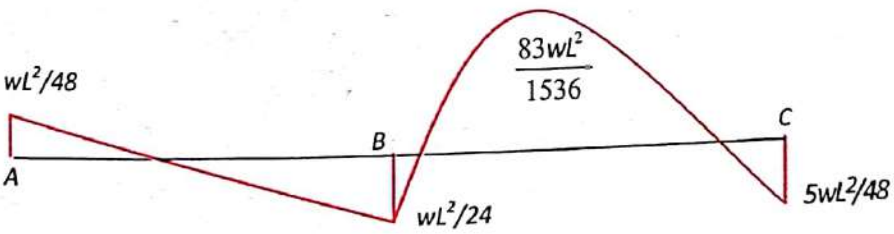
$$V_{BR} = \frac{wL}{2} + \frac{6EI}{L^2}u_1 = \frac{wL}{2} - \frac{6EI}{L^2}\frac{wL^3}{96EI} = \frac{7wL}{16}$$

$$V_C = \frac{wL}{2} - \frac{6EI}{L^2}u_1 = \frac{wL}{2} + \frac{6EI}{L^2}\frac{wL^3}{96EI} = \frac{9wL}{16}$$

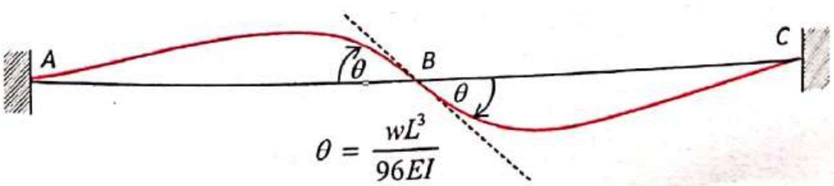
S.F.D



B.M.D



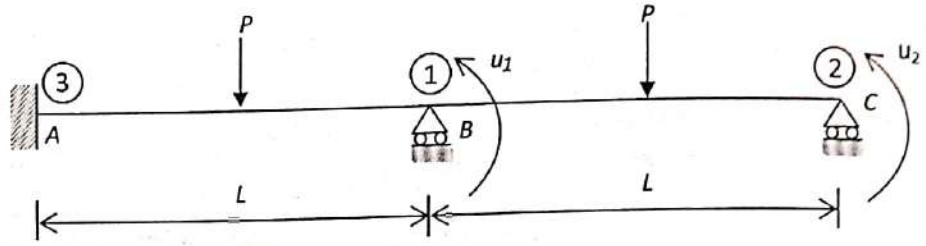
E.D.S



Example 3

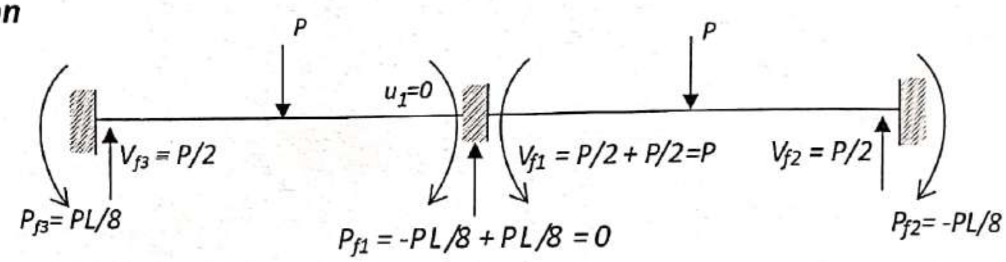
Find the unknown moments and reaction of the following continuous beam with point load P on each span at the middle.

d.o.k.i =2

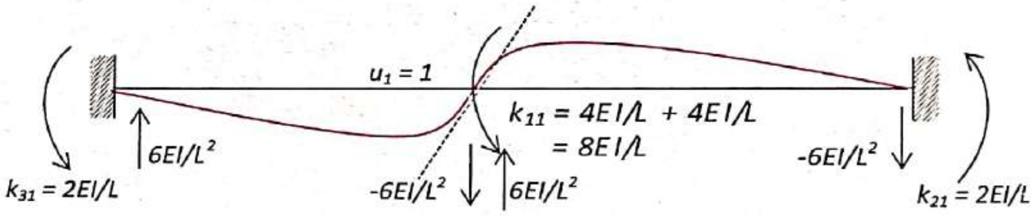


Solution

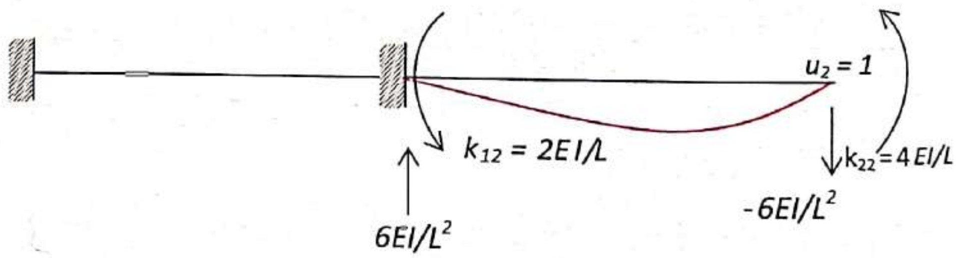
FEM



Apply
u_1=1
u_2=0



Apply
u_2=1
u_1=0



Equilibrium Equation corresponding to u_1:

$$P_{f1} + k_{11}u_1 + k_{12}u_2 = 0 \qquad (1)$$

Equilibrium Equation corresponding to u_2:

$$P_{f2} + k_{21}u_1 + k_{22}u_2 = 0 \qquad (2)$$

Putting Eq.(1) & Eq.(2) in Matrix form:

$$\begin{bmatrix} P_{f1} \\ P_{f2} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 \\ -\frac{PL}{8} \end{bmatrix} + \begin{bmatrix} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \qquad \{\text{notice that } k_{12} = k_{21}\}$$

Find $[K]^{-1}$ to determine unknown joint displacement,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = [K]^{-1} \begin{bmatrix} P_{f1} \\ P_{f2} \end{bmatrix} = \begin{bmatrix} \frac{L}{7EI} & -\frac{L}{14EI} \\ -\frac{L}{14EI} & \frac{2L}{7EI} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{PL}{8} \end{bmatrix}$$

$$u_1 = -\frac{PL^2}{112EI}$$

$$u_2 = \frac{PL^2}{28EI}$$

Member end forces can be found by, $[F]_m = [P_f]_m + [K]_m [u]$

$u_1 = 1 \quad u_2 = 1$

$$\begin{bmatrix} M_A \\ M_{BL} \\ M_{BR} \\ M_C \\ V_A \\ V_{BL} \\ V_{BR} \\ V_C \end{bmatrix} = \begin{bmatrix} \frac{PL}{8} \\ -\frac{PL}{8} \\ \frac{PL}{8} \\ -\frac{PL}{8} \\ \frac{P}{2} \\ \frac{P}{2} \\ \frac{P}{2} \\ \frac{P}{2} \end{bmatrix} \begin{bmatrix} \frac{2EI}{L} & 0 \\ \frac{4EI}{L} & 0 \\ \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \\ \frac{6EI}{L^2} & 0 \\ -\frac{6EI}{L^2} & 0 \\ \frac{6EI}{L^2} & \frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & -\frac{6EI}{L^2} \end{bmatrix} \begin{bmatrix} -\frac{PL^2}{112EI} \\ \frac{PL^2}{28EI} \end{bmatrix}$$

$$M_A = \frac{PL}{8} - \frac{2EI}{L}\frac{PL^2}{112EI} + 0.\frac{PL^2}{28EI} = \frac{3PL}{28}$$

$$M_{BL} = -\frac{PL}{8} - \frac{4EI}{L}\frac{PL^2}{112EI} + 0.\frac{PL^2}{28EI} = -\frac{9PL}{56}$$

$$M_{BR} = \frac{PL}{8} - \frac{4EI}{L}\frac{PL^2}{112EI} + \frac{2EI}{L}\frac{PL^2}{28EI} = \frac{9PL}{56}$$

$$M_C = -\frac{PL}{8} - \frac{2EI}{L}\frac{PL^2}{112EI} + \frac{4EI}{L}\frac{PL^2}{28EI} = 0$$

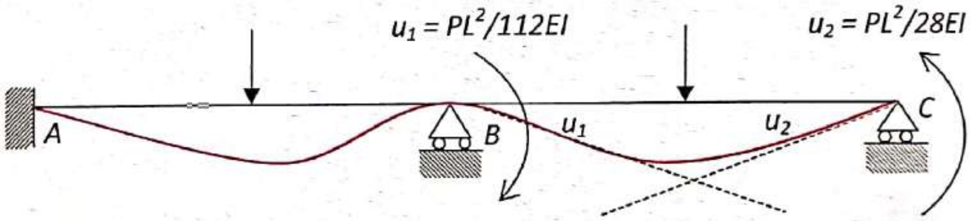
$$V_A = \frac{P}{2} - \frac{6EI}{L^2}\frac{PL^2}{112EI} + 0.\frac{PL^2}{28EI} = \frac{25P}{56}$$

$$V_{BL} = \frac{P}{2} + \frac{6EI}{L^2}\frac{PL^2}{112EI} + 0.\frac{PL^2}{28EI} = \frac{31P}{56}$$

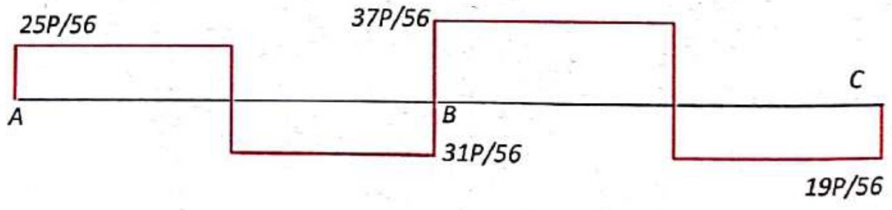
$$V_{BR} = \frac{P}{2} - \frac{6EI}{L^2}\frac{PL^2}{112EI} + \frac{6EI}{L^2}\frac{PL^2}{28EI} = \frac{37P}{56}$$

$$V_C = \frac{P}{2} + \frac{6EI}{L^2}\frac{PL^2}{112EI} - \frac{6EI}{L^2}\frac{PL^2}{28EI} = \frac{19P}{56}$$

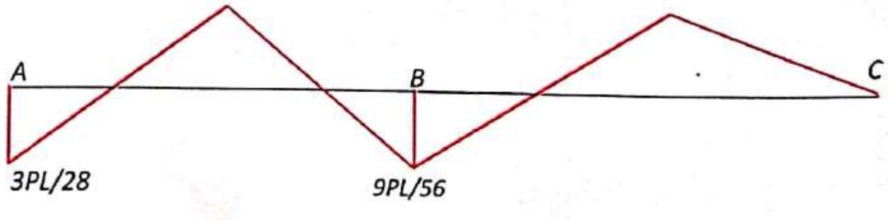
E.D.S



S.F.D



B.M.D



Method of Systematic Analysis

Classical Vs Systematic Analysis

The classical method of structure analysis is a handy tool with the help of basic structural principles. Only the simple problem with few degree of freedoms can be handled by classical method. Whereas the systematic computer analysis is also based on basic structural principles but can handle a large number of degree of freedoms. The solution technique is so straight forward that the engineers do not need to look at what is going on inside a large stiffness matrix [*K*]. The only concern here is the end results. But in the classical method, every step of calculation should be watched carefully and small matrix if needed have to be developed with cautious to avoid any mistake. At the time of systematic analysis care have to be taken only at the time of developing the joints (nodes) and elements with the proper degree of freedom per nodes.

Steps for Systematic Analysis

Step 1 Identify node and member (henceforth will be called as element)

Step 2 Designate the node and element in global and member coordinates

Step 3 Identify and denote the appropriate degree of freedom for each node and calculate the total number of degree of kinematic indeterminacy *(d.o.k.i)* u_1, u_2, u_3, u_4, u_5, u_6, ... in all nodes

Step 4 Generate element stiffness matrix $(k)_i$ in local coordinate for all elements, where *i=1,2,3............n.* and n is the total number of elements

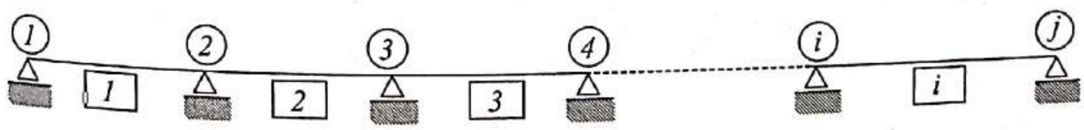
Step 5 Assemble the element stiffness matric in global stiffness matrix *(K)*

Step 6 Solve the global stiffness equilibrium equation for unknown displacement

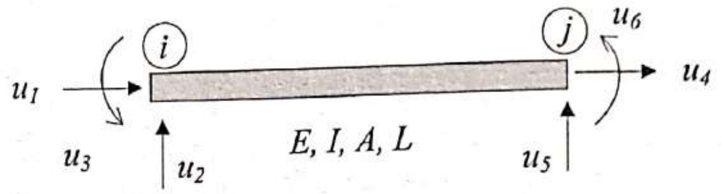
Step 7 Solve the member stiffness equilibrium equation for unknown member end forces

Generalized member stiffness matrix

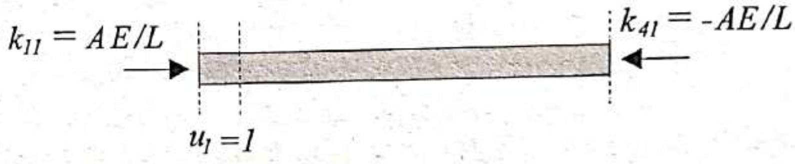
Consider a continuous beam of having multiple spans as shown in the following figure. For a typical span *i* supported by two nodes *i* and *j*, considering three degree of freedom for each node (two translation degree of freedom and one rotational degree of freedom), the element stiffness matrix for i^{th} element can be formed as follows:



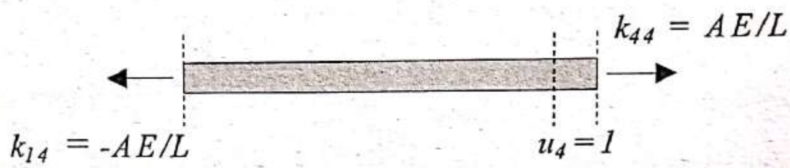
The six degree of freedoms can be identified as follows,



At first let's consider translational degree of freedom in axial direction, i.e. $u_1 = 1$ while $u_2 \ldots\ldots u_6 = 0$



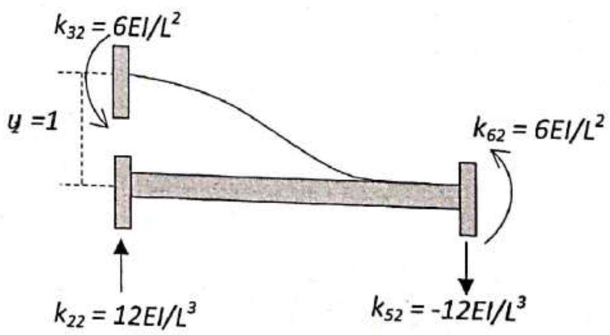
Again, $u_4 = 1$ while $u_1, u_2, u_3, u_5, u_6 = 0$



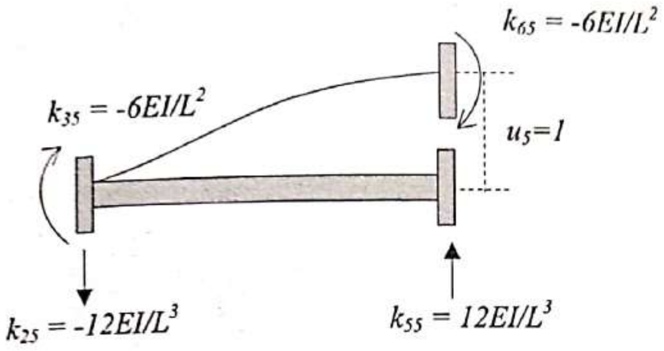
Since, there are six degree of freedom, the element stiffness matrix [k] will be 6×6 in dimension. Now putting the stiffness co-efficient for axial degree of freedom (u_1 and u_4) in appropriate address in element stiffness matrix [k], it will get the shape of;

$$[k] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ & & & & & \\ & & & & & \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ & & & & & \\ & & & & & \end{bmatrix} \end{array}$$

Now, let's take the translational degree of freedom in shear direction i.e. $u_2 = 1$, while $u_1, u_3, \ldots\ldots u_6 = 0$



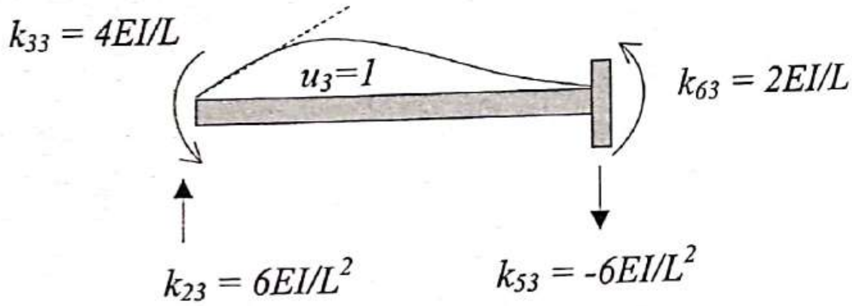
Again, $u_5 = 1$ while $u_1, u_2, u_3, u_4, u_6 = 0$



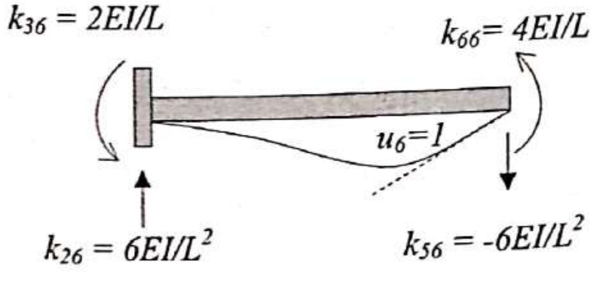
Now putting the stiffness co-efficient for these two degree of freedom *(u_2 and u_5)* in appropriate address in element stiffness matrix *[k]*, it will look like;

$$
[k] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
 & \frac{12EI}{L^3} & & & -\frac{12EI}{L^3} & \\
 & \frac{6EI}{L^2} & & & -\frac{6EI}{L^2} & \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
 & -\frac{12EI}{L^3} & & & \frac{12EI}{L^3} & \\
 & \frac{6EI}{L^2} & & & -\frac{6EI}{L^2} &
\end{bmatrix}
\end{array}
$$

Finally, take the rotational degree of freedom, i.e. $u_3 = 1$ while $u_1, u_2, u_4, u_5, u_6 = 0$



Again, $u_6 = 1$ while $u_1, u_2, u_3, u_4, u_5 = 0$



Finally, after getting all the stiffness co-efficient for all six degree of freedoms (u_1 *to* u_6) in appropriate addresses in element stiffness matrix *[k]*, it will have a look as follows;

$$[k]_{6\times6} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array}$$

If we neglect the axial degree of freedoms, stiffness matrix will be reduced to a 4x4 matrix.

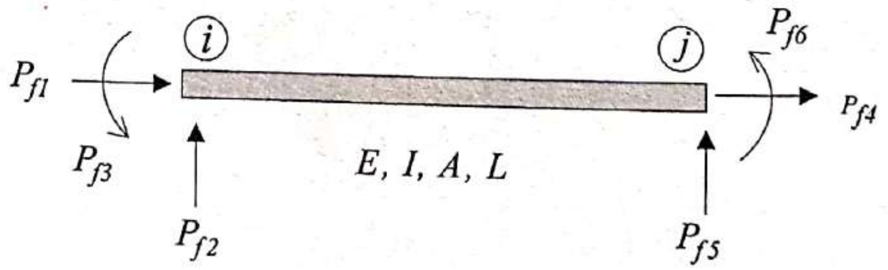
$$[k]_{4\times4} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array}$$

If only rotational degree of freedom is allowed a 2x2 matrix is enough.

$$[k]_{2\times2} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}$$

Member Equilibrium Equation

If the fixed end forces (FEF) are defined for an element as follows, the element stiffness equilibrium equation will be;



$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} + \begin{bmatrix} P_{f1} \\ P_{f2} \\ P_{f3} \\ P_{f4} \\ P_{f5} \\ P_{f6} \end{bmatrix}$$

[P_{f1} and P_{f6} are Moments, the rest are forces]

Displacement Matrix

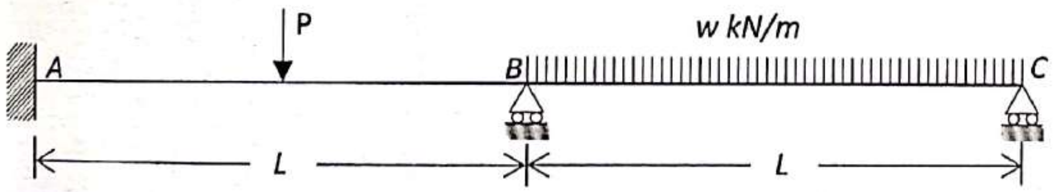
$$[P] = [k][u] + [P_f]$$

Fixed-end force Matrix

End force Matrix   Stiffness Matrix

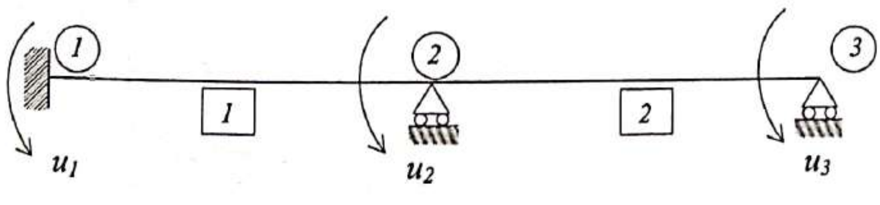
Assemble of Global Stiffness Matrix

Consider the following two span beam. It has three nodes and two elements.



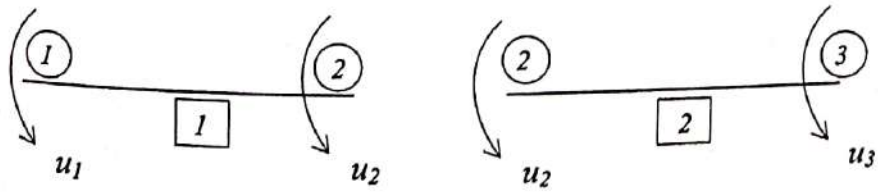
After identifying and designating one D.O.F per node, the beam will be globally look like as follows. Also note that there is no chance of translational degree of freedom for this beam, so only rotational degree of freedoms have been taken to each node.

Global



Locally it will be;

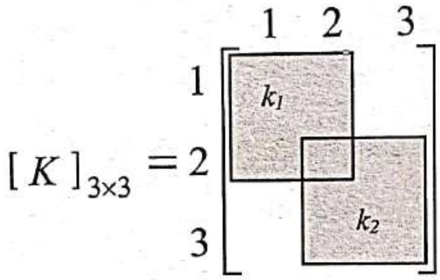
Local



Element stiffness matrix for element 1 and element 2 will be;

$$[k]_1 = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \end{array} \quad \text{and} \quad [k]_2 = \begin{array}{c} \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{bmatrix} \end{array}$$

These two element stiffness matric have to be assembled in the global stiffness matrix[K] in appropriate location as following. Note that for both the two element stiffness matric, DOF u_2 is common for both k_1 and k_2. So, in the global stiffness matrix the coefficient k_{22} will be the summation of k_{22} from element stiffness matrix k_1 and k_{22} from element stiffness matrix k_2. That is why, in the following figure k_1 overlap on k_2 at the location of k_{22} that is in the location of DOF u_2. It is also interesting to note that the gray areas in global K matrix have non-zero coefficients and all other white areas have coefficient of zero. So, k_{13} and k_{31} are zero coefficients.

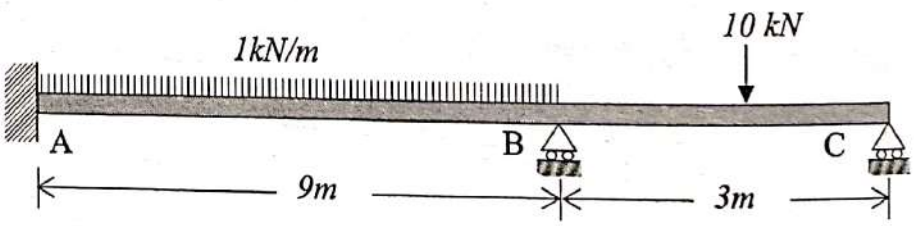


$$[K]_{3\times3} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22}+k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \end{array}$$

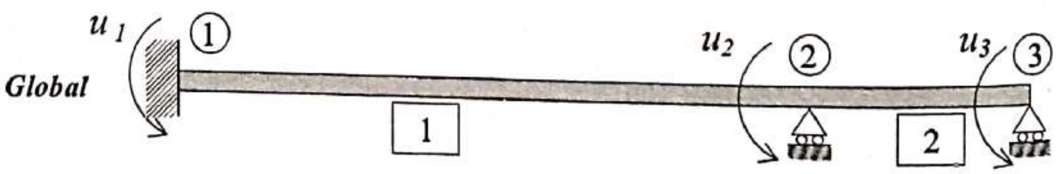
Worked out Examples

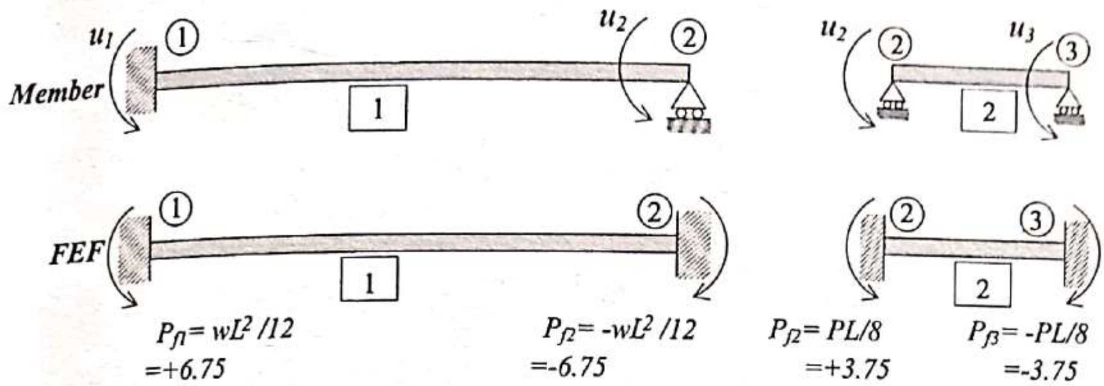
Example 1

For the beam shown, find out all support reaction and draw the SFD and BMD by stiffness method. Assume *EI* as constant.



Solution





Since only rotational displacements occur, the element (member) stiffness matrix will be a 2x2 matrix,

$$[k]_{2\times2}=\begin{bmatrix}\frac{4EI}{L} & \frac{2EI}{L}\\ \frac{2EI}{L} & \frac{4EI}{L}\end{bmatrix}$$

Stiffness matrix for element, 1 $\quad [k]_1 = EI\begin{bmatrix}4/9 & 2/9\\ 2/9 & 4/9\end{bmatrix}$

Stiffness matrix for element, 2 $\quad [k]_2 = EI\begin{bmatrix}4/3 & 2/3\\ 2/3 & 4/3\end{bmatrix}$

Now, assemble the element stiffness matrix into global stiffness matrix,

$$[K]=EI\begin{bmatrix}4/9 & 2/9 & 0\\ 2/9 & 4/9+4/3 & 2/3\\ 0 & 2/3 & 4/3\end{bmatrix}$$

Global equilibrium equation, $\quad [P]=[K][u]+[P_f]$

$$\begin{bmatrix}M_1\\ M_2\\ M_3\end{bmatrix}=EI\begin{bmatrix}4/9 & 2/9 & 0\\ 2/9 & 16/9 & 2/3\\ 0 & 2/3 & 4/3\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}+\begin{bmatrix}6.75\\ -6.75+3.75\\ -3.75\end{bmatrix}$$

Impose boundary conditions: $M_2 = 0; M_3 = 0; u_1 = 0$

[At any point, the internal moment is equal and opposite, so $M_2=0$, at end support $M_3=0$, at fixed support $u_1 = 0$]

$$\begin{bmatrix}M_1\\ 0\\ 0\end{bmatrix}=EI\begin{bmatrix}4/9 & 2/9 & 0\\ 2/9 & 16/9 & 2/3\\ 0 & 2/3 & 4/3\end{bmatrix}\begin{bmatrix}0\\ u_2\\ u_3\end{bmatrix}+\begin{bmatrix}6.75\\ -3\\ -3.75\end{bmatrix}$$

Solving the simultaneous equation,

$$u_2 = \frac{0.779}{EI}$$

$$u_3 = \frac{2.423}{EI}$$

$$M_1 = \frac{2EI}{9}u_2 + 6.75 = +6.923$$

For member 1, Equilibrium Equation, $[P_m] = [k_m][u] + [P_f]$

$$\begin{bmatrix} M_A \\ M_{BL} \end{bmatrix} = EI \begin{bmatrix} 4/9 & 2/9 \\ 2/9 & 4/9 \end{bmatrix} \begin{bmatrix} \overset{0}{u_1} \\ u_2 \end{bmatrix} + \begin{bmatrix} 6.75 \\ -6.75 \end{bmatrix}$$

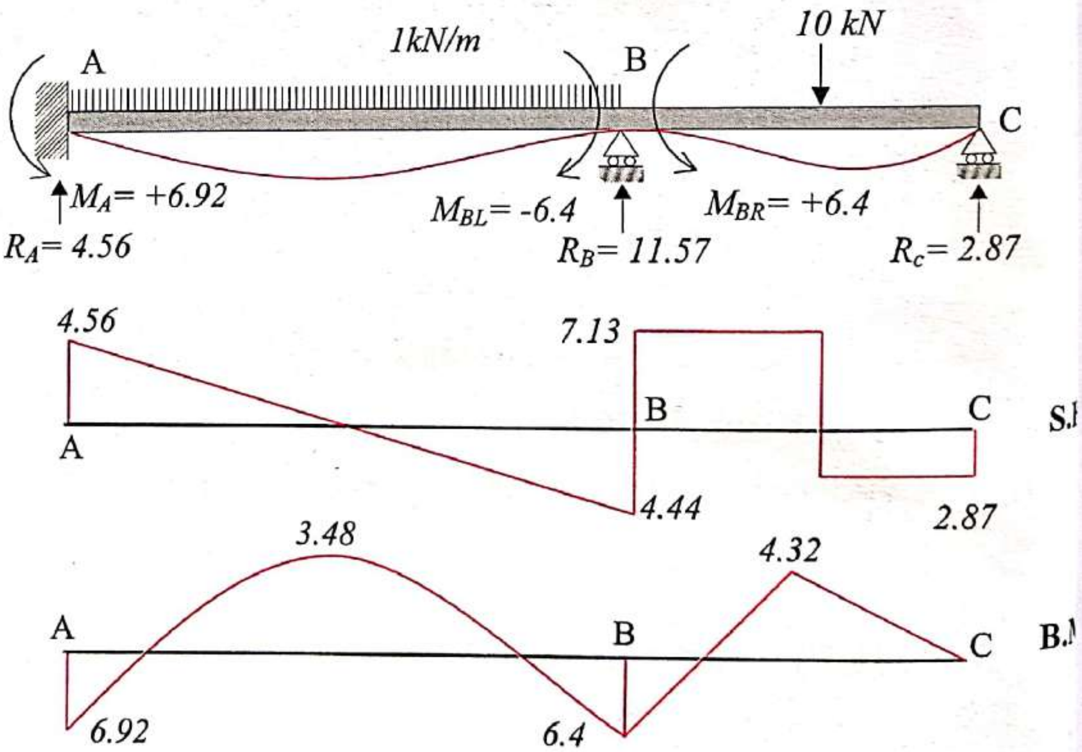
$\therefore M_{BL} = -6.4$ [M_{BL} = internal moment just left of B]

For member 2, equilibrium equation, $\begin{bmatrix} M_{BR} \\ M_C \end{bmatrix} = EI \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 3.75 \\ -3.75 \end{bmatrix}$

$\therefore M_{BR} = 6.4$ [M_{BR} = internal moment just right of B]

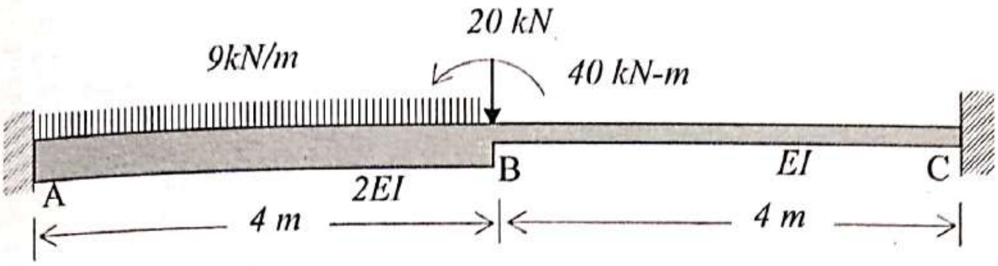
$M_C = 0$

[Here note that, M_{BL}=M_{BR} which, reiterates that the enternal moments at an section ar equal and opposite]



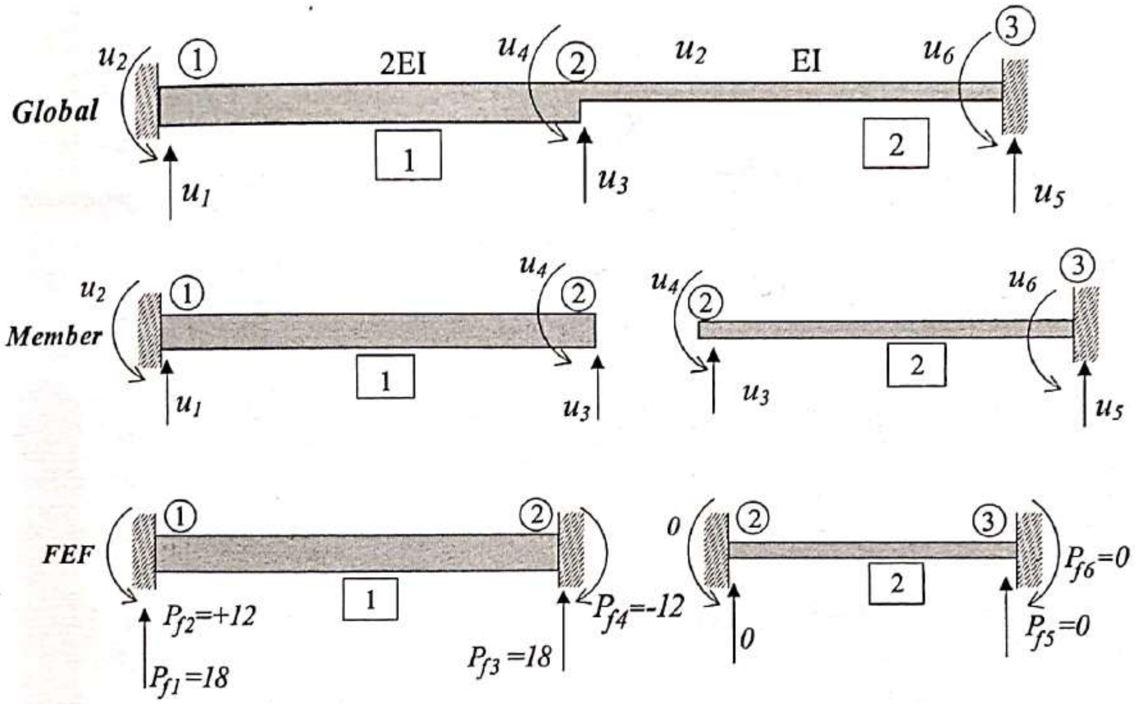
Example 2

For the beam shown, find the rotation and deflection at B also find all support reactions and draw the SFD and BMD by stiffness method.



Solution

At section B, there is a sectional change, so an additional node is to be taken at B. Since, at B there is no support, there will be a translational degree of freedom.



As, both the rotational and translational degree of freedoms are concern, a 4x4 element stiffness matrix will be needed.

$$
[k]_{4\times4} =
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4
\end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix}
\frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\end{array}
$$

$$
[k]_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix}
0.375EI & 0.75EI & -0.375EI & 0.75EI \\
0.75EI & 2.EI & -0.75EI & EI \\
-0.375EI & -0.75EI & 0.375EI & -0.75EI \\
0.75EI & EI & -0.75EI & 2EI
\end{bmatrix} \end{array}
$$

$$
[k]_2 = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\
0.375EI & EI & -0.375EI & 0.5EI \\
-0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\
0.375EI & 0.5EI & -0.375EI & EI
\end{bmatrix} \end{array}
$$

Assemble of global stiffness matrix,

$$
[K] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} EI
\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
0.375 & 0.75 & -0.375 & 0.75 & 0 & 0 \\
0.75 & 2 & -0.75 & 1 & 0 & 0 \\
-0.375 & -0.75 & 0.5625 & -0.375 & -0.1875 & -0.375 \\
0.75 & 1 & -0.375 & 3 & -0.375 & 0.5 \\
0 & 0 & -0.1875 & -0.375 & 0.1875 & -0.375 \\
0 & 0 & 0.375 & 0.5 & -0.375 & 1
\end{bmatrix} \end{array}
$$

Global stiffness equilibrium equation, $[P]=[K][u]+[P_f]$

$$
\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = EI
\begin{bmatrix}
0.375 & 0.75 & -0.375 & 0.75 & 0 & 0 \\
0.75 & 2 & -0.75 & 1 & 0 & 0 \\
-0.375 & -0.75 & 0.5625 & -0.375 & -0.1875 & -0.375 \\
0.75 & 1 & -0.375 & 3 & -0.375 & 0.5 \\
0 & 0 & -0.1875 & -0.375 & 0.1875 & -0.375 \\
0 & 0 & 0.375 & 0.5 & -0.375 & 1
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} +
\begin{bmatrix} P_{f1} \\ P_{f2} \\ P_{f3} \\ P_{f4} \\ P_{f5} \\ P_{f6} \end{bmatrix}
$$

Impose boundary conditions, $P_3 = -20; P_4 = +40; u_1 = 0; u_2 = 0; u_5 = 0; u_6 = 0$

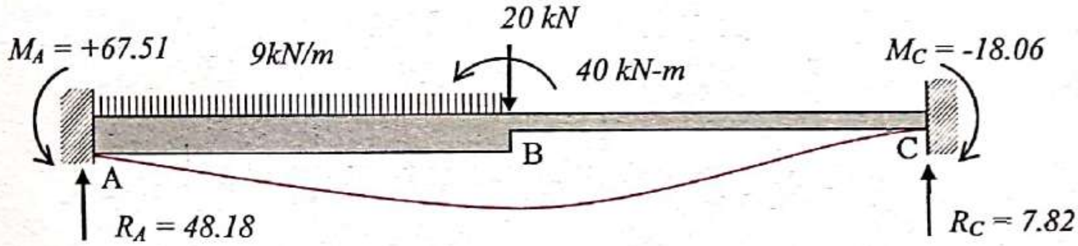
$$\begin{bmatrix} P_1 \\ P_2 \\ -20 \\ 40 \\ P_5 \\ P_6 \end{bmatrix} = EI \begin{bmatrix} 0.375 & 0.75 & -0.375 & 0.75 & 0 & 0 \\ 0.75 & 2 & -0.75 & 1 & 0 & 0 \\ -0.375 & -0.75 & 0.5625 & -0.375 & -0.1875 & -0.375 \\ 0.75 & 1 & -0.375 & 3 & -0.375 & 0.5 \\ 0 & 0 & -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0 & 0 & 0.375 & 0.5 & -0.375 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 18 \\ 12 \\ 18 \\ -12 \\ 0 \\ 0 \end{bmatrix}$$

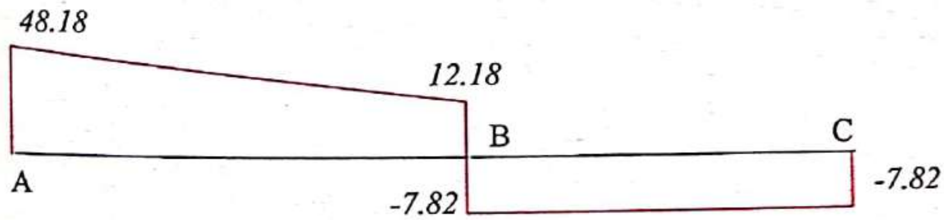
Solving for only u_3 and u_4,

$$u_3 = -\frac{61.09}{EI}$$

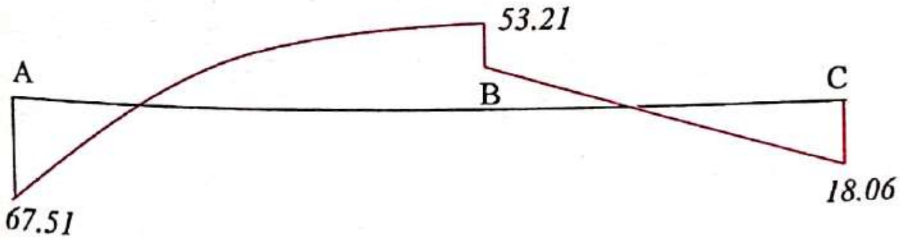
$$u_4 = \frac{9.697}{EI}$$

Get, $P_1 = 48.18$ kN, $P_2 = 67.51$ kN.m, $P_5 = 7.82$ kN, $P_6 = -18.06$ kN.m.





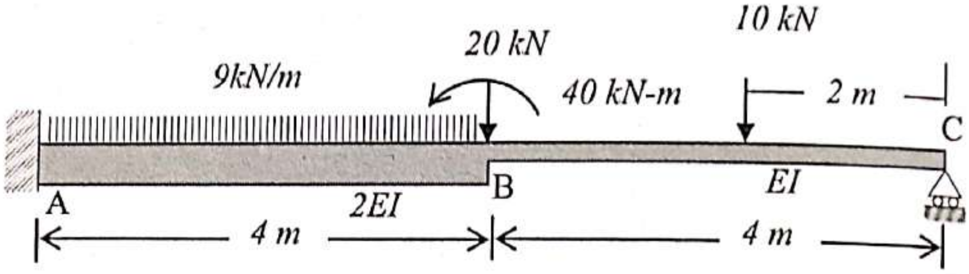
S.F.D



B.M.D

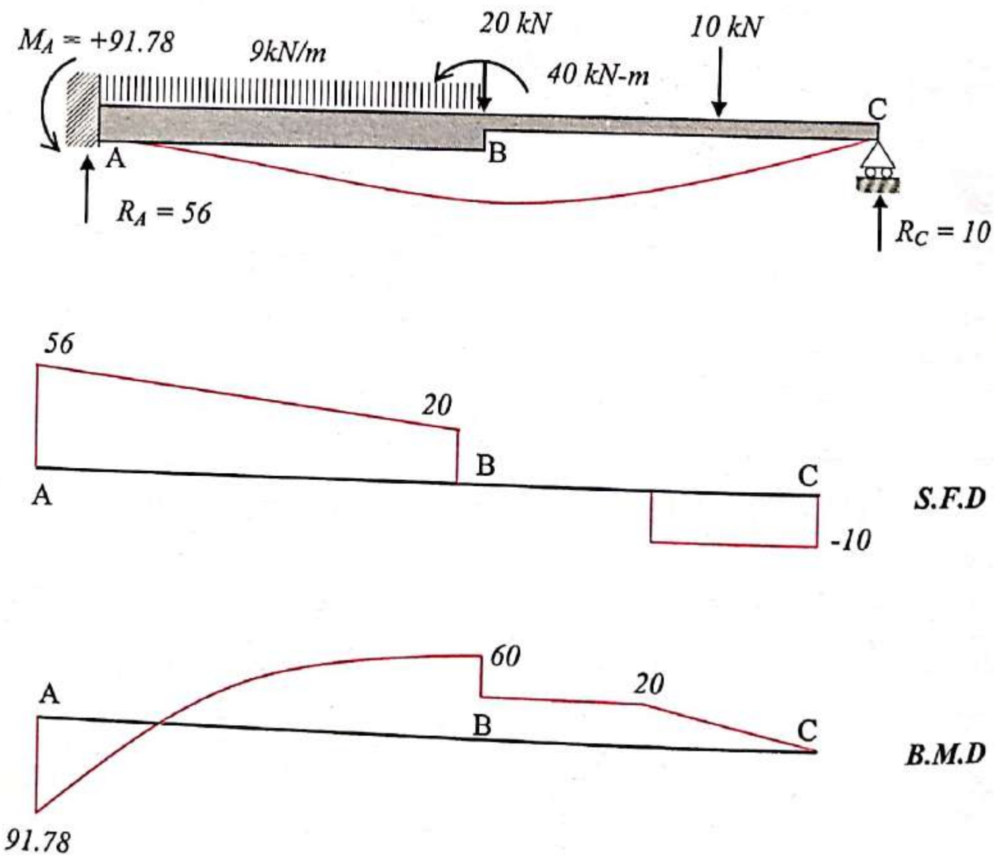
Example 3

For the beam shown, find the rotation and deflection at B also find out all support reaction and draw the SFD and BMD by stiffness method.



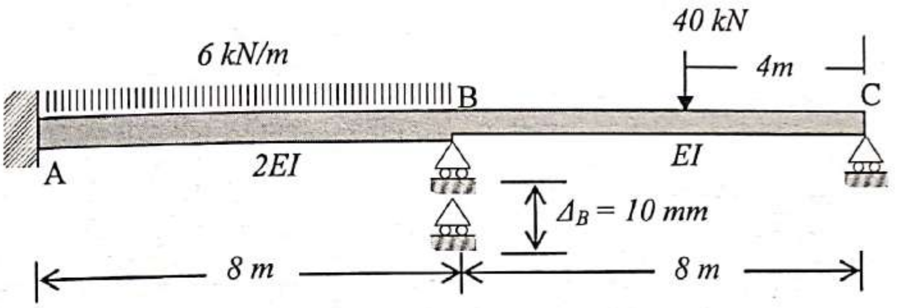
Solution

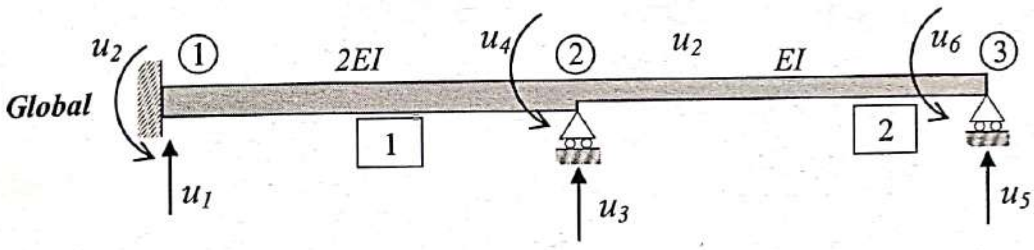
Solution procedure is same as ***Example 2*** above, try by yourself.

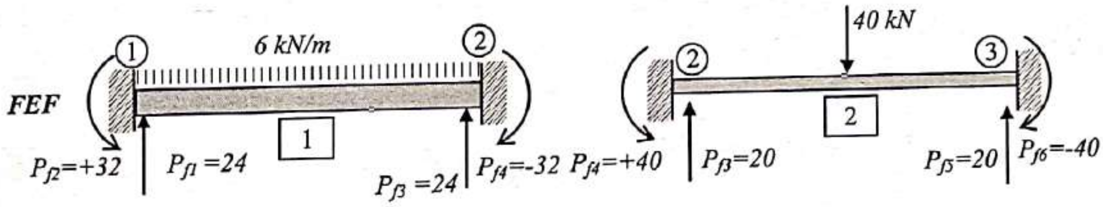


Example 4

For the following beam, support B settle 10 mm downward. Use the stiffness method to determine all the reactions at supports. Take I = $200x10^6$ mm^4 and $E = 200$ GPa







Due to the settlement at support B, a translational degree of freedom have to be considered there. So, a 4x4 element stiffness matrix will be needed for all nodes.

$$[k]_{4\times4}= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\ \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \end{array}$$

$$[k]_1 = \frac{EI}{8}\begin{array}{c} \\ 1\\2\\3\\4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.375 & 1.5 & -.0375 & 1.5 \\ 1.5 & 8 & -1.5 & 4 \\ -.0375 & -1.5 & 0.375 & -1.5 \\ 1.5 & 4 & -1.5 & 8 \end{bmatrix} \end{array}$$

$$[k]_2 = \begin{array}{c} \\ 3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.1875 & 0.75 & -0.1875 & 0.75 \\ 0.75 & 4 & -0.75 & 2 \\ -0.1875 & -0.75 & 0.1875 & -.075 \\ 0.75 & 2 & -.075 & 4 \end{bmatrix} \end{array}$$

Assemble of global stiffness matrix,

$$[k]_1 = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.375 & 1.5 & -0.375 & 1.5 & 0 & 0 \\ 1.5 & 8 & -1.5 & 4 & 0 & 0 \\ -0.375 & -1.5 & 0.5625 & -0.75 & -0.1875 & 0.75 \\ 1.5 & 4 & -0.75 & 12 & -0.75 & 2 \\ 0 & 0 & -0.1875 & -0.75 & 0.1875 & -0.75 \\ 0 & 0 & 0.75 & 2 & -0.75 & 4 \end{bmatrix} \end{array}$$

Global stiffness equilibrium equation, $[P]=[K][u]+[P_f]$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \frac{EI}{8}\begin{bmatrix} 0.375 & 1.5 & -0.375 & 1.5 & 0 & 0 \\ 1.5 & 8 & -1.5 & 4 & 0 & 0 \\ -0.375 & -1.5 & 0.5625 & -0.75 & -0.1875 & 0.75 \\ 1.5 & 4 & -0.75 & 12 & -0.75 & 2 \\ 0 & 0 & -0.1875 & -0.75 & 0.1875 & -0.75 \\ 0 & 0 & 0.75 & 2 & -0.75 & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} + \begin{bmatrix} P_{f1} \\ P_{f2} \\ P_{f3} \\ P_{f4} \\ P_{f5} \\ P_{f6} \end{bmatrix}$$

Impose boundary conditions: $P_4 = 0; P_6 = 0; u_1 = 0; u_2 = 0; u_3 = -0.01; u_5 = 0$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ 0 \\ P_5 \\ 0 \end{bmatrix} = \frac{EI}{8}\begin{bmatrix} 0.375 & 1.5 & -0.375 & 1.5 & 0 & 0 \\ 1.5 & 8 & -1.5 & 4 & 0 & 0 \\ -0.375 & -1.5 & 0.5625 & -0.75 & -0.1875 & 0.75 \\ 1.5 & 4 & -0.75 & 12 & -0.75 & 2 \\ 0 & 0 & -0.1875 & -0.75 & 0.1875 & -0.75 \\ 0 & 0 & 0.75 & 2 & -0.75 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ -0.01 \\ u_4 \\ 0 \\ u_6 \end{bmatrix} + \begin{bmatrix} 24 \\ 32 \\ 24+20 \\ -32+40 \\ 20 \\ -40 \end{bmatrix}$$

Solving for only u_4 and u_6:

$$u_4 = -\frac{61.27}{EI} = -\frac{61.27}{200\times 2000} = -1.532\times 10^{-3} \text{ rad}$$

$$u_6 = \frac{185.64}{EI} = \frac{185.64}{200\times 2000} = 4.641\times 10^{-3} \text{ rad}$$

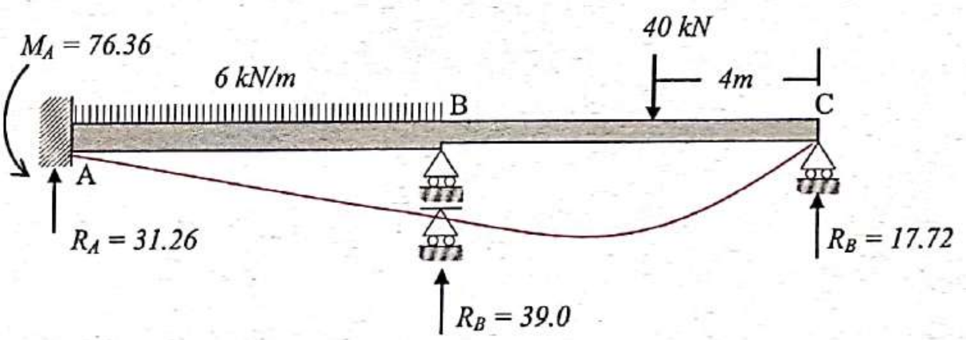
Putting back the value of u_4 and u_6 in the above matrix one gets,

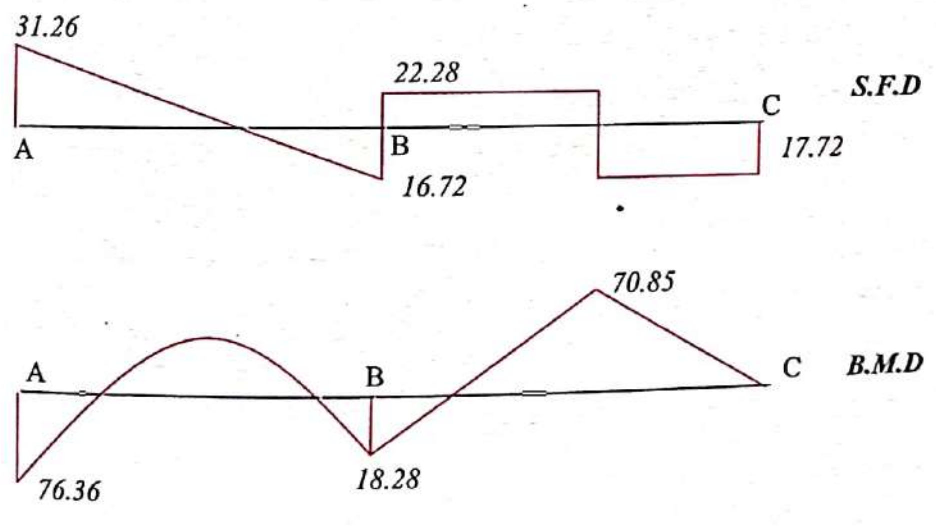
P_1 = 31.26 kN; P_2 = 76.37 kN.m; P_3 = 39.0 kN; P_5 = 17.72 kN.

Solving equilibrium equation for member 1, $[P]_m = [k_1][u] + [P_f]_m$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \frac{200\times 200}{8} \begin{bmatrix} 0.375 & 1.5 & -0.375 & 1.5 \\ 1.5 & 8 & -1.5 & 4 \\ -0.375 & -1.5 & 0.375 & -1.5 \\ 1.5 & 4 & -1.5 & 8 \end{bmatrix} \begin{bmatrix} u_1 = 0 \\ u_2 = 0 \\ u_3 = -0.01 \\ -1.532\times 10^{-3} \end{bmatrix} + \begin{bmatrix} 24 \\ 32 \\ 24 \\ -32 \end{bmatrix}$$

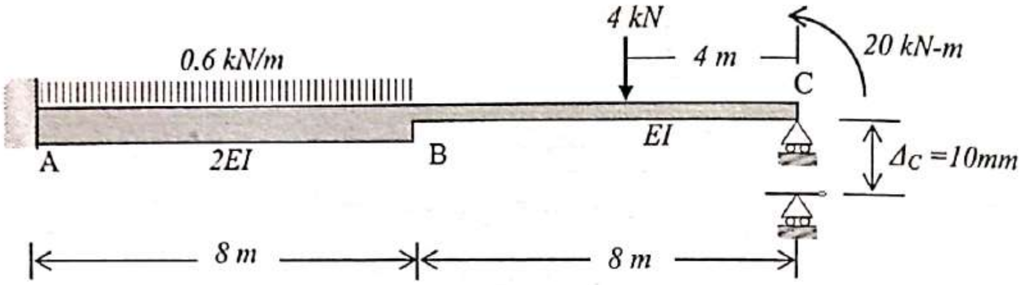
P_4 = -18.28 kN.m





Example 5

For the following beam, support C settle 10 mm downward. Use the stiffness method to determine all the reactions at supports. Take I = 200x10^6 mm^4 and E = *200* GPa



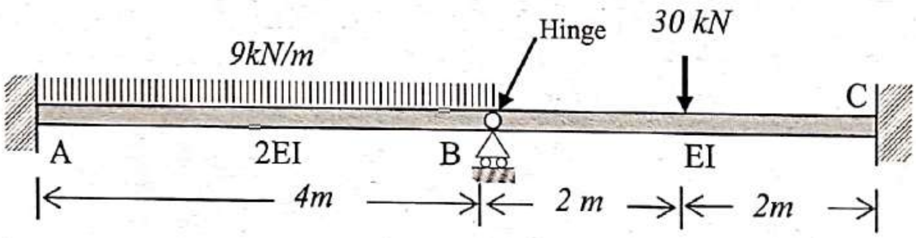
Solution

The solution procedure is same as the above *Example 4*. Do by yourself.
Ans: R_A = 8.55 kN, M_A = 43.19 kN-m,

Example 6

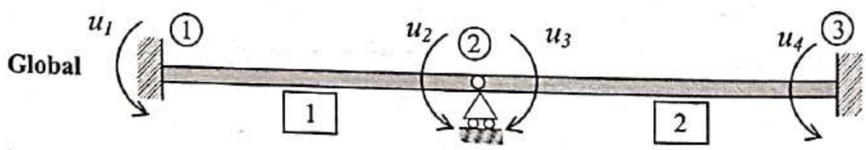
For the beam shown below, use the stiffness method to determine all the reactions at supports.

Take E = *200* GPa, I = 50x10^{-6} m^4.



Solution

Since there is an internal hinge at support B, the deflection curve will have two slopes at just left and just right of the hinge. So, two rotational degree of freedoms have to be taken at node 2 as follows,



Since only rotational degree of freedom are there, the element stiffness matrix will be a 2x2 matrix,

$$[k]_{2\times2} = \begin{bmatrix} \dfrac{4EI}{L} & \dfrac{2EI}{L} \\ \dfrac{2EI}{L} & \dfrac{4EI}{L} \end{bmatrix}$$

The element stiffness matric for element 1 and element 2 will be,

$$[k]_1 = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 2EI & EI \\ 2 & EI & 2EI \end{array} \; ; \; [k]_2 = \begin{array}{c|cc} & 3 & 4 \\ \hline 3 & EI & 0.5EI \\ 4 & 0.5EI & EI \end{array}$$

Now, assemble the element stiffness matrix into the global stiffness matrix,

$$[K] = EI \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0.5 \\ 4 & 0 & 0 & 0.5 & 1 \end{array}$$

Global stiffness equilibrium equation, $[P]=[K][u]+[P_f]$

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = EI \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 12 \\ -12 \\ 15 \\ -15 \end{bmatrix}$$

Applying boundary conditions: $M_2 = 0; M_3 = 0; u_1 = 0; u_4 = 0$ and solve for simultaneous equation,

$$\begin{bmatrix} M_1 \\ 0 \\ 0 \\ M_4 \end{bmatrix} = EI \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ u_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 12 \\ -12 \\ 15 \\ -15 \end{bmatrix}$$

$$u_2 = \frac{12}{2EI} = \frac{12}{2\times200\times50\times10^{-6}\times10^{6}} = 0.0006(rad)$$

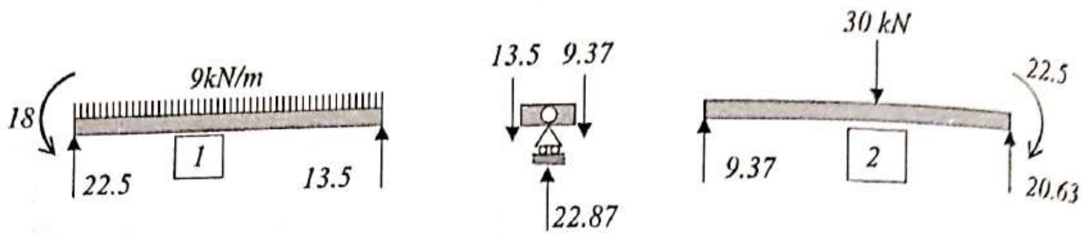
$$u_3 = -\frac{15}{EI} = -\frac{15}{200\times50\times10^{-6}\times10^{6}} = -0.0015(rad)$$

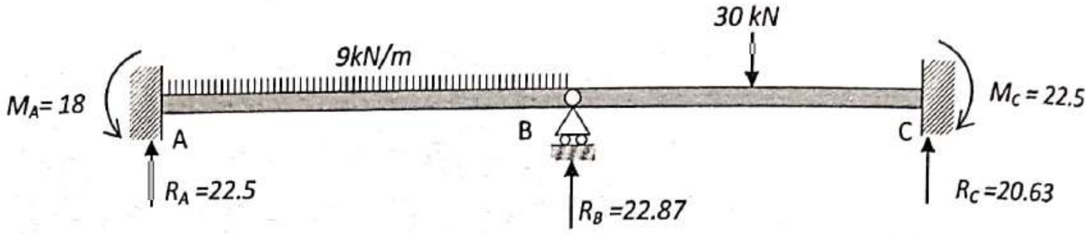
Now solve for M_1 *and* M_4

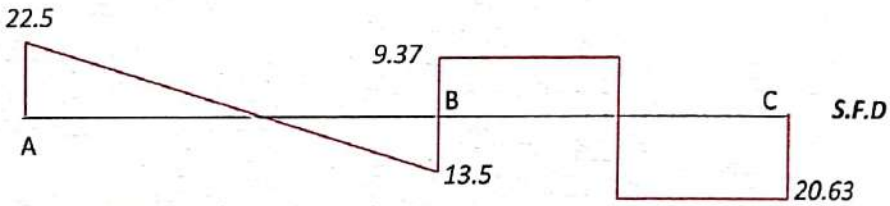
$$M_1 = EIu_1 + 12 = 0.0006\times200\times50+12 = 18\,kN.m$$

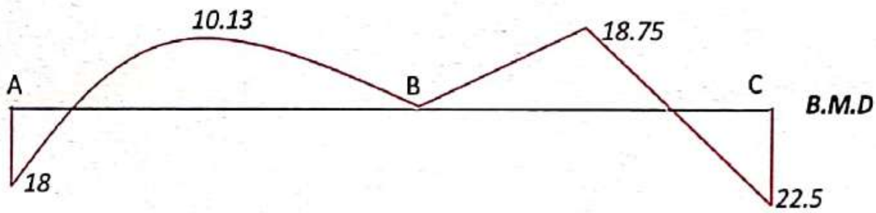
$$M_4 = 0.5EIu_2 - 15 = -0.0015\times200\times50-15 = -22.5\,kN.m$$

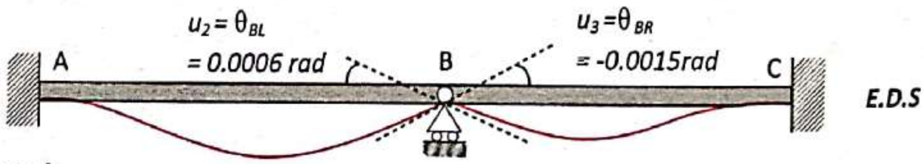
From the free-body of member 1 and member 2,







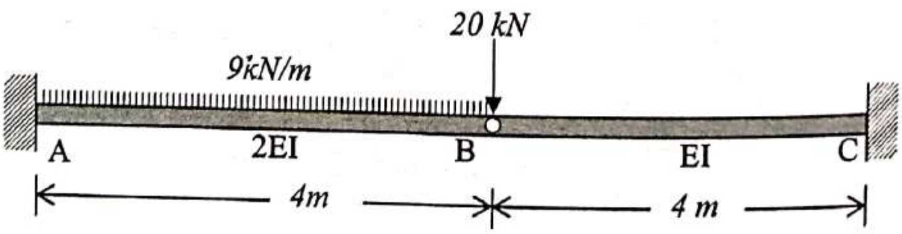




Exercise 1

For the beam shown below, use the stiffness method to determine all the reactions at supports.

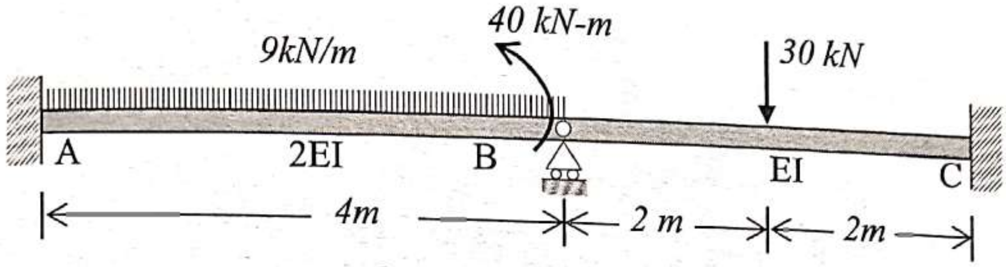
Assume, E = 200 GPa, I = 50x10^{-6}m ^{4}.



Ans: M_A =107.32 kN-m, M_C =44.66 kN-m, R_A = 44.83 kN, R_C =11.16 kN

Exercise 2

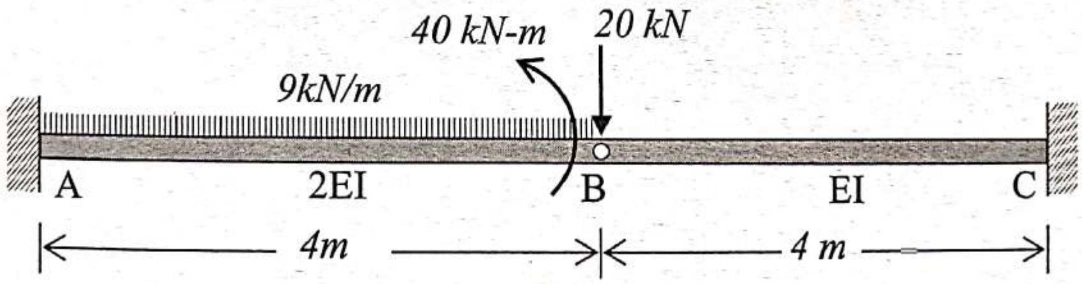
For the beam shown below, an external moment 40 kN-m is applied at end of member *AB*, use the stiffness method to determine all the reactions at supports. Assume, E = 200 *GPa*, I = $50x10^{-6}$ m^4.



Ans: M_A=38 kN-m , M_C =22.5 kN-m, R_A = 7.87 kN, R_B =20.63 kN, R_C =20.63 kN

Exercise 3

For the beam shown below, an external moment 40 kN-m is applied at end of member AB, use the stiffness method to determine all the reactions at supports. Assume, E =200 GPa, I = $50x10^{-6}$ m^4

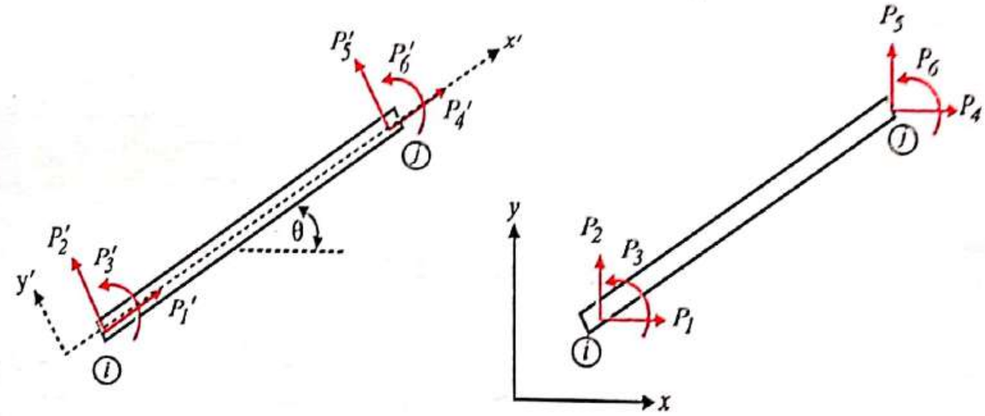


Ans: M_A =87.37 kN-m , M_C =24.7 kN-m, R_A = 49.85 kN, R_C =6.18 kN

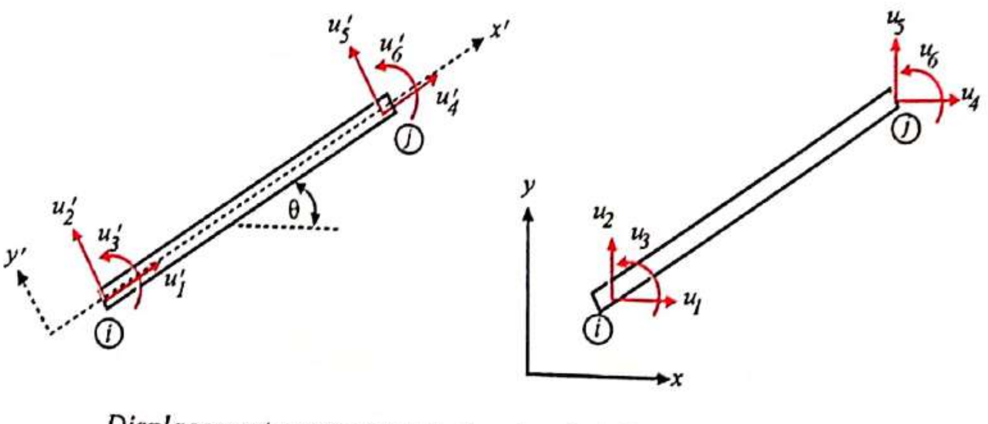
Analysis of Indeterminate Frames by Direct Stiffness Method

Introduction

Most common rigid frames are statically indeterminate. In planar frame, all the members lie in the same plane and are interconnected by rigid joints. The internal stress resultants at a cross-section of a plane frame member consist of bending moment, shear force and an axial force. The significant deformations in the plane frame are only flexural and axial. In this section, the analysis of plane frame by direct stiffness matrix method is discussed. In some of the frames where the axial load effects are insignificant is neglected to achieve a simpler and quicker solution. Initially, the stiffness matrix of a beam element of a plane frame is derived in its local co-ordinate axes and then it is transformed to global co-ordinate system. In the case of plane frames, members are oriented in different directions and hence before forming the global stiffness matrix it is necessary to refer all the member stiffness matrices to the same set of axes. This is achieved by transformation of forces and displacements to global co-ordinate system.



Force components in local and global coordinate system



Displacement components in local and global coordinate system

The Member stiffness equilibrium equation for local coordinate system can be written as,

$$\left[P'\right]=\left[k'\right]\left[u'\right]+\left[P_f'\right]$$

$$\begin{bmatrix} P_1' \\ P_2' \\ P_3' \\ P_4' \\ P_5' \\ P_6' \end{bmatrix}=\begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \\ u_5' \\ u_6' \end{bmatrix}+\begin{bmatrix} P_{f1}' \\ P_{f2}' \\ P_{f3}' \\ P_{f4}' \\ P_{f5}' \\ P_{f6}' \end{bmatrix}$$

Transformation from local to global co-ordinate system

Transformation of displacement matrix,

$$u_1' = u_1\cos\theta + u_2\sin\theta$$

$$u_2' = -u_1\sin\theta + u_2\cos\theta$$

$$u_3' = u_3$$

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \\ u_5' \\ u_6' \end{bmatrix}=\begin{bmatrix} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

$$l=\cos\theta=\frac{x_j-x_i}{L};\ \ \mathrm{m}=\sin\theta=\frac{y_j-y_i}{L};\ \ L=\sqrt{\left(x_j-x_i\right)^2+\left(y_j-y_i\right)^2}$$

In matrix notation, $\left[u'\right]=\left[T\right]\ \left[u\right]$; where [T] is the transformation matrix.

Transformation of force matrix, $P_1 = P_1' \cos\theta - P_2' \sin\theta$

$$P_2 = P_1' \sin\theta + P_2' \cos\theta$$

$$P_3 = P_3'$$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \begin{bmatrix} l & -m & 0 & 0 & 0 & 0 \\ m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & -m & 0 \\ 0 & 0 & 0 & m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_1' \\ P_2' \\ P_3' \\ P_4' \\ P_5' \\ P_6' \end{bmatrix}$$

In matrix notation, $[P]=[T]^T[P']$; where $[T]^T$ is the transformation matrix.

$$[P'] = [k'][u'] + [P_f']$$

again, $[P] = [T]^T [P']$

$$= [T]^T ([k'][u'] + [P_f'])$$

$$= [T]^T [k'][u'] + [T]^T [P_f']$$

$$= [T]^T [k'][T][u] + [T]^T [P_f']$$

$$= [k][u] + [P_f]$$

Therefore, $[k] = [T]^T [k'][T]$

$$[P_f] = [T]^T [P_f']$$

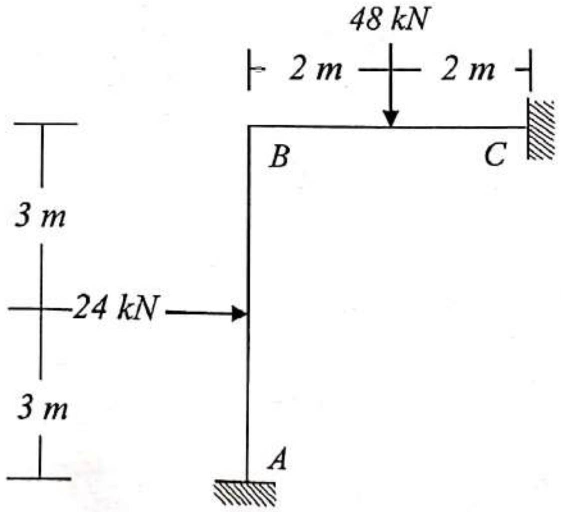
$$[P] = [T]^T [P']$$

$$[u'] = [T][u]$$

Worked out Examples

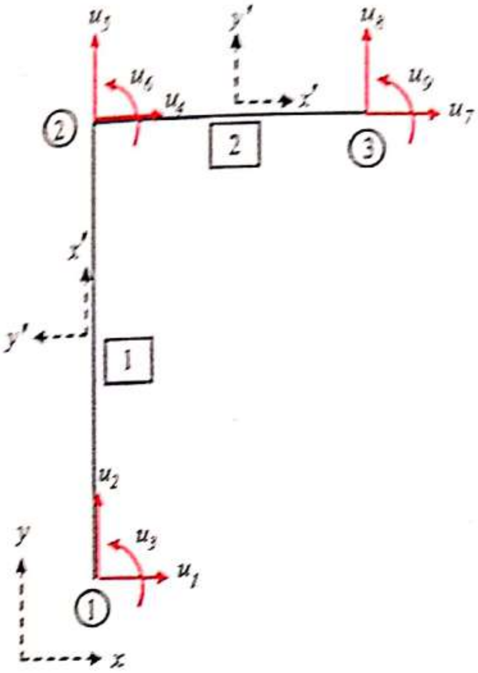
Example 1

Analyze the rigid frame shown in below by direct stiffness matrix method. Assum *E = 200GPa; I=1.33x10^{-4} m^4; A=0.04 m^2.*

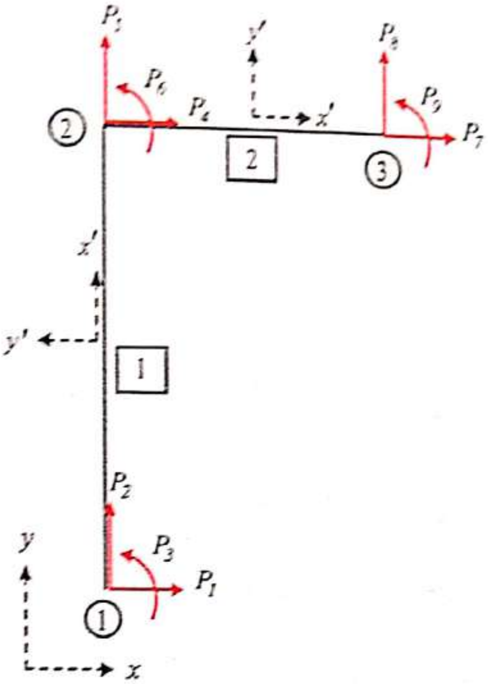


Solution by Systematic analysis

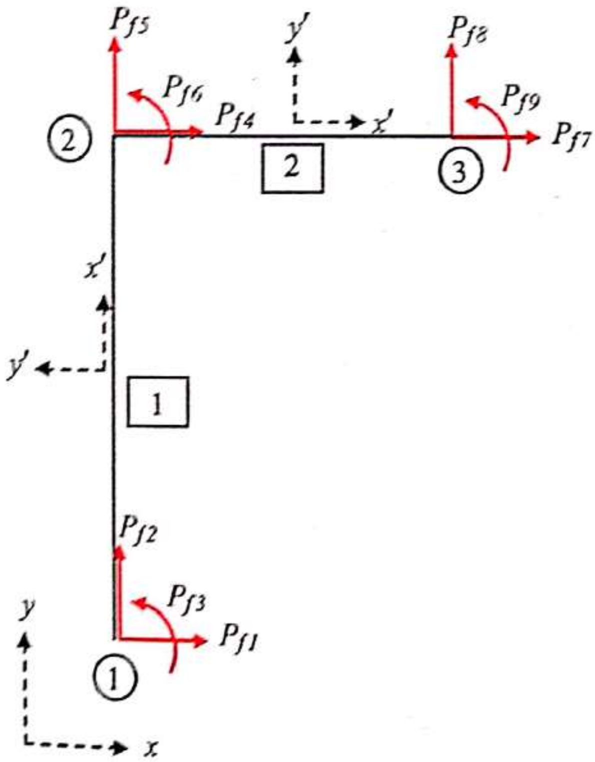
Designate the members, supports and joints by appropriate element and node numbers sequentially.



Displacement components in global axes



Force components in global axes



Fixed End Force (FEF) components in global axes

Construct element stiffness matrix,

For element 1, $L = 6\ m;\ q = 90^\circ;\ l = 0;\ m = 1$

Element stiffness matrix $[k']_1$ in local axis:

$$[k']_1=\begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix}$$

$$[k']_1=\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1330000 & 0 & 0 & -1330000 & 0 & 0 \\ 0 & 1480 & 4440 & 0 & -1480 & 4440 \\ 0 & 4440 & 17780 & 0 & -4440 & 8890 \\ -1330000 & 0 & 0 & 1330000 & 0 & 0 \\ 0 & -1480 & -4440 & 0 & 1480 & -4440 \\ 0 & 4440 & 8890 & 0 & -4440 & 17780 \end{bmatrix} \end{array}$$

Now, transfer the element stiffness matrix from local axis to global axis,

$[k]_1=[T]^T[k']_1[T]$; where $[T]$ = transformation matrix and $[T]^T$ = transpose of $[T]$

$$[T]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } [T]^T=\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, multiplying this metrics we get the element stiffness matrix in global axes for element 1,

$$[k]_1=\begin{array}{c} \\ 1\\2\\3\\4\\5\\6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1480 & 0 & -4440 & -1480 & 0 & -4440 \\ 0 & 1330000 & 0 & 0 & -1330000 & 0 \\ -4440 & 0 & 17780 & 4440 & 0 & 8890 \\ -1480 & 0 & 4440 & 1480 & 0 & 440 \\ 0 & -1330000 & 0 & 0 & 1330000 & 0 \\ -4440 & 0 & 8890 & 4440 & 0 & 17780 \end{bmatrix} \end{array}$$

For element 2, $L = 4\ m;\ \theta = 0^\circ;\ l = 1;\ m = 0$

$$[k']_2 = \begin{bmatrix} 2000000 & 0 & 0 & -2000000 & 0 & 0 \\ 0 & 5000 & 10000 & 0 & -5000 & 10000 \\ 0 & 10000 & 26600 & 0 & -10000 & 13300 \\ -2000000 & 0 & 0 & 2000000 & 0 & 0 \\ 0 & -5000 & -10000 & 0 & 5000 & -10000 \\ 0 & 10000 & 13300 & 0 & -10000 & 26600 \end{bmatrix}$$

$$[T] \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } [T]^T \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

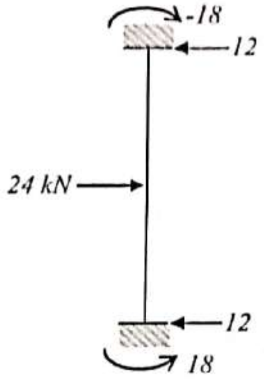
Now multiplying the matrices we get the element stiffness matrix in global axes for element 2,

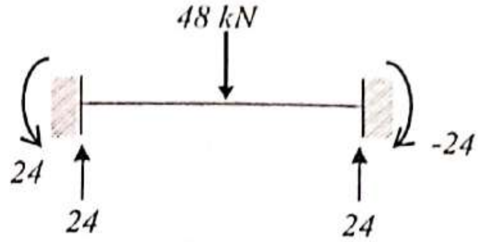
$$[k]_2 = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 2000000 & 0 & 0 & -2000000 & 0 & 0 \\ 0 & 5000 & 10000 & 0 & -5000 & 10000 \\ 0 & 10000 & 26600 & 0 & -10000 & 13300 \\ -2000000 & 0 & 0 & 2000000 & 0 & 0 \\ 0 & -5000 & -10000 & 0 & 5000 & -10000 \\ 0 & 10000 & 13300 & 0 & -10000 & 26600 \end{bmatrix} \end{array}$$

Now assemble the these two element stiffness matrices, i.e. *[k]₁ and [k]₂* in global stiffness matrix [*K*],

$$[K]_{9\times9} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{bmatrix} 1480 & 0 & -4440 & -1480 & 0 & -4440 & 0 & 0 & 0 \\ 0 & 1330000 & 0 & 0 & -1330000 & 0 & 0 & 0 & 0 \\ -4440 & 0 & 17780 & 4440 & 0 & 8890 & 0 & 0 & 0 \\ -1480 & 0 & 4440 & 2001480 & 0 & 4440 & -2000000 & 0 & 0 \\ 0 & -1330000 & 0 & 0 & 1335000 & 10000 & 0 & -5000 & 10000 \\ -4440 & 0 & 8890 & 4440 & 10000 & 44380 & 0 & -10000 & 13300 \\ 0 & 0 & 0 & -2000000 & 0 & 0 & -2000000 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5000 & -10000 & 0 & 5000 & -10000 \\ 0 & 0 & 0 & 0 & 10000 & 13300 & 0 & -10000 & 26600 \end{bmatrix}$$

Calculate the Fixed End Forces (FEF) for fully restrained elements,





Establish global equilibrium equation $[P]=[K][u]+[P_f]$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \end{bmatrix} = \begin{bmatrix} 1480 & 0 & -4440 & -1480 & 0 & -4440 & 0 & 0 & 0 \\ 0 & 1330000 & 0 & 0 & 1330000 & 0 & 0 & 0 & 0 \\ -4440 & 0 & 17780 & 4440 & 8890 & 0 & 0 & 0 & 0 \\ -1480 & 0 & 4440 & 2001480 & 0 & 4440 & -2000000 & 0 & 0 \\ 0 & -1330000 & 0 & 0 & 1335000 & 10000 & 0 & -5000 & 10000 \\ -4440 & 0 & 8890 & 4440 & 10000 & 44380 & 0 & -10000 & 13300 \\ 0 & 0 & 0 & -2000000 & 0 & 0 & 2000000 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5000 & -10000 & 0 & 5000 & -10000 \\ 0 & 0 & 0 & 0 & 10000 & 13300 & 0 & -10000 & 26600 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{bmatrix} + \begin{bmatrix} P_{f1} \\ P_{f2} \\ P_{f3} \\ P_{f4} \\ P_{f5} \\ P_{f6} \\ P_{f7} \\ P_{f8} \\ P_{f9} \end{bmatrix}$$

Apply boundary conditions,

$P_4 = 0;\ P_5 = 0;\ P_6 = 0;$

$u_1 = 0;\ u_2 = 0;\ u_3 = 0;\ u_7 = 0;\ u_8 = 0;\ u_9 = 0$

$P_{f1}=-12;\ P_{f2}=0;\ P_{f3}=18;\ P_{f4}=-12;\ P_{f5}=24;\ P_{f6}=6;\ P_{f7}=0;\ P_{f8}=24;\ P_{f9}=-24$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ 0 \\ 0 \\ 0 \\ P_7 \\ P_8 \\ P_9 \end{bmatrix} = \begin{bmatrix} 1480 & 0 & -4440 & -1480 & 0 & -4440 & 0 & 0 & 0 \\ 0 & 1330000 & 0 & 0 & 1330000 & 0 & 0 & 0 & 0 \\ -4440 & 0 & 17780 & 4440 & 8890 & 0 & 0 & 0 & 0 \\ -1480 & 0 & 4440 & 2001480 & 0 & 4440 & -2000000 & 0 & 0 \\ 0 & -1330000 & 0 & 0 & 1335000 & 10000 & 0 & -5000 & 10000 \\ -4440 & 0 & 8890 & 4440 & 10000 & 44380 & 0 & -10000 & 13300 \\ 0 & 0 & 0 & -2000000 & 0 & 0 & 2000000 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5000 & -10000 & 0 & 5000 & -10000 \\ 0 & 0 & 0 & 0 & 10000 & 13300 & 0 & -10000 & 26600 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ u_5 \\ u_6 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -12 \\ 0 \\ 18 \\ -12 \\ 24 \\ 6 \\ 0 \\ 24 \\ -24 \end{bmatrix}$$

Solve for u_4, u_5 and u_6

$u_4 = 6.29 \times 10^{-6} m;\ \ u_5 = 1.7 \times 10^{-5}\ m;\ \ u_6 = -0.13 \times 10^{-3}\ rad$

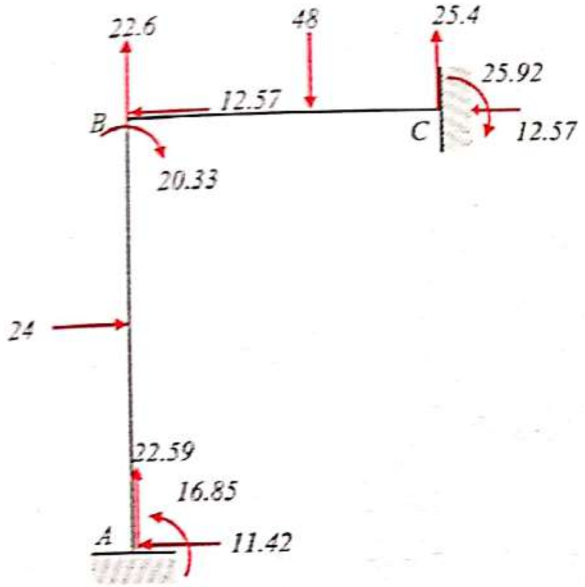
Now, get unknown Joint forces,

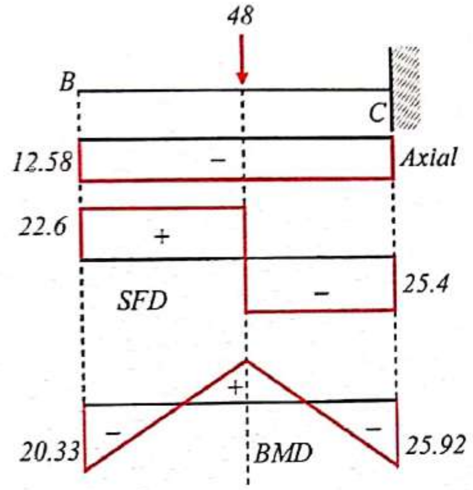
P_1 =-11.42 kN ; P_2 =22.59 kN; P_3=16.85 kN.m; P_7=-12.57 kN; P_8 =25.4 kN; P_9 =-25.92 kN.m

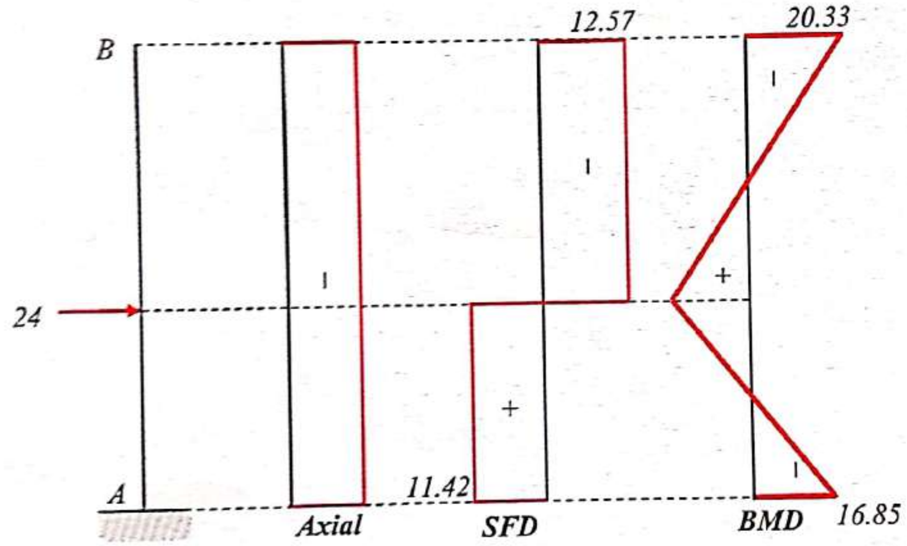
From the free body of Element 1 and Element 2, it gives,

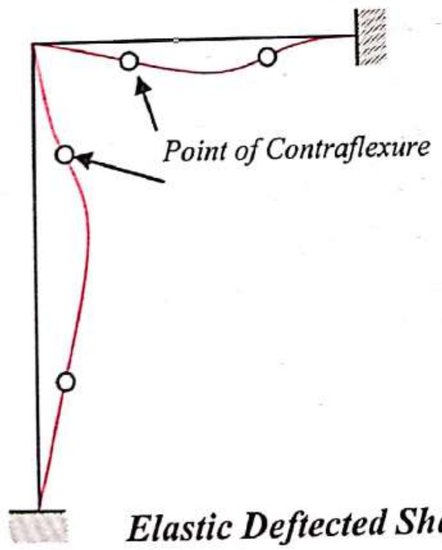
P_4 = 12.57 kN; P_5 = 22.6 kN; P_6 = 20.33 kN.m;

Now draw the axial, shear and bending moment diagrams for the whole structures.



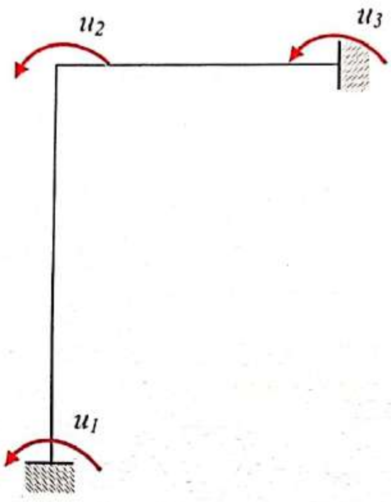


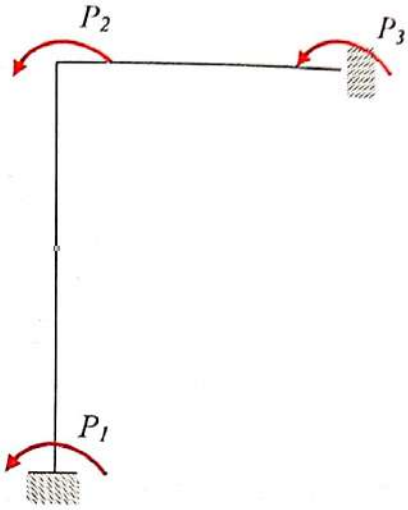




Elastic Deflected Shape

The above frame doesn't show any tendency of having side sway. In that case the axial and shear deformation can be neglected. So the solution becomes much simple.





Since, only rotational degree of freedoms are considered, there is no need of transformation between local and global coordinate system.
The element stiffness matrix will be in the form of 2 x 2.

$$[k] = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}; \; [k]_1 = \begin{bmatrix} 17780 & 8890 \\ 8890 & 17780 \end{bmatrix} \text{ and } [k]_2 = \begin{bmatrix} 26600 & 13300 \\ 13300 & 26600 \end{bmatrix}$$

The global stiffness matrix will be,

$$[K] = \begin{bmatrix} 17780 & 8890 & 0 \\ 8890 & 44380 & 13300 \\ 0 & 13300 & 26600 \end{bmatrix}$$

The global equilibrium equation will be,

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 17780 & 8890 & 0 \\ 8890 & 44380 & 13300 \\ 0 & 13300 & 26600 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} P_{f1} \\ P_{f2} \\ P_{f3} \end{bmatrix}$$

Apply boundary conditions, *$P_2 = 0$; $u_1 = 0$; $u_3 = 0$*

$$\begin{bmatrix} P_1 \\ 0 \\ P_3 \end{bmatrix} = \begin{bmatrix} 17780 & 8890 & 0 \\ 8890 & 44380 & 13300 \\ 0 & 13300 & 26600 \end{bmatrix} \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 18 \\ 6 \\ -24 \end{bmatrix}$$

Solving for u_2, $u_2 = -0.135 \times 10^{-3}$ rad
(Compare the result with the u_6 in previous solution, the difference is only 3%)
Calculate, *P_1 = 16.8 kN.m* and *P_2 = -25.8 kN.m*

$$M_{BL} = \frac{4EI}{L}u_2 - 18 = 17780(-0.135\times10^{-3}) - 18 = -20.4kN.m$$

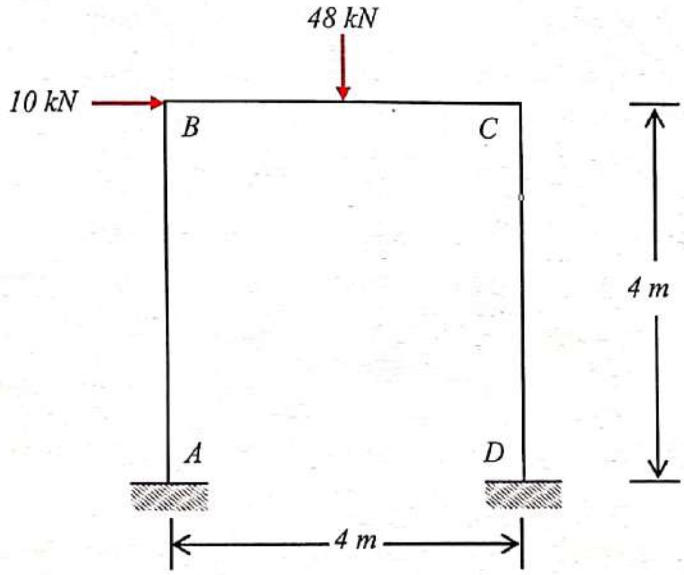
$$M_{BR} = \frac{4EI}{L}u_2 + 24 = 26600(-0.135\times10^{-3}) + 24 = 20.4kN.m$$

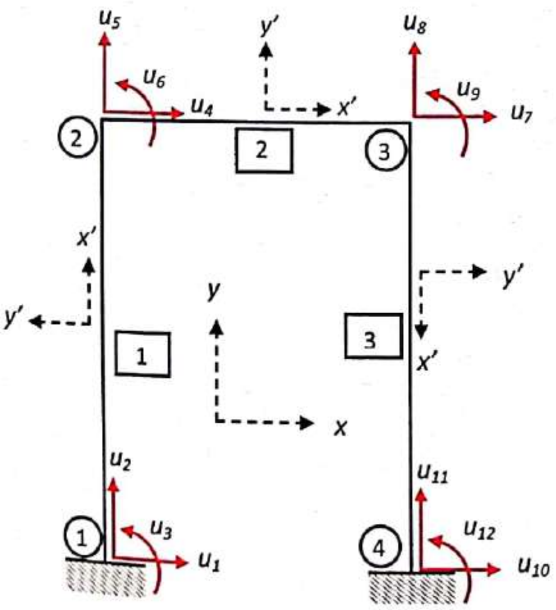
Now, from the free body, $V_A = 11.4kN$; $V_C = 25.35kN$

Example 2

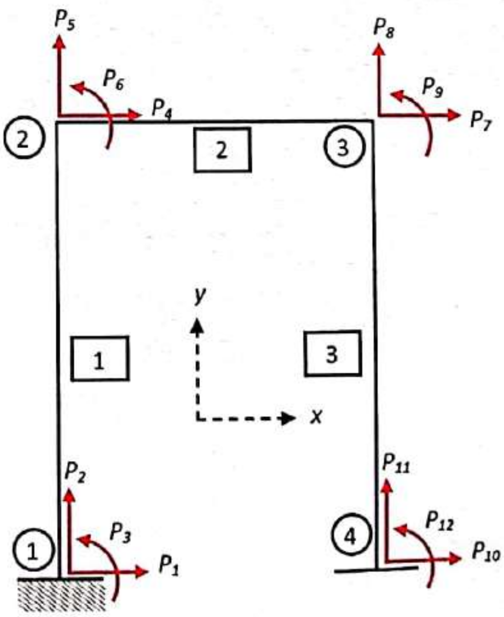
Analyze the rigid frame shown in below by direct stiffness matrix method. Assume *E = 200GPa; I=1.33x10^{-5} m^4; A=0.01 m^2*. Draw the Axial, SFD and BMD. Show a qualitative deflection shape.

Solution by Systematic Analysis





Displacement components in global axes



Force components in global axes

The following terms are common for all elements.

$$\frac{AE}{L}=5\times10^5;\quad \frac{6EI}{L^2}=998;\quad \frac{12EI}{L^3}=499;\quad \frac{4EI}{L}=2660;\quad \frac{2EI}{L}=1330$$

So the local stiffness matrix will be the same for all three elements.
Element stiffness Matrix [k'] in local axis:

$$[k'] = \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix}$$

$$[k'] = \begin{bmatrix} 500000 & 0 & 0 & -500000 & 0 & 0 \\ 0 & 499 & 998 & 0 & -499 & 998 \\ 0 & 998 & 2660 & 0 & -998 & 1330 \\ -500000 & 0 & 0 & 500000 & 0 & 0 \\ 0 & -499 & -998 & 0 & 499 & -998 \\ 0 & 998 & 1330 & 0 & -998 & 2660 \end{bmatrix}$$

Transfer the element stiffness matrix from local to global axes,
For Elemen t 1, *L = 4 m; θ = 90°; l = 0 ; m = 1*

$$[k]_1 = [T]^T [k']_1 [T]$$

$$[T] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } [T]^T = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now multiplying these metrics, will produce element stiffness matrix in global axes,

$$[k]_1=\begin{bmatrix} 499 & 0 & -998 & -499 & 0 & -998 \\ 0 & 500000 & 0 & 0 & -500000 & 0 \\ -998 & 0 & 2660 & 998 & 0 & 1330 \\ -499 & 0 & 998 & 499 & 0 & 998 \\ 0 & -500000 & 0 & 0 & 500000 & 0 \\ -998 & 0 & 1330 & 998 & 0 & 2660 \end{bmatrix}$$

For Element 2 : *$L = 4$ m; $\theta = 0^o$; $l = 1$; $m = 0$*

$$[T]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } [T]^T=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now multiplying the matrices, will produce the element stiffness matrix in global axes,

$$[k]_2=\begin{bmatrix} 500000 & 0 & 0 & -500000 & 0 & 0 \\ 0 & 499 & 998 & 0 & -499 & 998 \\ 0 & 998 & 2660 & 0 & -998 & 1330 \\ -500000 & 0 & 0 & 500000 & 0 & 0 \\ 0 & -499 & -998 & 0 & 499 & -998 \\ 0 & 998 & 1330 & 0 & -998 & 2660 \end{bmatrix}$$

For Element 3: *$L = 4$ m; $\theta = 270^o$; $l = 1$; $m = -1$*

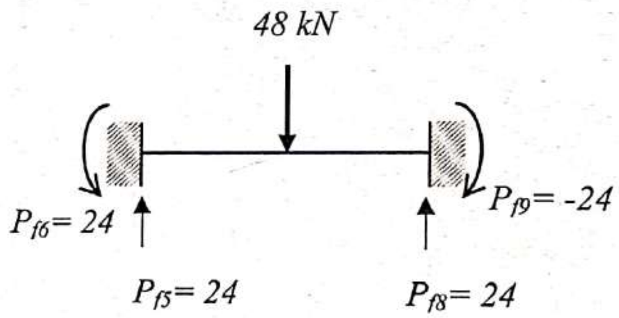
$$[T]=\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } [T]^T=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[k]_3=\begin{bmatrix} 499 & 0 & 998 & -499 & 0 & 998 \\ 0 & 500000 & 0 & 0 & -500000 & 0 \\ 998 & 0 & 2660 & -998 & 0 & 1330 \\ -499 & 0 & -998 & 499 & 0 & -998 \\ 0 & -500000 & 0 & 0 & 500000 & 0 \\ 998 & 0 & 1330 & -998 & 0 & 2660 \end{bmatrix}$$

Now assemble the element stiffness matrices *[k]1 , [k]2* and *[k]3* in global stiffness matrix [*K*],

$$
[K]=\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12 \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array}\\
\begin{bmatrix}
499 & 0 & -998 & -499 & 0 & -998 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 500000 & 0 & 0 & -500000 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-998 & 0 & 2660 & 998 & 0 & 1330 & 0 & 0 & 0 & 0 & 0 & 0\\
-499 & 0 & 998 & 500499 & 0 & 998 & -500000 & 0 & 0 & 0 & 0 & 0\\
0 & -500000 & 0 & 0 & 500499 & 998 & 0 & -499 & 998 & 0 & 0 & 0\\
-998 & 0 & 1330 & 998 & 998 & 5320 & 0 & -998 & 1330 & 0 & 0 & 0\\
0 & 0 & 0 & -500000 & 0 & 0 & 500499 & 0 & 998 & -499 & 0 & 998\\
0 & 0 & 0 & 0 & -499 & -998 & 0 & 500499 & -998 & 0 & -500000 & 0\\
0 & 0 & 0 & 0 & 998 & 1330 & 998 & -998 & 5320 & -998 & 0 & 1330\\
0 & 0 & 0 & 0 & 0 & 0 & -499 & 0 & -998 & 499 & 0 & -998\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -500000 & 0 & 0 & 500000 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 998 & 0 & 1330 & -998 & 0 & 2660
\end{bmatrix}
\end{array}
$$

Fixed End Forces (FEF) for fully restrained elements, Also note that the *10 kN* nodal load at joint 2 will directly go to force vector matrix as *P4,*



Establish global equilibrium equation, $[P]=[K][u]+[P_f]$

$$
\begin{bmatrix} P_1\\P_2\\P_3\\P_4\\P_5\\P_6\\P_7\\P_8\\P_9\\P_{10}\\P_{11}\\P_{12} \end{bmatrix}
=[K]
\begin{bmatrix} u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12} \end{bmatrix}
+
\begin{bmatrix} P_{f1}\\P_{f2}\\P_{f3}\\P_{f4}\\P_{f5}\\P_{f6}\\P_{f7}\\P_{f8}\\P_{f9}\\P_{f10}\\P_{f11}\\P_{f12} \end{bmatrix}
$$

Apply boundary conditions,

$P_4 = 10; P_5 = 0; P_6 = 0; P_7 = 0; P_8 = 0; P_9 = 0;$

$u_1 = 0; u_2 = 0; u_3 = 0; u_{10} = 0; u_{11} = 0; u_{12} = 0$

$P_{f1} = 0; P_{f2} = 0; P_{f3} = 0; P_{f4} = 0; P_{f5} = 24; P_{f6} = 24; P_{f7} = 0;$

$P_{f8} = 24; P_{f9} = -24; P_{f10} = 0; P_{f11} = 0; P_{f12} = 0;$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ P_{10} \\ P_{11} \\ P_{12} \end{bmatrix} = [K] \begin{bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 24 \\ 24 \\ 0 \\ 24 \\ -24 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve for u_4; u_5; u_6; u_7; u_8 and u_9,

$u_4 = 1.43 x 10^{-2}\ m$ $\quad$ $u_7 = 1.43 x 10^{-2}\ m$

$u_5 = -3.94 x 10^{-5}\ m$ $\quad$ $u_8 = -5.66 x 10^{-5}\ m$

$u_6 = -8.17 x 10^{-3}\ rad$ $\quad$ $u_9 = 3.86 x 10^{-3}\ rad$

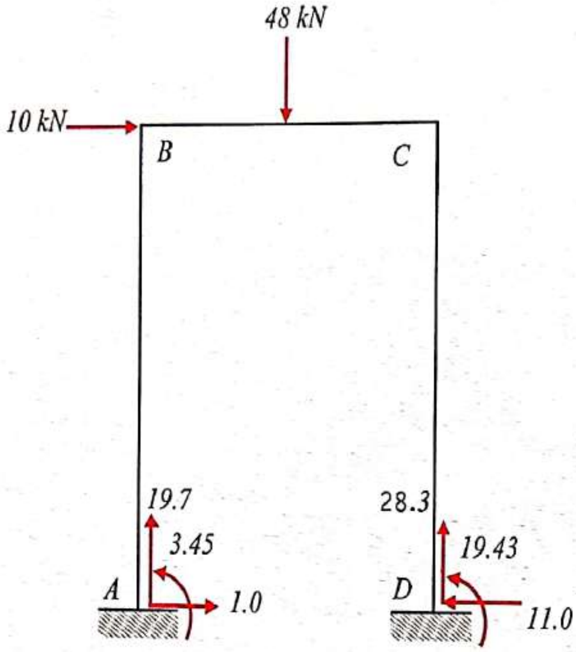
Now, get unknown Joint forces,

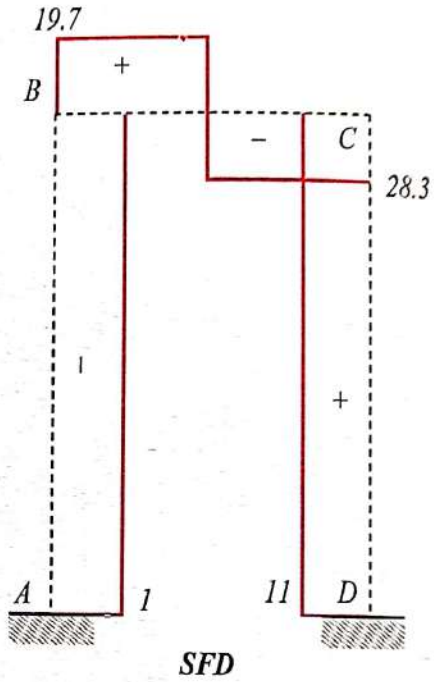
P1 = 1.0 kN ; P2 = 19.7 kN; P3 = 3.45 kN.m; P10 = -11.0 kN;
P11 = 28.3 kN; P12 = 19.43 kN.m

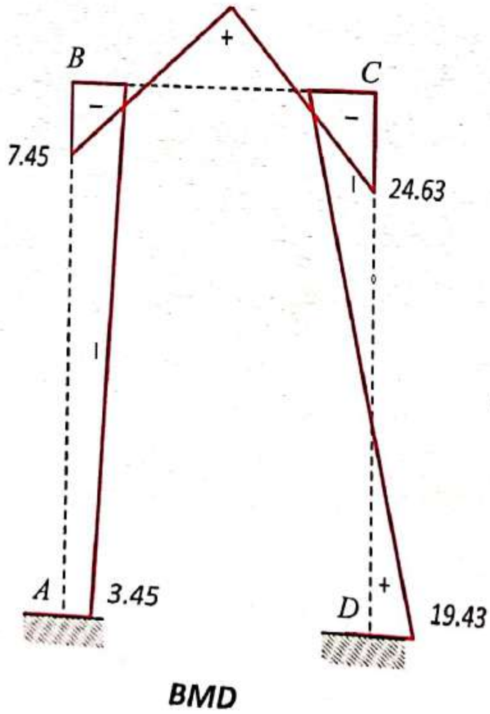
From the free body of Element 1 and Element 2 it gives,

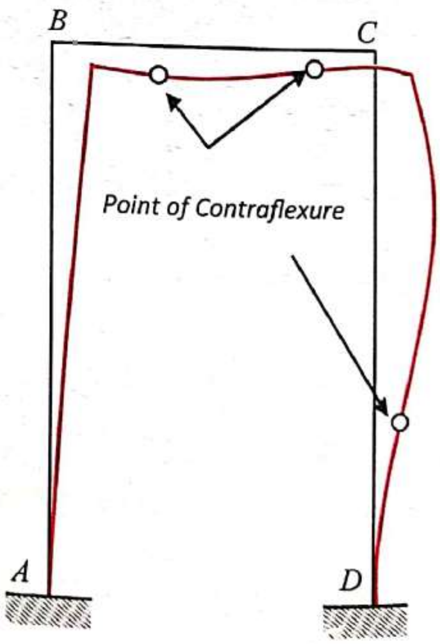
P4 = 11.0 kN; P5 = -19.7 kN; P6 = -7.45 kN.m; P7 = -11 kN;
P8 = 28.3 kN; P9 = -24.63 kN.m

Now, draw the axial force, shear force and bending moment diagrams for the whol structures.



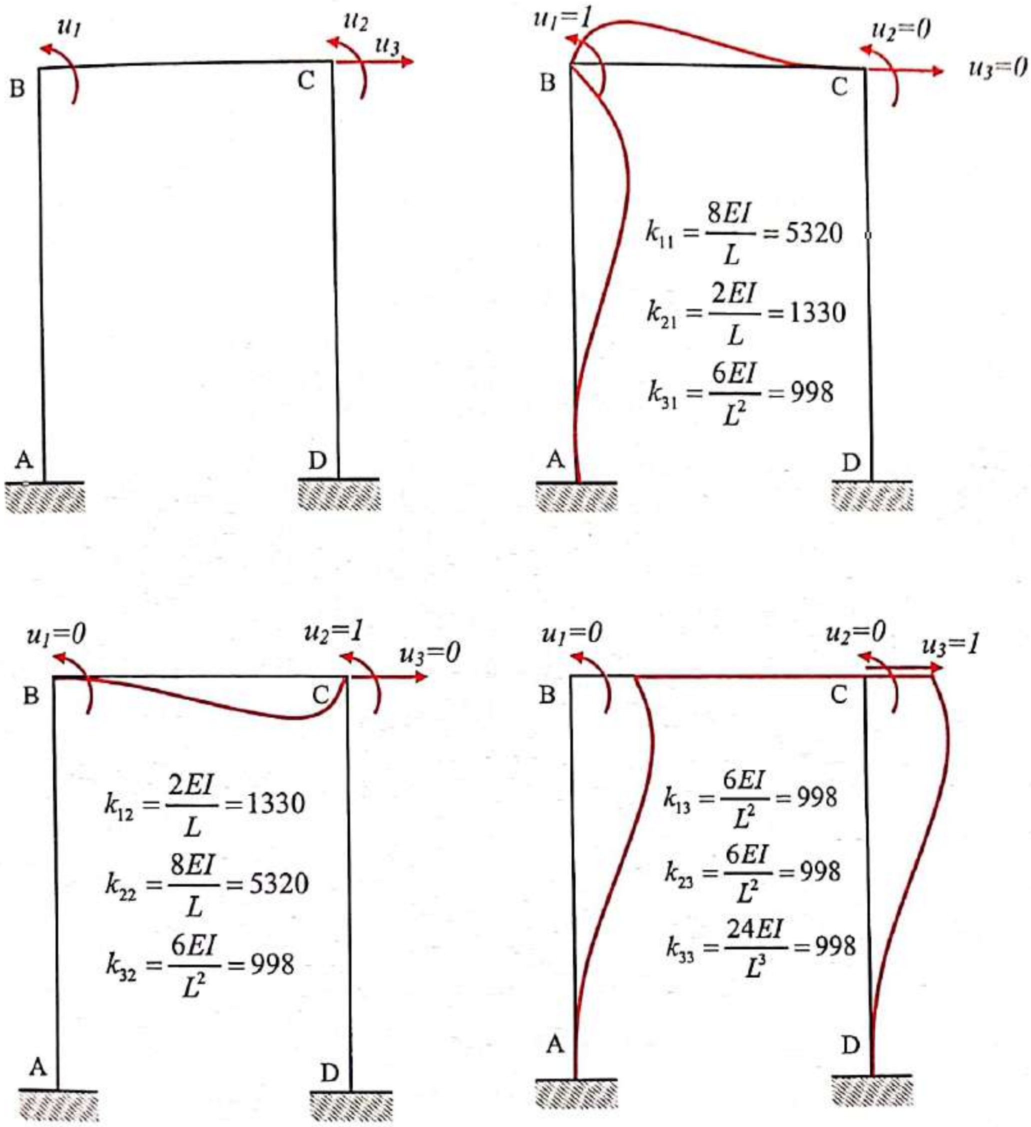






Solution by classical analysis

in this frame, there will be to rotations and one side sway. So, the total degree of freedoms will be three, *d.o.k.i = 3*



Equation of equilibrium:

$$[P]=[K][u]+[P_f]$$

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 5320 & 1330 & 998 \\ 1330 & 5320 & 998 \\ 998 & 998 & 998 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} P_{f1} \\ P_{f2} \\ P_{f3} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 5320 & 1330 & 998 \\ 1330 & 5320 & 998 \\ 998 & 998 & 998 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 24 \\ -24 \\ 0 \end{bmatrix}$$

Solve for u,

$$u_1 = -8.16\times10^{-3}\ rad$$

$$u_2 = 3.86\times10^{35}\ rad$$

$$u_3 = 1.43\times10^{-2}\ m$$

Now apply member stiffness equation to get unknown forces,

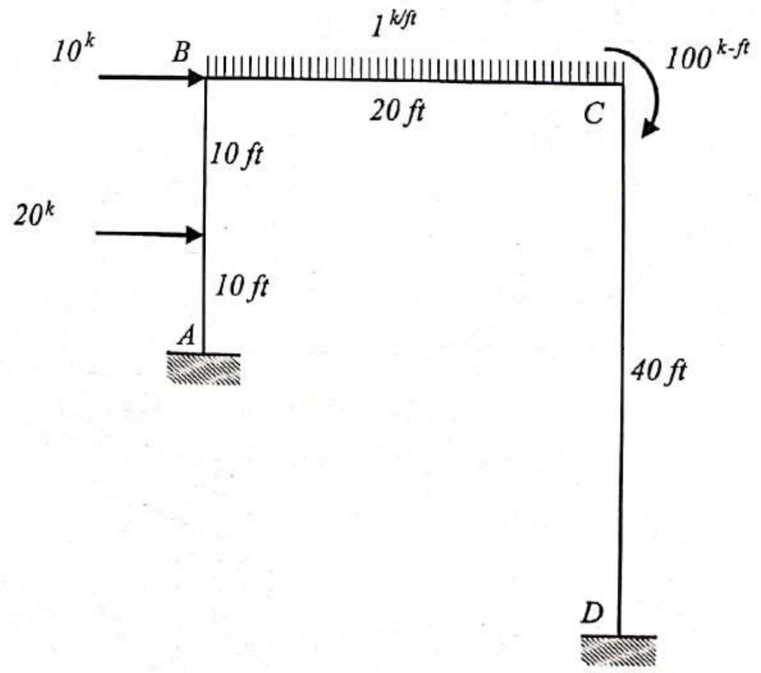
$$\begin{aligned} M_A &= k_1u_1 + k_3u_3 + P_f \\ &= \frac{2EI}{L}u_1 + \frac{6EI}{L^2}u_3 + 0 \\ &= -1330\times8.16\times10^{-3} + 998\times1.43\times10^{-2} \\ &= 3.42\ \text{kN.m} \end{aligned}$$

The rest of the forces find by yourself.

$$\begin{aligned} M_{BL} &= k_1u_1 + k_3u_3 + P_f \\ &= \frac{4EI}{L}u_1 + \frac{6EI}{L^2}u_3 + 0 \\ &= -2660\times8.16\times10^{-3} + 998\times1.43\times10^{-2} \\ &= -7.43\ \text{kN.m} \end{aligned}$$

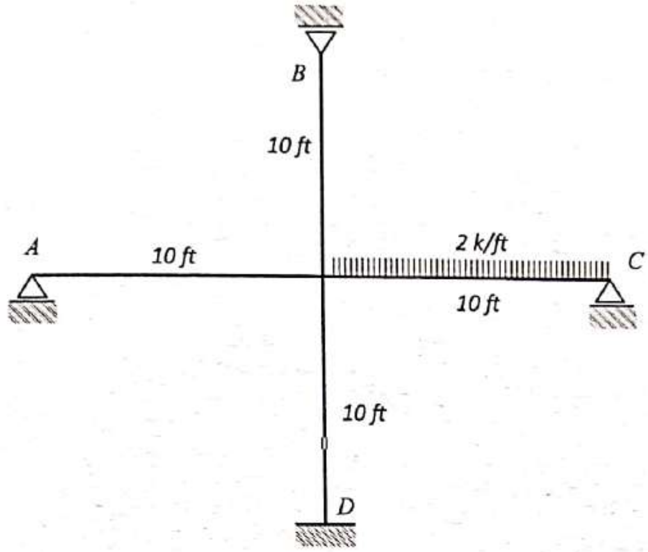
Exercise 1

Analyze the frame for *A = 4 in² ; I = 200 in⁴ ; E = 30 x 10³ ksi*



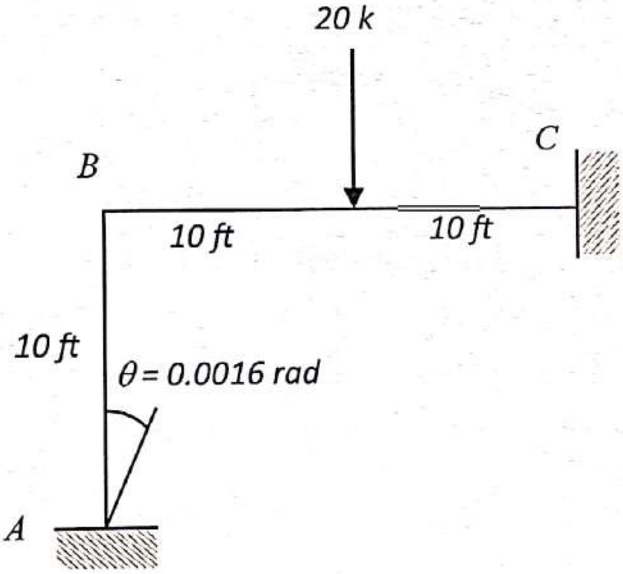
Exercise 2

Analyze the frame for $A = 5in^2$; $I = 400\ in^4$; $E = 2.5 \times 10^3$ ksi

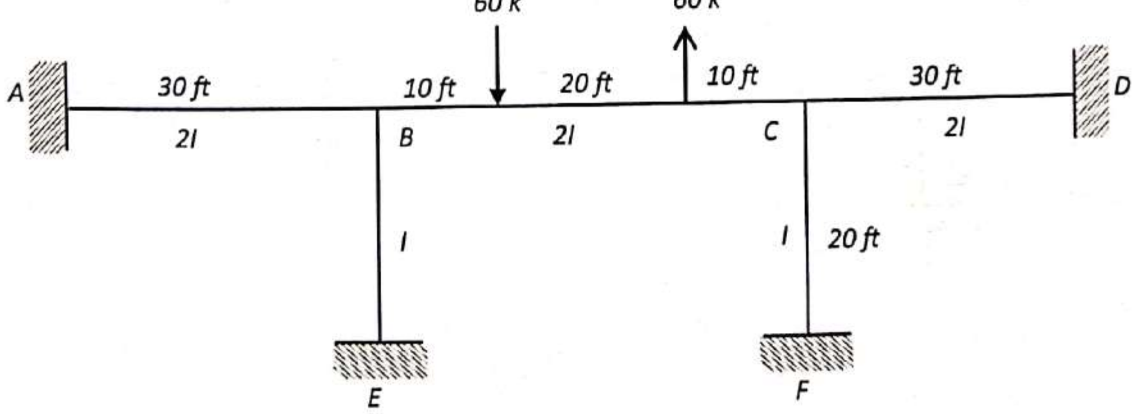


Exercise 3

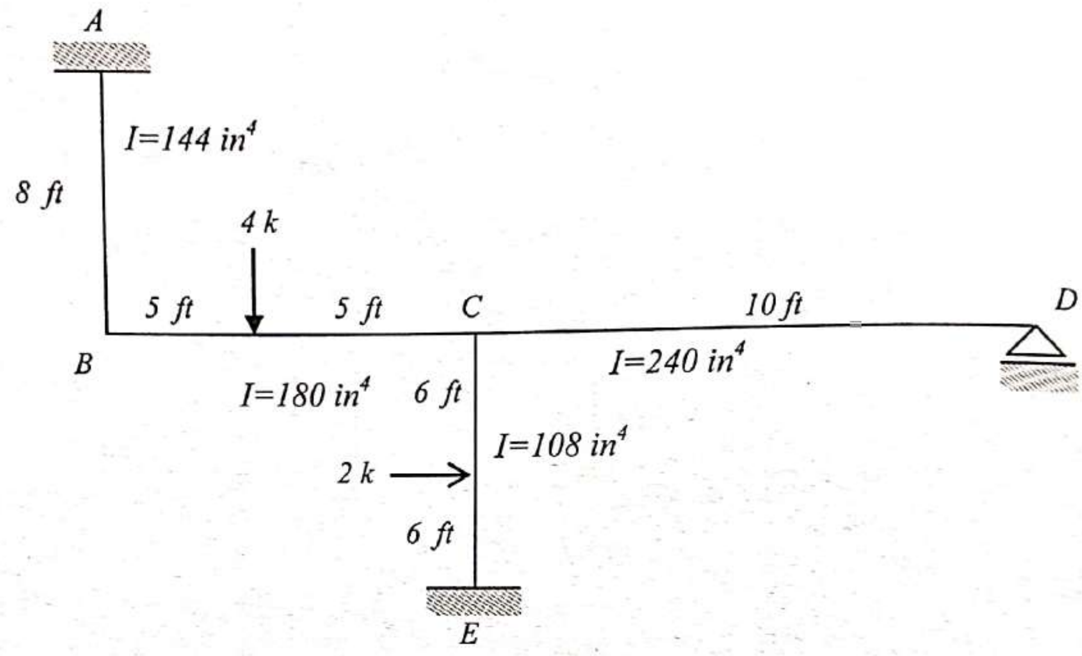
Analyze the frame for the applied load and a rotational yield of support A 0.0016 rad clockwise. for $A = 4.5\ in^2$; $I = 450\ in^4$; $E = 2.5 \times 10^3$ ksi



Exercise 4

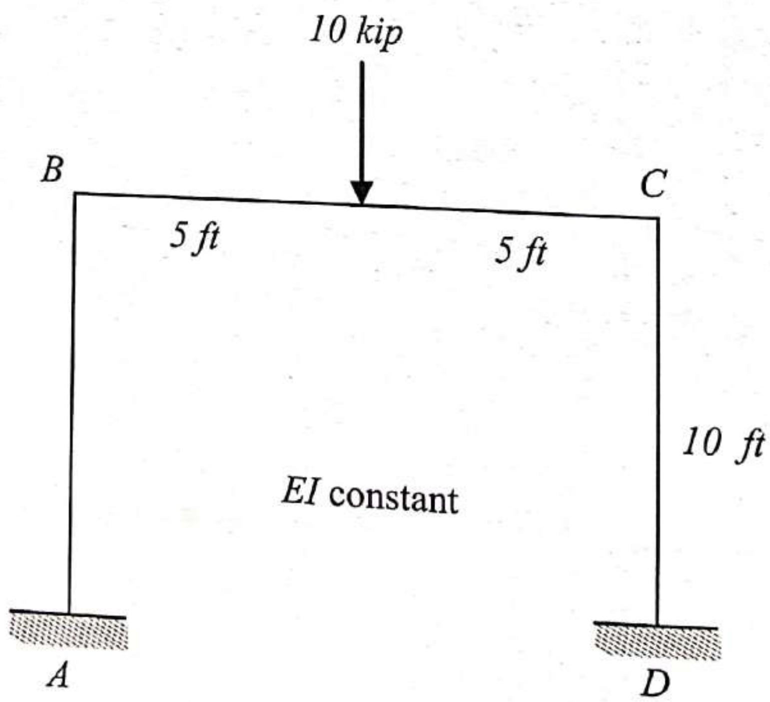


Exercise 5

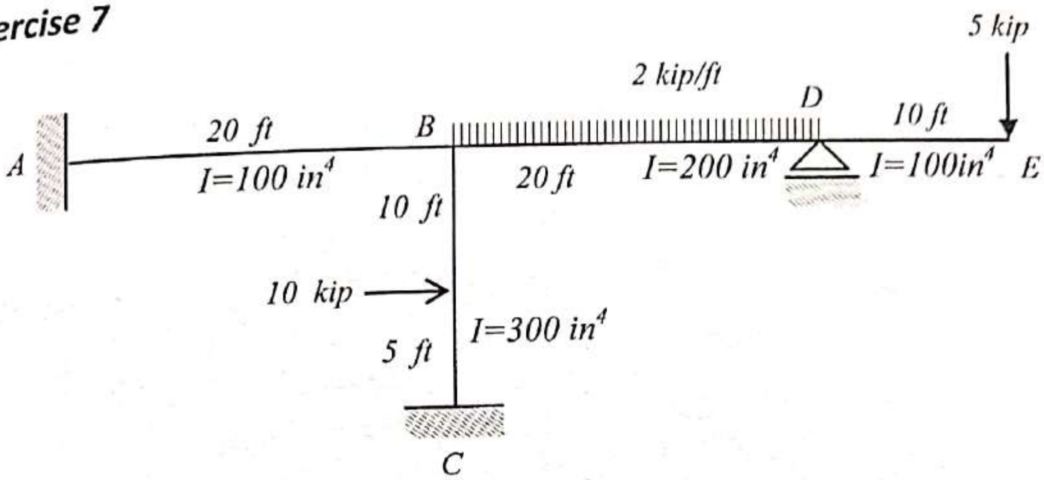


Ans: $M_A = -1.73$, $M_E = 3.96$, $M_B = 3.47$

Exercise 6

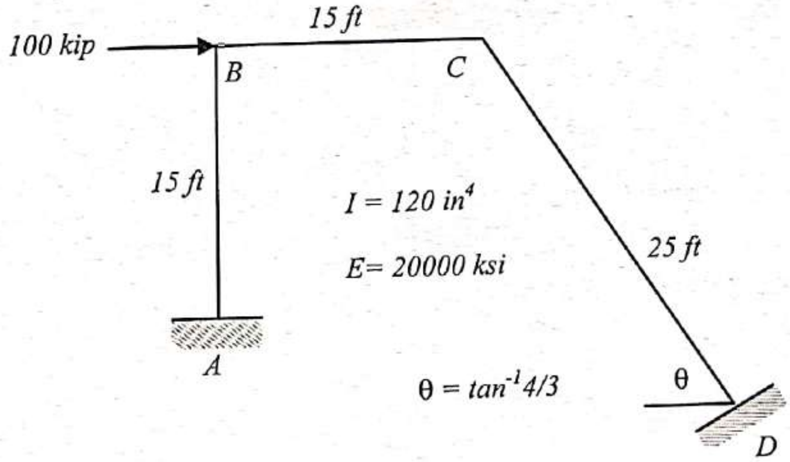


Exercise 7

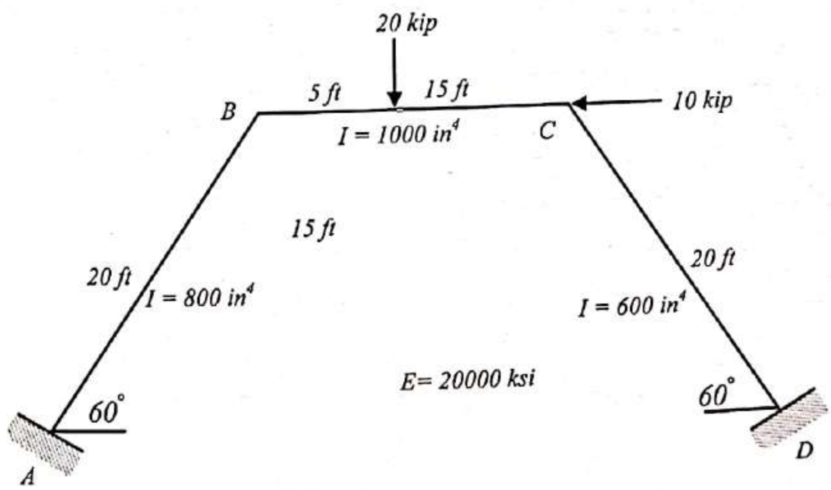


Ans: $M_A = -47$, $M_D = 50$, $M_{BA} = -31.5$, $M_{BC} = -53.6$, $M_{BD} = 85.1$, $M_C = 1.0$ *(in kip-ft)*

Exercise 8



Exercise 9



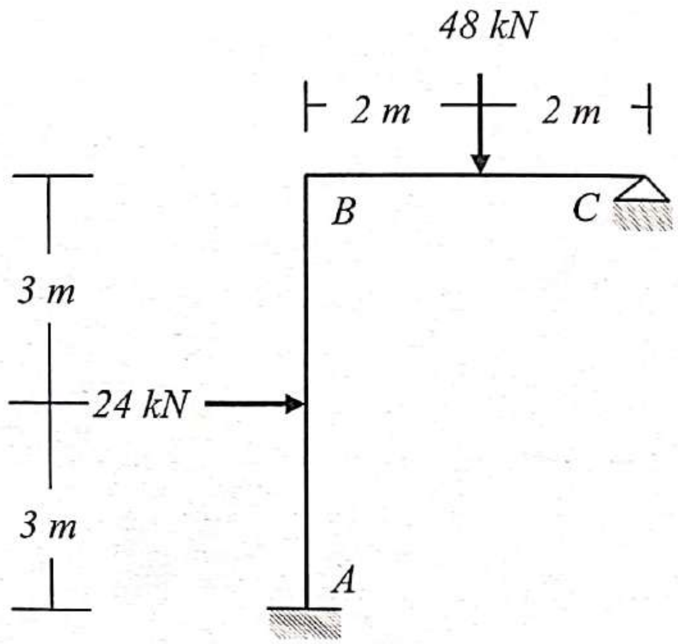
Ans: $M_A = -24.5$, $M_B = 25.1$, $M_C = 22.3$, $M_D = 20.1$ *(kip-ft)*

Exercise 10

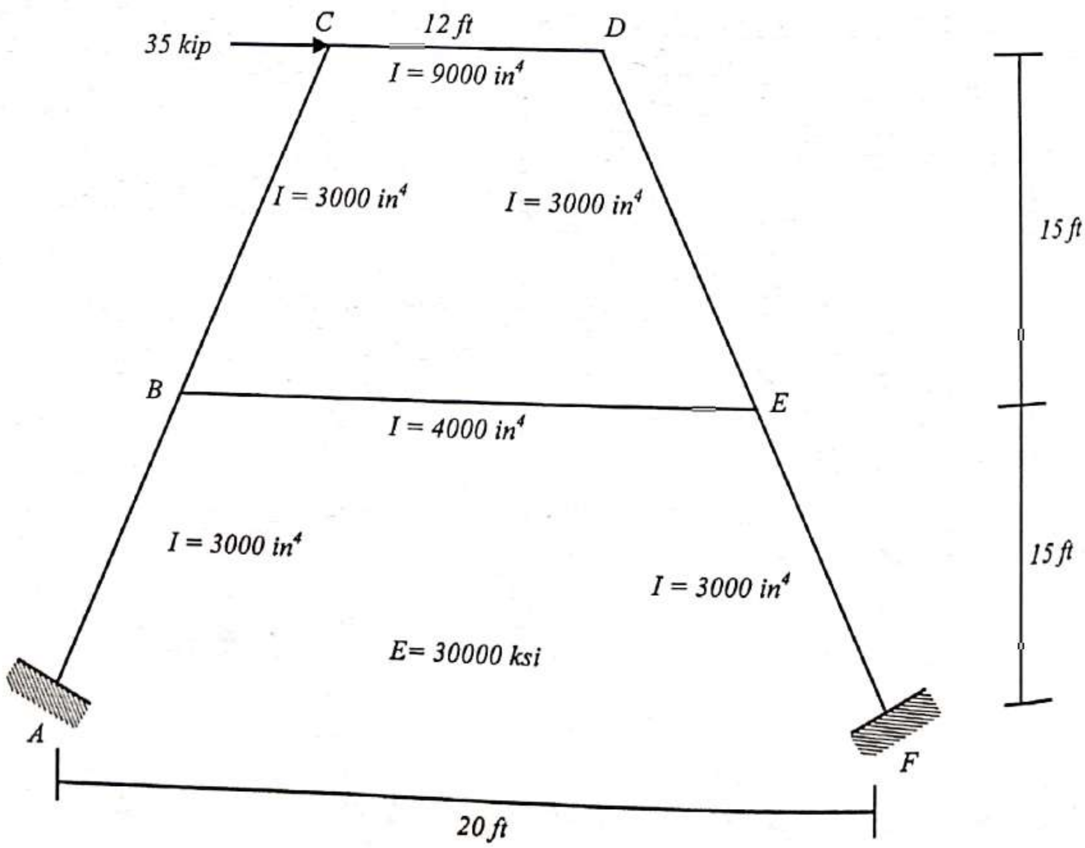
Analysis the following frame considering,

a) Axial shear and rotational deformation

b) Only rotational deformation



Exercise 11

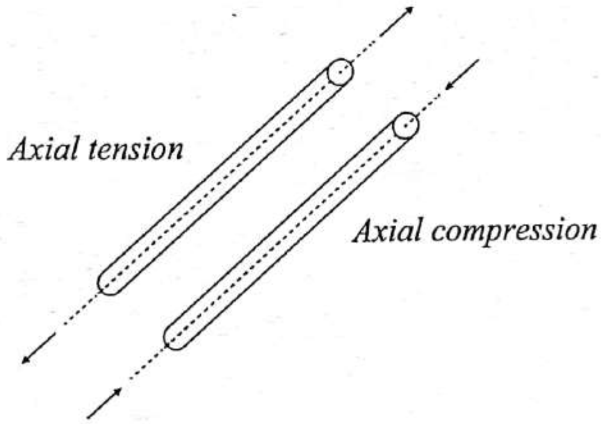


Analysis of Indeterminate Trusses by Direct Stiffness method

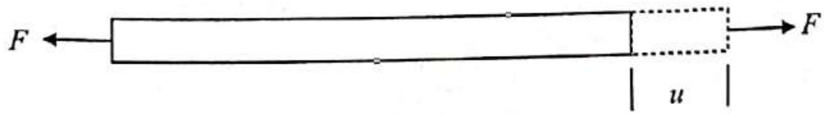
Introduction

The basic concept of the stiffness method of beam and frame analysis has been described in the first section of this chapter. The problems were solved with hand computation by classical method of analysis with the help of direct application of the basic principles. The classical method of analysis is not suitable for computer programming. It is necessary to keep hand computation to a minimum while implementing this procedure on the computer. The systematic analysis of structures by direct stiffness method is developed for the very aim of computer programming. In this section, the direct stiffness method for planar truss structure is discussed.

A plane truss is a structural system that is made up of straight, slender and prismatic short thin members interconnected at hinges to form triangulated patterns. A frictional pin connection can only transmit forces from one member to another but not the moment. In a truss, the loads are applied at the joints only. Therefore, a truss member is subjected to only axial forces and as the forces remain constant along the length of the member, they are called two-force member. The forces in the member at its two ends must be of the same magnitude but act in the opposite directions for equilibrium as shown below.

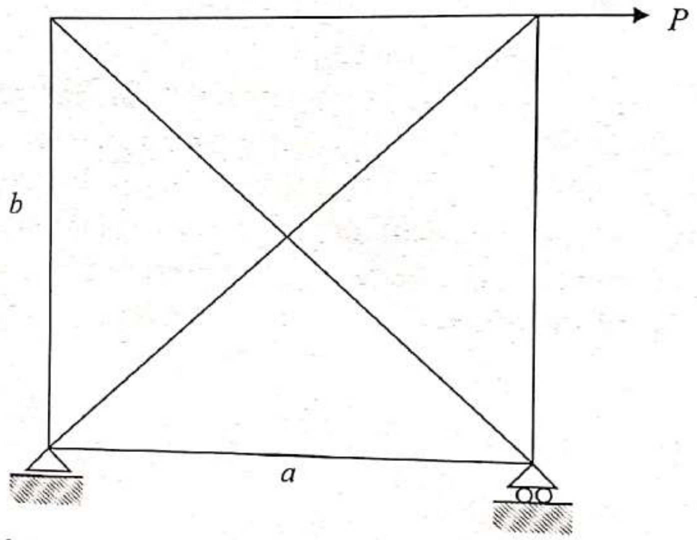


Consider a truss member having cross sectional area A, Young's modulus of material E, and length of the member L. Let the member be subjected to an axial tensile force as shown under the action of constant axial force F, applied at each end, the member gets an elongation in the order of u.



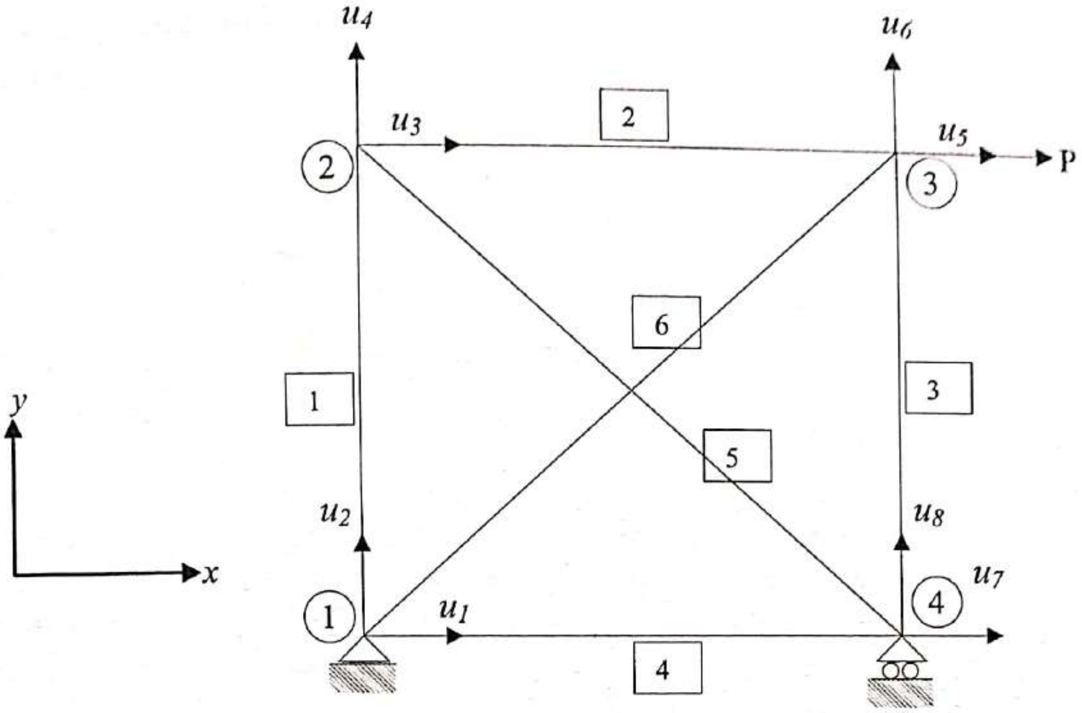
The axial elongation can be given by, $u = \frac{FL}{AE}$ Now, the force displacement relationship for the truss member can be written as, $F = \frac{AE}{L} u$ If the stiffness is defined as k ,it can be written as, $F = ku$ where, $k = \frac{AE}{L}$ is the stiffness of the truss member and is defined as the force required for unit deformation of the structure.

In a real truss structure, there are many such members. For example, consider a planar truss shown in figure below. For each member of the truss, one equation can be written of the type shown above, along its axial direction (which is called as local coordinate system). Each truss member has different orientation with global coordinate system. To analyze a planar truss, it is therefore required to write a force-displacement relation for the complete truss in a coordinate system common to all members. Such a coordinate system is referred to as global coordinate system.



Local and Global Co-ordinate System

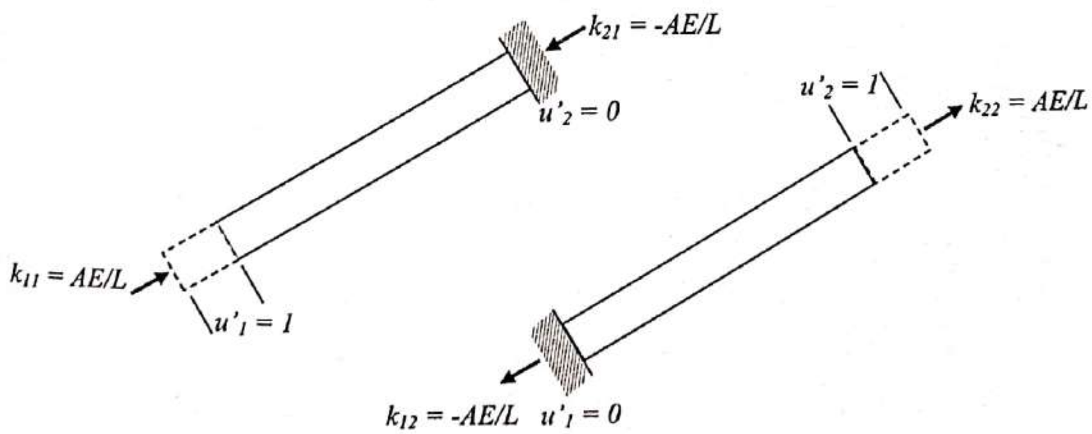
Loads and displacements are vector quantities and therefore a proper coordinate system is required to specify their correct sense of direction. Consider a planar truss as shown in the figure below. In this truss, each node is identified by a number enclosed in a circle and each member is identified by a number enclosed in a rectangle. The displacements and loads acting on the truss are defined with respect to global coordinate system xyz. In a global coordinate system, each node of a planar truss can have only two displacements: one along x-axis and another along y-axis. The truss shown in this figure has eight displacements degree of freedom. So, the degree of kinematic indeterminacy ($d.o.k.i$) of this truss is eight. Each displacement (degree of freedom) ina truss is shown by a number and direction arrow in the figure below. However out of eight displacements, five are unknown.



To analyze the truss shown in the above figure, the global stiffness matrix K need to be evaluated for the given truss. This may be achieved by suitably putting all the member stiffness matrices into the global stiffness matrix. Since all members are oriented at different directions, it is required to transform member displacements and forces from the local coordinate system to global coordinate system so that a global load-displacement relation may be written for the complete truss.

Member Stiffness Matrix

Unlike a member of a frame, the member of a truss only under goes axial deformation due to axial forces; they are either in tension or in compression. So, in each node there will be a single degree of freedom in local coordinate system, as shown in figure below. And obviously it is displacement degree of freedom.



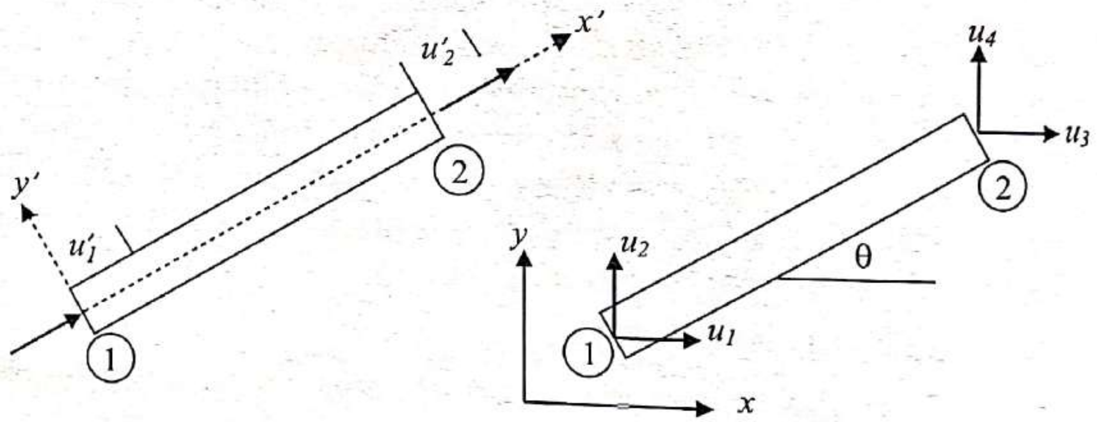
Thus, the member (element) stiffness matrix in local coordinate system is given by,

$$\left[k'\right]=\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Transformation of element stiffness matrix from local to global coordinate System

Displacement transformation matrix

A truss member is shown in the local and global coordinate system in figure below. Let $x'y'$ be the local coordinate system and xy be the global coordinate system. Let the truss member is inclined to xy axis by an angle θ as shown in the figure. It is observed from the figure that u'_1 is equal to the projection of u_1 on x' axis plus projection u_2 of on x'-axis. Thus,



$$u_1' = u_1\cos\theta + u_2\sin\theta$$
$$u_2' = u_3\cos\theta + u_4\sin\theta$$

In matrix form, $\begin{bmatrix}u_1'\\ u_2'\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta & 0 & 0\\ 0 & 0 & \cos\theta & \sin\theta\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix}$

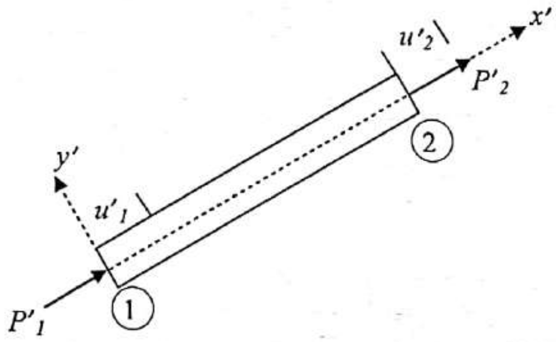
$$\begin{bmatrix}u_1'\\ u_2'\end{bmatrix}=\begin{bmatrix}l & m & 0 & 0\\ 0 & 0 & l & m\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix}$$ [*where,* $l=\cos\theta$; $m=\sin\theta$]

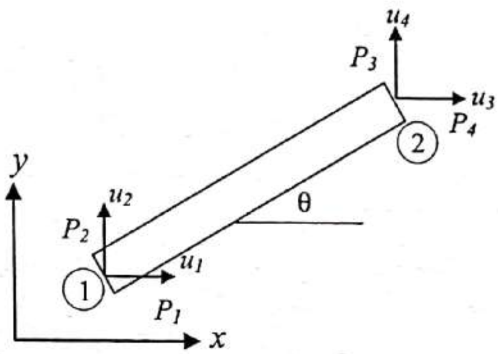
or $\left[u'\right]=[T][u]$

In the above equation, $[T]$ is the displacement transformation matrix which transforms the four global displacement degree of freedom into two displacement degree of freedom in local coordinate system.

Force transformation matrix

Let P'_1 , P'_2 be the forces in a truss member at node 1 and 2 respectively producing u'_1, u'_2 displacements in the local co-ordinate system $x'y'$ and P_1, P_2, P_3, P_4 be the forces in global co-ordinate system at node 1 and 2 respectively producing displacements u_1, u_2 and u_3, u_4.





$P_1 = P_1^{'} \cos\theta;\ P_2 = P_1^{'} \sin\theta;\ P_3 = P_2^{'} \cos\theta;\ P_4 = P_2^{'} \sin\theta$; In matrix form,

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{bmatrix} \begin{bmatrix} P_1^{'} \\ P_2^{'} \end{bmatrix}$$

or, $[P] = [T]^T \left[P^{'}\right]$

Element Stiffness Matrix in Global Coordinate System

$$\begin{aligned} [P'] &= [k'][u'] \\ \textit{again,}\ [P] &= [T]^T [P'] \\ &= [T]^T [k'][u'] \\ &= [T]^T [k'][T][u] \\ &= [k][u] \end{aligned}$$

$$\text{Therefore, } [k]=[T]^T[k'][T]$$
$$[P]=[T]^T[P']$$
$$[u']=[T][u]$$

$[k]=[T]^T[k'][T]$; Multiplying $[T]$ and $[T]^T$ with $[k']$, the member stiffness matrix $[k]$ in global coordinate becomes,

$$[k]=\frac{EA}{L}\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$[k]=\frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

The global load-displacement equilibrium equations for a truss element are therefore written as,

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix}=\frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \qquad [P]=[K][u]$$

The major steps in solving planar truss problems are:

Step 1 Watch out the problem carefully and set up the local coordinate system for each element. Identify and label the nodes and the elements. For each element select a start node (node 1) and an end node (node 2). This establishes the local coordinate system for each element. The orientation of local x' axis should be starting from lower node to upper node and the local y' axis will be at right angle with local x' in counter-clockwise direction. Label the two global DOF at each node starting from node 1 and proceeding sequentially.

Step 2 Find out the member orientation of each element. It is the angle, the global x axis made with an element's local x' axis in counter-clockwise direction. Then Construct the element stiffness matrix $[k]$ for all the elements of the truss in global coordinate system.

Step 3 Assemble the element stiffness matrices $[k]$ into the global stiffness matrix $[K]$

Step 4 Construct the stiffness equilibrium equation $[P]_{i\times 1}=[K]_{i\times 1}[u]_{i\times 1}$ where i is the total number of degree of freedom (DOF) in the truss.

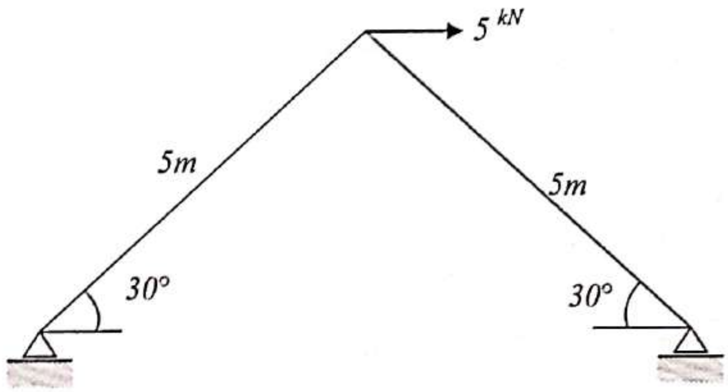
Step 5 Impose the boundary conditions.

Step 6 Solve the system equations $[P]=[K][u]$ for the nodal displacements u.

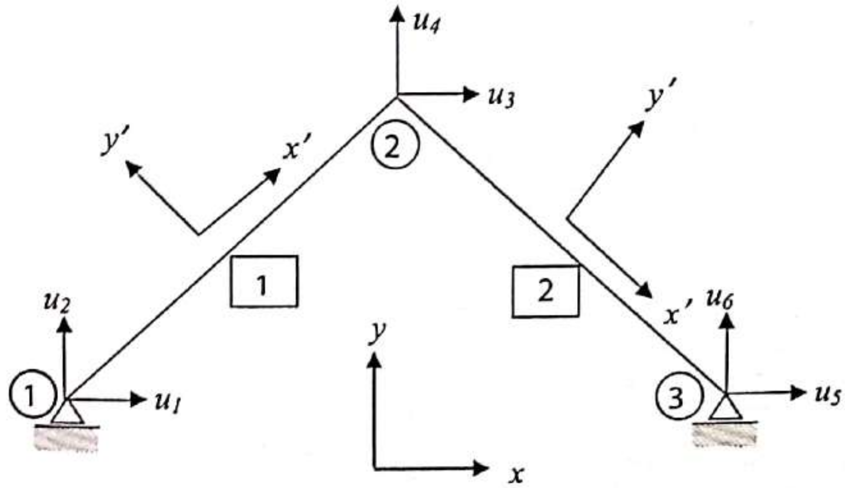
Step 7 For each member, using the nodal displacements, compute the element nodal forces from element stiffness equilibrium equation $[P]_m=[K]_m[u]_m$

Example 1

Analyze the two member truss shown in figure below. Assume EA to be constant for all members. The length of each member is $5m$.



Solution



Now construct the member stiffness matrix for each member in global co-ordinate system,

For member 1, $\theta = 30^\circ$ and for member 2, $\theta = 330^\circ$

$$[k]_1 = \frac{EA}{5}\begin{bmatrix} 0.75 & 0.433 & -0.75 & -0.433 \\ 0.433 & 0.25 & -0.433 & -0.25 \\ -0.75 & -0.433 & 0.75 & 0.433 \\ -0.433 & -0.25 & 0.433 & 0.25 \end{bmatrix}$$

$$[k]_2 = \frac{EA}{5}\begin{bmatrix} 0.75 & -0.433 & -0.75 & 0.433 \\ -0.433 & 0.25 & 0.433 & -0.25 \\ -0.75 & 0.433 & 0.75 & -0.433 \\ 0.433 & -0.25 & -0.433 & 0.25 \end{bmatrix}$$

The global stiffness matrix of the truss can be obtained by assembling the two element stiffness matrices. Thus,

$$[K] = \frac{EA}{5}\begin{bmatrix} 0.75 & 0.433 & -0.75 & -0.433 & 0 & 0 \\ 0.433 & 0.25 & -0.433 & -0.25 & 0 & 0 \\ -0.75 & -0.433 & 1.5 & 0 & -0.75 & 0.433 \\ -0.433 & -0.25 & 0 & 0.5 & 0.433 & -0.25 \\ 0 & 0 & -0.75 & 0.433 & 0.75 & -0.433 \\ 0 & 0 & 0.433 & -0.25 & -0.433 & 0.25 \end{bmatrix}$$

The global stiffness equilibrium equation can be given as,

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} = \frac{EA}{5}\begin{bmatrix} 0.75 & 0.433 & -0.75 & -0.433 & 0 & 0 \\ 0.433 & 0.25 & -0.433 & -0.25 & 0 & 0 \\ -0.75 & -0.433 & 1.5 & 0 & -0.75 & 0.433 \\ -0.433 & -0.25 & 0 & 0.5 & 0.433 & -0.25 \\ 0 & 0 & -0.75 & 0.433 & 0.75 & -0.433 \\ 0 & 0 & 0.433 & -0.25 & -0.433 & 0.25 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

Applying boundary conditions,

$$u_1 = 0;\ u_2 = 0;\ u_5 = 0;\ u_6 = 0 \text{ and}$$
$$P_3 = 5;\ P_4 = 0$$

$$\begin{bmatrix} P_1 \\ P_2 \\ 5 \\ 0 \\ P_5 \\ P_6 \end{bmatrix} = \frac{EA}{5}\begin{bmatrix} 0.75 & 0.433 & -0.75 & -0.433 & 0 & 0 \\ 0.433 & 0.25 & -0.433 & -0.25 & 0 & 0 \\ -0.75 & -0.433 & 1.5 & 0 & -0.75 & 0.433 \\ -0.433 & -0.25 & 0 & 0.5 & 0.433 & -0.25 \\ 0 & 0 & -0.75 & 0.433 & 0.75 & -0.433 \\ 0 & 0 & 0.433 & -0.25 & -0.433 & 0.25 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ 0 \\ 0 \end{bmatrix}$$

Solving for u_3 and u_4,

$$u_3 = \frac{16.667}{EA}; \quad u_4 = 0$$

Solve for unknown nodal force, P_1, P_2, P_5 and P_6

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} -2.5 \\ -1.443 \\ -2.5 \\ +1.443 \end{Bmatrix} kN$$

Now force in each member can be calculated by either solving the element load-displacement equilibrium equation $[P'] = [k'][u']$ or just static nodal analysis.

For member 1, $L = 5m$; $l = 0.866$, $m = 0.5$

$$[P'] = [k'][u'] = [k'][T][u]$$

$$\begin{bmatrix} P_1' \\ P_2' \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\begin{bmatrix} P_1' \\ P_2' \end{bmatrix} = \frac{AE}{5}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0.866 & 0.5 & 0 & 0 \\ 0 & 0 & 0.866 & 0.5 \end{bmatrix}\begin{bmatrix} 0 \\ 16.667/EA \\ 0 \end{bmatrix}$$

$$\left[P_1'\right]=\frac{AE}{5}\left[0.866 \quad 0.5 \quad -0.866 \quad -0.5\right]\begin{bmatrix}0\\0\\16.667/EA\\0\end{bmatrix}$$

$$= -2.88 \text{ kN } [T]$$

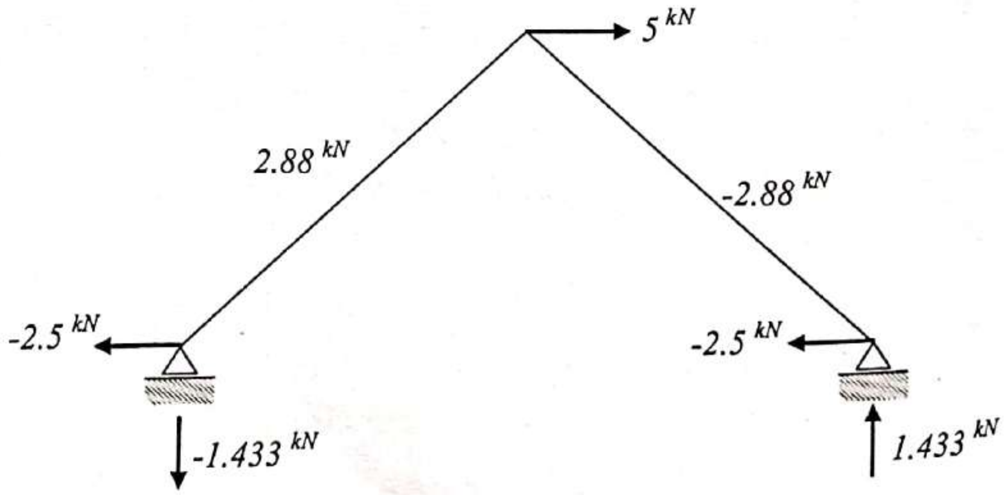
For member 2, *L = 5m; l = 0.866, m =- 0.5*

$$\begin{bmatrix}P_3'\\P_4'\end{bmatrix}=\frac{AE}{L}\begin{bmatrix}P & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}l & m & 0 & 0\\0 & 0 & l & m\end{bmatrix}\begin{bmatrix}u_3\\u_4\\u_5\\u_6\end{bmatrix}$$

$$\begin{bmatrix}P_3'\\P_4'\end{bmatrix}=\frac{AE}{5}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}0.866 & -0.5 & 0 & 0\\0 & 0 & 0.866 & -0.5\end{bmatrix}\begin{bmatrix}16.667/EA\\0\\0\\0\end{bmatrix}$$

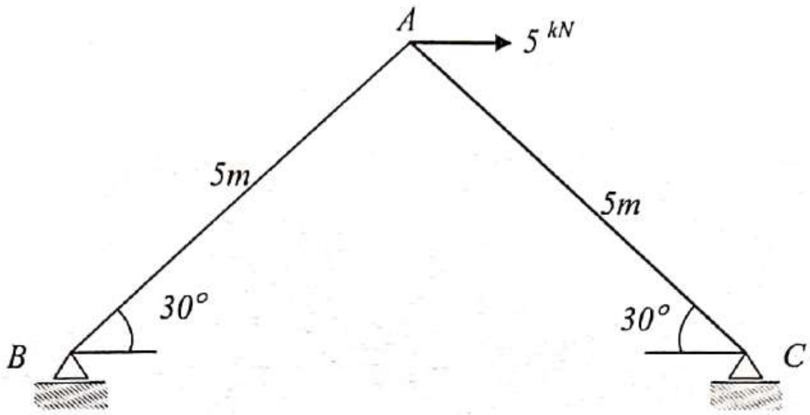
$$\left[P_4'\right]=\frac{AE}{5}\left[-0.866 \quad +0.5 \quad +0.866 -0.5\right]\begin{bmatrix}16.667/EA\\0\\0\\0\end{bmatrix}$$

$$=2.88 \text{ kN } [C]$$



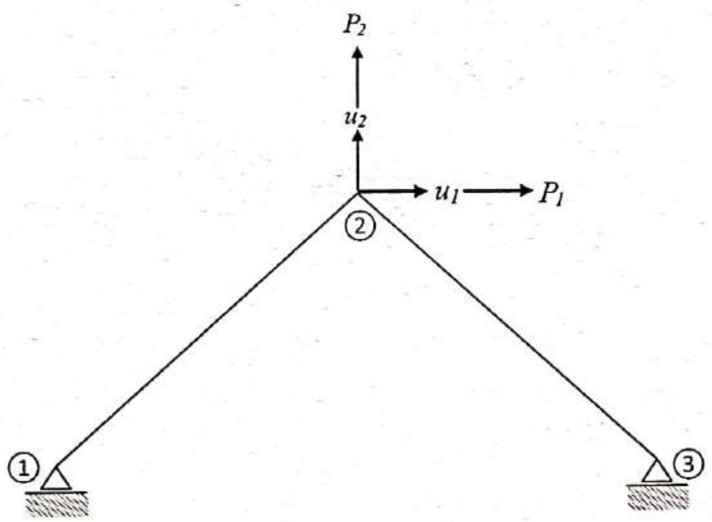
Example 1 (alternate solution)

Solution by classical method

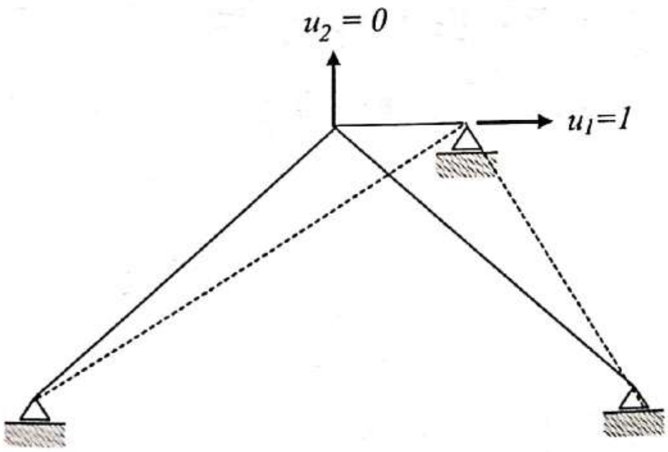


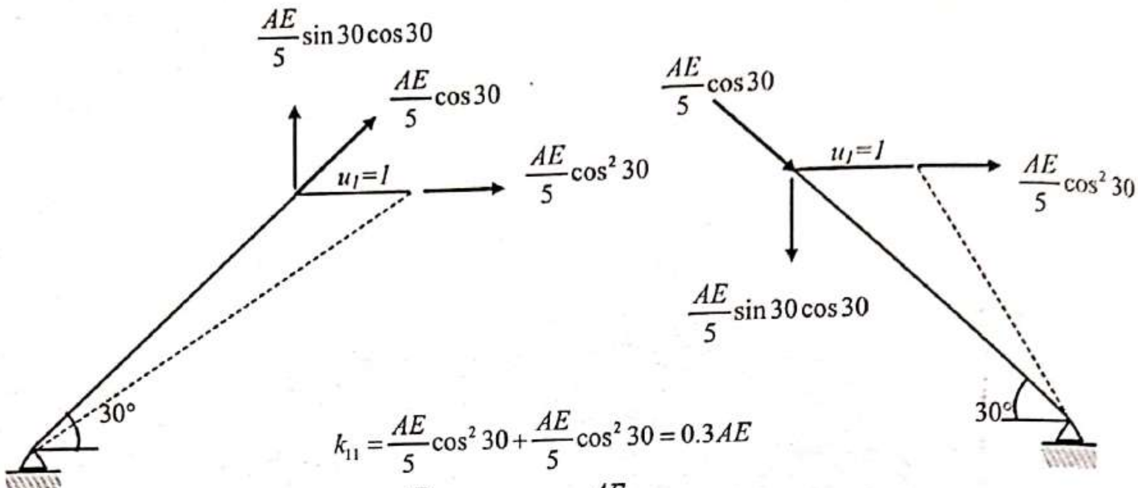
In classical method, the drgree of freedoms are only taken where there will be nodal translations. So, at node 2 there are two degree of freedoms.

d.o.k.i. = 2



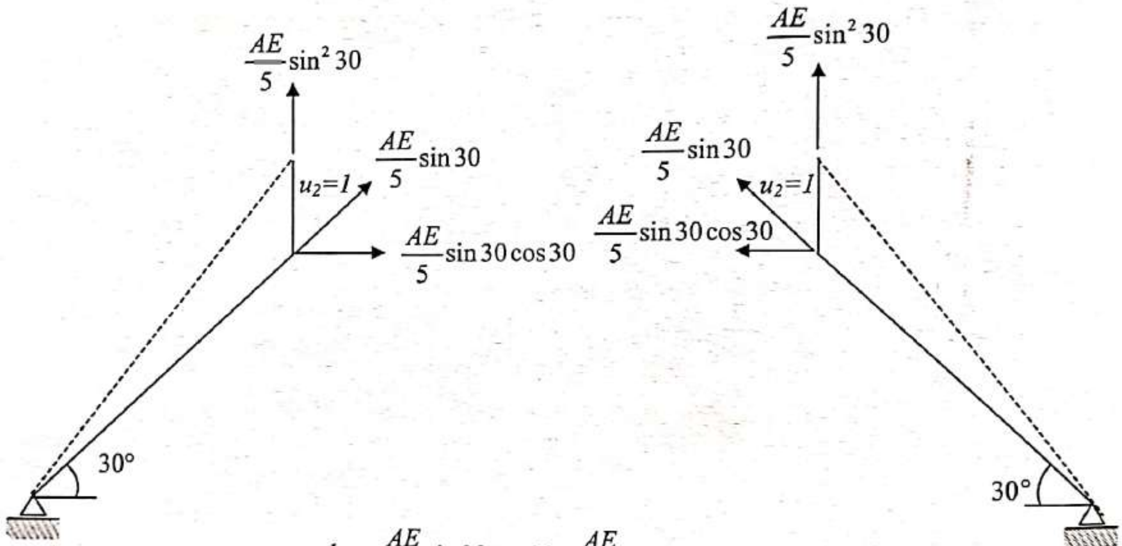
Apply unit displacement along one DOF at a time, and calculate the stiffness coefficients;





$$k_{11}=\frac{AE}{5}\cos^2 30+\frac{AE}{5}\cos^2 30=0.3AE$$

$$k_{21}=\frac{AE}{5}\sin 30\cos 30-\frac{AE}{5}\sin 30\cos 30=0$$



$$k_{12}=\frac{AE}{5}\sin 30\cos 30-\frac{AE}{5}\sin 30\cos 30=0$$

$$k_{22}=\frac{AE}{5}\sin^2 30+\frac{AE}{5}\sin^2 30=0.1AE$$

The stiffness equilibrium equation can be written as,

$$[P]=[k][u]$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}=AE\begin{bmatrix} 0.3 & 0 \\ 0 & 0.1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Apply boundary conditions, $P_1 = 5; P_2 = 0,$

$$\begin{bmatrix} 5 \\ 0 \end{bmatrix} = AE \begin{bmatrix} 0.3 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

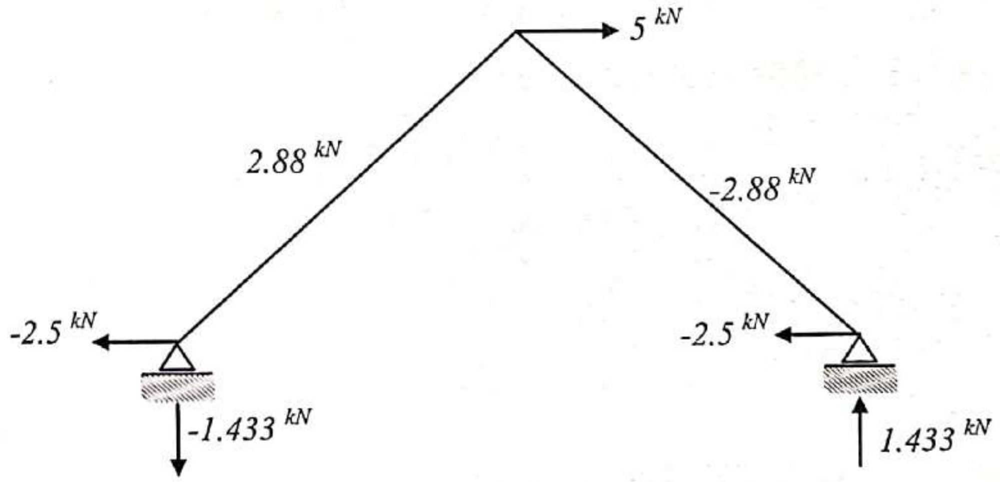
Solving for u_1 and u_2, $u_1 = \dfrac{16.667}{EA}$; $u_2 = 0$

Calculation of bar forces,

$$F_{AB} = \frac{AE}{5}\cos 30 \times u_1 + \frac{AE}{5}\sin 30 \times u_2 = \frac{AE}{5}\cos 30 \times \frac{16.667}{AE} = 2.88(T)$$

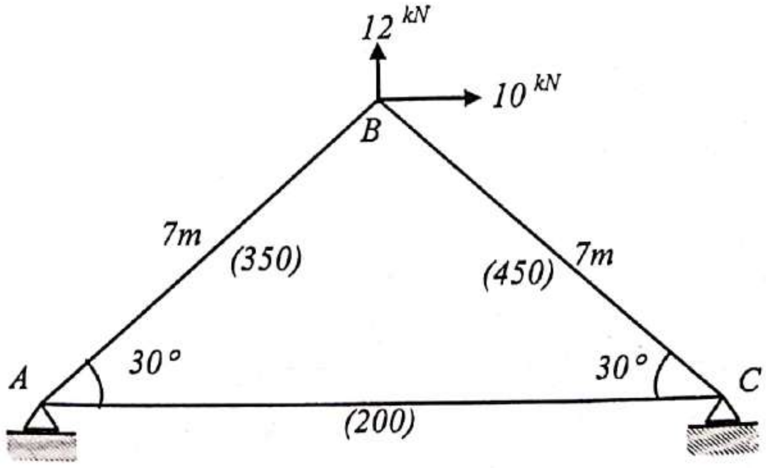
$$F_{AC} = -\frac{AE}{5}\cos 30 \times u_1 + \frac{AE}{5}\sin 30 \times u_2 = -\frac{AE}{5}\cos 30 \times \frac{16.667}{AE} = -2.88(C)$$

The external reaction can be found by simply static analysis of the truss,

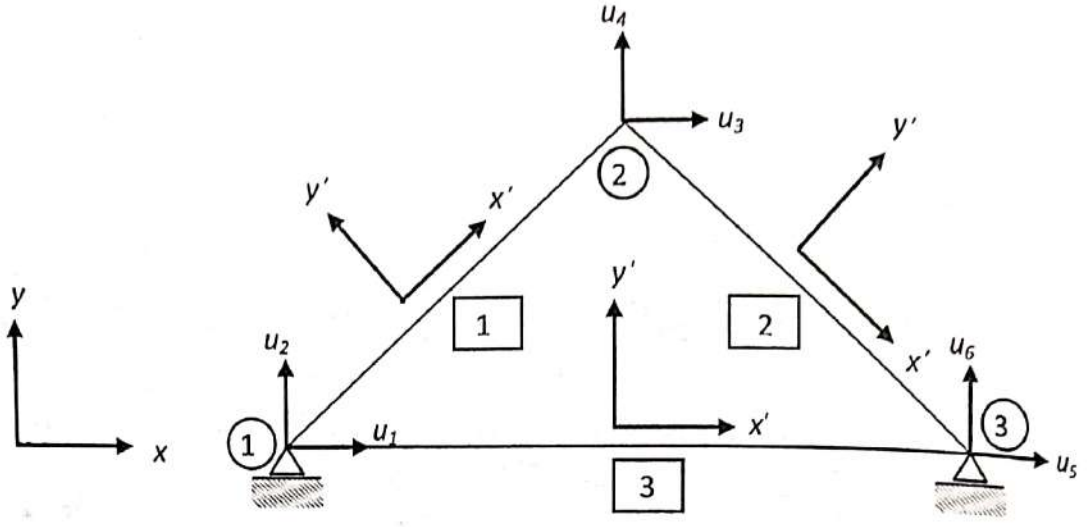


Example 2

Analyze the following indeterminate truss and find the unknown reaction and bar forces by direct stiffness method. Figure in parenthesis gives the corresponding bar cross-sectional area in mm². Assume *E* as constant.



Solution:



Now member stiffness matrix for each member in global co-ordinate system,

For member 1, $\theta = 30^o$ A = 350 mm^2, L= 7m.

$$[k]_1 = \frac{E}{10^6} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 37.5 & 21.65 & -37.5 & -21.65 \\ 21.65 & 12.5 & -21.65 & -12.5 \\ -35.5 & -21.65 & 37.5 & 21.65 \\ -21.65 & -12.5 & 21.65 & 12.5 \end{bmatrix} \end{array}$$

For member 2, $\theta = 330^o$ A = 450 mm^2, L= 7m.

$$[k]_2 = \frac{E}{10^6} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 48.21 & -27.84 & -48.21 & 27.84 \\ -27.84 & 16 & 27.84 & -16 \\ -48.21 & 27.84 & 48.21 & -27.84 \\ 27.84 & -16 & -27.84 & 16 \end{bmatrix} \end{array}$$

For member 3, $\theta = 0^o$ A = 200 mm^2, L= 12.12m

$$[k]_3 = \frac{E}{10^6} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} 16.5 & 0 & 16.5 & 0 \\ 0 & 0 & 0 & 0 \\ -16.5 & 0 & 16.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The global stiffness matrix of the truss can be obtained by assembling these three element stiffness matrices. Thus,

$$[K]=\frac{E}{10^6}\begin{array}{cc} & \begin{array}{cccccc}1 & 2 & 3 & 4 & 5 & 6\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\\6\end{array} & \begin{bmatrix} 54 & 21.65 & -37.5 & -21.65 & -16.5 & 0 \\ 21.65 & 12.52 & -21.65 & -12.5 & 0 & 0 \\ -37.5 & -21.65 & 85.75 & -6.19 & -48.21 & 27.84 \\ -21.65 & -12.5 & -6.19 & 28.5 & 27.84 & -16 \\ -16.5 & 0 & -48.21 & 27.84 & 64.71 & -27.84 \\ 0 & 0 & 27.84 & -16 & -27.84 & 16 \end{bmatrix}\end{array}$$

The global stiffness equilibrium equation can be given as,

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}=\frac{E}{10^6}\begin{bmatrix} 54 & 21.65 & -37.5 & -21.65 & -16.5 & 0 \\ 21.65 & 12.52 & -21.65 & -12.5 & 0 & 0 \\ -37.5 & -21.65 & 85.75 & -6.19 & -48.21 & 27.84 \\ -21.65 & -12.5 & -6.19 & 28.5 & 27.84 & -16 \\ -16.5 & 0 & -48.21 & 27.84 & 64.71 & -27.84 \\ 0 & 0 & 27.84 & -16 & -27.84 & 16 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

Applying boundary conditions, $u_1 = 0;\ u_2 = 0;\ u_5 = 0;\ u_6 = 0$ and $P_3 = 10;\ P_4 = 12$

$$\begin{bmatrix} P_1 \\ P_2 \\ 10 \\ 12 \\ P_5 \\ P_6 \end{bmatrix}=\frac{E}{10^6}\begin{bmatrix} 54 & 21.65 & -37.5 & -21.65 & -16.5 & 0 \\ 21.65 & 12.52 & -21.65 & -12.5 & 0 & 0 \\ -37.5 & -21.65 & 85.75 & -6.19 & -48.21 & 27.84 \\ -21.65 & -12.5 & -6.19 & 28.5 & 27.84 & -16 \\ -16.5 & 0 & -48.21 & 27.84 & 64.71 & -27.84 \\ 0 & 0 & 27.84 & -16 & -27.84 & 16 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ 0 \\ 0 \end{bmatrix}$$

Solving for u_3 and u_4,

$$u_3=\frac{0.149\times10^6}{E};\quad u_4=\frac{0.453\times10^6}{E}$$

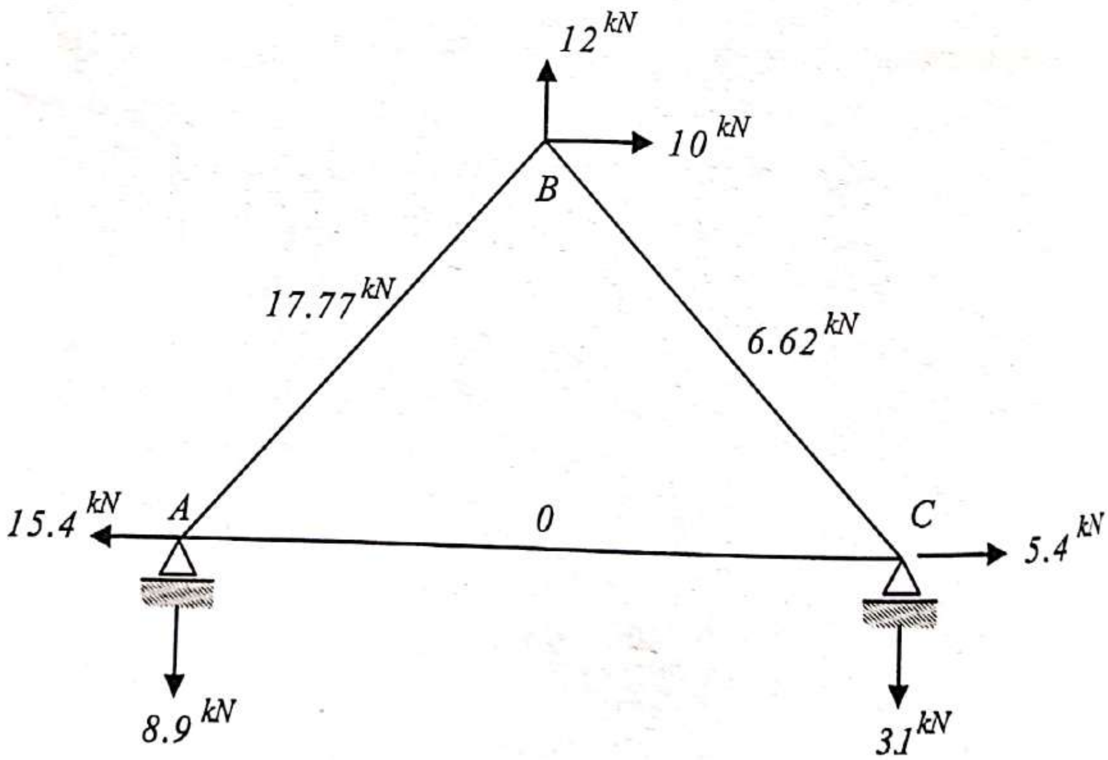
Solve for unknown nodal force, P_1, P_2, P_5 and P_6

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} -15.39 \\ -8.88 \\ -5.42 \\ -3.13 \end{Bmatrix} kN$$

Now force in each member can be found out by either solving the element load-displacement equilibrium equation $[P'] = [k'][u']$ or just static nodal analysis.

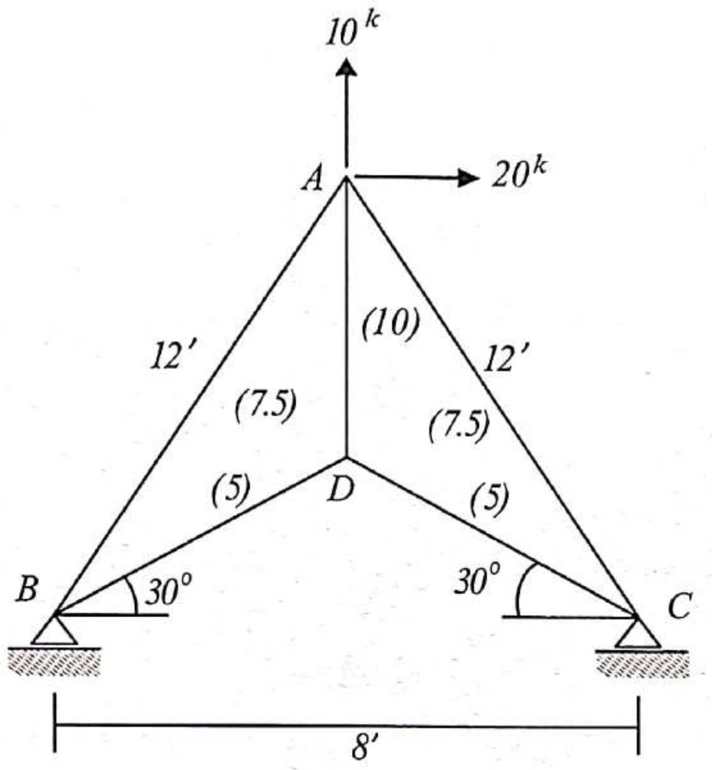
By static analysis,

$$\frac{F_{AB} \times 6.06}{7} = 15.39 \Rightarrow F_{AB} = 17.77(T)$$
$$\frac{F_{BC} \times 6.06}{7} = 5.42 \Rightarrow F_{BC} = 6.26(T)$$
$$F_{AC} = 0$$

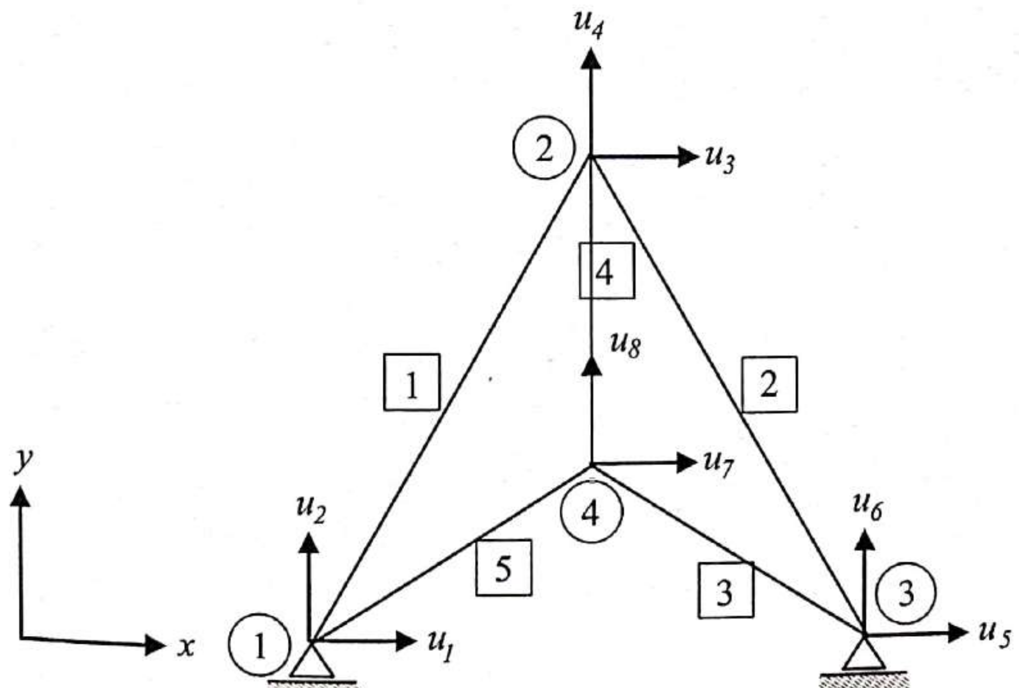


Example 3

For the following indeterminate truss, assemble the global stiffness matrix and find the bar forces and external reactions. Figure in parenthesis gives the corresponding bar cross-sectional area in in^2. Assume E as constant.



Solution



Now member stiffness matrix for each member in global co-ordinate system,

For member 1, $\theta = 70.53°$, A = 7.5 in^2, L= 12 ft.

$$[k]_1 = \frac{E}{144}\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0.069 & 0.196 & -0.069 & -0.196 \\ 2 & 0.196 & 0.556 & -0.196 & -0.556 \\ 3 & -0.069 & -0.196 & 0.069 & 0.196 \\ 4 & -0.196 & -0.556 & 0.196 & 0.556 \end{array}$$

For member 2, $\theta = 289.47°$, A = 7.5in^2, L= 12 ft.

$$[k]_2 = \frac{E}{144}\begin{array}{c|cccc} & 3 & 4 & 5 & 6 \\ \hline 3 & 0.069 & -0.196 & -0.069 & 0.196 \\ 4 & -0.196 & 0.556 & 0.196 & -0.556 \\ 5 & -0.069 & 0.196 & 0.069 & -0.196 \\ 6 & 0.196 & -0.556 & -0.196 & 0.556 \end{array}$$

For member 3, $\theta = 150°$, A = 5 in^2, L= 4.62 ft.

$$[k]_3 = \frac{E}{144}\begin{array}{c|cccc} & 5 & 6 & 7 & 8 \\ \hline 5 & 0.812 & -0.469 & -0.812 & 0.469 \\ 6 & -0.469 & 0.271 & 0.469 & -0.271 \\ 7 & -0.812 & 0.469 & 0.812 & -0.469 \\ 8 & 0.469 & -0.271 & -0.469 & 0.271 \end{array}$$

For member 4, $\theta = 270$, A = 10 in^2, L= 9 ft.

$$[k]_4 = \frac{E}{144} \begin{array}{c} \\ 3 \\ 4 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 7 & 8 \end{array} \\ \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 1.111 & 0.000 & -1.111 \\ 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & -1.111 & 0.000 & 1.111 \end{bmatrix} \end{array}$$

For member 5, $\theta = 30°$, A = 5 in^2, L= 4.62 ft.

$$[k]_5 = \frac{E}{144} \begin{array}{c} \\ 1 \\ 2 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\ \begin{bmatrix} 0.812 & 0.469 & -0.812 & -0.469 \\ 0.469 & 0.271 & -0.469 & -0.271 \\ -0.812 & -0.469 & 0.812 & 0.469 \\ -0.469 & -0.271 & 0.469 & 0.271 \end{bmatrix} \end{array}$$

The global stiffness matrix of the truss can be obtained by assembling these five element stiffness matrices. Thus,

$$[K] = \frac{E}{144} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0.881 & 0.665 & -0.069 & -0.196 & 0.000 & 0.000 & -0.812 & -0.469 \\ 0.665 & 0.827 & -0.196 & -0.556 & 0.000 & 0.000 & -0.469 & -0.271 \\ -0.069 & -0.196 & 0.139 & 0.000 & -0.069 & 0.196 & 0.000 & 0.000 \\ -0.196 & -0.556 & 0.000 & 2.222 & 0.196 & -0.556 & 0.000 & -1.111 \\ 0.000 & 0.000 & -0.069 & 0.196 & 0.881 & -0.665 & -0.812 & 0.469 \\ 0.000 & 0.000 & 0.196 & -0.556 & -0.665 & 0.826 & 0.469 & -0.271 \\ -0.812 & -0.469 & 0.000 & 0.000 & -0.812 & 0.469 & 1.624 & 0.000 \\ -0.469 & -0.271 & 0.000 & -1.111 & 0.469 & -0.271 & 0.000 & 1.653 \end{bmatrix} \end{array}$$

The global stiffness equilibrium equation can be given as,

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \end{bmatrix} = \frac{E}{144} \begin{bmatrix} 0.881 & 0.665 & -0.069 & -0.196 & 0.000 & 0.000 & -0.812 & -0.469 \\ 0.665 & 0.827 & -0.196 & -0.556 & 0.000 & 0.000 & -0.469 & -0.271 \\ -0.069 & -0.196 & 0.139 & 0.000 & -0.069 & 0.196 & 0.000 & 0.000 \\ -0.196 & -0.556 & 0.000 & 2.222 & 0.196 & -0.556 & 0.000 & -1.111 \\ 0.000 & 0.000 & -0.069 & 0.196 & 0.881 & -0.665 & -0.812 & 0.469 \\ 0.000 & 0.000 & 0.196 & -0.556 & -0.665 & 0.826 & 0.469 & -0.271 \\ -0.812 & -0.469 & 0.000 & 0.000 & -0.812 & 0.469 & 1.624 & 0.000 \\ -0.469 & -0.271 & 0.000 & -1.111 & 0.469 & -0.271 & 0.000 & 1.653 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}$$

Applying boundary conditions, $u_1 = 0;\ u_2 = 0;\ u_5 = 0;\ u_6 = 0$

and $P_3 = 20;\ P_4 = 10;\ P_7 = 0;\ P_8 = 0;$

$$\begin{bmatrix} P_1 \\ P_2 \\ 20 \\ 10 \\ P_5 \\ P_6 \\ 0 \\ 0 \end{bmatrix} = \frac{E}{144}\begin{bmatrix} 0.881 & 0.665 & -0.069 & -0.196 & 0.000 & 0.000 & -0.812 & -0.469 \\ 0.665 & 0.827 & -0.196 & -0.556 & 0.000 & 0.000 & -0.469 & -0.271 \\ -0.069 & -0.196 & 0.139 & 0.000 & -0.069 & 0.196 & 0.000 & 0.000 \\ -0.196 & -0.556 & 0.000 & 2.222 & 0.196 & -0.556 & 0.000 & -1.111 \\ 0.000 & 0.000 & -0.069 & 0.196 & 0.881 & -0.665 & -0.812 & 0.469 \\ 0.000 & 0.000 & 0.196 & -0.556 & -0.665 & 0.826 & 0.469 & -0.271 \\ -0.812 & -0.469 & 0.000 & 0.000 & -0.812 & 0.469 & 1.624 & 0.000 \\ -0.469 & -0.271 & 0.000 & -1.111 & 0.469 & -0.271 & 0.000 & 1.653 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ 0 \\ 0 \\ u_7 \\ u_8 \end{bmatrix}$$

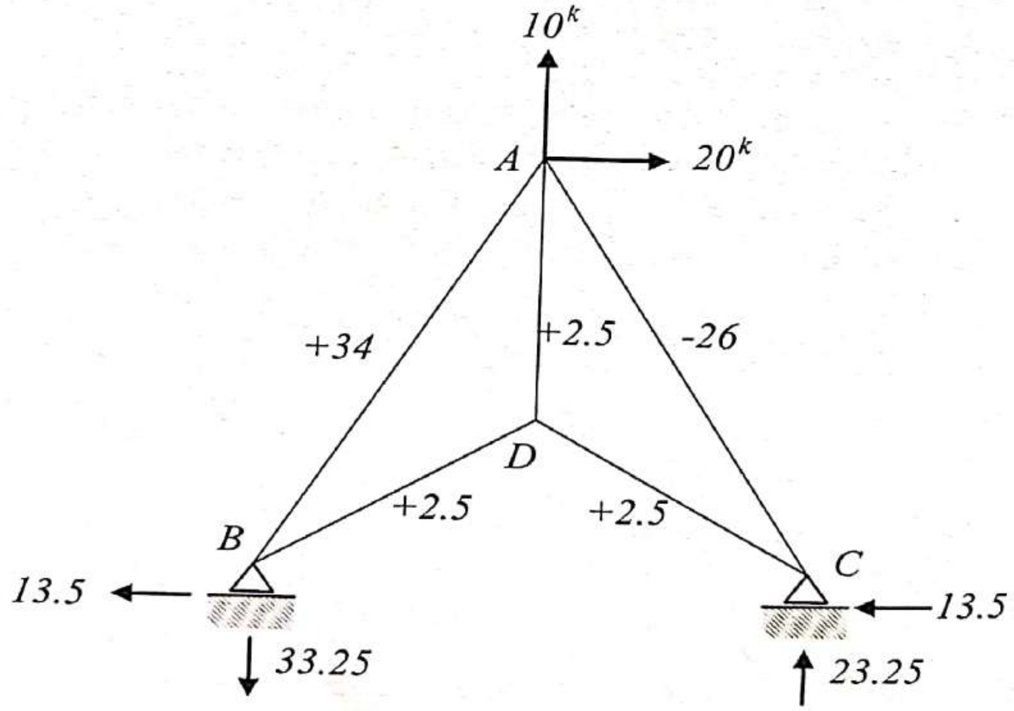
Solving for u_3, u_4, u_3 and u_4

$$u_3 = \frac{20719.42}{\mathrm{E}}; \quad u_4 = \frac{1008}{\mathrm{E}}; \ u_7 = 0; \quad u_8 = \frac{720}{\mathrm{E}}$$

Solve for unknown nodal force, P_1, P_2, P_5 and P_6

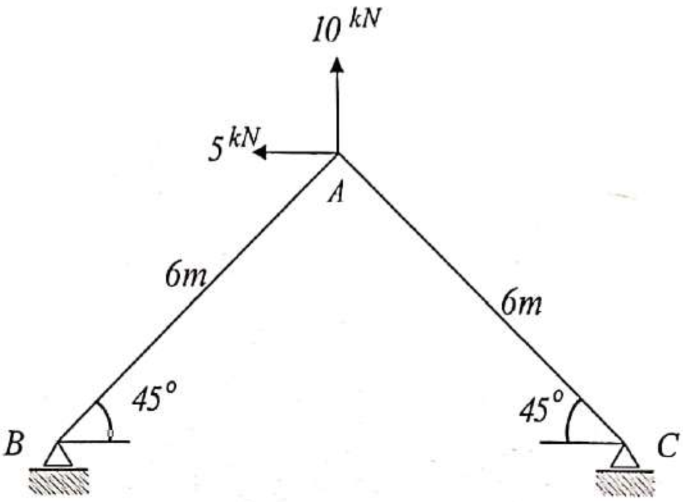
$$\begin{Bmatrix} P_1 \\ P_2 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} -13.5 \\ -33.25 \\ -6.5 \\ +23.25 \end{Bmatrix} kN$$

The bar forces can also be found by simply do a static analysis of the truss.

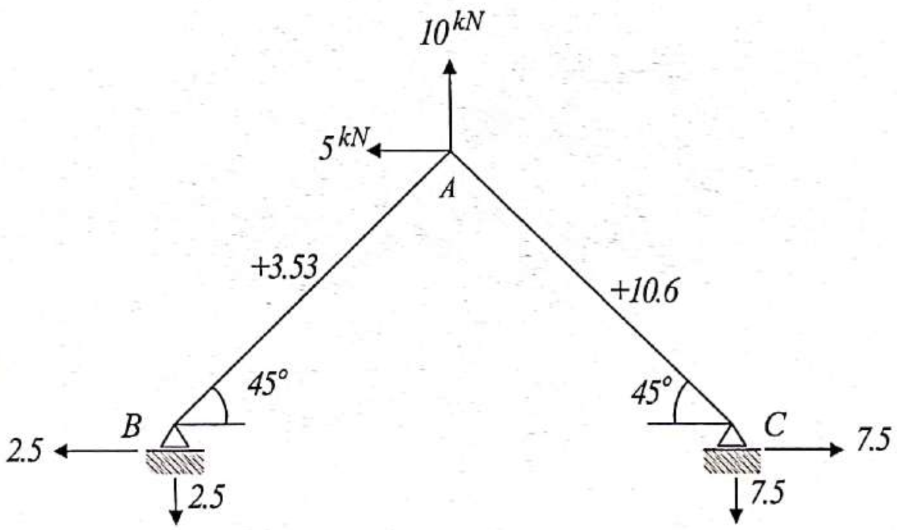


Exercise 1

Analyze the following truss by direct stiffness method and find the bar forces. Assume AE constant.

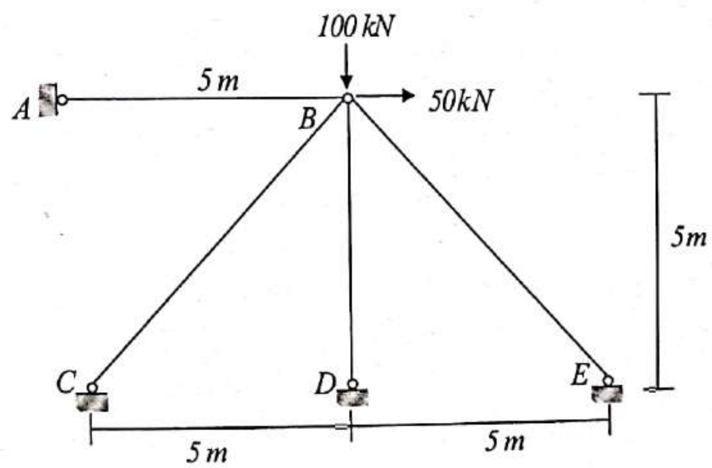


Ans:



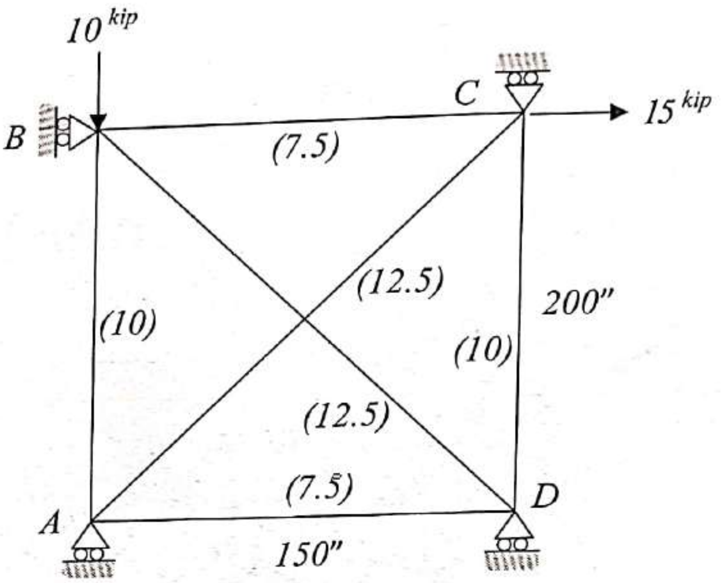
Exercise 2

Analyze the following truss by direct stiffness method and find the bar forces. Assume AE constant.



Exercise 3

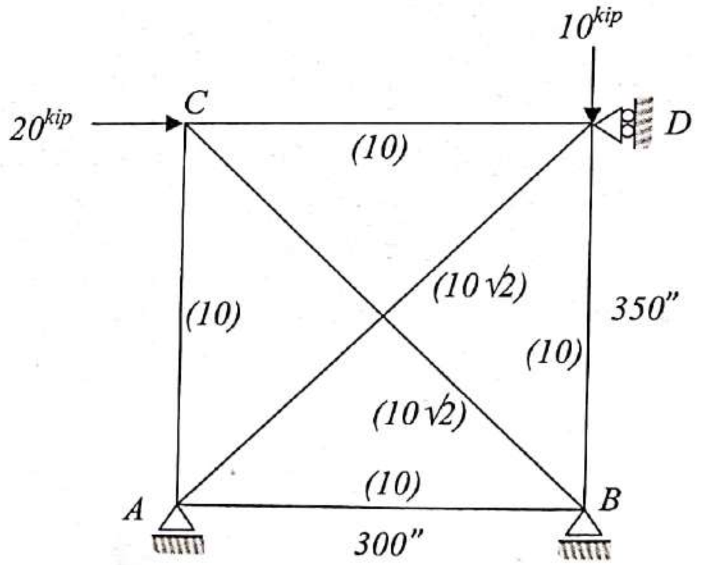
Assemble the global stiffness matrix for the following truss and find out the bar forces. Figure in parenthesis gives the cross sectional area *(A)* of the respective bar in in^2. Assume E is constant.



(Ans: $F_{AB} = -6.1^k$; $F_{BC} = 11.86^k$; ; $F_{CD} = ?$; $F_{AD} = 3.14^k$; $F_{AC} = 6.23^k$; $F_{BD} = -4.88^k$)

Exercise 4

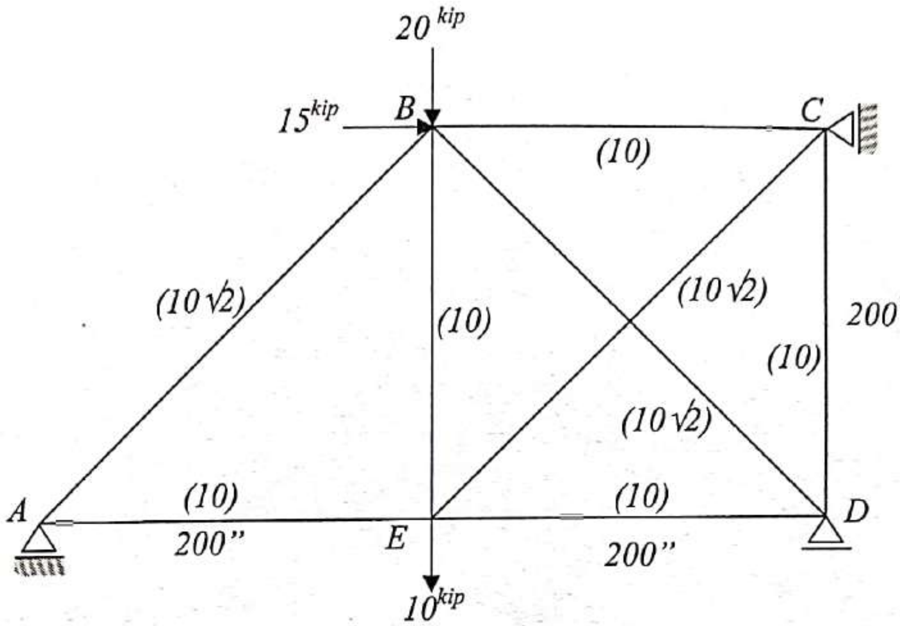
Analyze the following indeterminate truss and find the unknown reaction and bar forces by direct stiffness method. Assume E = 30000 *ksi*



(Ans: $F_{AB} = ?$; $F_{BC} = -7.07^k$; ; $F_{CD} = -15^k$; $F_{AD} = ?$; $F_{AC} = ?$; $F_{BD} = -6.67^k$)

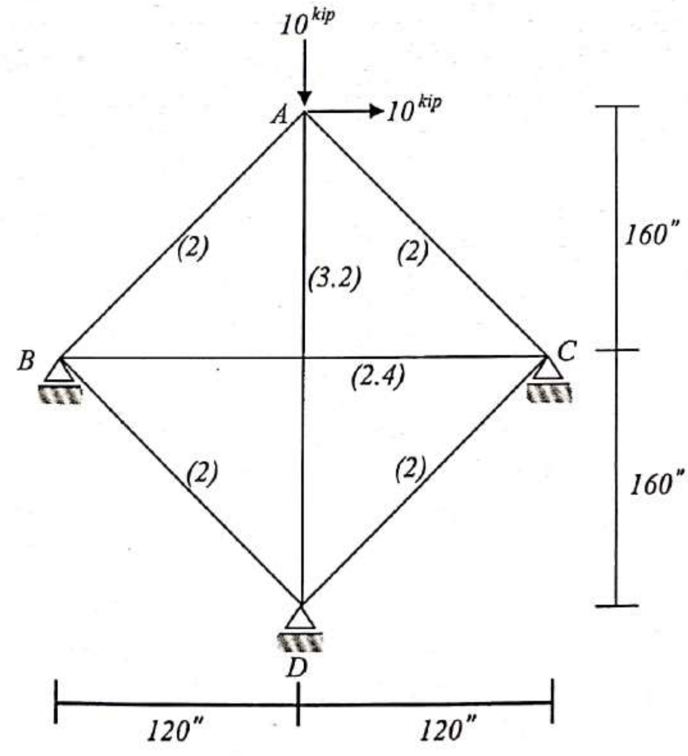
Exercise 5

Analyze the following indeterminate truss and find the unknown reaction and bar forces by direct stiffness method. Assume E = 30000 ksi



Exercise 6

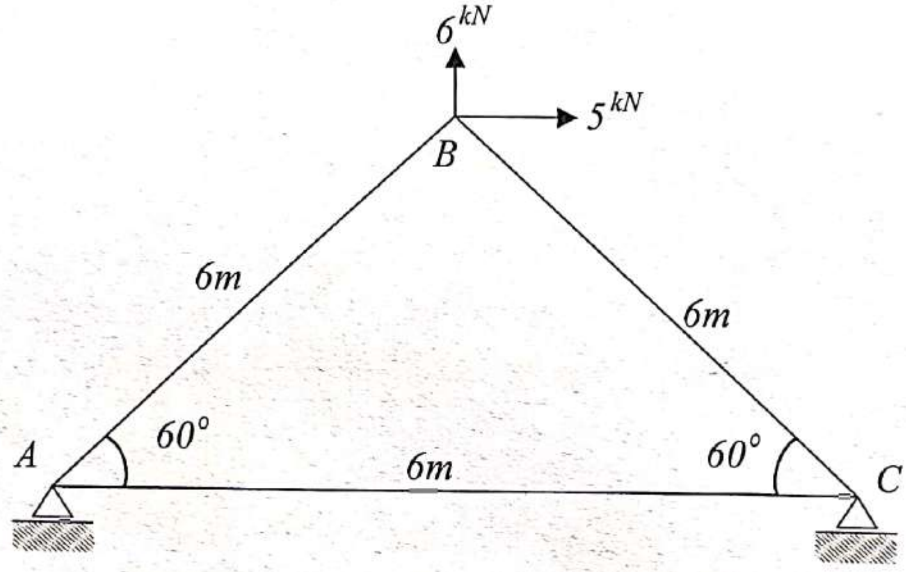
Analyze the following indeterminate truss and find the unknown reaction and bar forces by direct stiffness method. Assume E = 30000 ksi



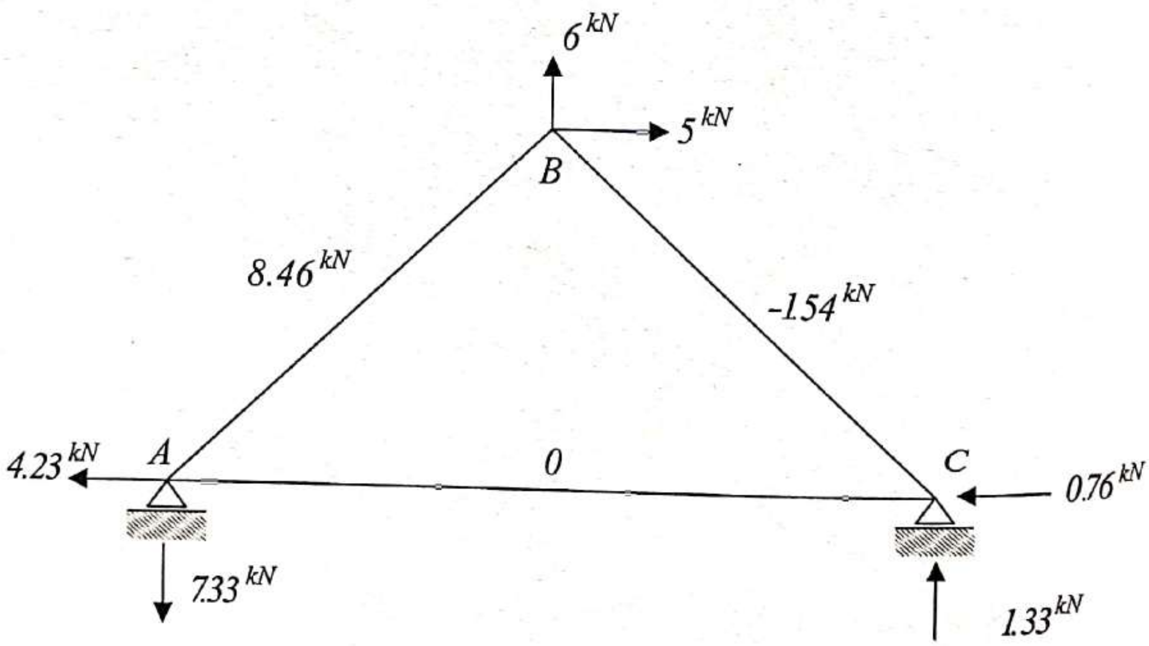
(Ans: F_{AB} =5.47^k; F_{BC} = ?; ; F_{CD} = ? ; F_{AD} = ?; F_{AC} = -11.16^k; F_{BD}=?)

Exercise 7

Analyze the following truss by direct stifness method and find the bar forces. Assume AE constant.



Ans.



CHAPTER 3

Moment Distribution Method

CHAPTER 3

MOMENT DISTRIBUTION METHOD

General

Moment Distribution is an iterative method of solving an indeterminate structure. It was originally developed by Prof. Hardy Cross in the US in the 1932 in response to the highly indeterminate skyscrapers being built. It is the method normally used to analyze all types of statically indeterminate beams and rigid frames in which the members are normally subjected to bending. The method of moment distribution can be applied to structures composed of prismatic or nonprismatic members with or without joint translation.

While the advancement of computer based analysis continues to grow exponentially within the field of structural engineering, the tools that are used to analyze structures by hand are no less important. Many would argue that such tools are even more vital today than they have ever been if we are to fully understand the output of analysis applications.

Moment distribution is a method by which statically indeterminate structures are analyzed elastically. It's based on the relative stiffness of elements that make up a structure and shifts bending moments from one section of the structure to another until they become balanced. Once this balance has been achieved, the shear forces and bending moments within the structure are drawn.

Analysis principles

The principle of moment distribution is based on creating fixed end moments at joints in a structure and then releasing them sequentially in order to derive the bending moments within it. This is done via an iterative process that relies on achieving equilibrium as the joints in the structure are released. Consider Fig. 1, which illustrates a 2 span beam that has fully fixed supports at each end. This is an indeterminate structure, which can quite easily be analyzed using the moment distribution method. This is done by placing artificial fixity at the point where the structure can rotate such as B (Fig. 2). The unbalanced bending moment *Mo* generated at these fixed ends, is distributed between both of the spans. These additional moments are then distributed again until they are dissipated to the point where equilibrium is achieved.

$$M_o = M_{BA} - M_{BC}$$

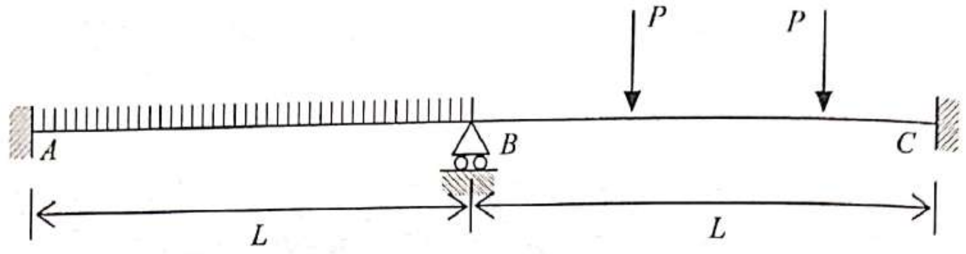


Fig.1 Two span continuous beam

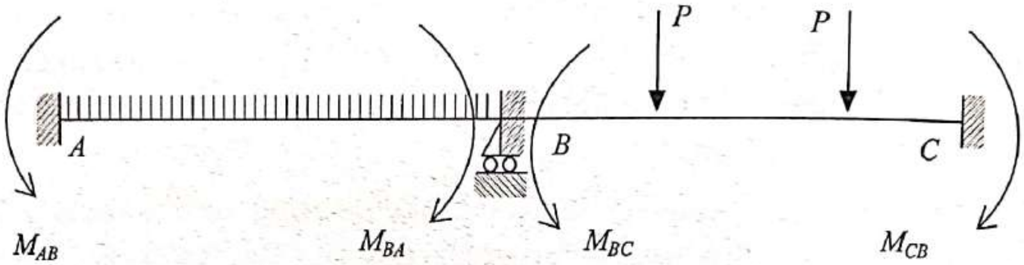


Fig.2 Fixed end moments with fixity at joint B

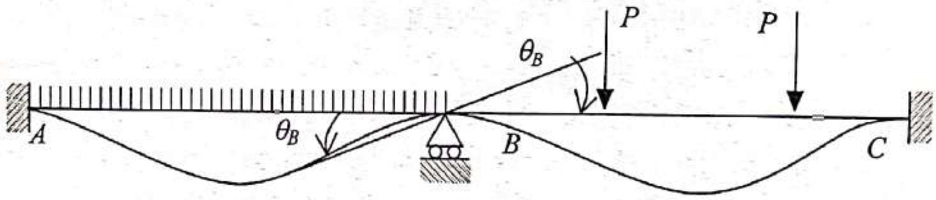


Fig.3 Elastic deflection of the beam after equilibrium is achieved

Steps of analysis

The moment distribution method essentially involves the following steps to find the end moments of all members:

Step 1 Lock all joints against rotation and find out the fixed end moments (*FEM*) corresponding to the joints. (*Use Table 1*)

Step 2 Calculate relative stiffness factors (*K*)

Step 3 Calculate distribution factors (*DF*)

Step 4 Calculate the distribution moment (*DM*)

Step 5 Find the carryover factor (*COF*)

Step 6 Calculate the carryover moment (*COM*)

Step 7 Repeat step 4-6 until equilibrium is achieved

Step 8 Sum up all the moments at each end will give the final end moments.

Step 1 The fixed end moments (*FEM*) is calculated using the following table for different loading conditions. The counter-clockwise moment will be taken as positive.

Table 1 Fixed end moments for different loading conditions

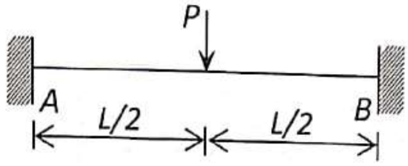
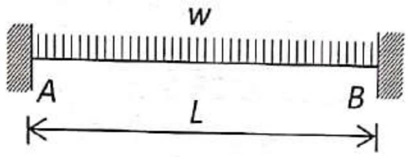
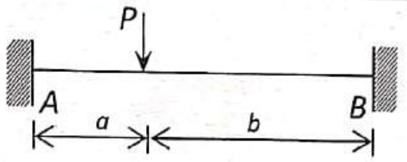
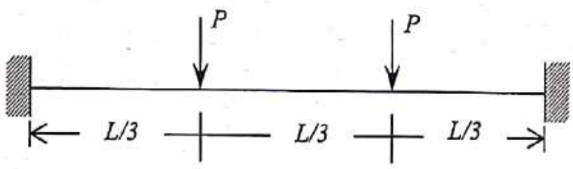
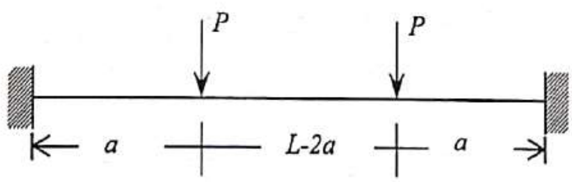
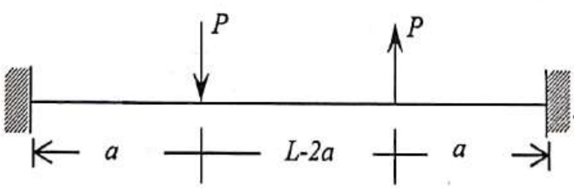
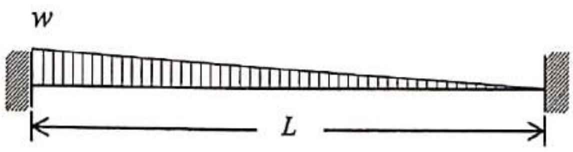
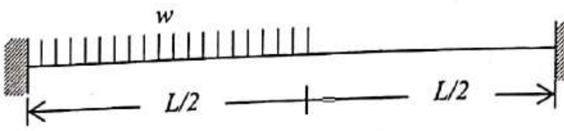
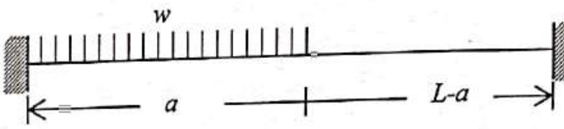
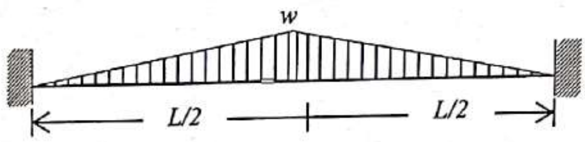
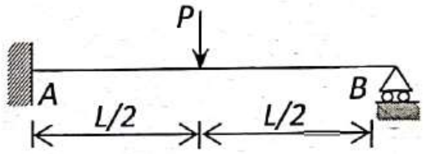
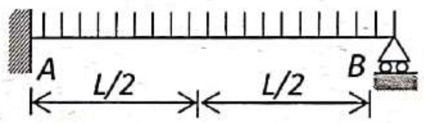
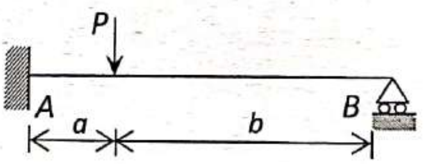
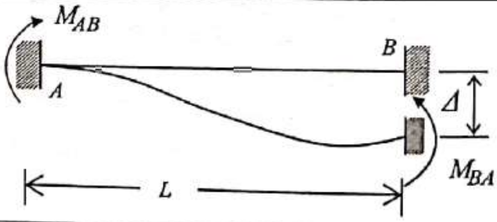
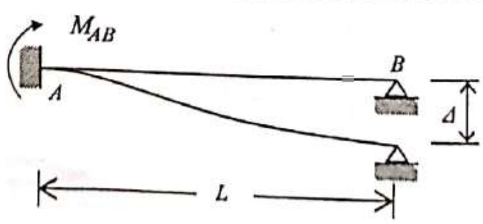
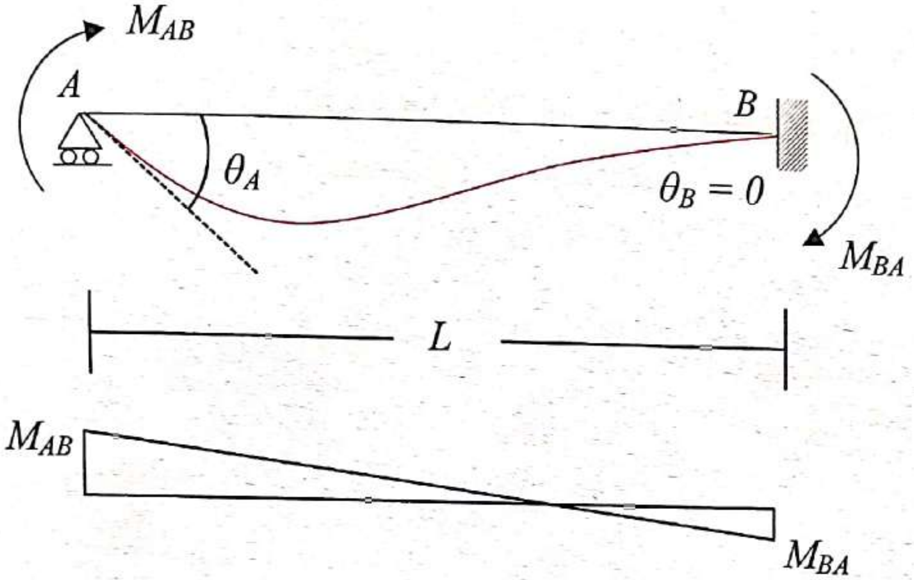
M_{AB}	Load case	M_{BA}
$+\dfrac{PL}{8}$		$-\dfrac{PL}{8}$
$+\dfrac{wL^2}{12}$		$-\dfrac{wL^2}{12}$
$+\dfrac{Pab^2}{L^2}$		$-\dfrac{Pa^2b}{L^2}$
$+\dfrac{2PL}{9}$		$-\dfrac{2PL}{9}$
$+\dfrac{Pa(L-a)}{L}$		$-\dfrac{Pa(L-a)}{L}$
$+\dfrac{Pa(L-a)(L-2a)}{L^2}$		$+\dfrac{Pa(L-a)(L-2a)}{L^2}$
$+\dfrac{wL^2}{20}$		$-\dfrac{wL^2}{30}$

Table 1 Continued

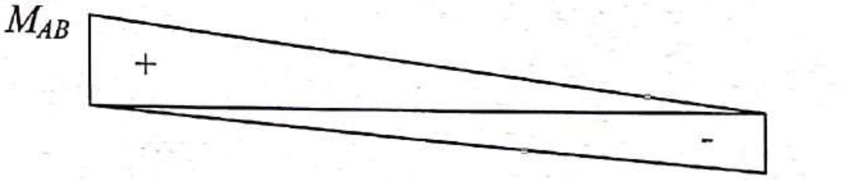
M_{AB}	*Load case*	M_{BA}
$+\dfrac{11wL^2}{192}$		$-\dfrac{5wL^2}{192}$
Find by yourself		Find by yourself
$+\dfrac{5wL^2}{96}$		$-\dfrac{5wL^2}{96}$
$+\dfrac{3PL}{16}$		
$+\dfrac{wL^2}{8}$		
$+\dfrac{Pab(2L-a)}{{}^{2}2L}$		
$+\dfrac{6EI\Delta}{L^2}$		$+\dfrac{6EI\Delta}{L^2}$
$+\dfrac{3EI\Delta}{L^2}$		

Step 2

Stiffness of all members meeting at a joint is formulated first. The relative stiffness of those members are evaluated by vanishing the common terms. For a member of uniform section and rigidity (Constant EI), the rotational stiffness K is defined as the end moment required to produce a unit rotation at one end of the member while the other end is fixed. Consider the following where moment M_{AB} is applied at the end A in such an amount that it will produce unit rotation at that end. So, it can be written,



The displacement of A with respect to B is zero. From the 2nd area moment theorem, the tangential distance of A from the tangent drawn at B is also zero, so we can write,



$$EI\left(\frac{1}{2}M_{AB}L\cdot\frac{1}{3}L-\frac{1}{2}M_{BA}L\cdot\frac{2}{3}L\right)=0$$

$$\therefore\ M_{BA}=\frac{1}{2}M_{AB} \qquad (1)$$

Again, $d\theta_{AB}=\theta_A-\theta_B=\theta_A$

From 1^{st} area moment theorem,

$$EId\theta_{AB}=\frac{1}{2}M_{AB}L-\frac{1}{2}M_{BA}L=\frac{1}{2}M_{AB}L-\frac{1}{2}\cdot\frac{1}{2}M_{AB}L=\frac{M_{AB}L}{4}$$

$$\theta_A = \frac{L}{4EI} M_{AB}$$

$$\therefore \quad M_{AB} = \frac{4EI}{L}\theta_A$$

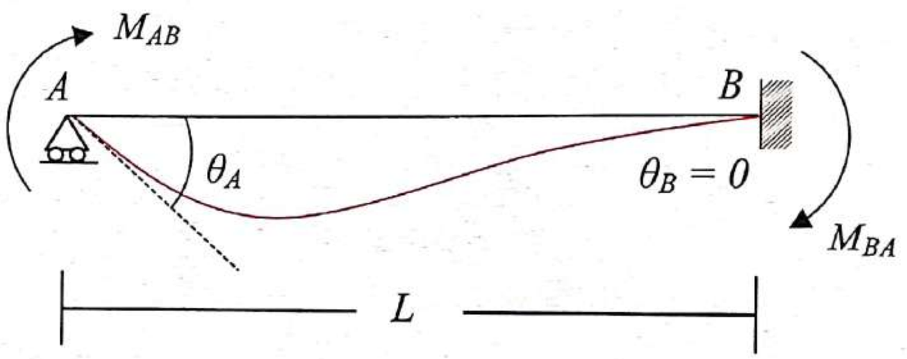
If $\theta_A = 1$, for unit rotation a joint A,

$$M_{AB} = K_{AB} = \frac{4EI}{L} \qquad (2)$$

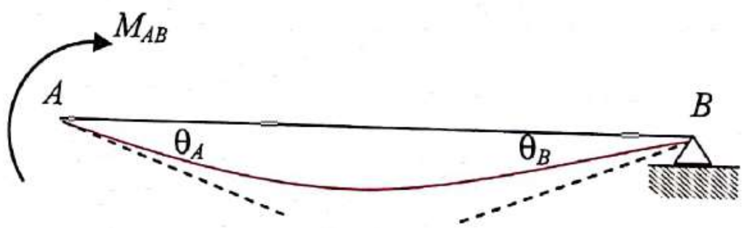
K_{AB} is called the stiffness of member AB.

Member stiffness for different end conditions

1. Rotation at near end A while far end B is fixed, $K_{AB} = \dfrac{4EI}{L}$



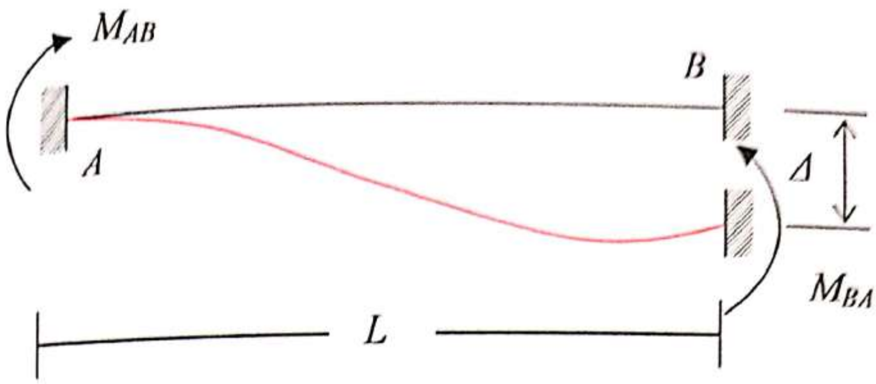
2. Rotation at near end while far end is hinged (or roller), $K_{AB} = \dfrac{3EI}{L}$



$$M_{AB} = M_{BA} = \frac{6EI\Delta}{L^2}$$

if, $\Delta = 1$,

$$K_{AB} = K_{BA} = \frac{6EI}{L^2}$$

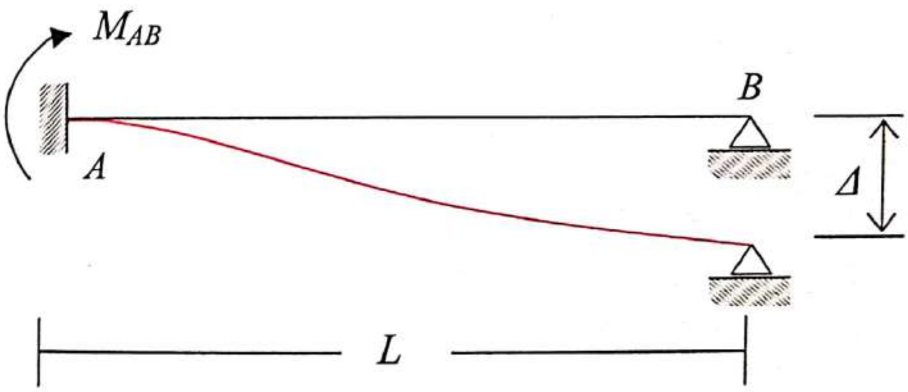


4. When one end is fixed and far end is hinged but one end is displaced perpendicularly at a distance Δ with respect to other, the stiffness at fixed end is,

$$M_{AB} = \frac{3EI\,\Delta}{L^2}$$

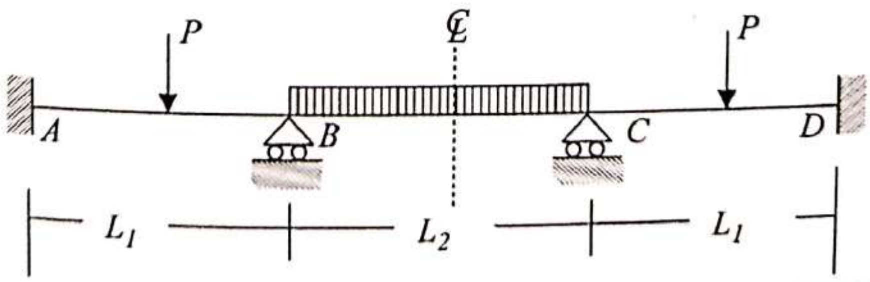
$$\text{if, } \Delta = 1,$$

$$K_{AB} = \frac{3EI}{L^2}$$



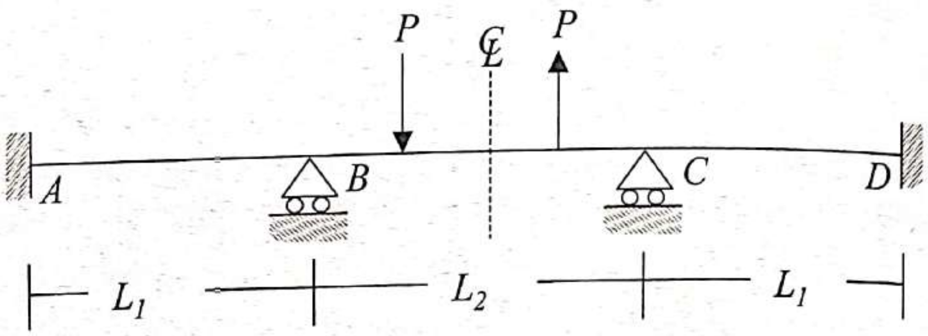
5. If the concerned member is symmetrical in nature in terms of loading, geometry and support conditions.

$$K_{BC} = K_{CB} = \frac{2EI}{L}$$



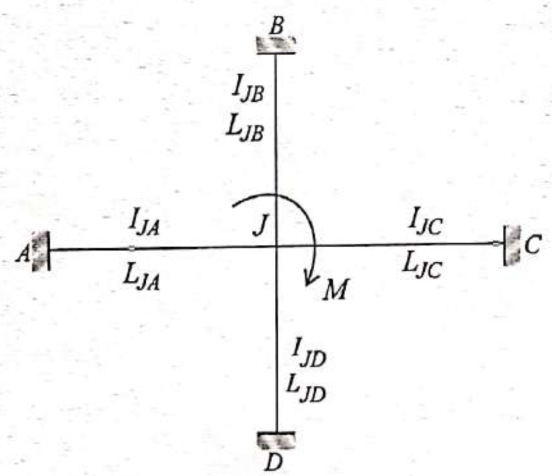
6. If the concerned member is anti-symmetrical in nature in terms of loading but the structure is symmetrical in geometry and support conditions.

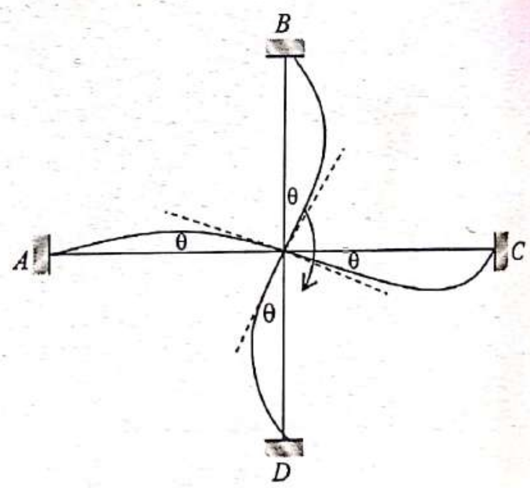
$$K_{BC} = K_{CB} = \frac{6EI}{L}$$

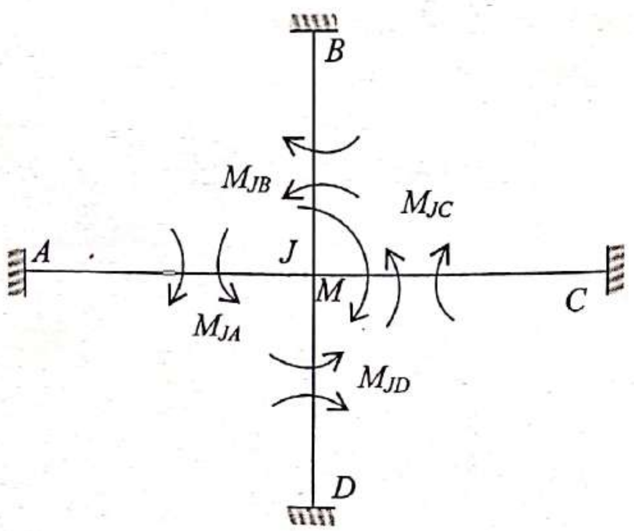


Step 3, STEP 4

Calculation of distribution factors (DF) based on relative stiffness of those members meet at a joint.







The applied moment M will be resisted by the four members meeting at joint J. The resisting moment M_{JA}, M_{JB}, M_{JC} and M_{JD} will be induced at the ends of the four members to balance the effect of the external moment M as shown in the Figure above.

Equilibrium of the joint requires that,

$$M_{JA} + M_{JB} + M_{JC} + M_{JD} = M$$

$$Now, \quad \theta_{JA} = \theta_{JB} = \theta_{JC} = \theta_{JD} = \theta$$

$$Hence, \quad \theta = \frac{M_{JA}}{K_{JA}} = \frac{M_{JB}}{K_{JB}} = \frac{M_{JC}}{K_{JC}} = \frac{M_{JD}}{K_{JD}} = \frac{M}{K_{JA} + K_{JB} + K_{JC} + K_{JD}} = \frac{M}{\sum K}$$

From the above relationship, it can be written that,

$$M_{JA} = \frac{K_{JA}}{\sum K} \cdot M = DF_{JA} \cdot M$$

$$M_{JB} = \frac{K_{JB}}{\sum K} \cdot M = DF_{JB} \cdot M$$

$$M_{JC} = \frac{K_{JC}}{\sum K} \cdot M = DF_{JC} \cdot M$$

$$M_{JD} = \frac{K_{JD}}{\sum K} \cdot M = DF_{JD} \cdot M$$

In the above equation, the ratio $\frac{K_{JI}}{\sum K}$ is defined as distribution factor DF_I ($I = A, B, C, D$). Thus, a moment resisted by a joint will be distributed among the connecting members in proportion to their distribution factors, DF. In determining the distribution factors, only the relative K values for the connecting members are needed as the common terms are cancelled out from the equation. It is important to note that the summation of all stiffness at a joint is called joint stiffness i.e $\sum K$ in the above equation. If the end span is supported by hinge, the DF will be =1 and if it is fixed the $DF = 0$

Step 5, Step 6

The moment distributed at the near end will be carried over to far end in an amount equal to one-half of that at near end. The induced moments at far ends named as carryover moment (*COM*) are therefore given by,

$$M_{AJ} = \frac{1}{2} M_{JA}$$

$$M_{BJ} = \frac{1}{2} M_{JB}$$

$$M_{CJ} = \frac{1}{2} M_{JC}$$

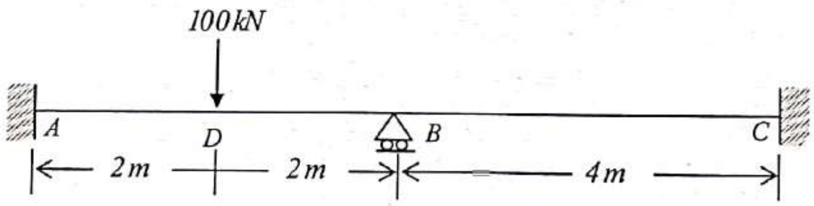
$$M_{DJ} = \frac{1}{2} M_{JD}$$

In the above equation the carry over factor (*COF*) is equal to ½. If the far end is hinged, the carryover factor is zero. Similarly no moment will be carried over to far end if the near end is originally fix supported. Following examples will further illustrate these matters.

Worked out Examples

Example 1

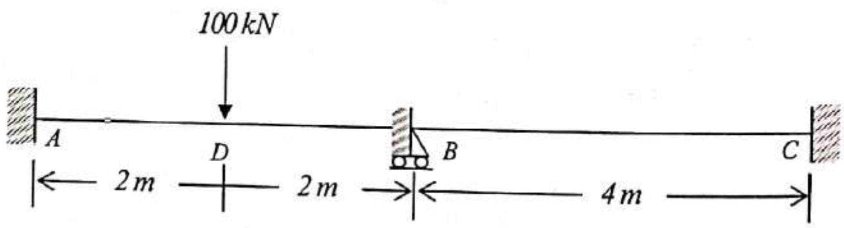
Find the end moments and draw the bending Moment diagram for the following prismatic beam.



Solution

Step 1

Lock the joint B and find out the fixed end moments *(FEM)*



$$FEM_{AB} = +\frac{PL}{8} = +50kN.m$$

$$FEM_{BA} = -\frac{PL}{8} = -50kN.m$$

Since, joint B was originally free to rotate and there was no fixity at this joint, the internal moment at joint B should be equal and opposite in sign to maintain equilibrium. After imposing artificial clamped at joint B, a moment of *-50 kNm* has been induced at this joint. For the sake of equilibrium and to allow the joint for rotation, this *-50 kN.m* moment have to be distributed along the each of the two members i.e BA and BC in accordance to their relative stiffness and distribution factors.

Step 2

Calculate the relative stiffness of member BA and BC.

$$K_{BA} = \frac{4EI}{L} = \frac{4.1}{4} = 1 \qquad \text{[assuming } EI = 1\text{]}$$
$$K_{BC} = \frac{4EI}{} = \frac{4.1}{4} = 1$$

The summation of stiffness of these two members,

$\sum K = K_{BA} + K_{BC} = 1 + 1 = 2$, which is also called the stiffness of joint B.

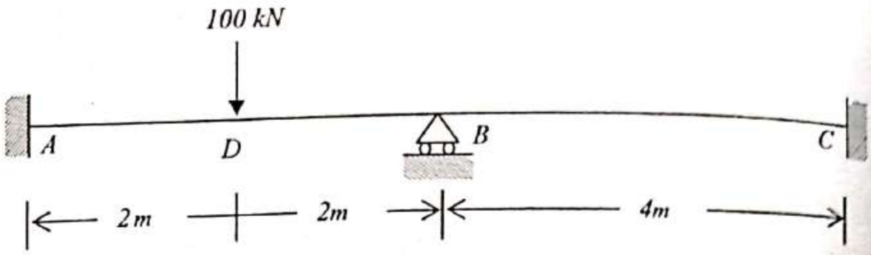
Step 3

The unbalanced *-50 kN-m* moment induced at joint B due to artificial fixity have to be distributed in a ratio of distribution factors (*DF*) in opposite sign. So,

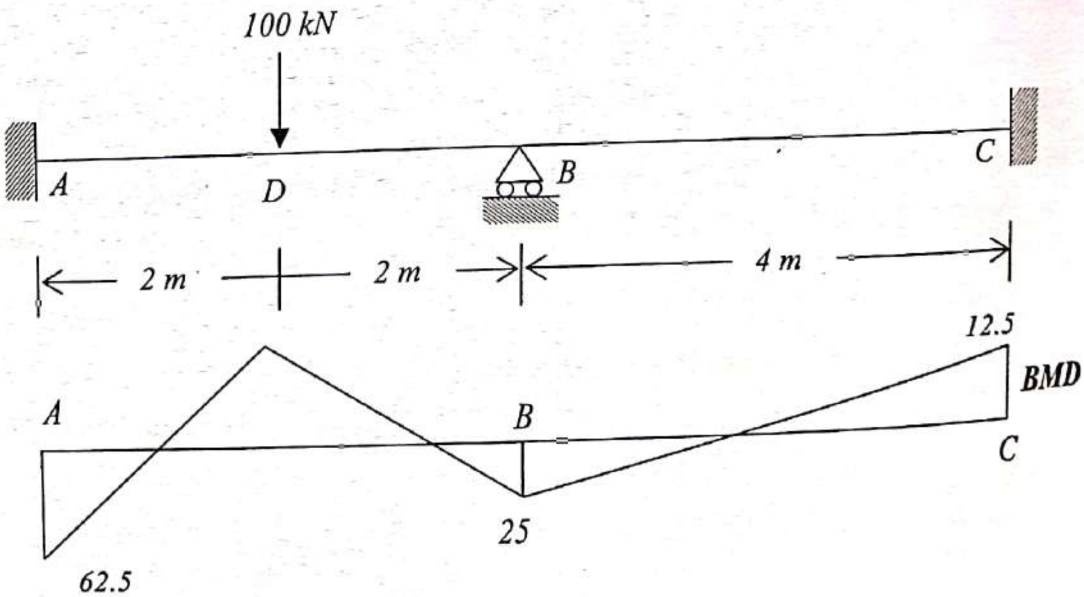
$$DF_{BA} = \frac{K_{BA}}{\sum K} = \frac{1}{2} = 0.5$$
$$DF_{BC} = \frac{K_{BC}}{\sum K} = \frac{1}{2} = 0.5$$

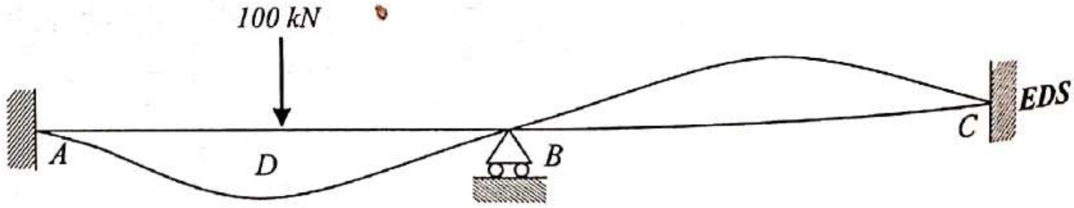
Step 4

The distributed moment *DM* can now be calculated using the *DFs* given above. The rest of the steps are shown in a tabular form below.



Joint	A		B		C
Member	AB	BA	BC	CB
DF	0	0.5	0.5	0
FEM	+50	-50		
DM		+25	+25	
COM	+12.5			+12.5
Final Moment = Σ	+62.5	-25	+25	+12.5

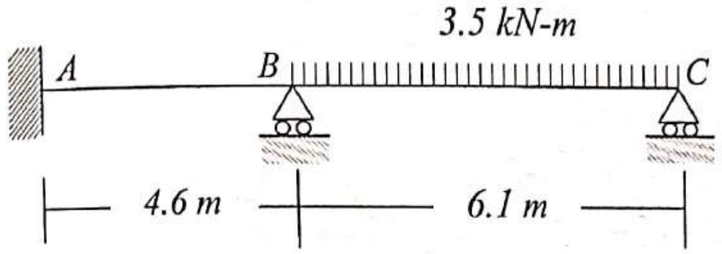




Example 2

Find the end moments and draw the bending Moment diagram for the following prismatic beam.

$I_{AB} = 1.249 \times 10^{-4}\ m^4$; $I_{BC} = 2.497 \times 10^{-4}\ m^4$. E is constant for all members.

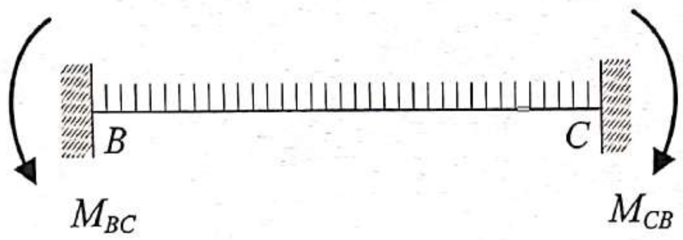


Solution (Method 1)

Fixing the end joint C and calculate the fixed end moment (FEM) and distribution factor (DF)

Step 1

Lock the joint B and C and find out the fixed end moments (FEM)



$$M_{BC} = +\frac{wL^2}{12} = +\frac{3.5\times 6.1^2}{12} = +10.85\ kN-m$$

$$M_{CB} = -\frac{wL^2}{12} = -\frac{3.5\times 6.1^2}{12} = -10.85\ kN-m$$

Step 2

Calculate the stiffness factors of member BA and BC.

$$K_{BA} = \frac{4EI}{L} = \frac{4E.1.249\times 10^{-4}}{4.6} = 1.086\times 10^{-4}\ \text{(Assume } E=1\text{)}$$

$$K_{BC} = \frac{4EI}{L} = \frac{4E.2.497\times 10^{-4}}{6.1} = 1.637\times 10^{-4}$$

Step 3

Find the distribution factors (DF).

Since, the left of support A is fixed, the stiffness of left of A is infinity. Similarly the right of support C has no member, therefore the stiffness of right of C is zero. Also note that the summation of DFs at a joint is equal to *1. So,* $DF_{BA} + DF_{BC} = 1$.

$$DF_{AB} = \frac{K_{AB}}{\infty + K_{AB}} = 0$$

$$DF_{BA} = \frac{K_{BA}}{\sum K} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{1.086}{1.086 + 1.637} = 0.4$$

$$DF_{BC} = \frac{K_{BC}}{\sum K} = \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{1.637}{1.086 + 1.637} = 0.6$$

$$DF_{BC} = \frac{K_{CB}}{K_{CB} + 0} = 1$$

Step 4

The distributed moment DM can now be calculated using the DF given above. The rest of the steps are shown in a tabular form below.

Table for Method 1

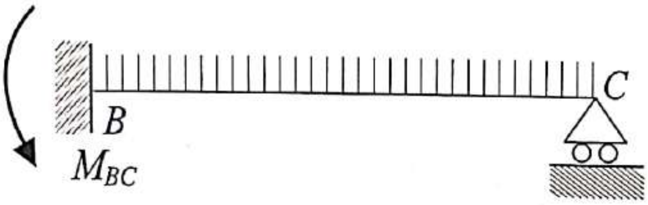
Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.4	0.6	1
FEM			+10.85	-10.85
DM		-4.34	-6.51	+10.85
COM	-2.17		+5.425	-3.255
DM		-2.17	-3.255	+3.255
COM	-1.085		+1.628	-1.628
DM		-0.651	-0.977	+1.628
COM	-0.326		+0.814	-0.488
DM		-0.326	-0.488	+0.488
COM	-0.163		+0.244	-0.244
DM		-0.098	0.146	+0.244
COM	-0.049		+0.122	-0.073
DM		-0.049	-0.073	
Final Moment = Σ	-3.8	-7.634	+7.734	0

Method 2 (Modified Method)

Keeping the end joint C as hinged and calculate the fixed end moment (FEM) and distribution factors (*DF*).

Step 1

Lock only the joint B (C will remain free against rotation) and calculate the fixed end moments (*FEM*).



$$M_{BC} = +\frac{wL^2}{8} = +\frac{3.5\times 6.1^2}{8} = +16.28\ kN-m$$

$$M_{CB} = 0$$

Step 2

Calculate the stiffness factors of member BA and BC. Since far end is free for member BC, the stiffness factor will be *3EI/L* (*see* ***Table 1***)

$$K_{BA} = \frac{4EI}{L} = \frac{4E\times 1.249\times 10^{-4}}{4.6} = 1.086\times 10^{-4}\ (\text{Assume } E=1)$$

$$K_{BC} = \frac{3EI}{L} = \frac{3E\times 2.497\times 10^{-4}}{6.1} = 1.228\times 10^{-4}$$

Step 3

Find the distribution factors (DF).

$$DF_{AB} = \frac{K_{AB}}{\infty + K_{AB}} = 0$$

$$DF_{BA} = \frac{K_{BA}}{\sum K} = \frac{K_{BA}}{K_{BA}+K_{BC}} = \frac{1.086}{1.086+1.228} = 0.47$$

$$DF_{BC} = \frac{K_{BC}}{\sum K} = \frac{K_{BC}}{K_{BA}+K_{BC}} = \frac{1.228}{1.086+1.228} = 0.53$$

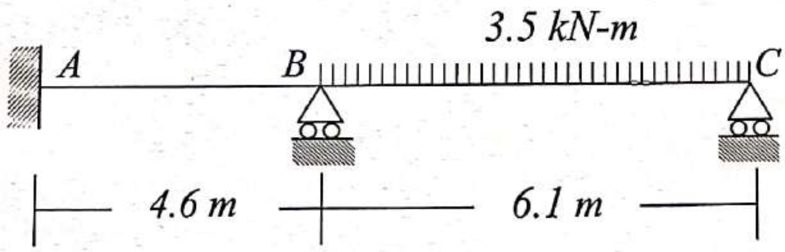
$$DF_{BC} = \frac{K_{CB}}{K_{CB}+0} = 1$$

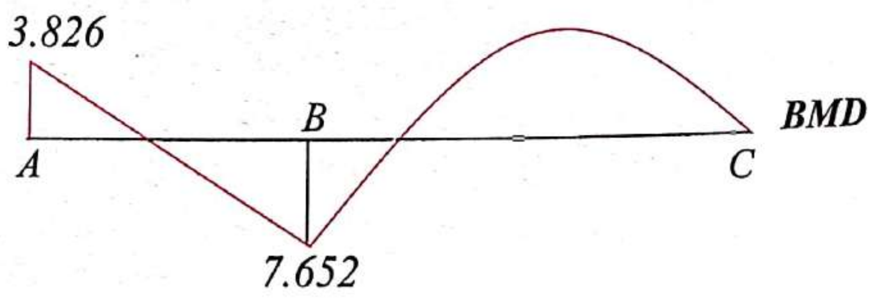
Step 4

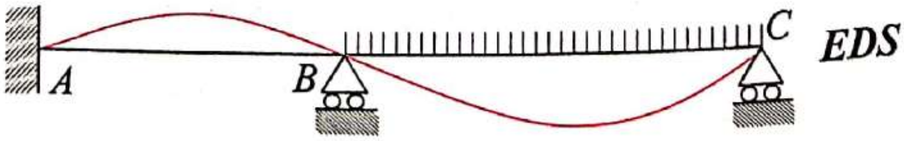
The distributed moment *DM* can now be calculated using the *DF* given above. The rest of the steps are shown in a tabular form below. Note that the difference between *method 1* and *method 2* is obvious. As the joint *C* is treated as free for *method 2,* no distributed moment will go to the end *C* and hence has reduced the calculation time and effort to a great extent for *method 2*.

Table for Method 2

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.47	0.53	1
FEM			+16.28	
DM COM	 -3.826	-7.652	-8.628	
Final Moment = Σ	-3.826	-7.652	+7.652	0

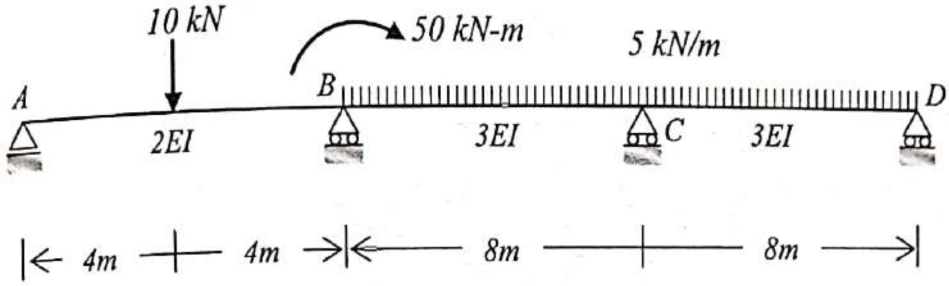






Example 3

Find the end moments and draw the bending Moment diagram for the following prismatic beam.



Solution

FEM

$$M_{BA}=-\frac{3PL}{16}=-15;\ M_{BC}=+\frac{wL^2}{12}=+26.67;\ M_{CB}=-\frac{wL^2}{12}=-26.67;\ M_{CD}=+\frac{wL^2}{8}=+40$$

DF

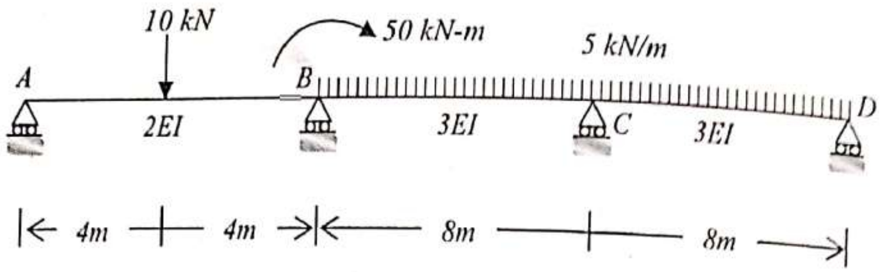
$$DF_{BA}=\frac{K_{BA}}{K_{BA}+K_{BC}}=\frac{3EI/L}{3EI/L+4EI/L}=\frac{6}{6+12}=0.333;$$

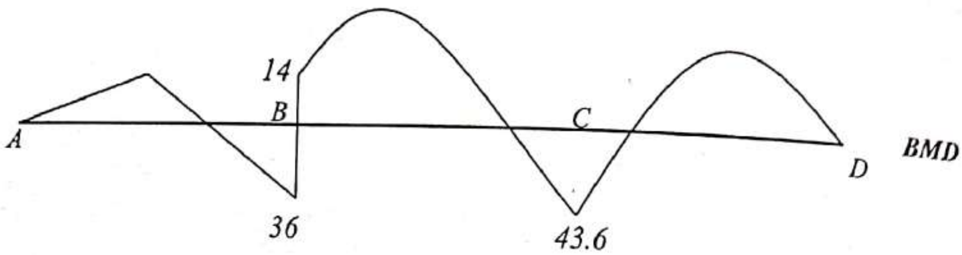
$$DF_{BC}=\frac{K_{BC}}{K_{BA}+K_{BC}}=\frac{4EI/L}{3EI/L+4EI/L}=\frac{12}{6+12}=0.667$$

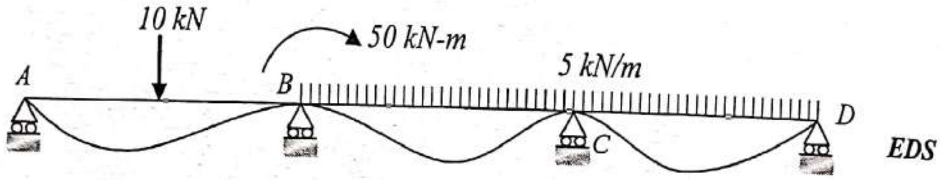
$$DF_{CB}=\frac{K_{CB}}{K_{CB}+K_{CD}}=\frac{4EI/L}{4EI/L+3EI/L}=\frac{4}{7}=0.57;$$

$$DF_{CD}=\frac{K_{CD}}{K_{CB}+K_{CD}}=\frac{3EI/L}{4EI/L+3EI/L}=\frac{3}{7}=0.43$$

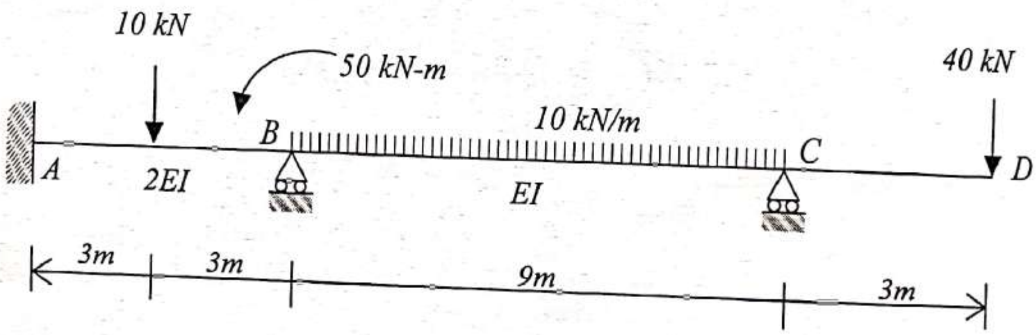
Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	1	0.33	0.67	0.57	0.43	1
Applied Moment		-16.5	-33.5			
COM				-16.75		
FEM		-15	+26.67	-26.67	+40	
DM		-3.85	-7.82	+1.95	+1.47	
COM			+0.98	-3.91		
DM		-0.32	-0.66	+2.23	+1.68	
COM			+1.11	-0.33		
DM		-0.37	-0.74	+0.19	+0.14	
COM			+0.095	-0.37		
Final Moment = Σ	0	-36.04	-13.87	-43.66	+43.29	





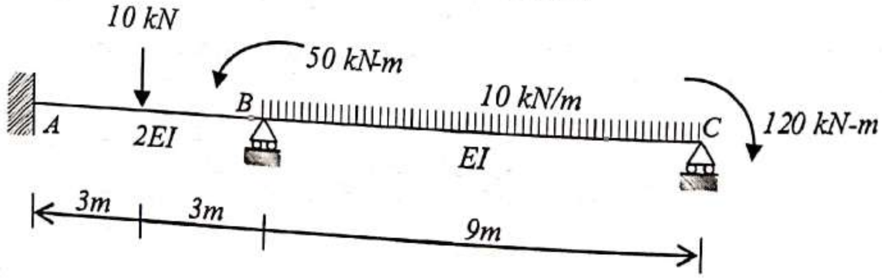


Example 4



Solution

In this problem the overhanging part is determinate and the moment due to *40 kN* load can be applied at support *C* as shown in figure below. This *120 kN-m* moment is treated as external applied moment at this support *C*. The rest of the solution is same as *Example 2*.



FEM

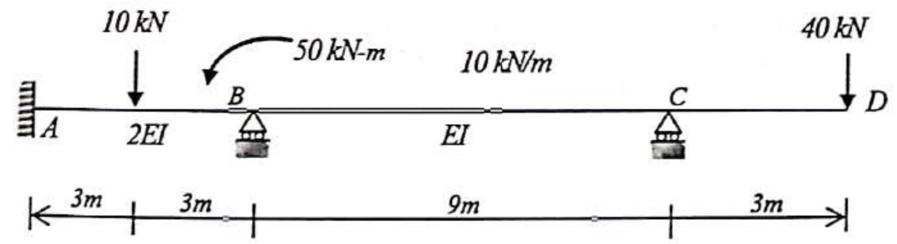
$$M_{AB} = +\frac{PL}{8} = +\frac{40\times 6}{8} = +30;\; M_{BA} = -\frac{PL}{8} = -\frac{40\times 6}{8} = -30;\; M_{BC} = +\frac{wL^2}{8} = +\frac{10\times 9^2}{8} = +101.25$$

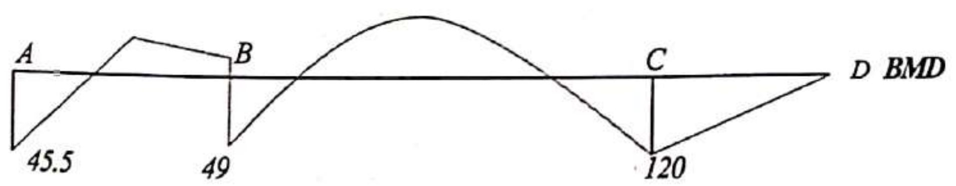
DF

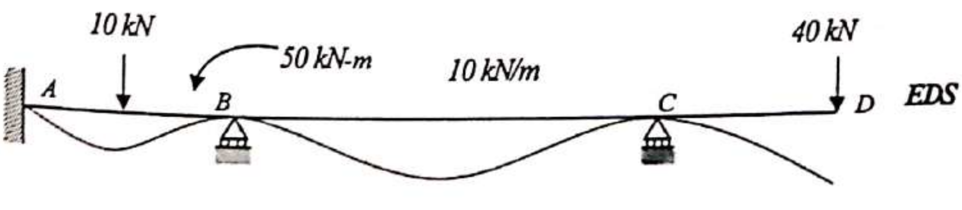
$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{4 \times 2EI/L}{8EI/L + 3EI/L} = \frac{4 \times 2EI/6}{8EI/6 + 3EI/9} = \frac{\frac{8}{6}}{\frac{8}{6} + \frac{3}{9}} = 0.8$$

$$DF_{BC} = 1 - 0.8 = 0.2$$

Joint	A		B	C
Member	AB	BA	BC	CB
DF	0	0.8	0.2	1
Applied Moment		+40	+10	-120
COM	+20		-60	
FEM	+30	-30	+101.25	
DM		-9	-2.25	
COM	-4.5			
Final Moment = Σ	+45.5	+1	+49	-120



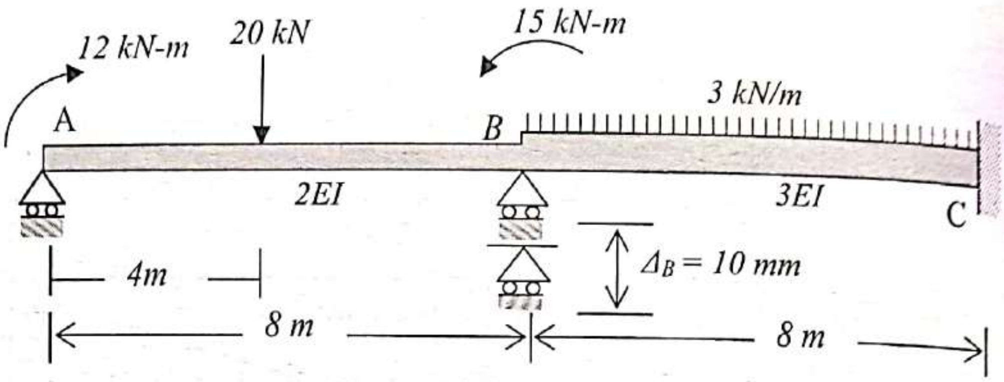




Examples of Beam with Support Settlement

Example 5

For the following beam, support B settles 10 mm downward. Use the moment distributions method to determine all the unknown moments at supports. Take $I = 50x10^6$ mm^4 and $E = 200$ GPa



Solution

FEM

For support settlement:

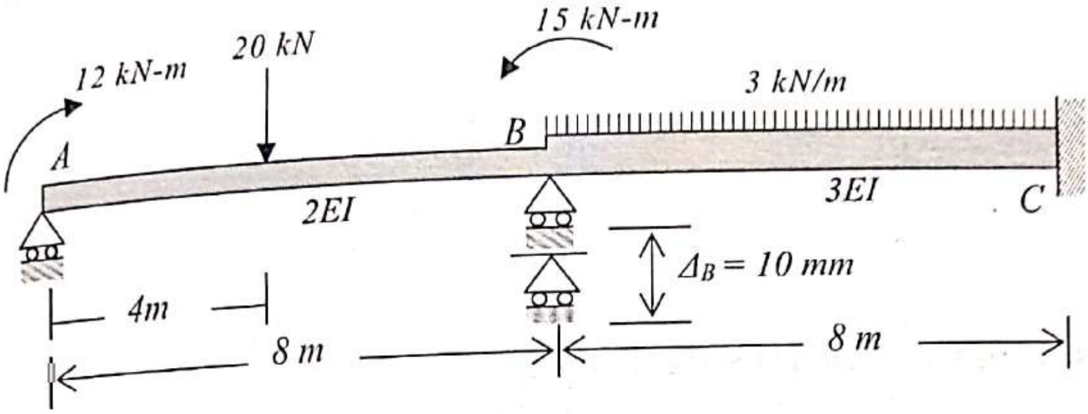
$$M_{BA} = +\frac{3EI\Delta}{L^2} = +\frac{3\times2\times200\times50\times10^6\times10}{8^2} = +9.375kN\cdot m$$

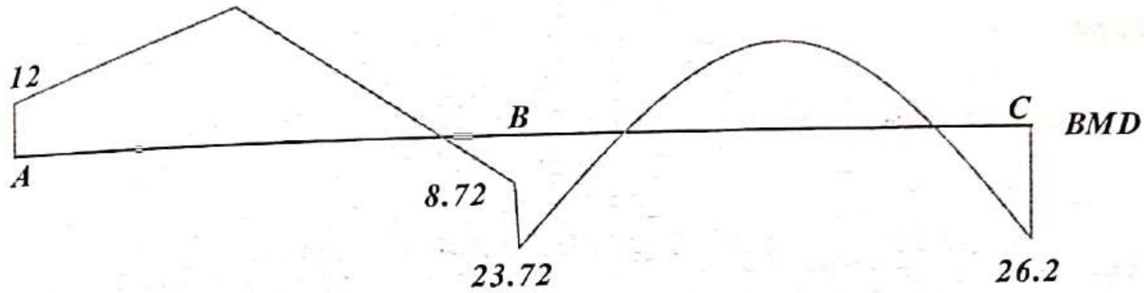
$$M_{BC} = M_{CB} = -\frac{6EI\Delta}{L^2} = -\frac{6\times3\times200\times50\times10^6\times10}{8^2} = -28.125kN\cdot m$$

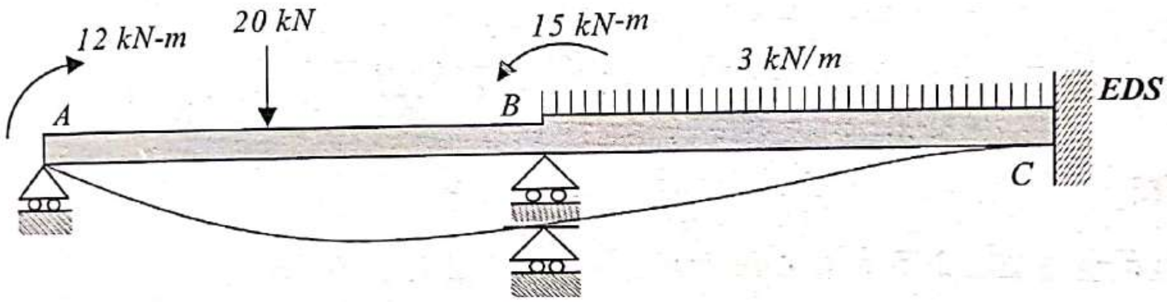
For External loading

$$M_{BA} = -\frac{3PL}{16} = -30kN\cdot m;\quad M_{BC} = +\frac{wL^2}{12} = +16kN\cdot m;\quad M_{CB} = -\frac{wL^2}{12} = -16kN.m$$

Joint	A	B		C
Member	AB	BA	BC	CB
DF	1	0.33	0.67	0
Applied Moment	-12	+5	+10	
COM		-6		+5
$[FEM]_{\Delta}$		+9.375	-28.125	-28.125
$[FEM]_{Load}$		-30	+16	-16
DM		+12.788	+25.96	
COM				+12.98
Final Moment = Σ	-12	-8.72	+23.72	-26.2

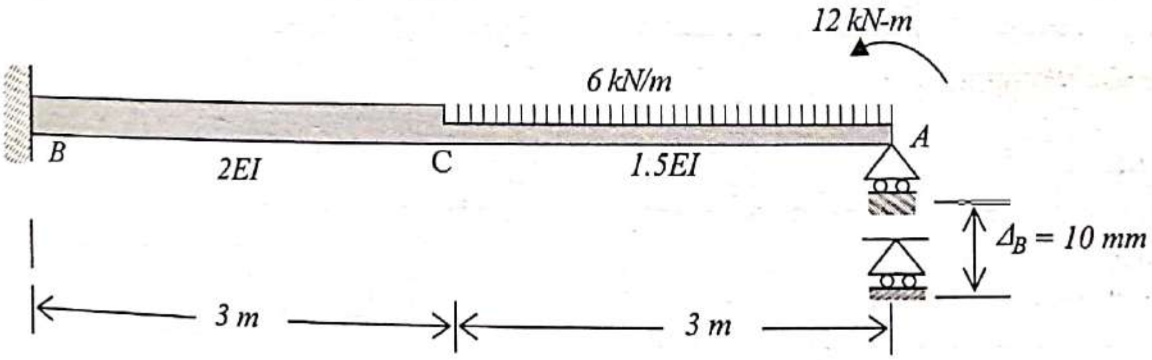






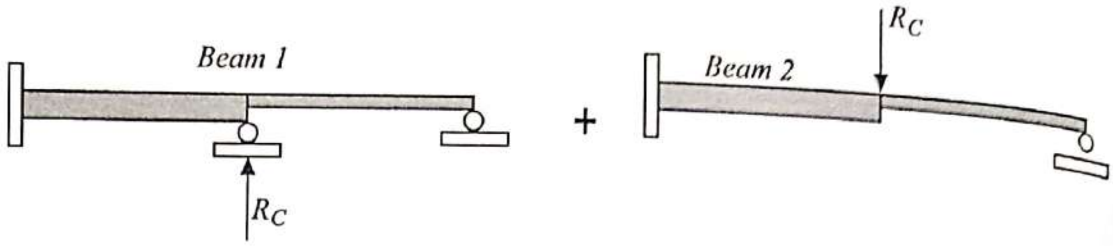
Example 6

For the following beam, support A settles 10 mm downward. Use the moment distributions method to determine all the unknown moments at supports. Take $I = 50x10^6\ mm^4$ and $E = 200\ GPa$



Solution

Since at point C there is a difference of cross-section, put an artificial support at C. The equivalent structure will be the summation of the following two beams.



At first in beam 1, find the reaction R_C,

FEM

For support settlement,

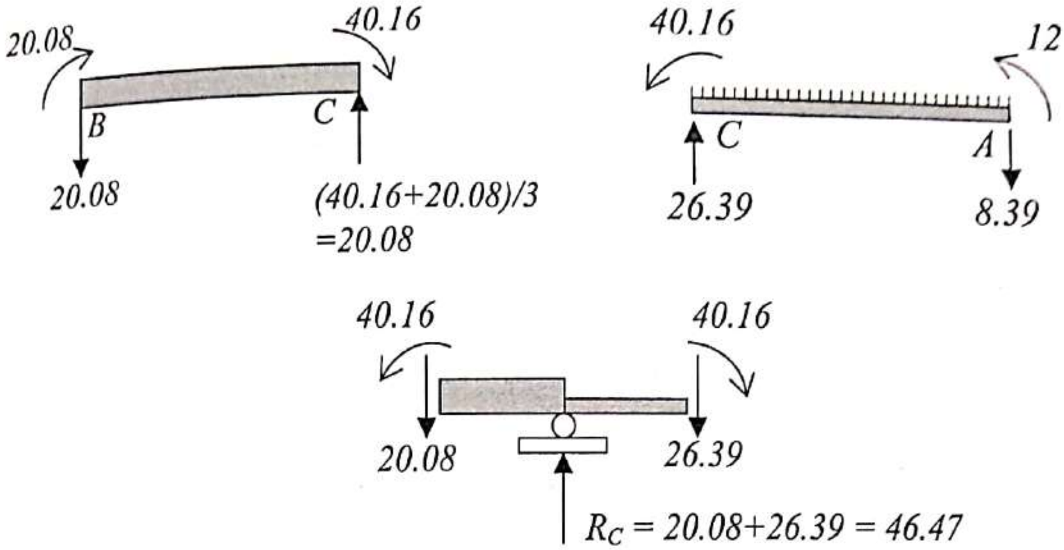
$$M_{CA} = +\frac{3EI\Delta}{L^2} = +\frac{3\times1.5\times200\times50\times10^6\times10}{3^2} = +50kN\cdot m$$

For External loading, $M_{CA} = +\frac{wL^2}{8} = +\frac{6\times 9}{8} = +6.75kN\cdot m$

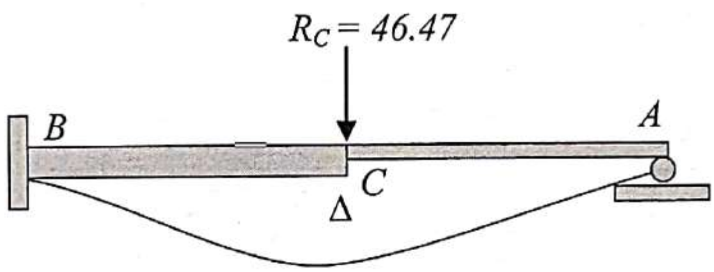
DF

$DF_{CB} = 8/12.5 = 0.64;\ DF_{CA} = 4.5/12.5 = 0.36$

Joint	B	C		A
Member	BC	CB	CA	AC
DF	0	0.64	0.36	1
Applied Moment				+12
COM			+6	
$[FEM]_\Delta$ $[FEM]_{Load}$			+50 +6.75	
DM		-40.16	-22.59	
COM	-20.08			
Final Moment = Σ	-20.08	-40.16	+40.16	+12



Now, apply R_C on *Beam 2*,



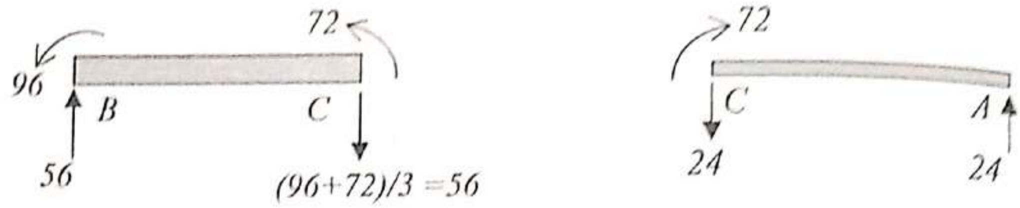
FEM for vertical displacement at C,

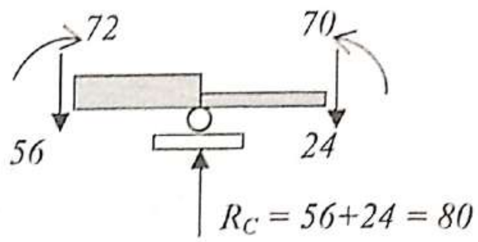
$$M_{BC} = +M_{CB} = +\frac{6EI}{L^2}\Delta = +\frac{6\times2\times90}{3^2} = +120$$

[Suppose, $\Delta = 90$]

$$M_{CA} = -\frac{3EI}{L^2}\Delta = -\frac{3\times1.4\times90}{3^2} = -45$$

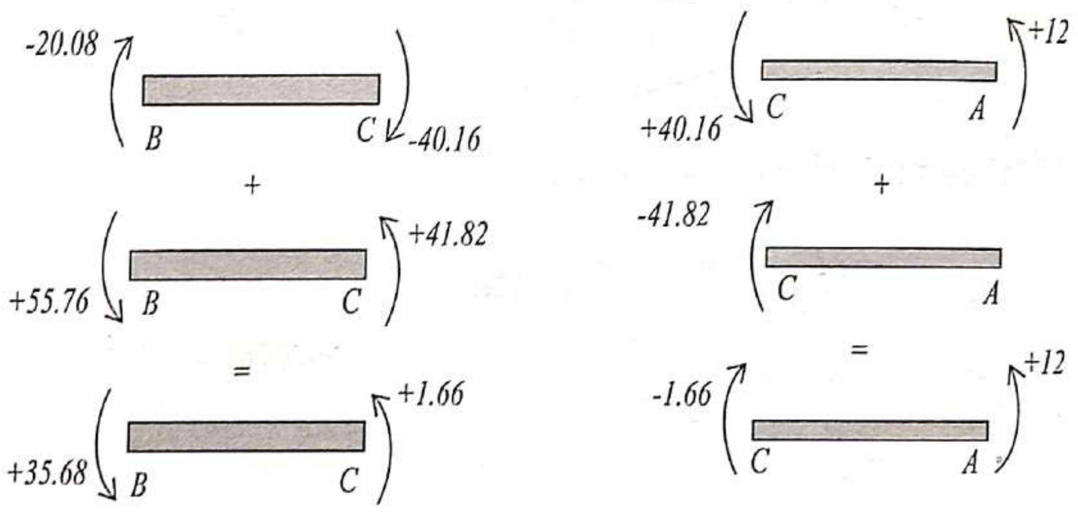
Joint	B	C		A
Member	BC	CB	CA	AC
DF	0	0.64	0.36	1
$[FEM]_\Delta$	+120	+120	-45	
DM		-48	-27	
COM	-24			
Final Moment = Σ	+96	+72	-72	

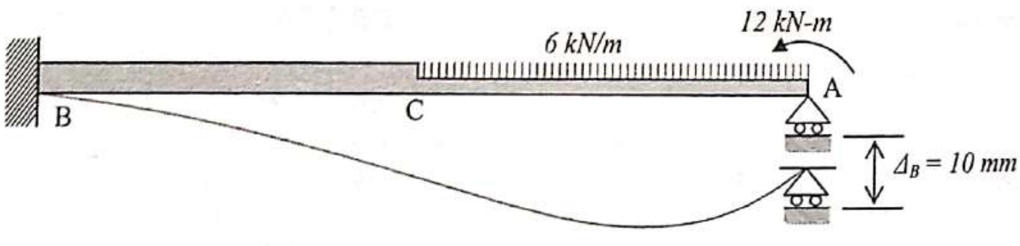


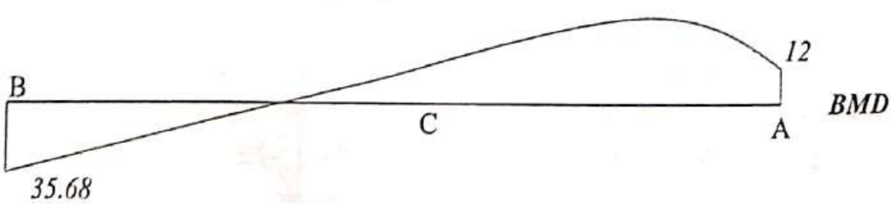


So, the real moment will be, $M_{BC} = 96\times46.47/80 = 55.76$;
$M_{CB} = 72\times46.47/80 = 41.82 = M_{CA}$

The resultant moment will be the moment of Beam 1 + moment of Beam 2 as follows,







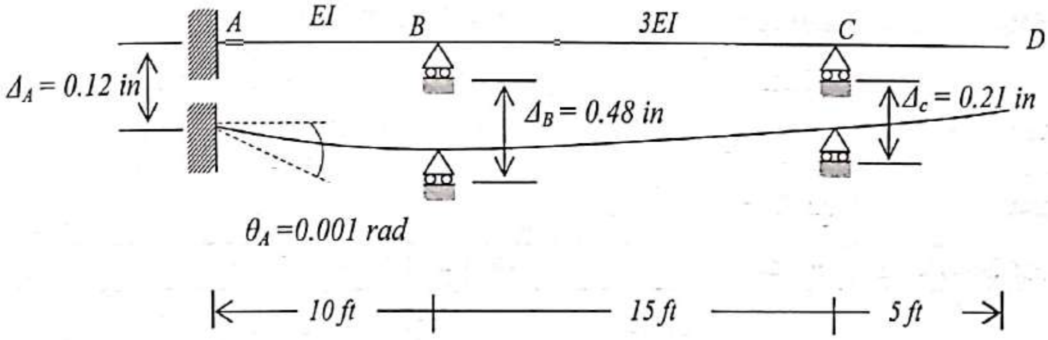
Beam with support settlement and support rotation

Example 7

Find support moment for the following continuous beam. Given that;

$$\theta_A = 0.001 \text{ radian clockwise}$$
$$\Delta_A = 0.12 \text{ in downward}$$
$$\Delta_B = 0.48 \text{ in downward}$$
$$\Delta_C = 0.21 \text{ in downward}$$

Assume $E = 30\times10^3$ ksi and $I = 1000\ in^4$



Solution

Note that the support A undergoes a clockwise rotation, so the induced moment due to this rotation is negative and the resultant settlement between support A and B is $(\Delta_A - \Delta_B)$ and induced moment due to this settlement is positive.

FEM

$$M_{AB} = -\frac{4EI\theta}{L} + \frac{6EI(\Delta_B - \Delta_A)}{L^2} = -\frac{4\times30\times10^3\times1000\times0.001}{10\times144} + \frac{6\times30\times10^3\times1000(0.48-0.12)}{10^2\times144\times12}$$
$$= -83.3 + 375 = +291.7 \text{ k-ft}$$

$$M_{BA} = -\frac{2EI\theta}{L} + \frac{6EI(\Delta_B - \Delta_A)}{L^2} = -\frac{2\times30\times10^3\times1000\times0.001}{10\times144} + \frac{6\times30\times10^3\times1000(0.48-0.12)}{10^2\times144\times12}$$
$$= -41.65 + 375 = +333.4 \text{ k-ft}$$

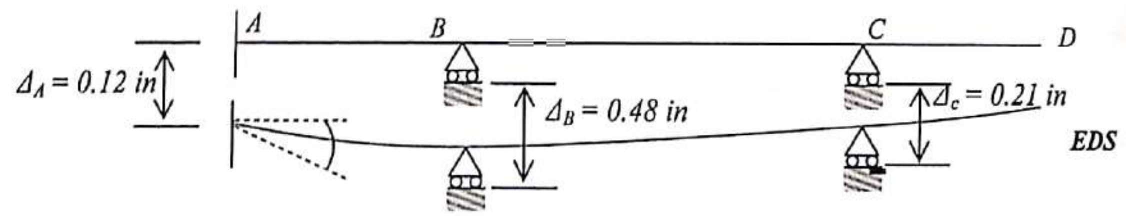
$$M_{BC} = -\frac{3\times3EI(\Delta_B - \Delta_C)}{L^2} = -\frac{3\times3\times30\times10^3\times1000(0.48-0.21)}{15^2\ \ 144\ \ 12} = -187.5 \text{ k-ft}$$

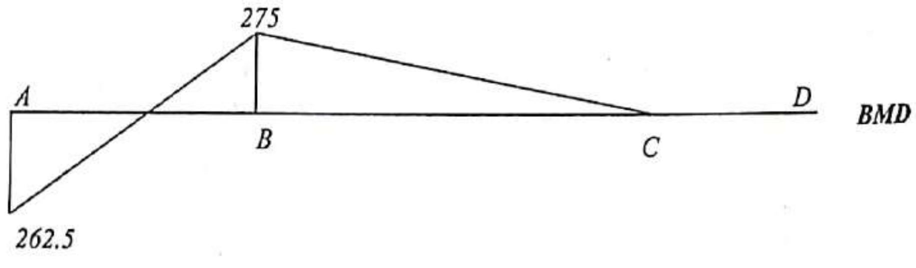
DF

$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{4EI/L}{4EI/L + 3\times 3EI/L} = \frac{4EI/10}{4EI/10 + 9EI/15} = \frac{\frac{2}{5}}{\frac{2}{5}+\frac{3}{5}} = 0.4$$

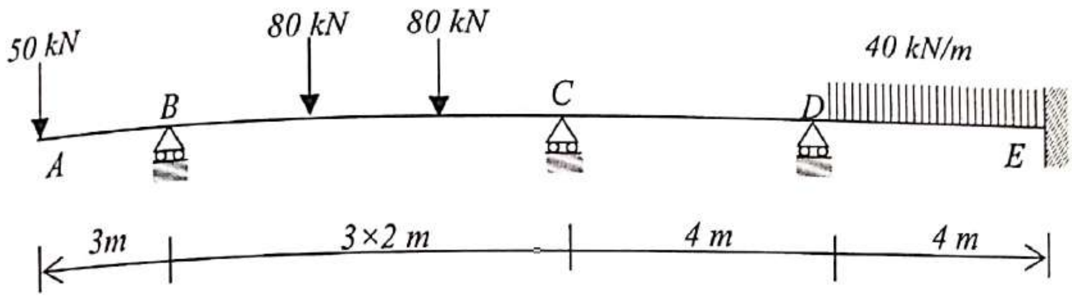
$$DF_{BC} = 1 - 0.4 = 0.6$$

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.4	0.6	1
$[FEM]_\Delta$	+291.7	+333.4	-187.5	
DM		-58.36	-87.57	
COM	-29.18			
Final Moment = Σ	+262.52	+275	-275	0



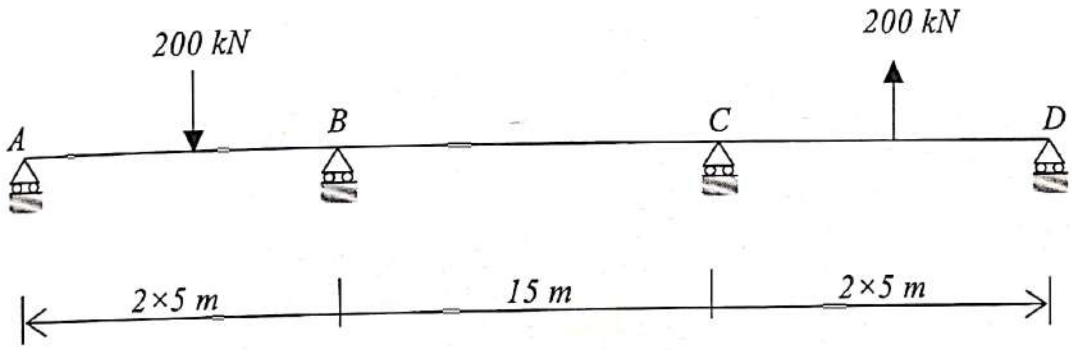


Exercise 1



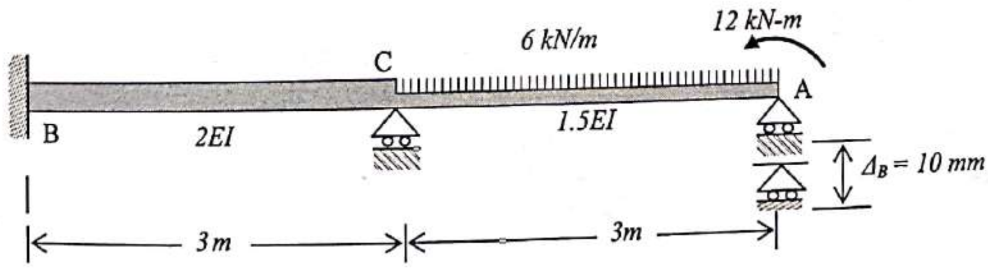
Ans: *At support C, $M = 49.23$, at support D, $M = 8.79$ and support E, $M = 75.61$*

Exercise 2



Ans: *At point B, $M = 213.75$*

Exercise 3



Ans: $M_B = -2$ kN-m, $M_C = 40.16$ kN-m

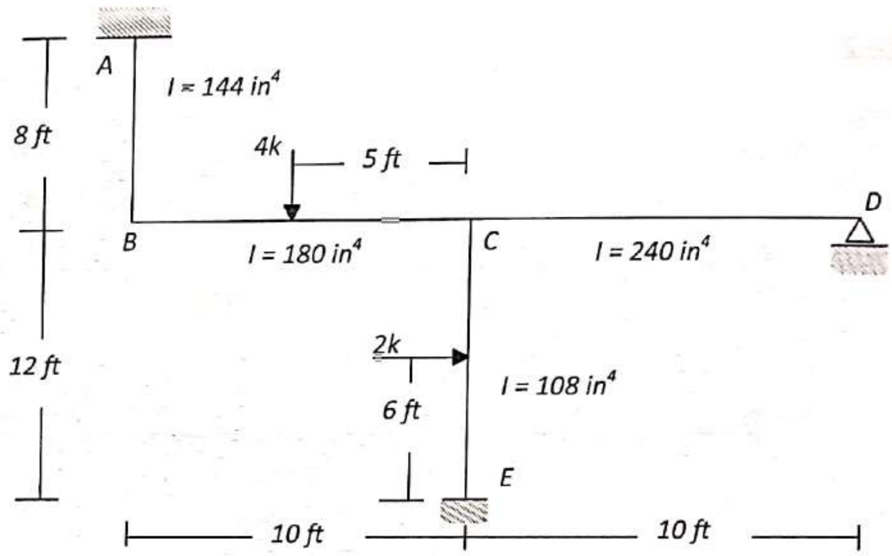
Analysis of Indeterminate Frame by Moment Distribution Method

Examples of frame without side way

If the frame does not experience any joint translation or side sway, the solution method is as simple as beam. No, special consideration is necessary. Following examples will show the step-by-step solution procedure.

Example 1

Analyze the following frame by moment distribution method, and find all unknown support moment



Solution

FEM

$$M_{BC} = -\frac{PL}{8} = -5.0;\ M_{CB} = +5.0;\ M_{CE} = +\frac{PL}{8} = +3.0;\ M_{EC} = -3.0$$

Relative Stiffness

$$K_{BA} = \frac{4EI}{L} = \frac{4E144}{8} = 2;\ K_{BC} = \frac{4EI}{L} = \frac{4E180}{10} = 2$$

$$K_{CD} = \frac{3EI}{L} = \frac{3E240}{10} = 2;\ K_{CE} = \frac{4EI}{L} = \frac{4E108}{12} = 1$$

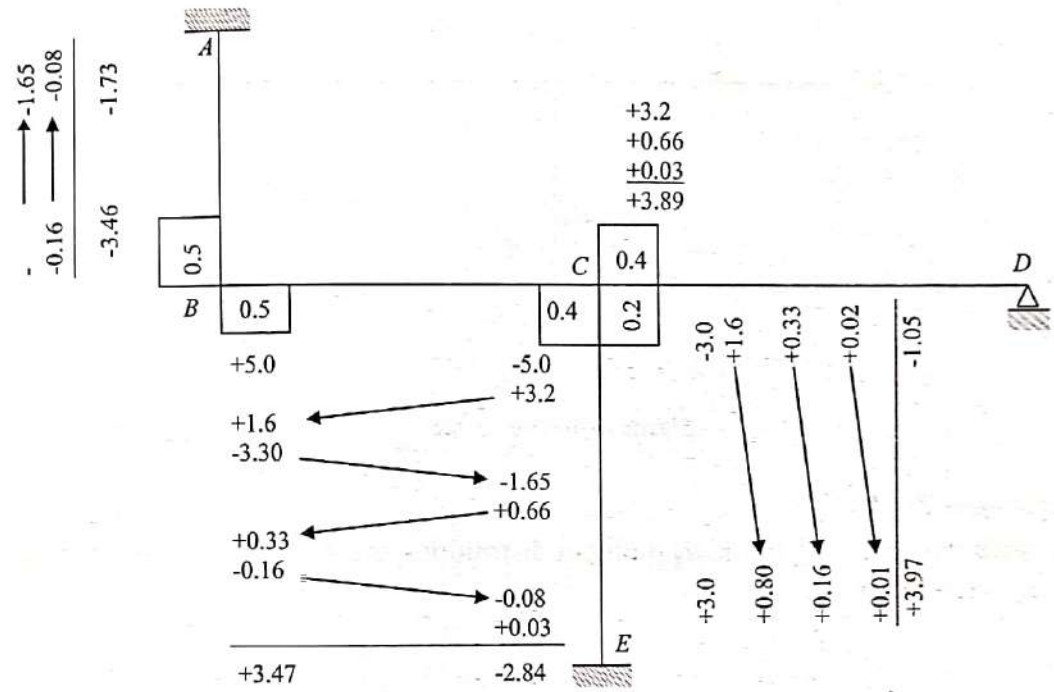
DF

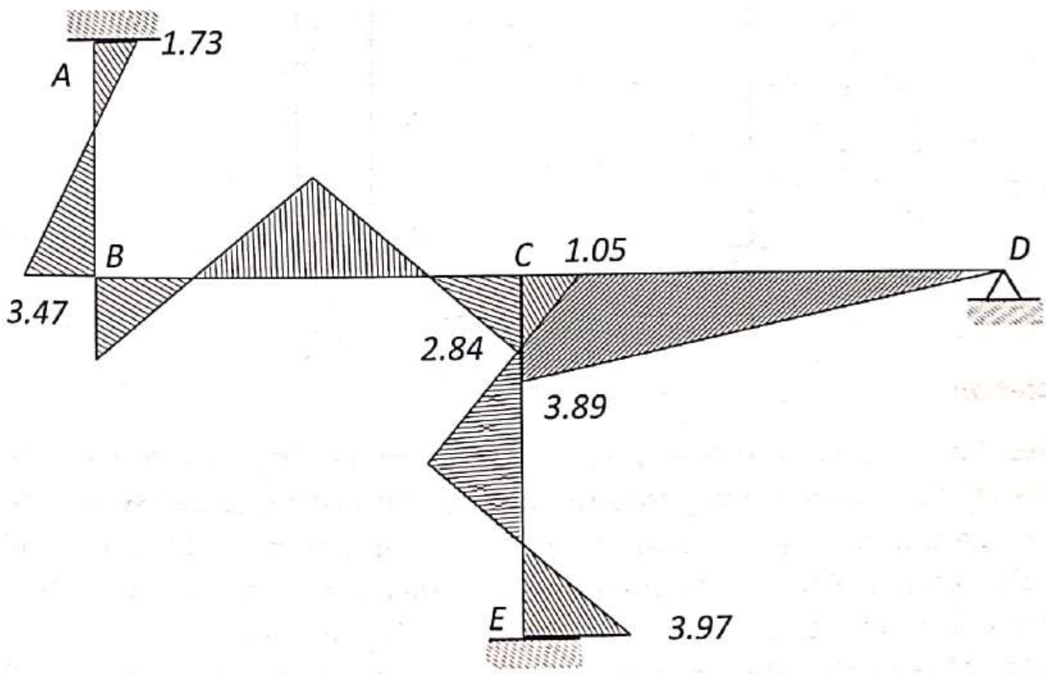
$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{2}{2+2} = 0.5$$

$$DF_{CB} = DF_{CD} = \frac{2}{2+2+1} = 0.4$$

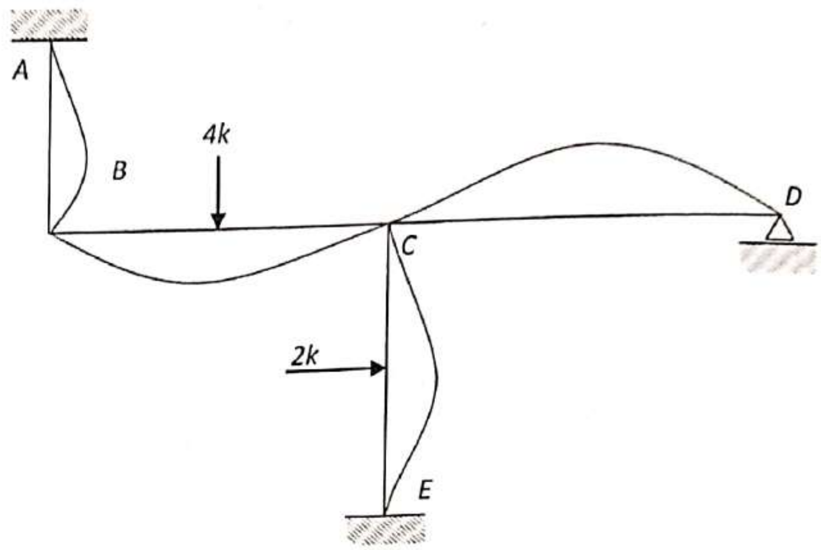
$$DF_{CE} = \frac{1}{5} = 0.2$$

For this type of frame problem, it is advantageous to solve the problem on the frame itself rather than in a separate table.





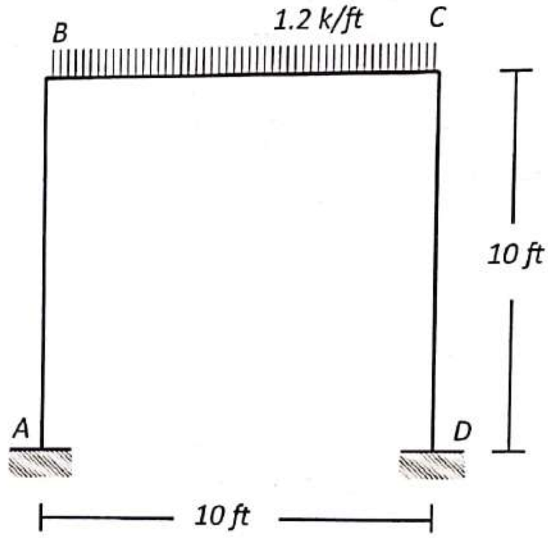
Bending moment diagram



Elastic deflection shape

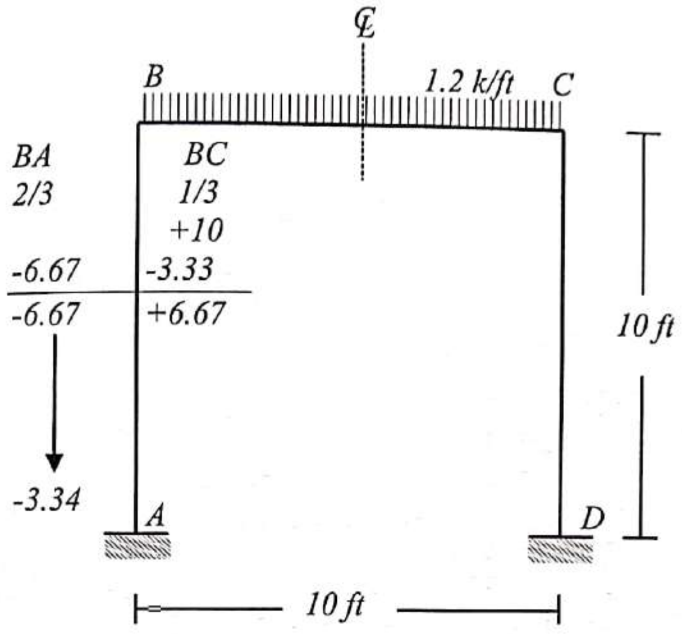
Example 2

Analyze the following frame by moment distribution method, and find all unknown support moments.

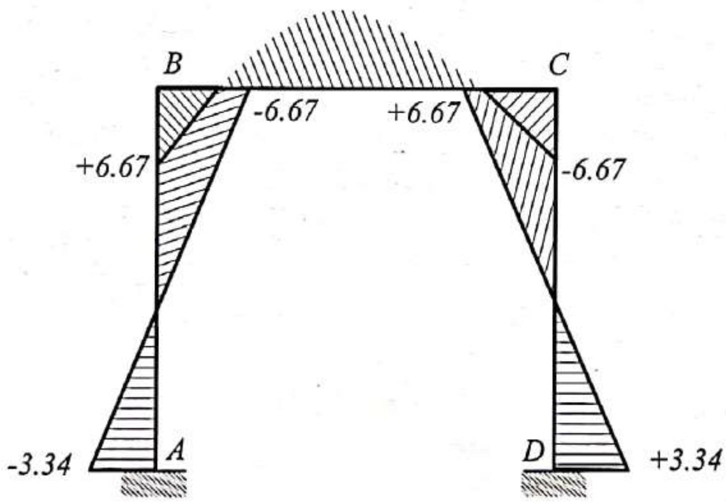


Solution

This frame is symmetrical about its centerline. So, it is prudent to analyze half of the frame, considering the stiffness factor for symmetrical case and find the unknown moments. The moments on the other half of the frame will be exactly the same in value but opposite in sign of that of the left-hala, as shown in the moment diagram below. For brevity, detail of the calculation of relative stiffness and distribution factors are not shown here, and they are the same calculation procedure as were in the previous examples.

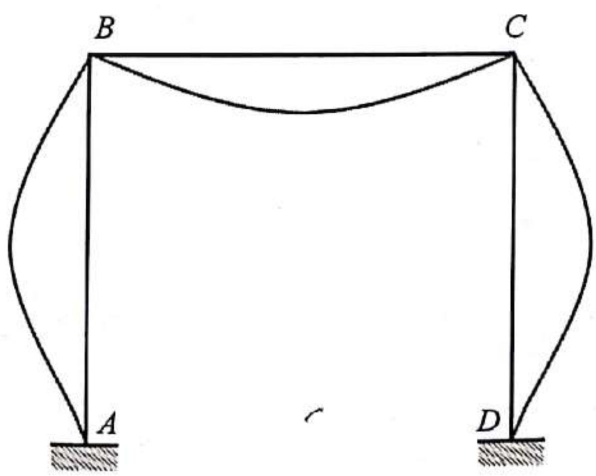


℄
B
1.2 k/ft
C
BA
2/3
BC
1/3
+10
-6.67
-3.33
-6.67
+6.67
-3.34
A
D
10 ft
10 ft



B
C
-6.67
+6.67
+6.67
-6.67
-3.34
A
D
+3.34

BMD

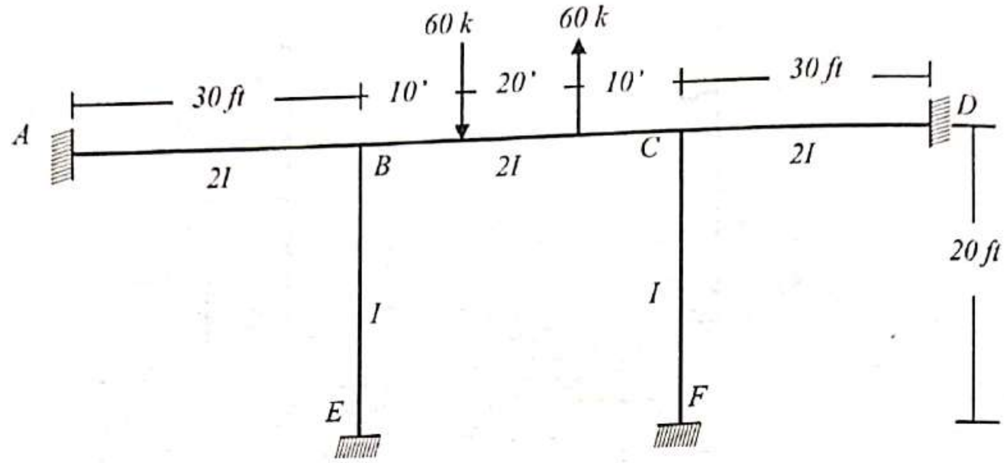


B
C
A
D

EDS

Example 3

Analyze the following frame by moment distribution method, and find all unknown support moments.



Solution

This frame is anti-symmetrical about its centerline. So, it is prudent to analyze half of the frame considering the stiffness factor for anti-symmetry case, and find the unknown moments. The moments on the other half of the frame will be exactly the same in value and same in sign of that of the left-half, as shown in the moment diagram below.

FEM

$$M_{BC} = +\frac{Pa(L-a)(L-2a)}{L^2} = \frac{60\times10\times30\times20}{40^2} = +225 \text{ k-ft}$$

Relative Stiffness

$$K_{BA} = \frac{4EI}{L} = \frac{4E\times2I}{30} = \frac{8}{30}$$

$$K_{BC} = \frac{6EI}{L} = \frac{6E\times2I}{40} = \frac{9}{30}$$

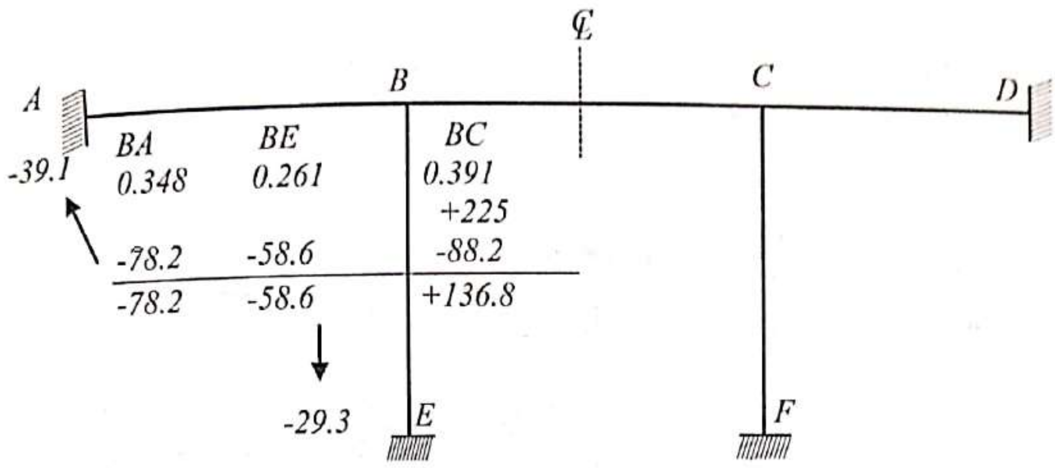
$$K_{BE} = \frac{4EI}{L} = \frac{4E\times I}{20} = \frac{6}{30}$$

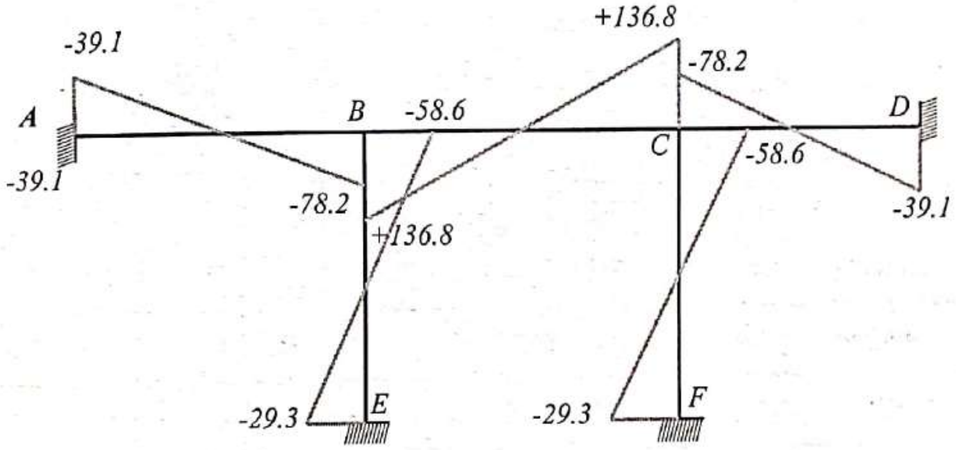
DF

$$DF_{BC} = \frac{K_{BA}}{K_{BA}+K_{BC}+K_{BE}} = \frac{8}{8+9+6} = 0.348$$

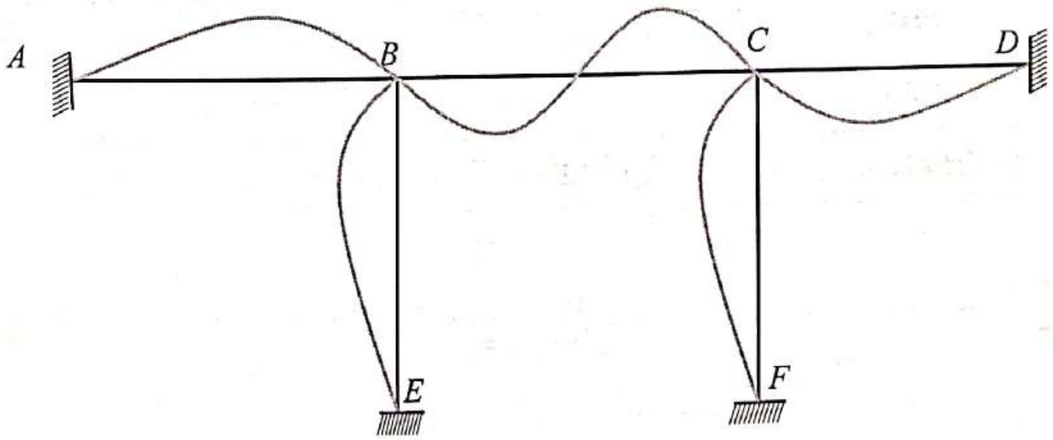
$$DF_{BC} = \frac{K_{BC}}{K_{BA}+K_{BC}+K_{BE}} = \frac{9}{8+9+6} = 0.391$$

$$DF_{BE} = 1-(0.348+0.391) = 0.261$$





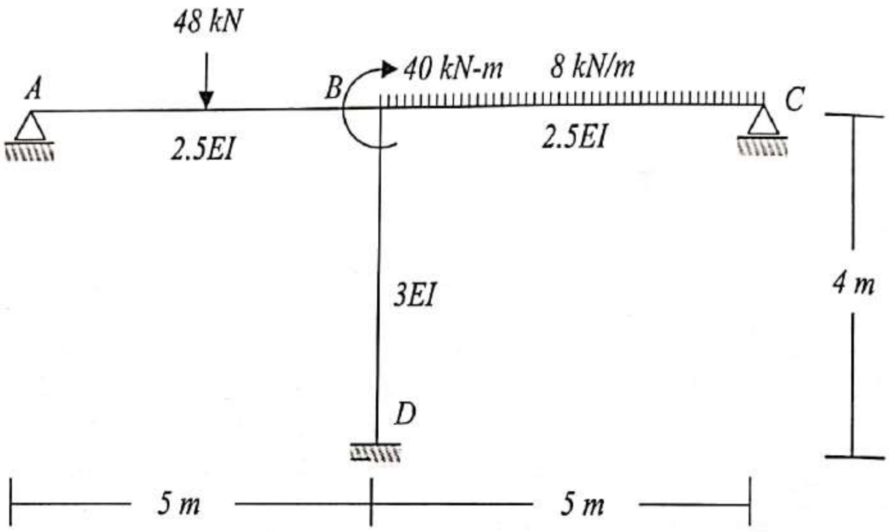
BMD



Elastic Deflection Shape

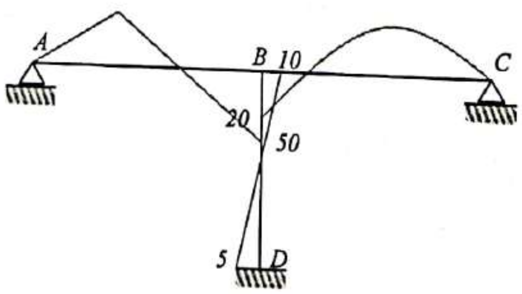
Example 4

Analyze the following frame by moment distribution method, and find all unknown support moments.

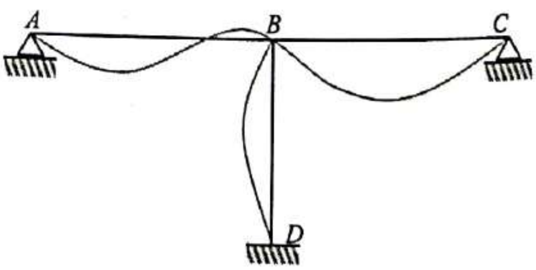


Solution

Joint	A	B			D	C
Member	AB	BA	BC	BD	DB	CB
DF	1	0.25	0.25	0.5	0	1
Applied M COM		-10	-10	-20	 -10	
FEM		-45	+25			
DM COM		+5	+5	+10	 +5	
Total M, Σ	0	-50	+20	-10	-5	0



BMD

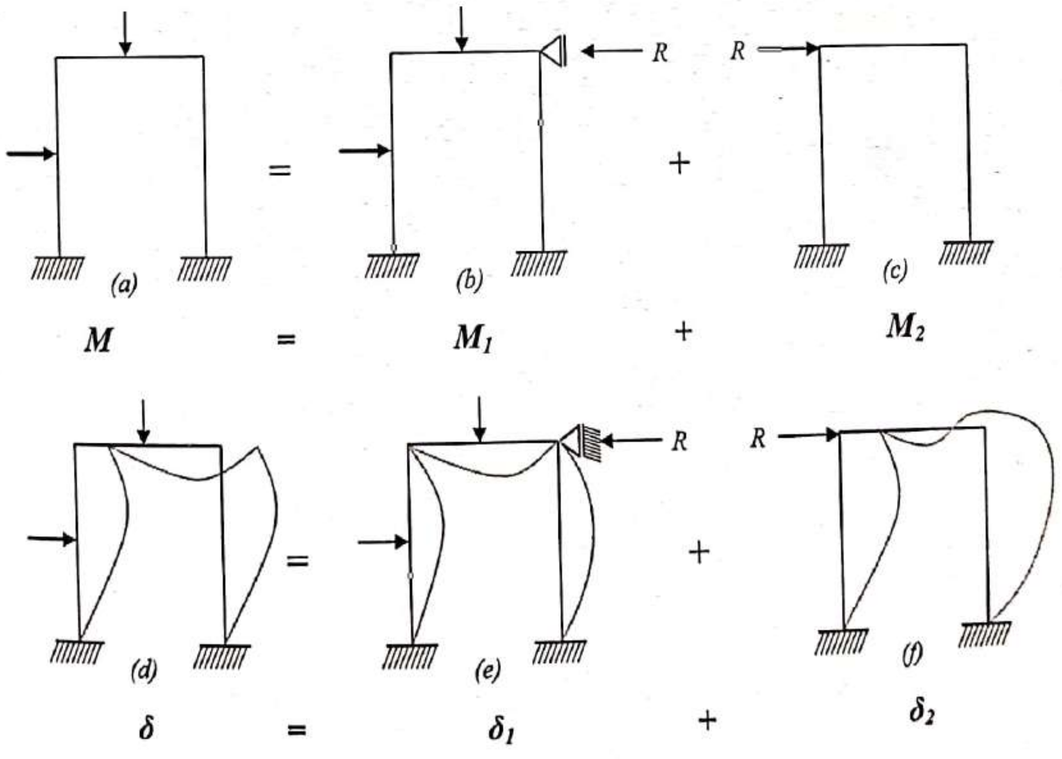


EDS

Examples of frames with inclined leg and side sway

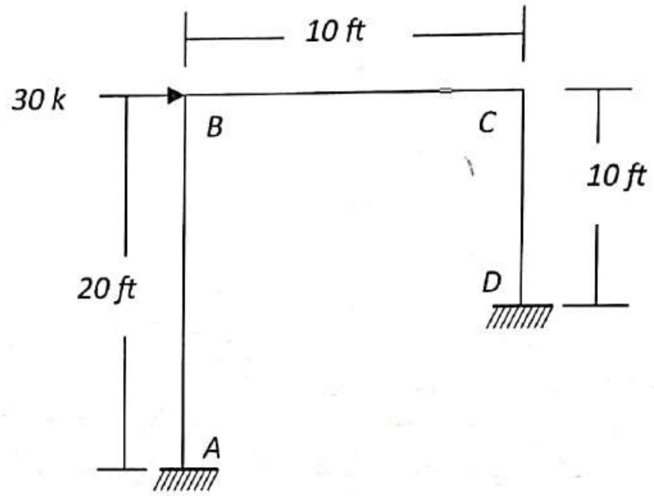
Many indeterminate frames experience joint translation along with joint rotation, and therefore, the process of moment distribution, described previously, cannot be applied directly without certain adjustments and operations.

Consider the following frame *(a)*, where the frame undergoes certain amount of joint translation along with the joint rotation, due to the applied load. A two-stage analysis procedure is required to account for the additional moments caused by the sway of the frame. In the first stage of analysis procedure, an artificial holding or prop is applied at the joint where the sway is assumed to happen. The whole purpose of the artificial holding is to prevent side sway as shown if figure *(b)*. While this artificial restraint is in place, the frame is analyzed as usual by moment distribution method. The artificial holding force R can then easily be found from its static analysis and free-body. As the frame was restrained against lateral sway with the help of this holding force R, equal and opposite force have to be applied on the frame to ensure original behavior of the frame. Figure *(c)* below shows the frame with the holding force R. Now, the frame is analyzed again by moment distribution method for this joint load R, and this is the second stage of the analysis. The resultant moments M of the frame will then be the summation of moment M_1 and M_2 from figure *(b)* and *(c)* respectively, based on the principle of superposition. The principle of superposition can also be applied to get the resultant deformation of the frame (figure *d, e, f*). Following examples will explain the technique more clearly.



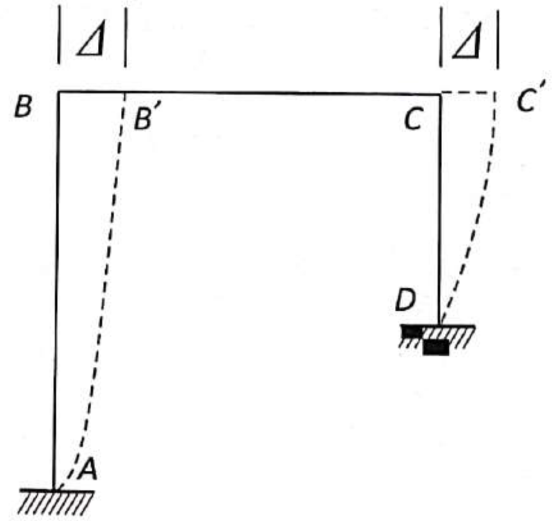
Example 5 (Frame with joint load)

Analyze the following unequal legged frame by moment distribution method, and find all unknown support moments.



Solution

Since only the joint load is present here, the second stage of solution alone will be necessary to get the final moment. A tentative joint translation of the frame will be as follows (without joint rotation). An arbitrary lateral displacement (sway) Δ = 40000 will induce the following fixed-end moments,



FEM

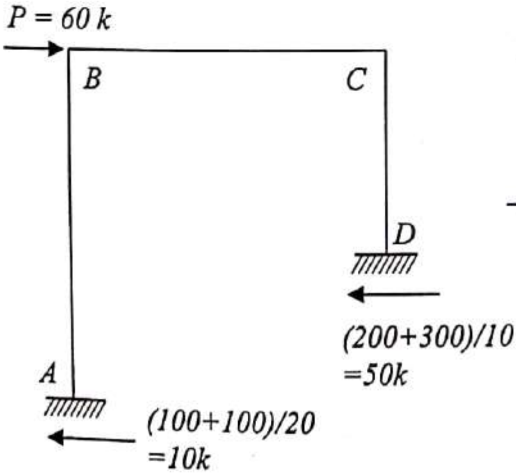
$$M_{AB} = M_{BA} = +\frac{6EI\Delta}{L^2} = \frac{6EI\Delta}{400} = +100$$

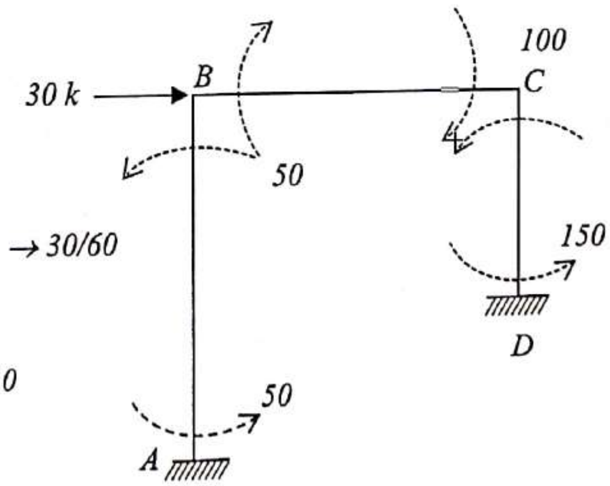
$$M_{CD} = M_{DC} = +\frac{6EI\Delta}{L^2} = \frac{6EI\Delta}{100} = +400$$

Now, distribute this fixed-end moment to get all the end moments.

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.33	0.67	0.5	0.5	0
FEM	+100	+100			+400	+400
DM		-33.3	-66.7	-200	-200	
COM	-16.7		-100	-33.3		-100
DM		+33.3	+66.7	+16.6	+16.7	
COM	+16.7		+8.3	+33.3		+8.3
DM		-2.8	-5.5	-16.6	-16.7	
COM	-1.4		-8.3	-2.8		-8.3
DM		+2.8	+5.5	+1.4	+1.4	
COM	+1.4		+0.7	+2.8		+0.7
DM		-0.2	-0.5	-1.4	-1.4	
Final Moment, k-ft = Σ	+100	+100	-100	-200	+200	+300

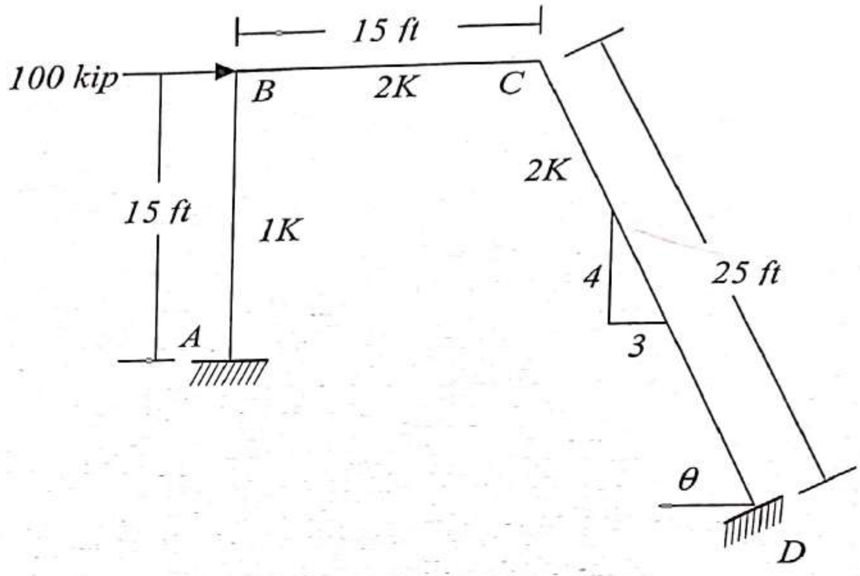
From the free-body, the lateral force P is found to be $60\ k$. Due to arbitrary lateral displacement ($\Delta = 40000$), an arbitrary lateral force $P = 60\ k$ is generated here. As, the original lateral joint force was $30\ k$, the exact moment can simply be the moment got by moment distribution in the above table multiplied by the ratio of 30/60 as shown in figure below.





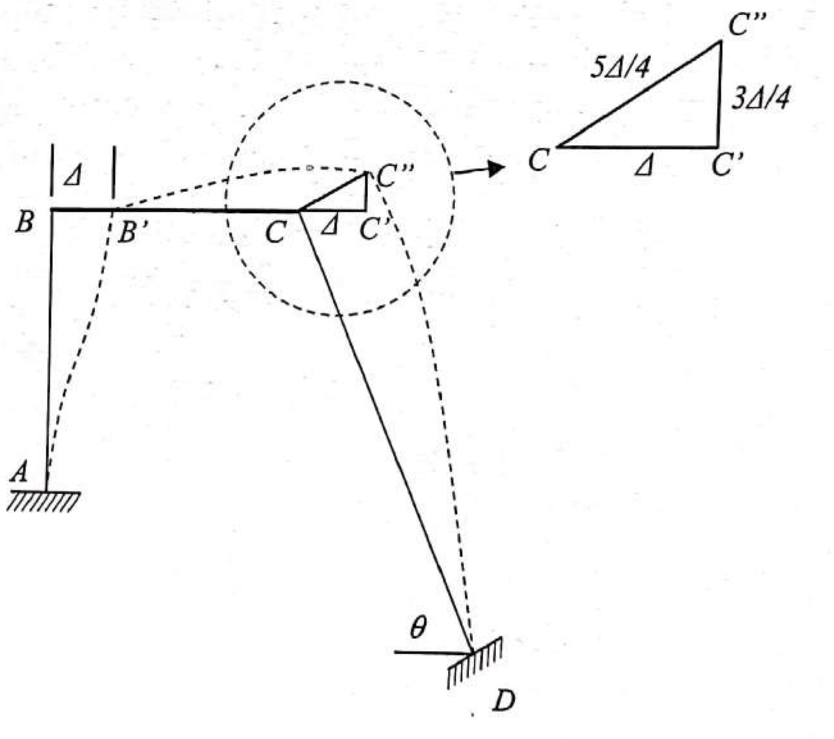
Example 6

Analyze the following inclined legged frame by moment distribution method, and find all unknown support moments.



Solution

A tentative joint translation of the frame will be as follows (without joint rotation). An arbitrary lateral displacement (sway) $\Delta = 250$ will induce the following fixed-end moments,



FEM

$$M_{AB} = M_{BA} = +\frac{6EI\Delta}{L^2} = +\frac{6EI\Delta}{L \cdot L} = +\frac{6 \cdot 1 \cdot \Delta}{15} = +100$$

$$M_{BC} = M_{CB} = -\frac{6EI\Delta}{L^2} = -\frac{6 \cdot 2 \cdot \frac{3}{4}\Delta}{15} = -150$$

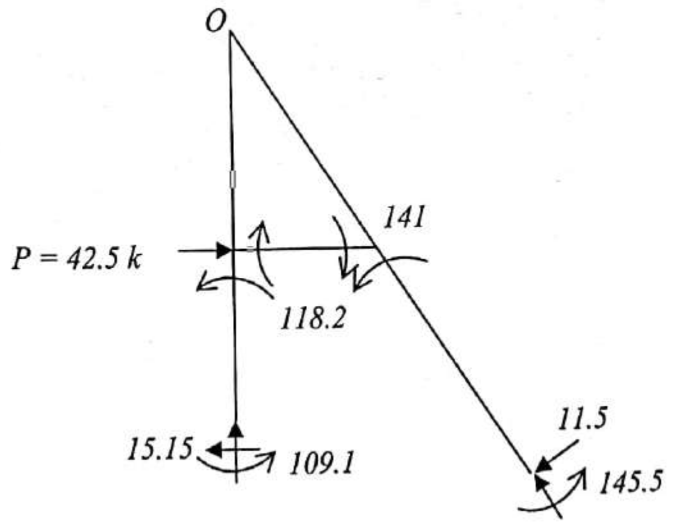
$$M_{CD} = M_{DC} = +\frac{6EI\Delta}{L^2} = +\frac{6 \cdot 2 \cdot \frac{5}{4}\Delta}{25} = +150$$

[Note that flexural stiffness, $k = EI/L$]

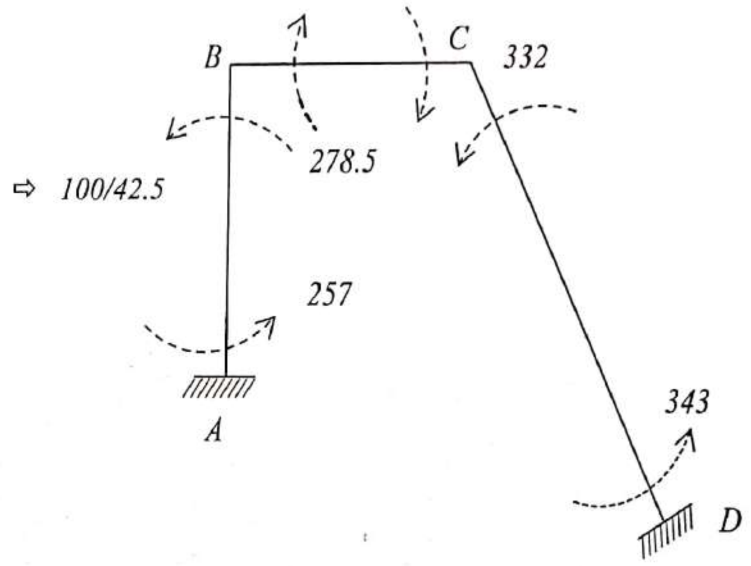
Distribute the FEM

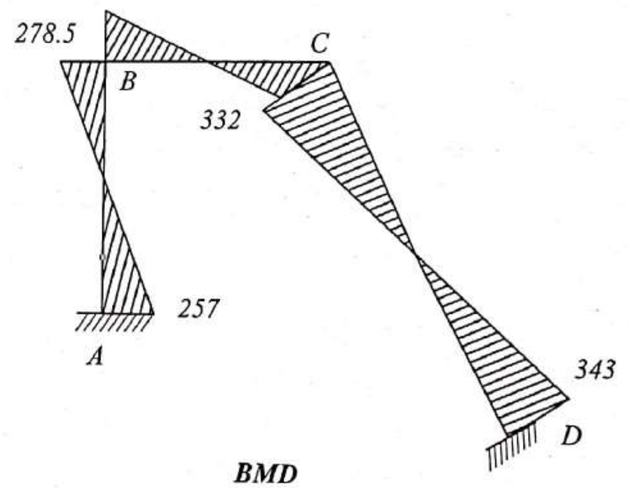
Joint	A	B		D		C
Member	AB	BA	BC	CB	CD	DC
DF	0	0.33	0.67	0.5	0.5	0
FEM	+100	+100	-150	-150	+150	+150
DM		+16.7	+33.3	0	0	
COM	+8.35		0	+16.6		
DM				-8.3	-8.3	
COM			-4.2			-4.2
DM		+1.4	+2.8			
COM	+0.7			+1.4		
DM				-0.7	-0.7	
COM			-0.35			-0.35
DM		+0.15	+0.2			
Final Moment, k-ft = Σ	+109.1	+118.2	-118.2	-141	+141	+145.5

from the free-body, calculate the lateral force, P for arbitrary lateral sway.

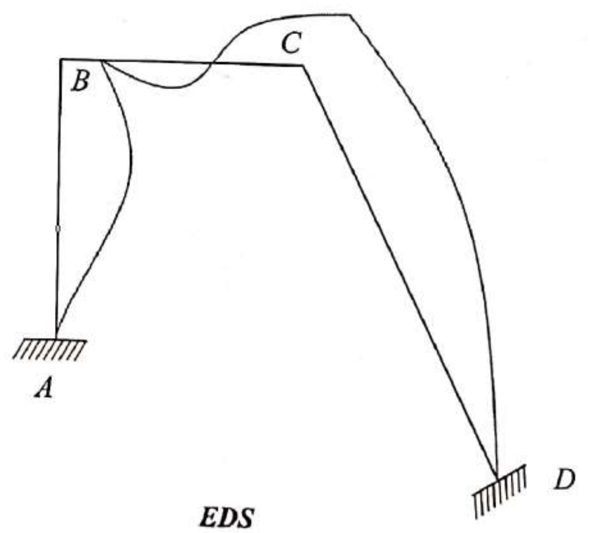


As, the original lateral joint force was $100\ k$, the exact moment can simply be the moment got by moment distribution in the above table multiplied by the ratio of $100/42.5$ as shown in figure below.





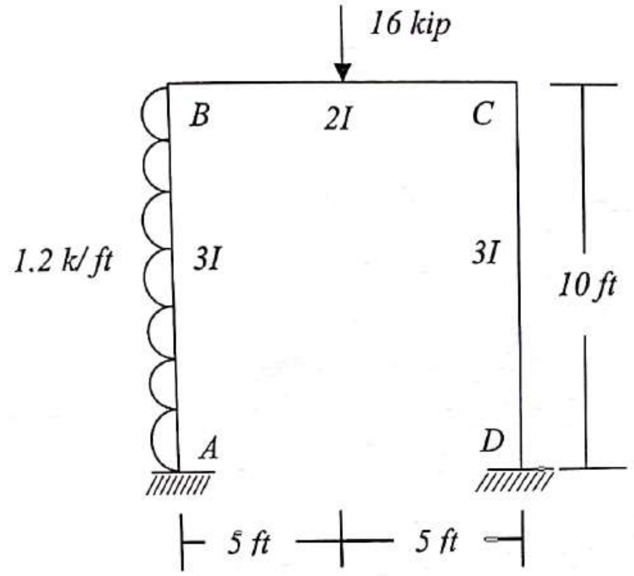
BMD



EDS

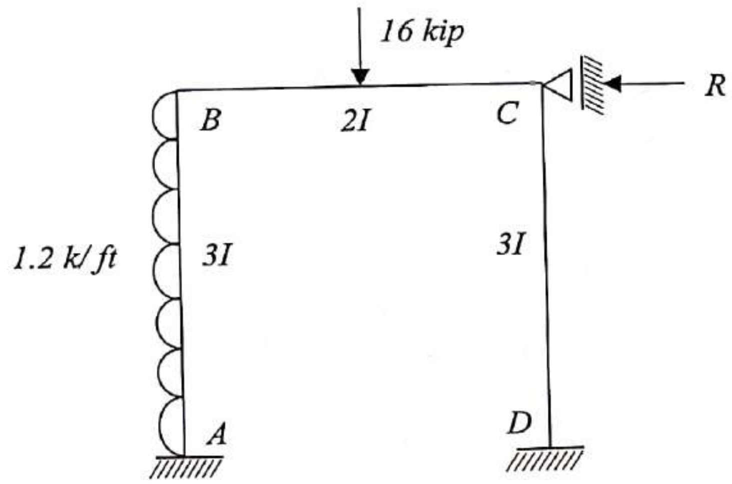
Example 7

Analyze the following equal legged portal frame by moment distribution method, and find all unknown support moments.



Solution

In this frame the loads are applied on the member, unlike the previous problems, where the load was applied at the joint. So, two stage solution procedure is needed for this problem, as discussed earlier. At the first stage of the solution, put an artificial holding at the top of the frame at C to prevent side sway. With this holding in place, find out the end moments (M_1) by moment distribution procedure.



FEM

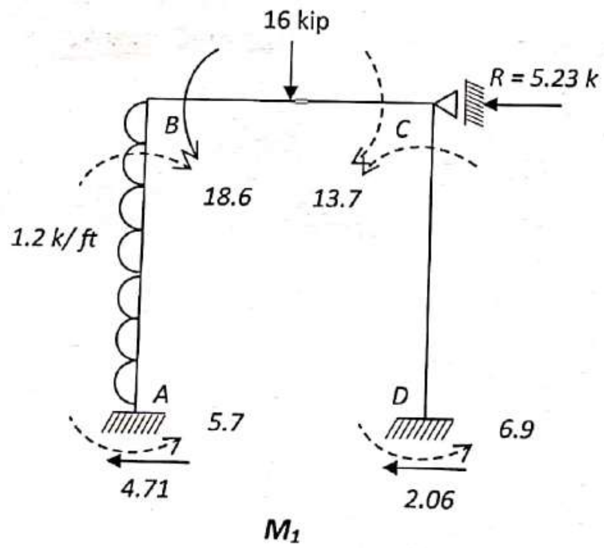
$$M_{AB} = +\frac{wL^2}{12} = +10; \; M_{BA} = -\frac{wL^2}{12} = -10$$

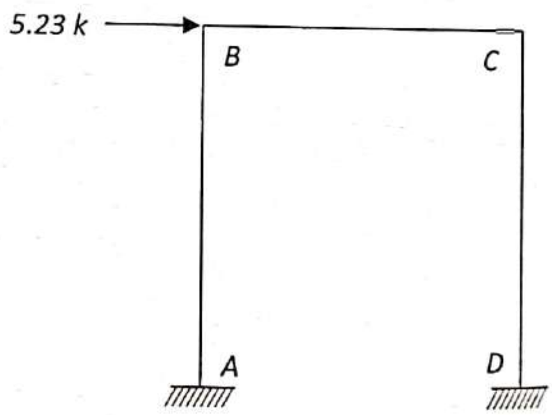
$$M_{BC} = +\frac{PL}{8} = +20; M_{CB} = -\frac{PL}{8} = -20$$

Distribute this fixed-end moment by moment distribution method,

Joint	A	B		C		D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.6	0.4	0.4	0.6	0
FEM	+10	-10	+20	-20		
DM		-6	-4	+8	+12	
COM	-3		+4	-2		+6
DM		-2.4	-1.6	+0.8	+1.2	
COM	-1.2		+0.4	-0.8		+0.6
DM		-0.24	-0.16	+0.32	+0.48	
COM	-0.12		+0.16	-0.08		+0.24
Final Moment, k-ft = Σ	+5.7	-18.6	+18.6	-13.7	+13.7	+6.9

From the free-body and from the static analysis, find the lateral holding force R at joint C. Apply the force R at joint B in opposite direction and again calculate the end moments (M_2) for side sway as follows.





Assume an arbitrary joint translation, $\Delta = 1800$

FEM

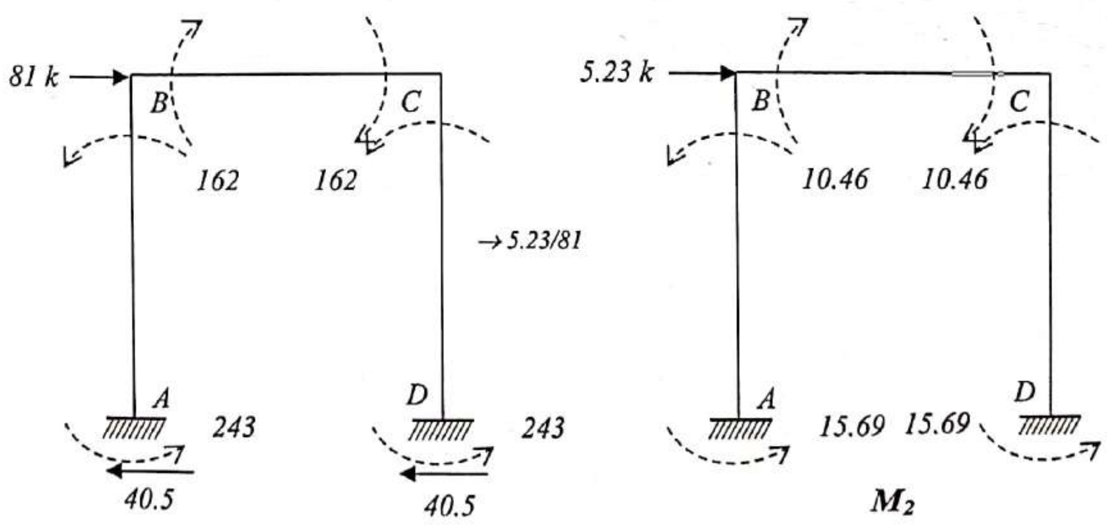
$$M_{AB} = M_{BA} = +\frac{6EI\Delta}{L^2} = +\frac{6 \cdot 3 \cdot \Delta}{100} = +324$$

$$M_{CD} = M_{DC} = +\frac{6EI\Delta}{L^2} = +\frac{6 \cdot 3 \cdot \Delta}{100} = +324$$

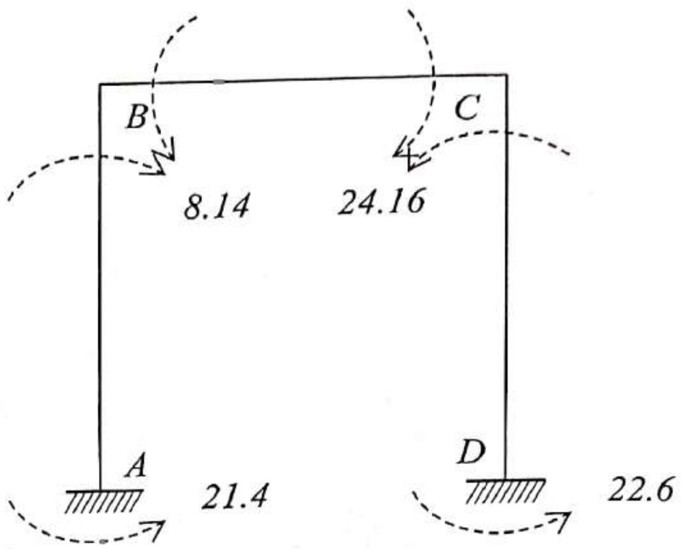
Distribute the FEM

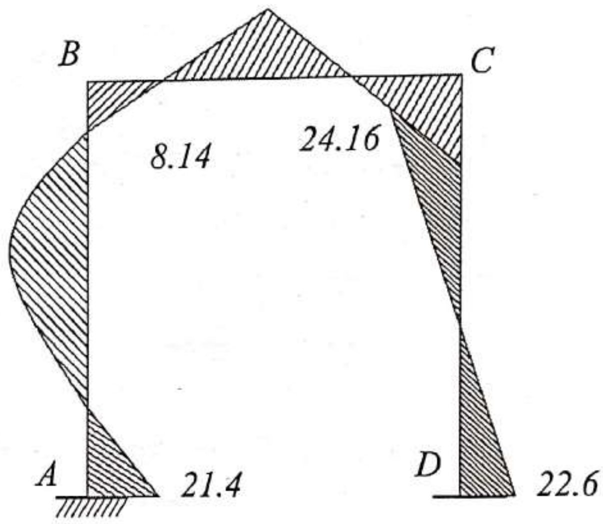
Joint	A		B		C	D
Member	AB	BA	BC	CB	CD	DC
DF	0	0.6	0.4	0.4	0.6	0
FEM	+324	+324			+324	+324
DM		-194.4	-129.6	-129.6	-194.4	
COM	-97.2		-64.8	-64.8		-97.2
DM		+38.9	+25.9	+25.9	+38.9	
COM	+19.5		+13	+13		+19.5
DM		-7.8	-5.2	-5.2	-7.8	
COM	-3.9		-2.6	-2.6		-3.9
DM		+1.6	+1.0	+1.0	+1.6	
COM	+0.8		+0.5	+0.5		+0.8
DM		-0.3	-0.2	-0.2	-0.3	
COM	-0.15		-0.1	-0.1		-0.15
Final Moment, k-ft = Σ	+243	+162	-162	-162	+162	+243

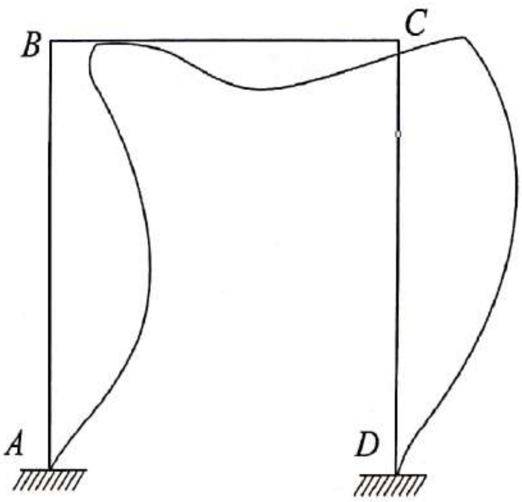
The exact moment M_2 will be the distributed moments found in the above table multiplied by the ratio of *5.23/81* as follows,



The resultant final moment, $M = M_1 + M_2$



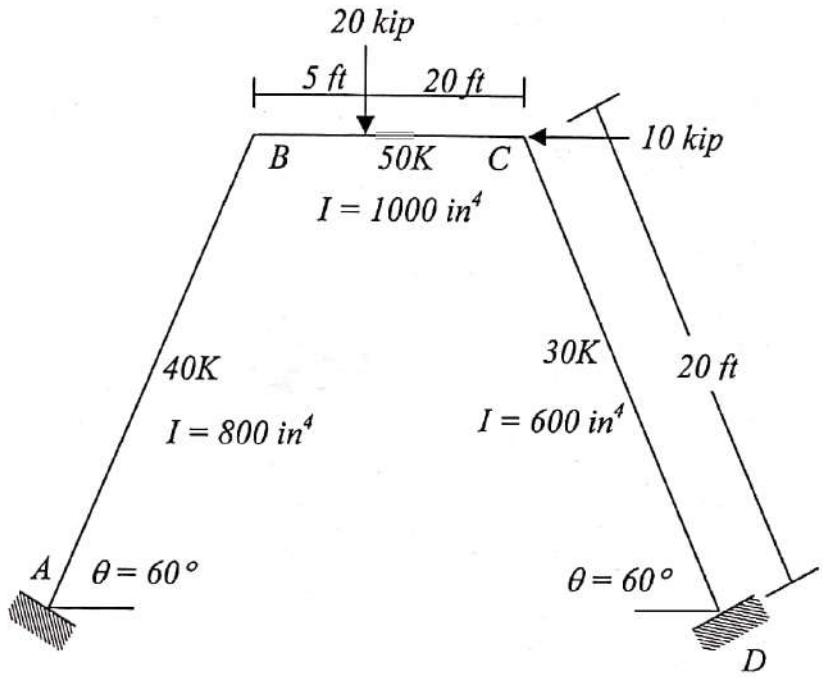




Elastic deflection shape of the frame

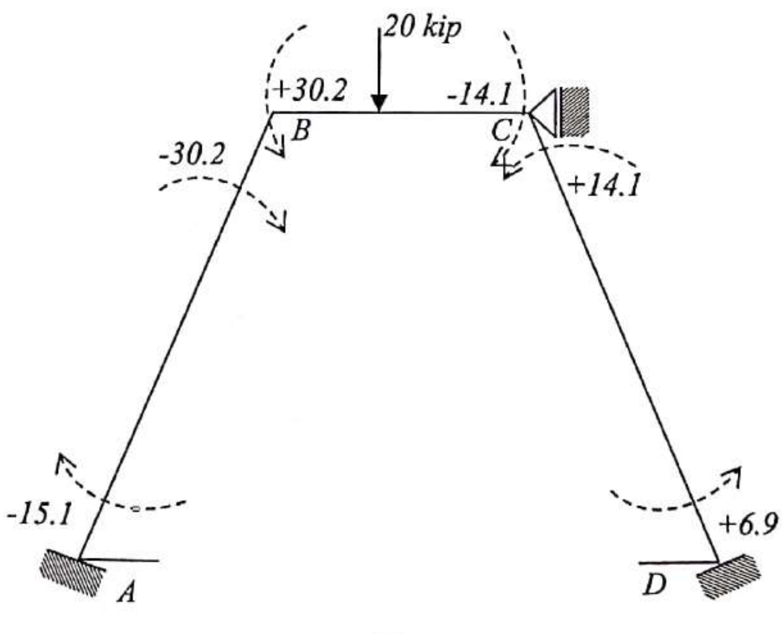
Example 8

Analyze the following both legged inclined frame by moment distribution method, and find all unknown support moments.



Solution

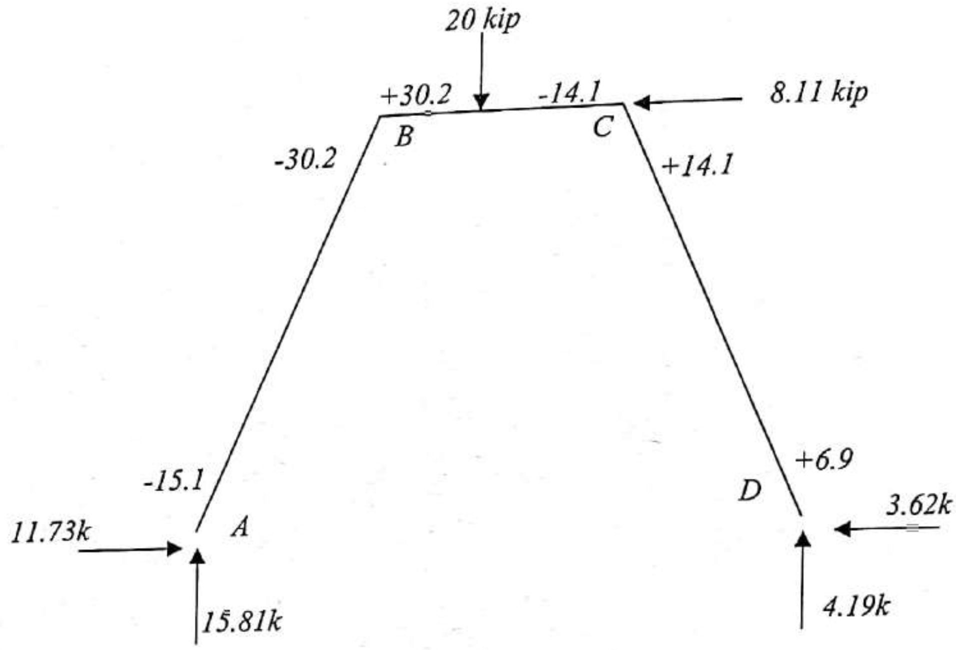
Put an artificial holding at the top of the frame at C to prevent side sway. With this holding in place, find out the end moments M_1 by moment distribution procedure as shown in figure below. (Detailed tabular value is not shown here, do it by yourself)



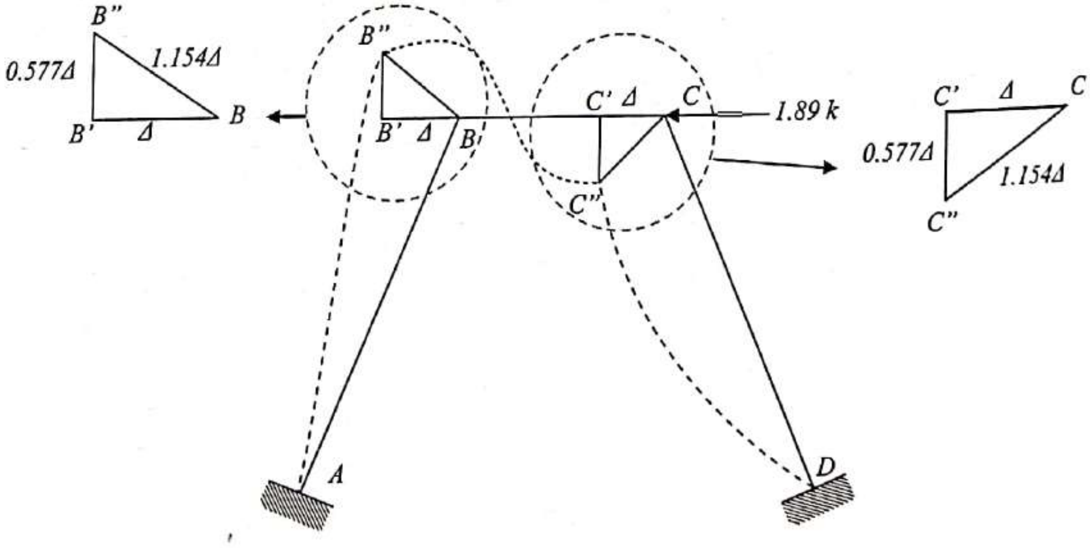
M_1

Draw the free-body and find out the lateral holding force R at joint C. Apply the force R at joint C in opposite direction and find out the end moments M_2 for side sway as follows.

Here, $R = 8.11\ kip$. So, an equal and opposite force have to be applied at joint C. A $10\ kip$ joint force is already there. So the resultant joint force at joint C will be = $10\text{-}8.11 = 1.89\ kip$. Now end moments have to be calculated for the side sway produced by this joint force at R.



Assume an arbitrary joint translation $\Delta = 10.4$ is induced due to this joint force of *1.89* kip, as shown in figure below,

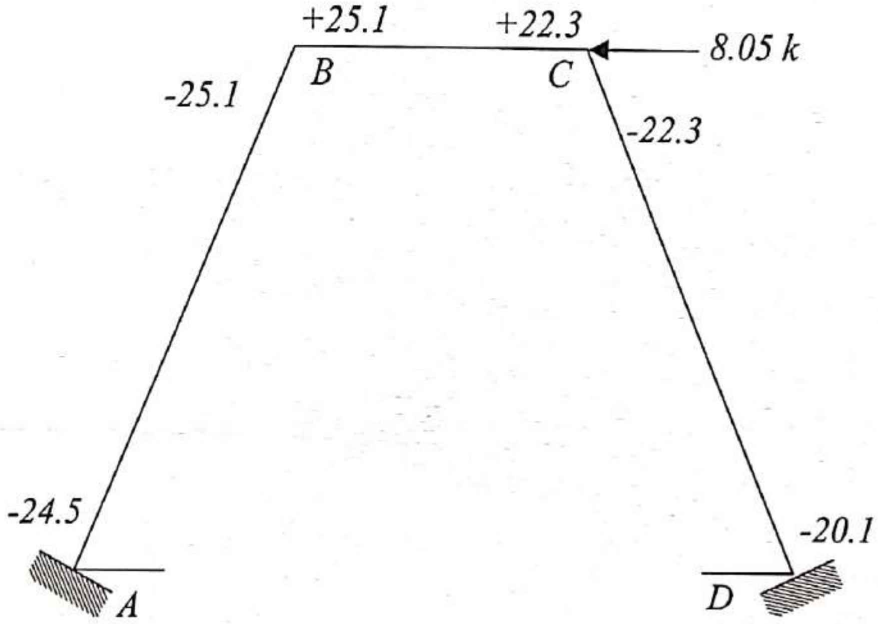


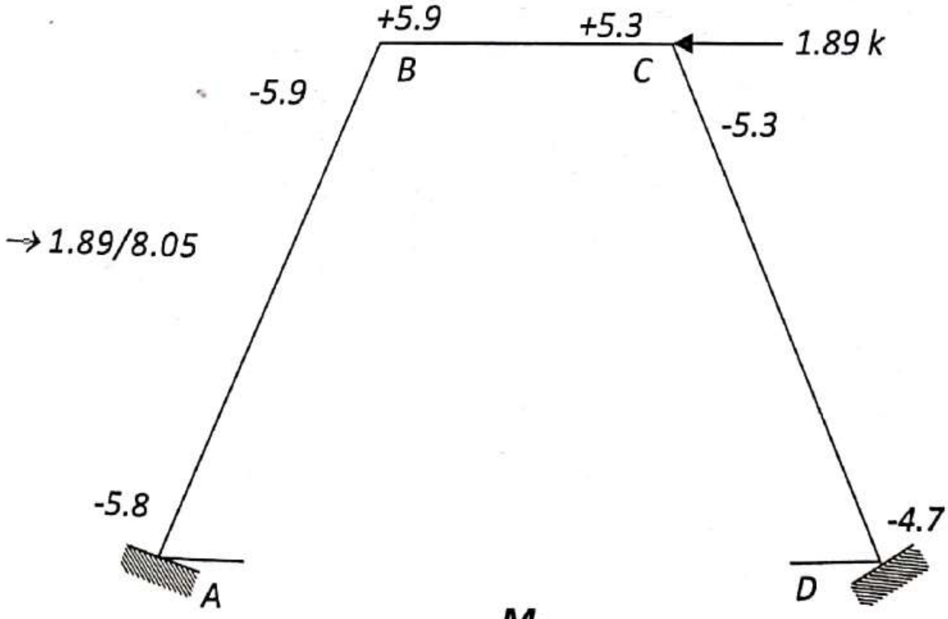
FEM

$$M_{AB} = M_{BA} = -\frac{6EI\Delta}{L^2} = \frac{800 \times 1.154\Delta}{20^2} = -24$$

$$M_{BC} = M_{CB} = +\frac{6EI\Delta}{L^2} = +\frac{1000 \times 2 \times 0.577\Delta}{20^2} = +30$$

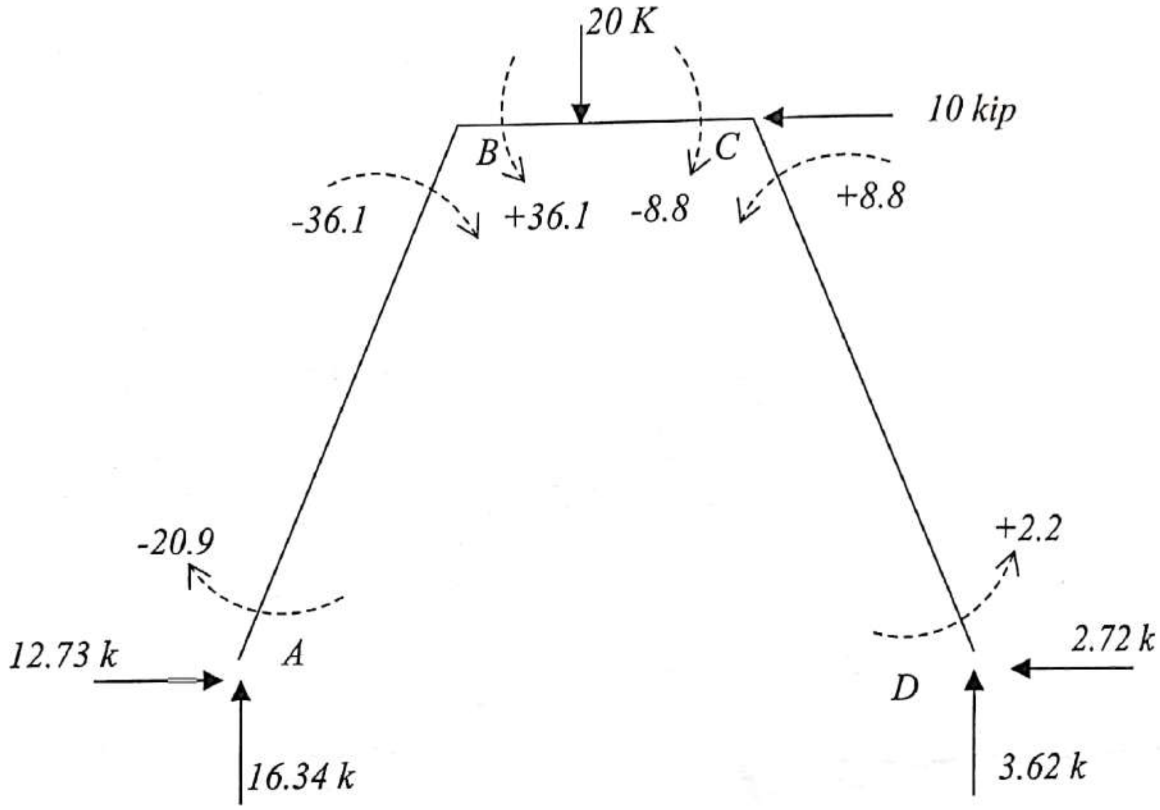
$$M_{CD} = M_{DC} = -\frac{6EI\Delta}{L^2} = \frac{600 \times 1.154\Delta}{20^2} = -18$$

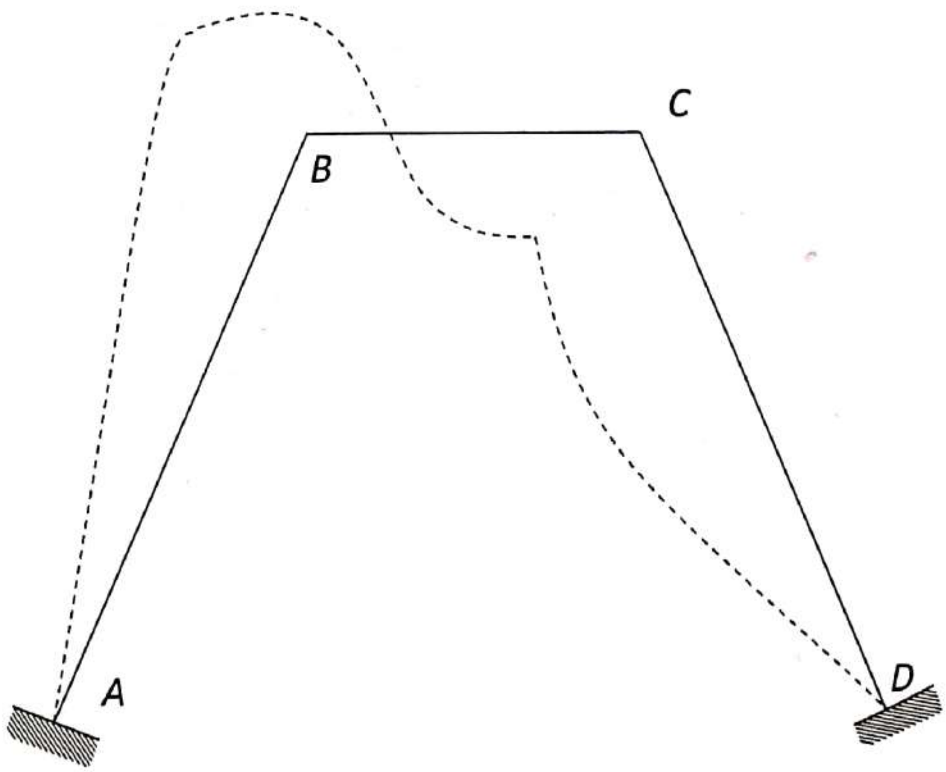




M_2

The final moment will be, $M = M_1 + M_2$

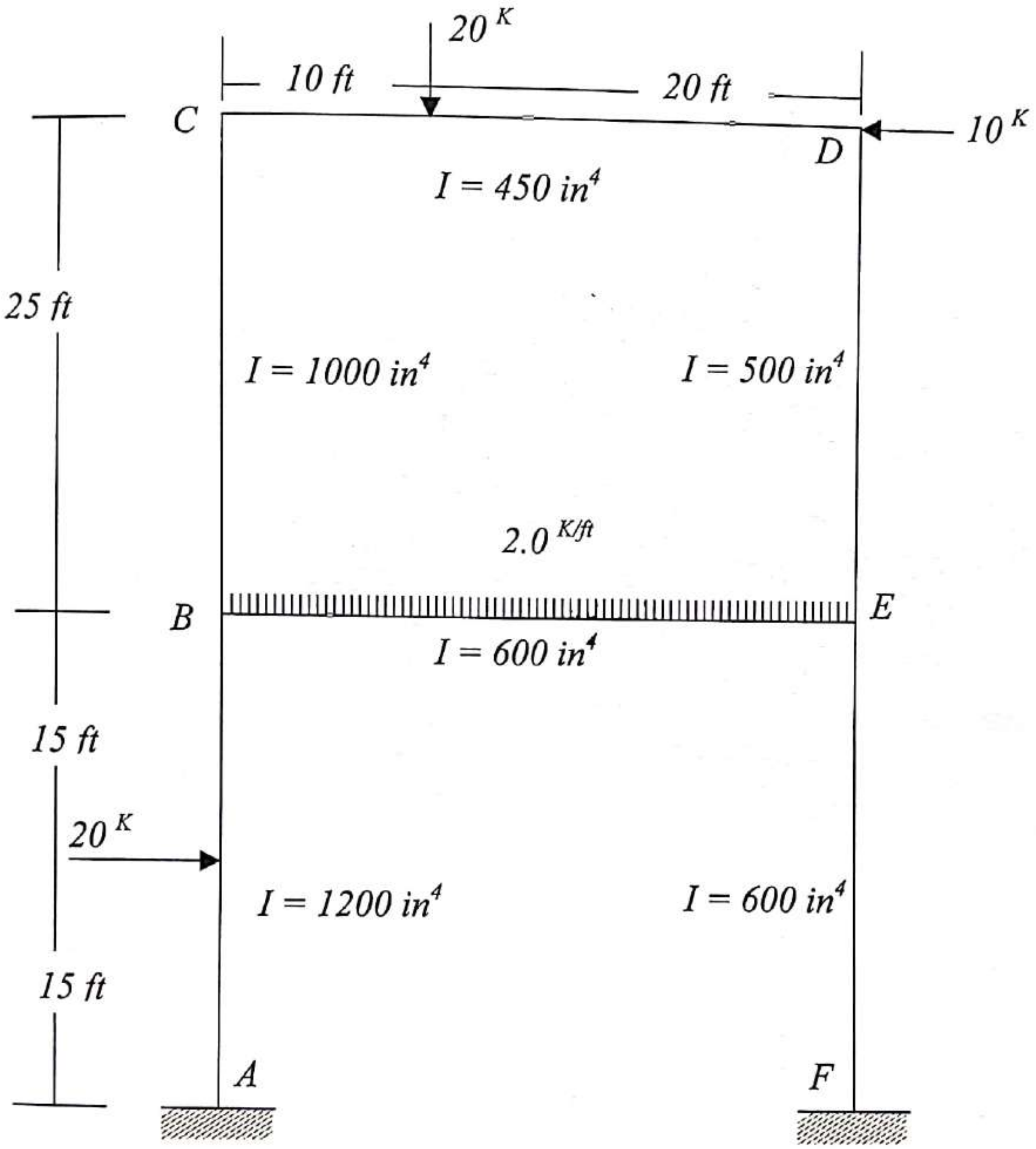




EDS

Example 9

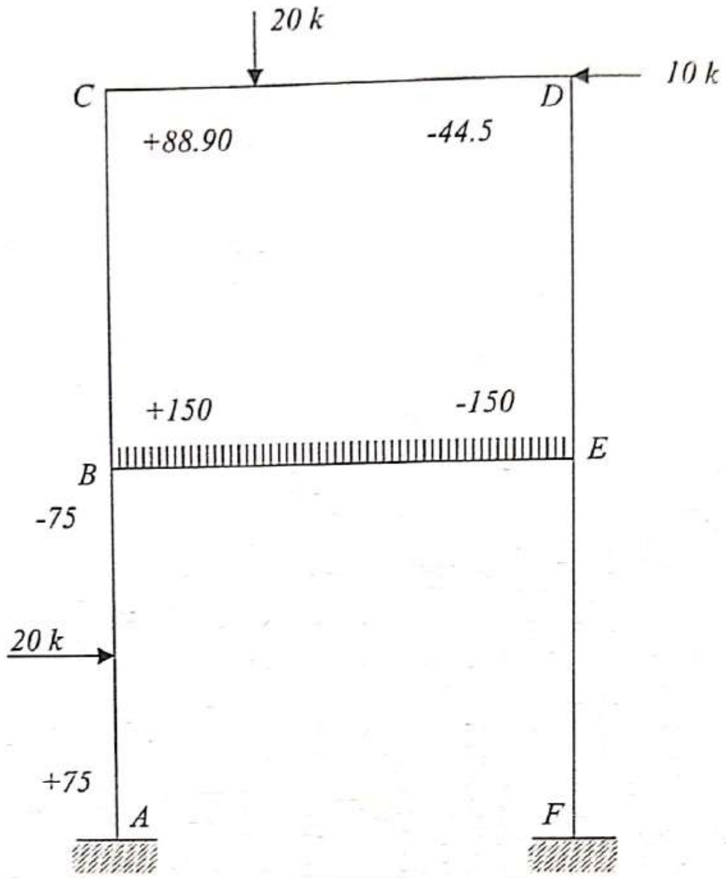
Analyze the following two story frame by moment distribution method, and find all unknown support moments.



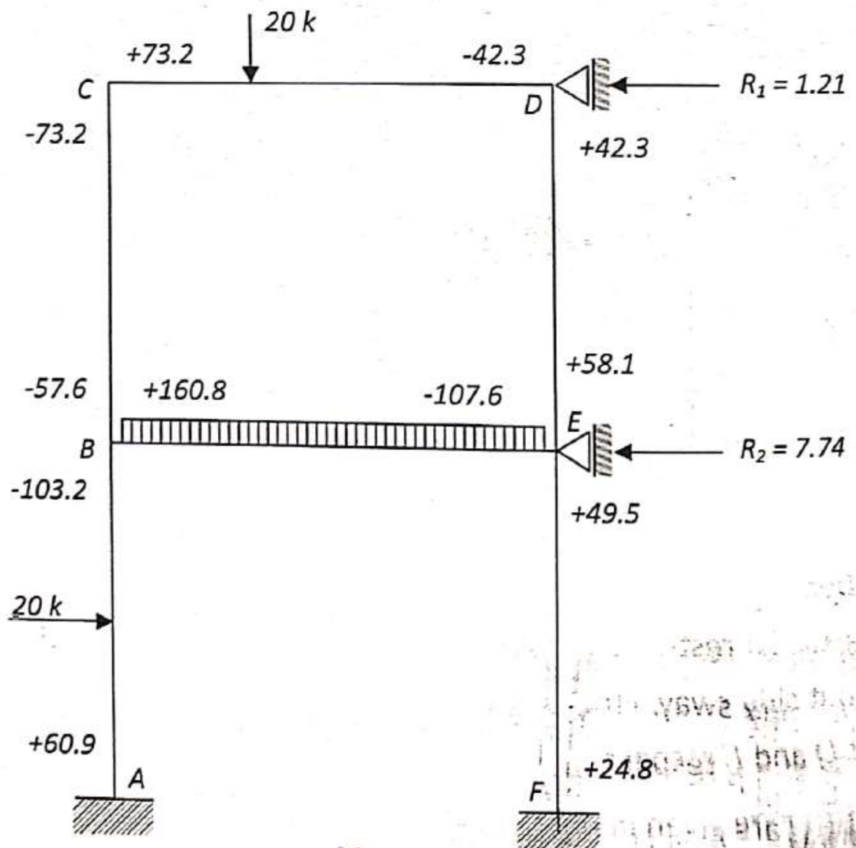
Solution

Put artificial restrains at joint D and E and find out the end moments M_1 without side sway. From the free body calculate the holding forces R_1 and R_2 at D and E respectively.

The FEM are given in the following figure,

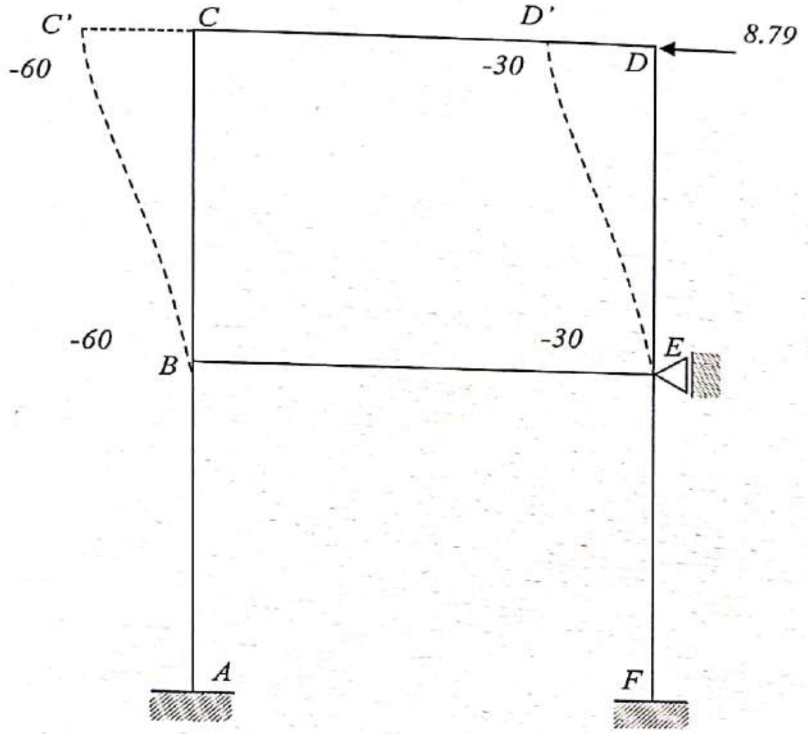

20 k
10 k
C
D
+88.90
-44.5
+150
-150
E
B
-75
20 k
+75
A
F

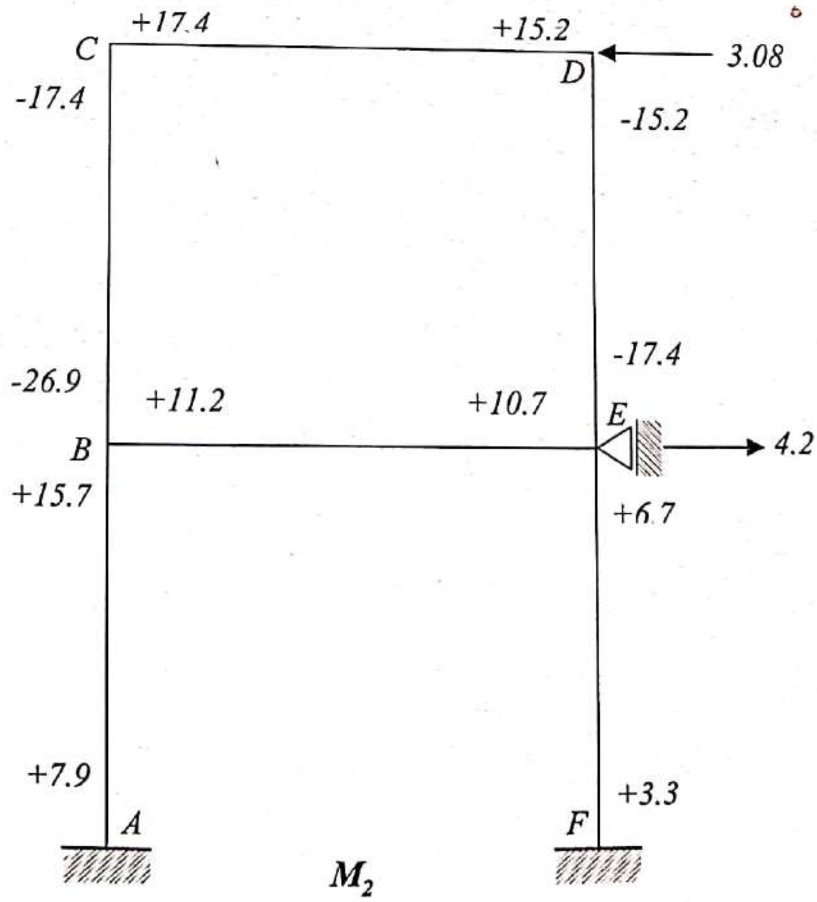
FEM


20 k
+73.2
-42.3
C
D
$R_1 = 1.21$
-73.2
+42.3
+58.1
-57.6
+160.8
-107.6
E
B
$R_2 = 7.74$
-103.2
+49.5
20 k
+60.9
A
+24.8
F

M_1

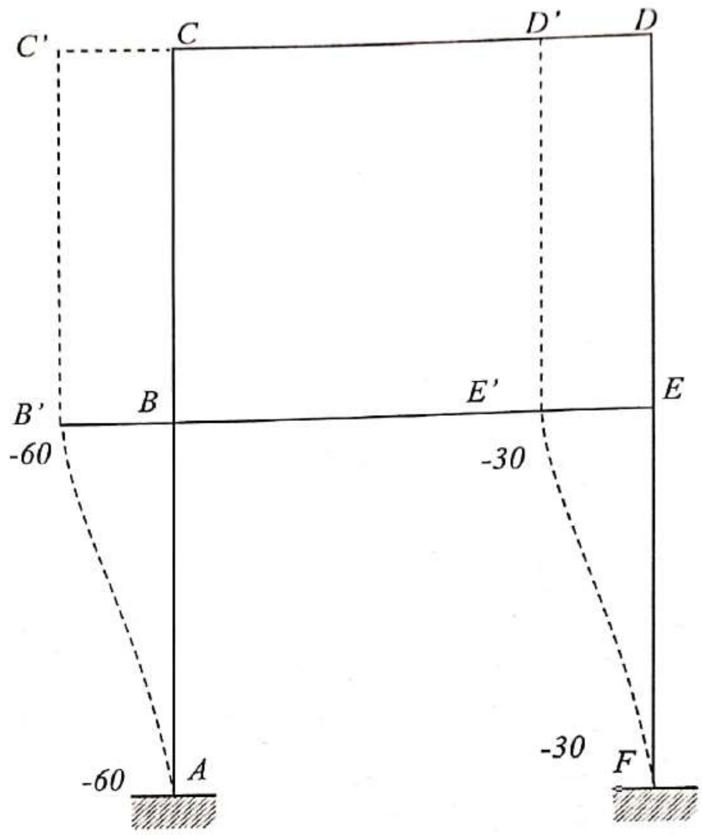
Now, release the artificial holding at joint D while the holding at E will be at place and apply the $R_1 = 1.21$ in opposite direction. The net force at D will be $(10\text{-}1.21) = 8.79\ k$. Find out the end moments M_2.

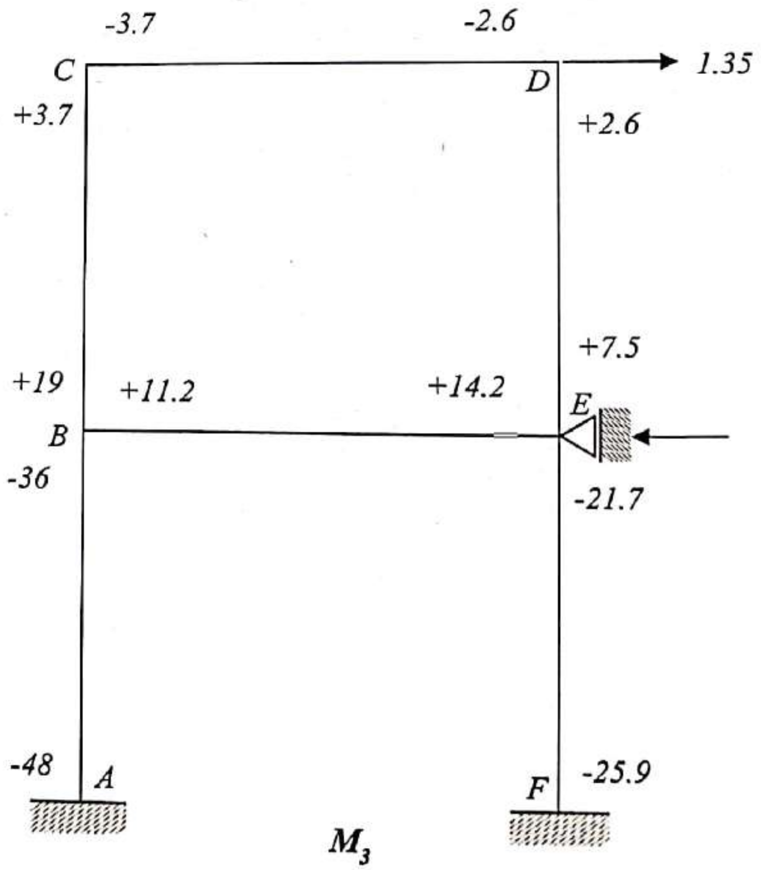




M_2

Next, release the holding at joint E and find out the end moments M_3.





M_3

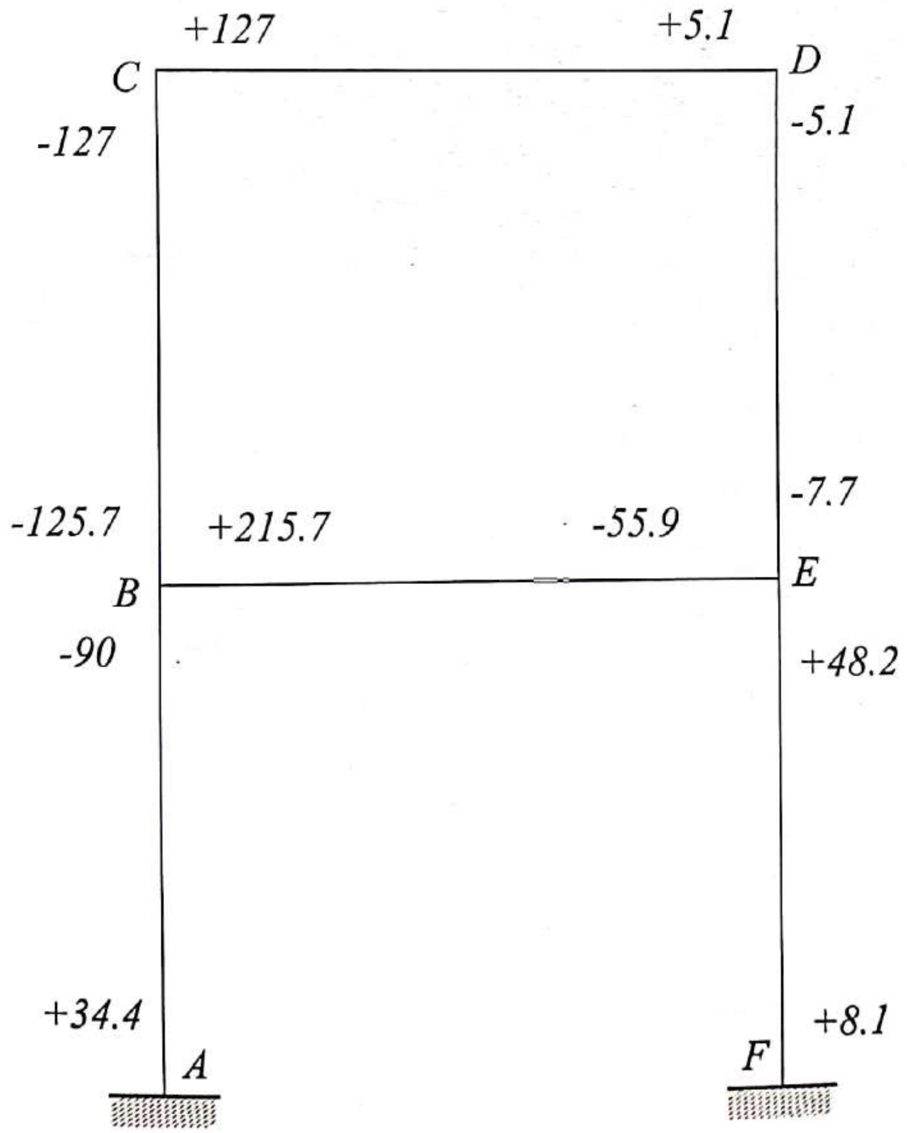
Since the moment after release of holdings at joint D and E were arbitrary moments induced for the arbitrary side sway at D and E, a multiplying factor X and Y have to be equated to find out the exact moment as follows:

$$1.21 + 3.08X - 1.35Y = 10$$
$$7.74 - 4.20X + 5.74Y = 0$$

Solving these two simultaneous equation will result, $X = +3.34$ *and* $Y = +1.1$. The final moment diagram will be as follows.

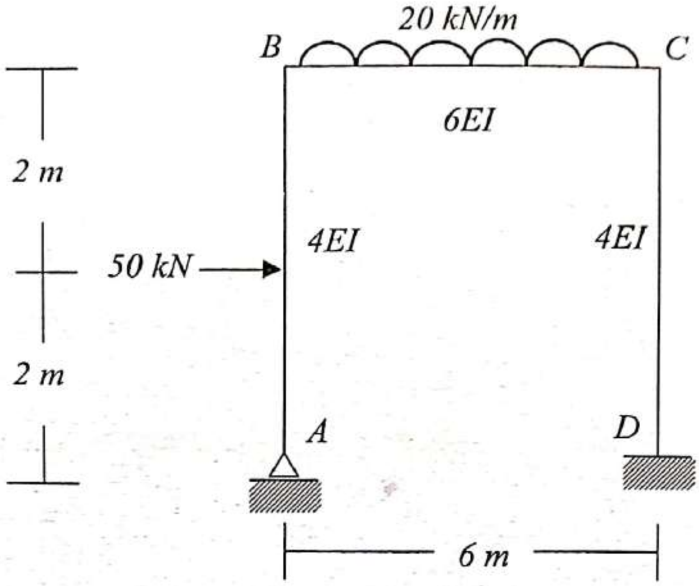
$$M = M_1 + M_2X + M_3Y$$

For example, the moment at joint A, = + *60.9* –+*7.9*×*3.34* -*48*×*1.1* = +*34.4 k-ft*



Exercise 1

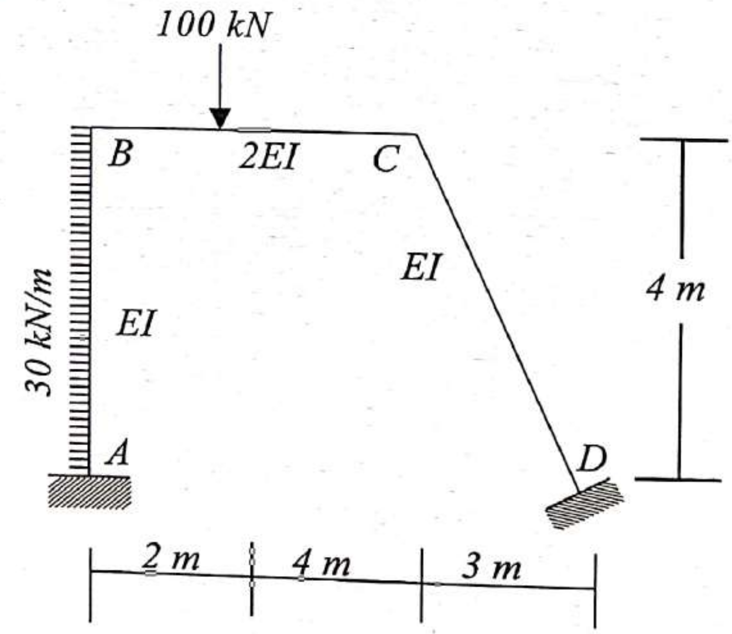
Analyze the following frame by moment distribution method, and find all unknown support moment.



Ans. $M_B = 31.87, M_C = 67.93, M_D = 63.87\ kN\text{-}m$

Exercise 2

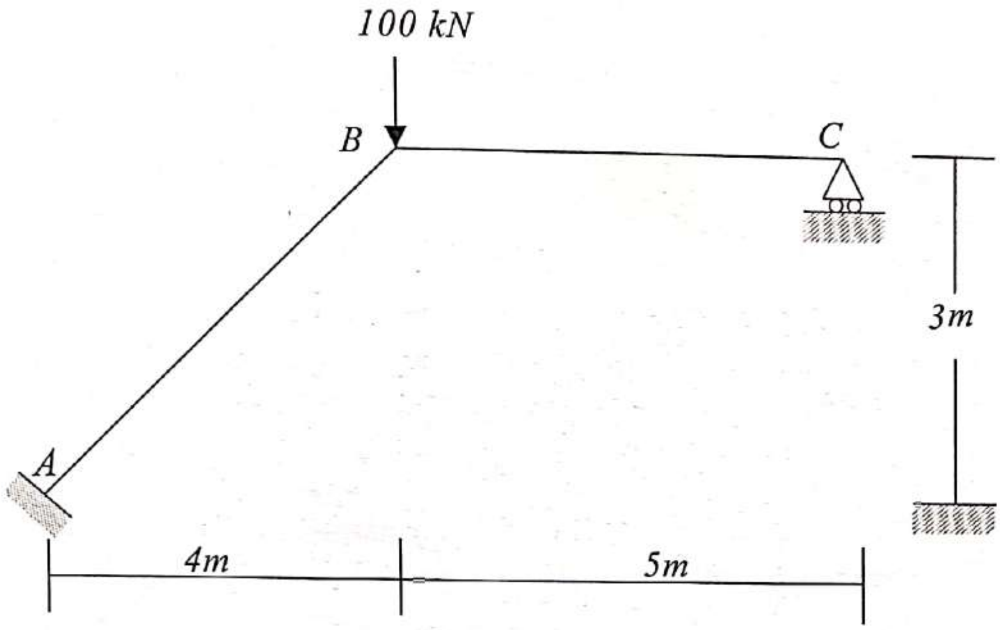
Analyze the following frame by moment distribution method, and find all unknown support moment.



Ans. $M_A = +63.2\ M_B = 34.35,\ M_C = 56.03,\ M_D = +44.32 kN\text{-}m$

Exercise 3

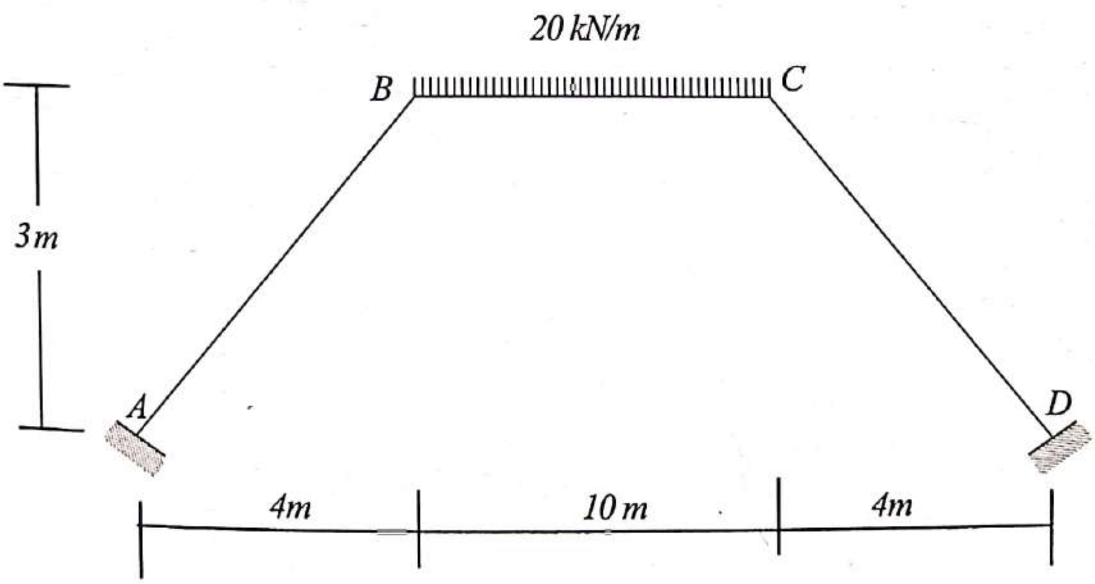
Analyze the following frame by moment distribution method, and find all unknown support moment.



Ans. $M_A = +164.73$ $M_B = 130.68$ $kN\text{-}m$

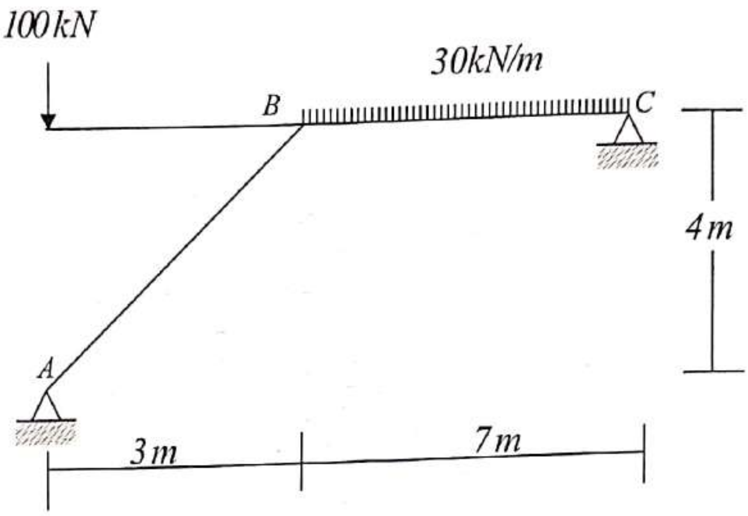
Exercise 4

Analyze the following frame by moment distribution method, and find all unknown support moment.



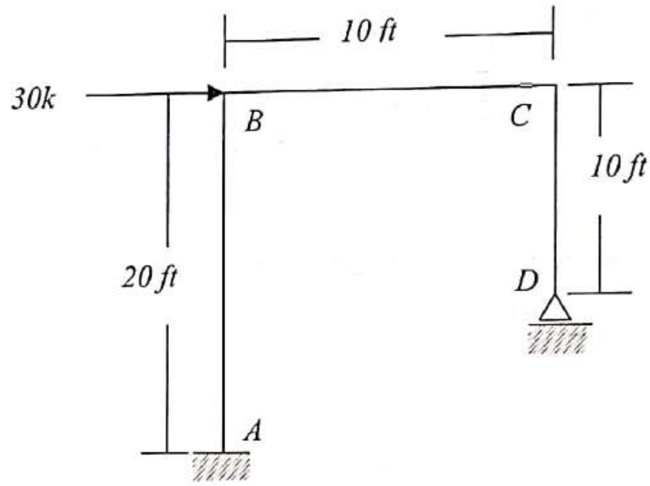
Exercise 5

Analyze the following frame by moment distribution method, and find all unknown support moment.

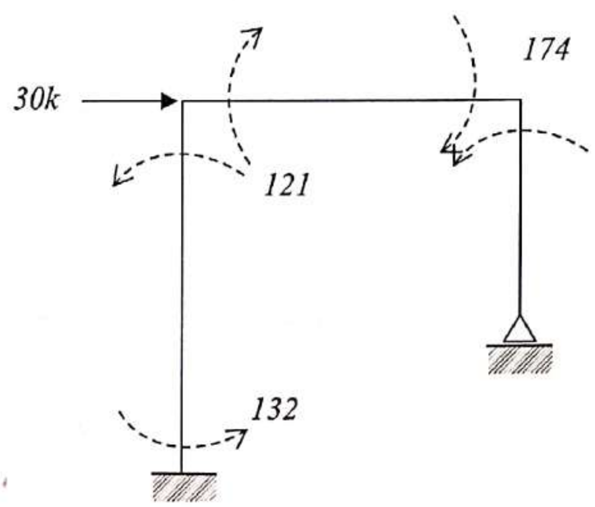


Exercise 6

Analyze the following frame by moment distribution method, and find all unknown support moment.



Ans:



CHAPTER 4

INFLUENCE LINE OF STATICALLY INDETERMINATE STRUCTURES

What is influence line?

An influence line is a graphical representation of how the movement of a unit load across a structure influence a force effect (Reaction, Shear, Bending Moment, Axial load, deflection) at one point of the structure.

The main purpose of influence lines for both statically determine and indeterminate structures is to determine where to position moving loads to cause the maximum effects of a design function. For example, an influence line can be used to forecast where the design live load should be placed on a continuous beam of a building floor system to cause maximum positive bending moment; another influence line can be made for a bridge truss member to determine where to place the live loads that will result in maximum member force.

Maxwell's law of Reciprocal Deflections

The law state that, $\Delta_{12} = \Delta_{21}$

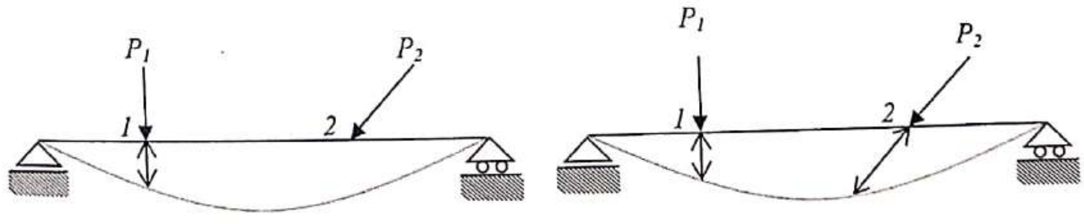
Where, Δ_{12} = the deflection at point *1* due to load P at point *2*

Δ_{21} = the deflection at point *2* along the original line of action of load P at point *2* due to load P at point *1* which is applied along the original deflection at point *1*

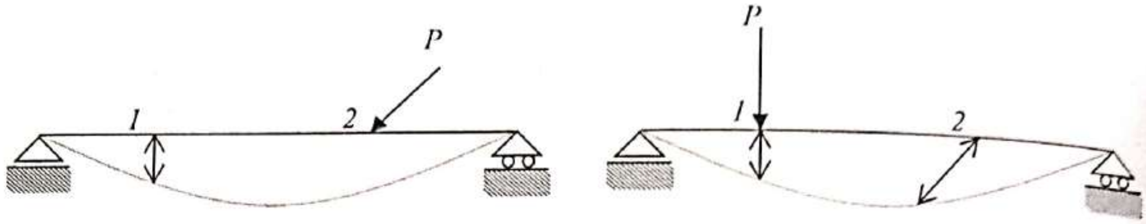
This come from the derivation of Betti's law. The law state that,

"The virtual work done by a force system P_1 through the deflection caused by force system P_2 is equal to the virtual work done by a force system P_2 through the deflection caused by force system P_1"

$$\sum P_1\delta_{12} = \sum P_2\delta_{21}$$



If $P_1 = P_2 = P$, then it proves the Maxwell's law of reciprocal deflection.



$$\Delta_{12} = \int_0^l \frac{M_2 m_1}{EI} dx$$

$$\Delta_{21} = \int_0^l \frac{M_1 m_2}{EI} dx$$

but $M_1 = Pm_1$ and $M_2 = Pm_2$

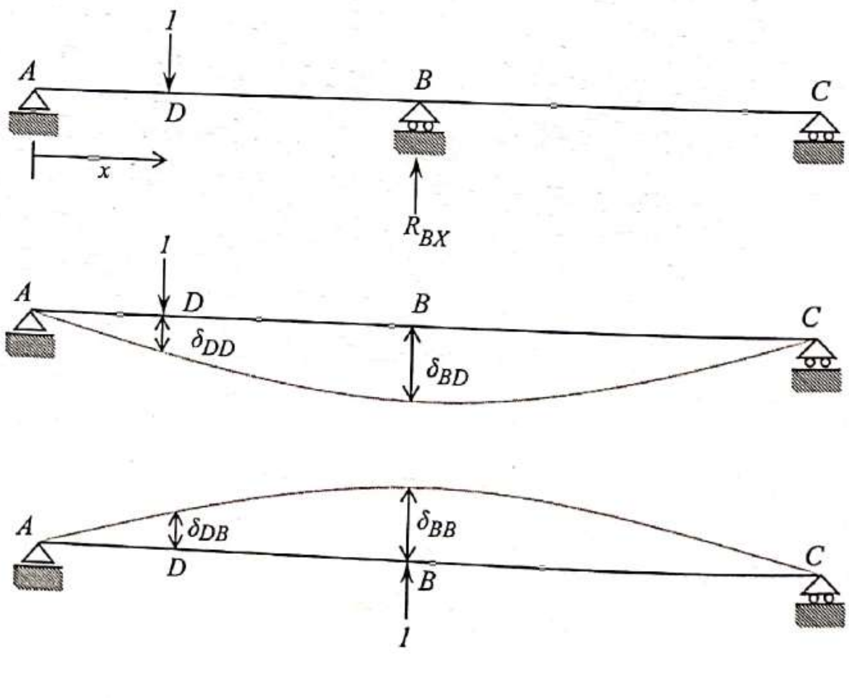
$$\text{Therefore, } \Delta_{12} = \int_0^l \frac{M_2 m_1}{EI} dx = \int_0^l \frac{Pm_2 m_1}{EI} dx = \int_0^l \frac{(Pm_1)m_2}{EI} dx = \int_0^l \frac{M_1 m_2}{EI} dx = \Delta_{21}$$

Special case when, $P = 1$

$$\delta_{12} = \delta_{21}$$

Müller-Breslau principle

"The ordinates of the influence line for any action (Reaction, Shear, Bending Moment, Axial load) of any structure are proportional to those of the deflection curve that is obtained by removing the restraint corresponding to that action from the structure and introducing in its place a corresponding deformation into primary structure that remains"



From the principle of consistent deformation and deformation superposition,

$$\delta_{BB} R_{BX} - \delta_{BD} = 0$$

$$\therefore \ R_{BX} = \frac{\delta_{BD}}{\delta_{BB}}$$

From the Maxwell law of reciprocal deflection, $\delta_{BD} = \delta_{DB}$

$$\therefore \ R_{BX} = \frac{\delta_{DB}}{\delta_{BB}}$$

Since, point D locates at a distance x from the origin,

δ_{DB} can be written as δ_{xB} or δ_{Bx}

$$\therefore \ R_{BX} = \frac{\delta_{Bx}}{\delta_{BB}}$$

Now find the δ_{xB} and δ_{BB}

Then all the ordinates of R_{BX} can easily be calculated from the above equation.

To get the influence line for moment, put an imaginary hinge at the point where the IL should be evaluated and then the above equation will be,

$$\therefore \ M_{Bx} = \frac{\theta_{Bx}}{\theta_{BB}} = \frac{\text{rotation at hinge B due to unit load at x}}{\text{rotation at B due to unit couple at B}}$$

From the law of reciprocal deflection,

rotation at hinge B due to unit load at x = deflection at x due to unit couple at A,

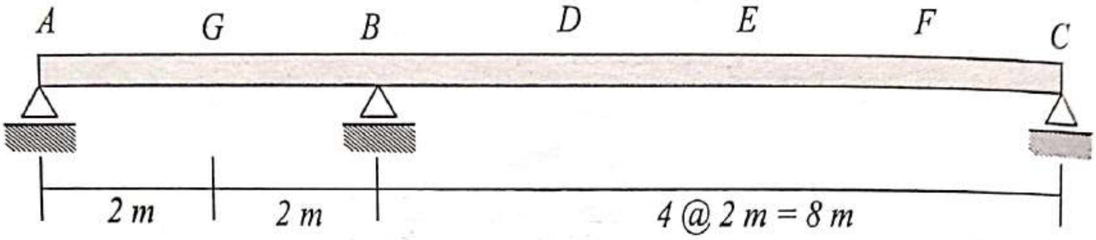
$$\therefore \ M_{Bx} = \frac{\delta_{xB}}{\theta_{BB}} = \frac{\text{deflection at x due to unit couple at B}}{\text{rotation at B due to unit couple at B}} = \frac{\delta_{Bx}}{\theta_{BB}}$$

Worked out Examples

Example 1

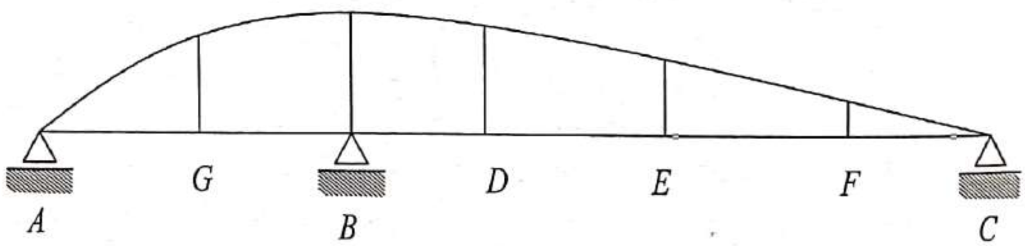
Draw the influence line for the vertical reaction at B- shear at E and bending moment at G.

Plot the ordinate at every $2\ \mathrm{m}$ interval. EI is constant



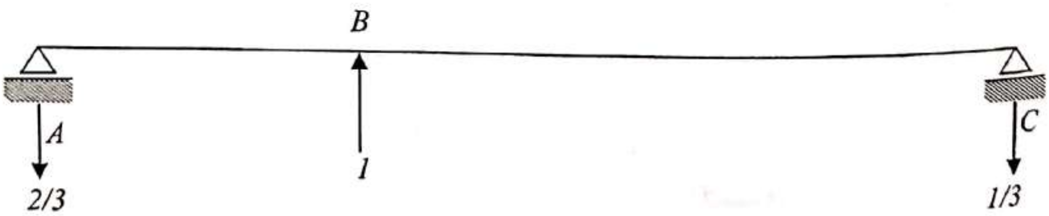
Solution

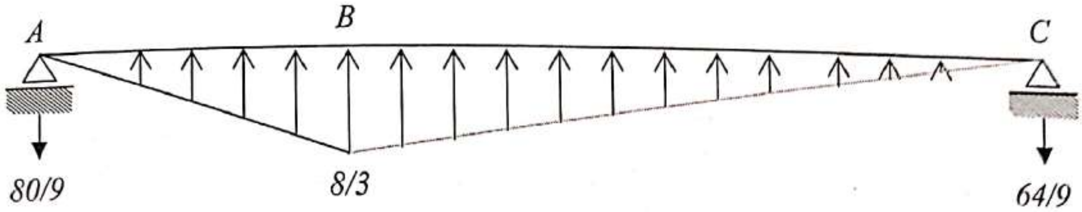
Draw the Influence line of R_B by Muller Breslau Principle. Pull the beam upward at support B by unity. The deflected shape will be the qualitative shape of the influence line for R_B.



To get the ordinate at G, D, E and F calculate the δ_{BB} and corresponding deflection at those points δ_{xB}. The ratio of δ_{xB}/δ_{BB} will give the ordinate at the respective points.

Find δ_{BB} and δ_{xB} by conjugate beam method.





Elastic load, $L_e = M/EI$

Assume, $EI = 1$

So, $L_e = M$

From A-B

$$L_e = \tfrac{2}{3}x$$

$$V_e = \int L_e dx = \int \tfrac{2}{3}x dx = \frac{2x^2}{6} + C = \frac{2x^2}{6} - \frac{80}{9}$$

$$M_e = \int V_e dx = \int \left(\frac{2x^2}{6} - \frac{80}{9} \right) dx = \frac{2x^3}{18} - \frac{80}{9}x + \cancel{C}^0$$

$$at\ x = 4m \Rightarrow \ \delta_{BB} = -\frac{256}{9}$$

$$at\ x = 2m \Rightarrow \ \delta_{GB} = -\frac{152}{9}$$

From C-B

$$L_e = \tfrac{1}{3}x$$

$$V_e = \int L_e dx = \int \tfrac{1}{3}x dx = \frac{x^2}{6} + C = \frac{x^2}{6} - \frac{64}{9}$$

$$M_e = \int V_e dx = \int \left(\frac{x^2}{6} - \frac{64}{9} \right) dx = \frac{x^3}{18} - \frac{64}{9}x + \cancel{C}^0$$

$$at\ x = 8m \Rightarrow \ \delta_{BB} = -\frac{256}{9}$$

$$at\ x = 2m \Rightarrow \ \delta_{FB} = -\frac{124}{9}$$

$$at\ x = 4m \Rightarrow \ \delta_{EB} = -\frac{224}{9}$$

$$at\ x = 6m \Rightarrow \ \delta_{DB} = -\frac{276}{9}$$

Point	x (m)	δ_{xB}	δ_{xB}/δ_{BB}	R_B	R_A
A	0	0	0	0	1
G	2	-152/9	152/256	0.594	0.437
B	4	-256/9	256/256	1	0
D	6	-276/9	276/256	1.078	-0.218
E	8	-224/9	224/256	0.875	-0.25
F	10	-124/9	124/256	0.484	-0.156
C	12	0	0	0	0

In the above table, $R_A = 1 - \frac{x}{12} - \frac{8R_B}{12}$

$R_C = 1 - R_A - R_B$

Check, at any point $R_A + R_B + R_C = 1$

Shear at E

1 unit load is on left of E,

$V_E = -R_C$

1 unit load is on right of E,

$V_E = 1 - R_C$

Check, Shear at left of E+ Shear at right of E = 1

0.625+0.375 = 1

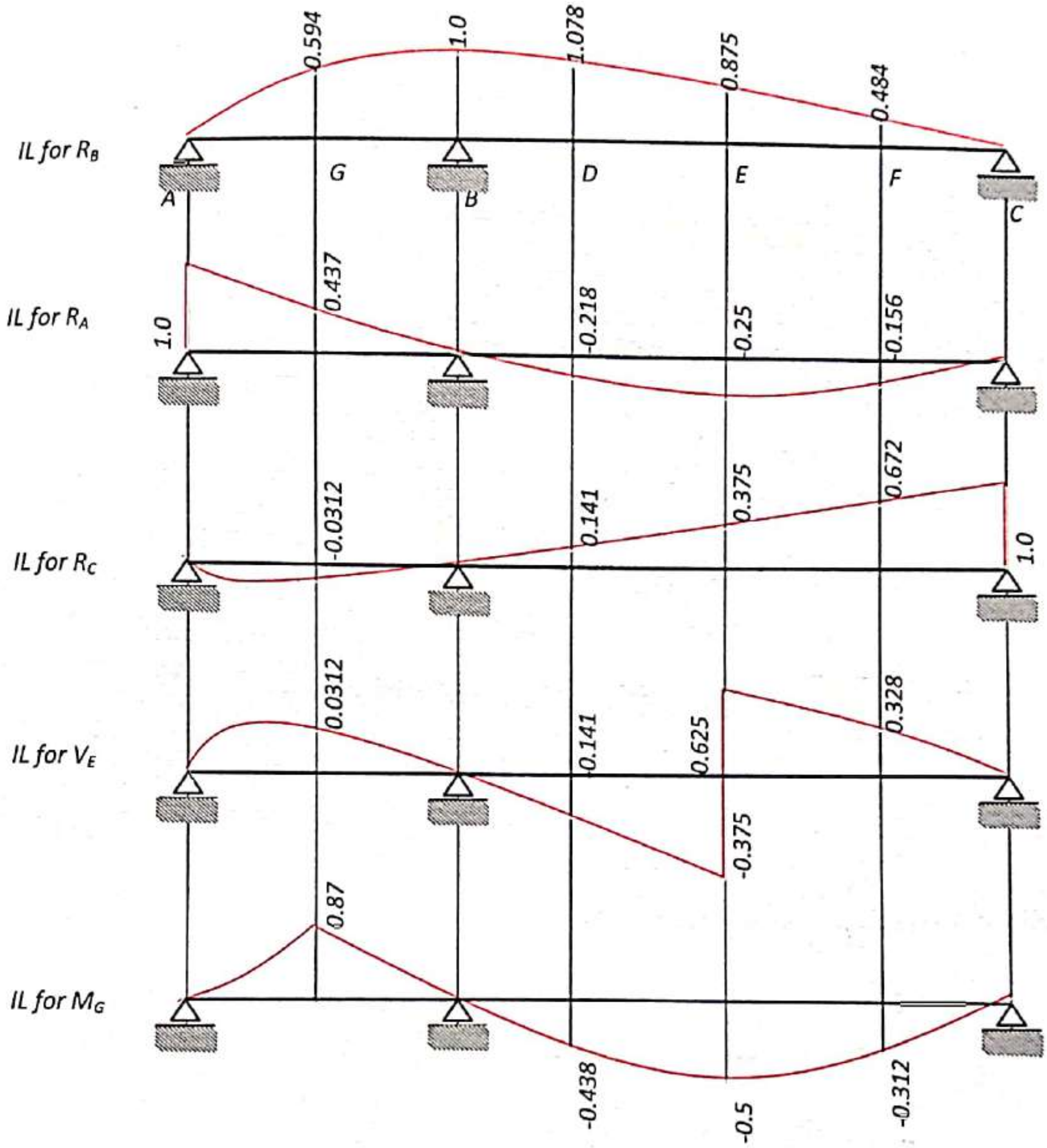
Moment at G

1 unit load is on left of G,

$M_G = 2R_A + (x-2)\times 1$

1 unit load is on right of G,

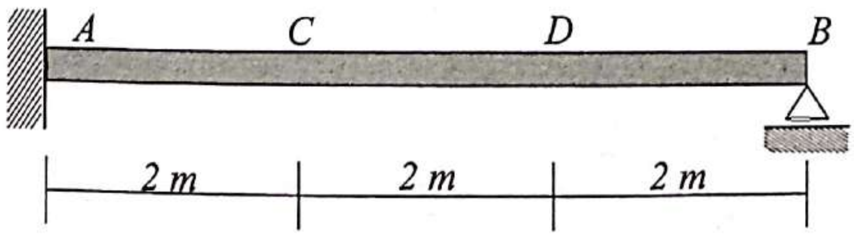
$M_G = 2R_A$



Example 2

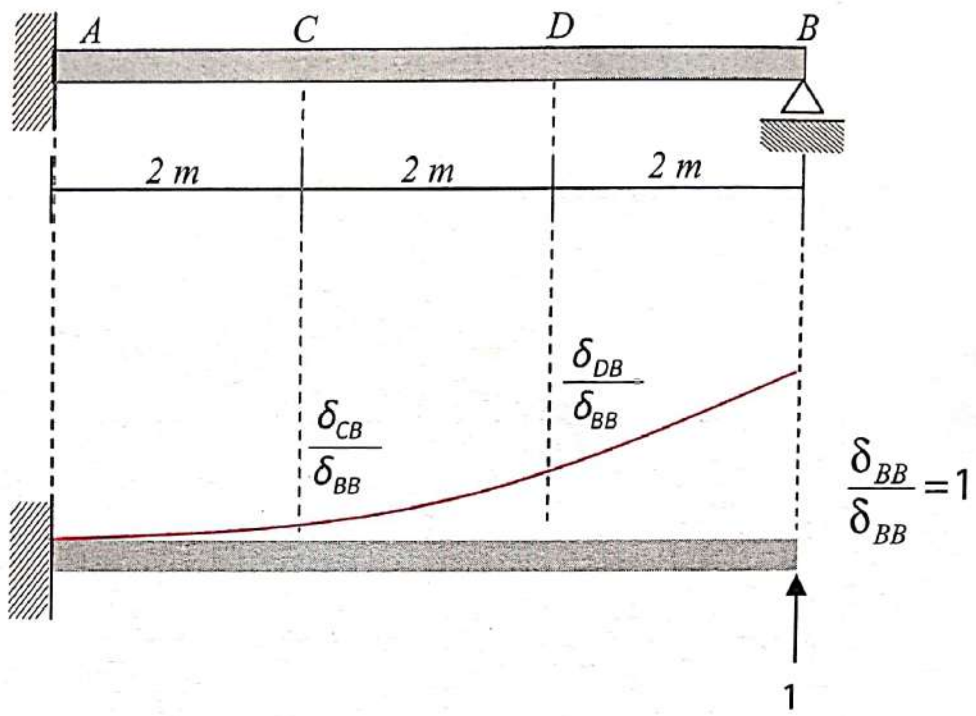
Draw the influence line for, a) the vertical reaction at A and B, b) shear at C, c) bending moment at A and C

Assume, EI is constant . Plot numerical values in every 2 m.

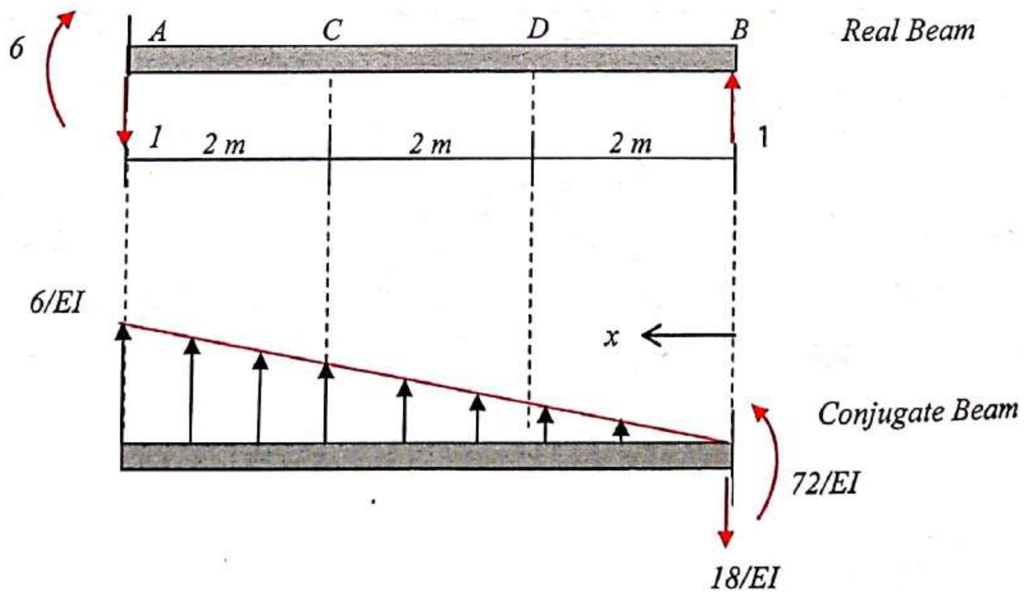


Solution

Influence line of R_B



Now, find $\delta_x B$, by conjugate beam,



Elastic load, $L_e = M/EI = x/EI$

Assume, $EI = 1$

So, $L_e = x$

$L_e = x$

$$V_e = \int L_e dx = \int x dx = \frac{x^2}{2} + C = \frac{x^2}{2} - 18$$

$$M_e = \int V_e dx = \int \left(\frac{x^2}{2} - 18 \right) dx = \frac{x^3}{6} - 18x + C = \frac{x^3}{6} - 18x + 72$$

$at\ x = 0m \Rightarrow\ \delta_{BB} = +72$

$at\ x = 2m \Rightarrow\ \delta_{DB} = +\frac{112}{3}$

$at\ x = 4m \Rightarrow\ \delta_{CB} = +\frac{32}{3}$

$at\ x = 6m \Rightarrow\ \delta_{AB} = 0$

Point	x (m)	δ_{xB}	δ_{xB}/δ_{BB}	R_B	R_A	V_C	M_A	M_C
B	0	72	1	1	0	0	0	0
D	2	112/3	0.518	0.518	0.482	0.482	-0.892	0.072
C	4	32/3	0.148	0.148	0.852	(-0.148) 0.852	-1.112	0.592
A	6	0	0	0	1	0	0	0

In the above table,

Shear at *C*

1 unit load is on left of C ,

$V_C = -R_B$

1 unit load is on right of C ,

$V_C = 1 - R_B$

Check, Shear at left of *C*+ Shear at right of *C* = 1

0.148+0.852 = 1

Moment at A

1 unit load is on beam,

$M_A = x - 6 + 6R_B$

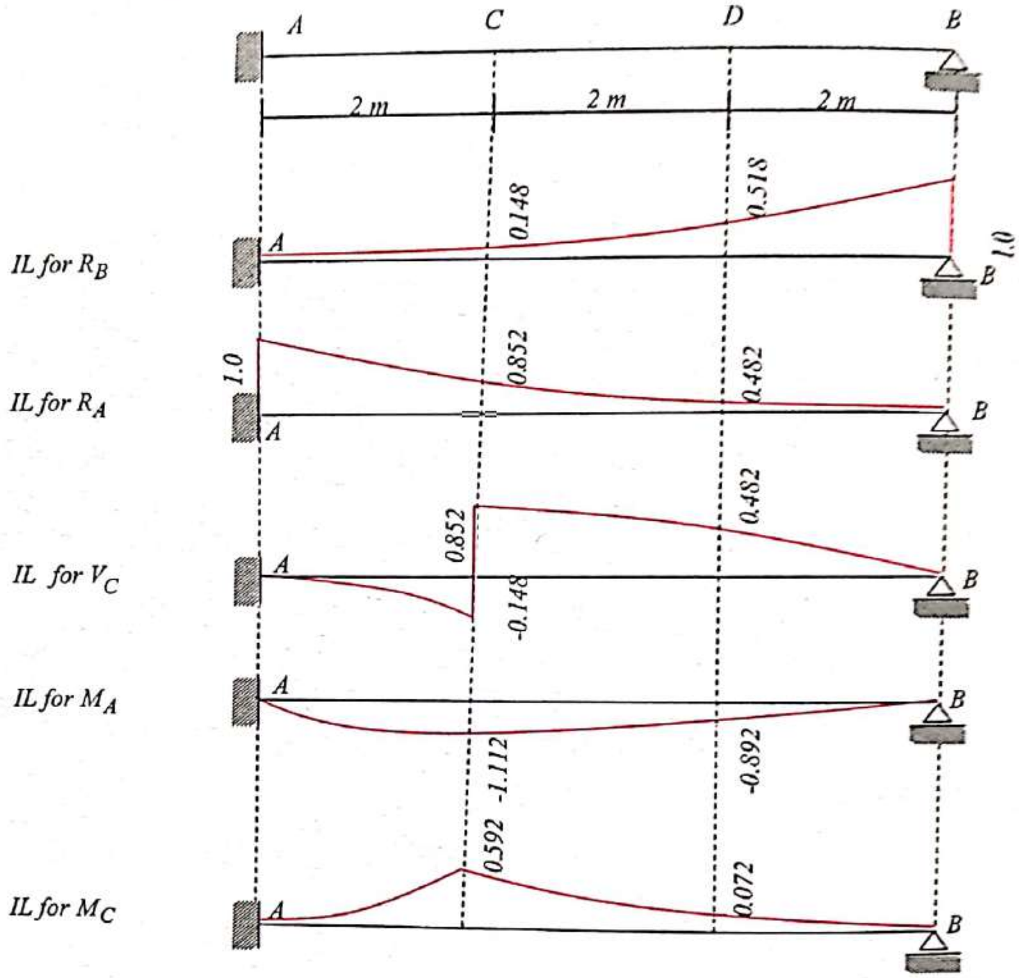
Moment at C

1 unit load is on left of C,

$M_C = 4R_B$

1 unit load is on right of C,

$M_C = x - 4 + 4R_B$

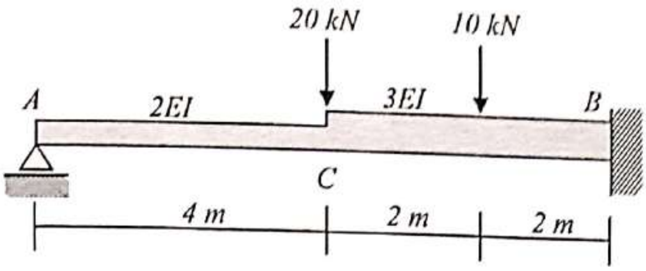


Example 3

For the beam shown below,

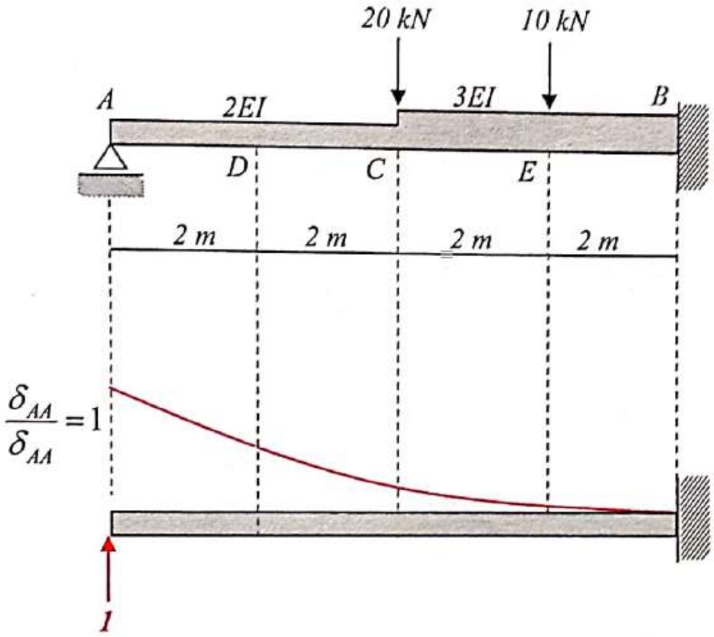
(a) Draw quantitative influence lines for the reaction at supports A and B, and bending moment at B. Shear at B and C and bending moment at C.

(b) Determine all the reactions at supports, and also draw its quantitative shear, bending moment diagrams, and qualitative deflected curve for,
- Only 10 kN downward at 6 m from A
- Both 10 kN downward at 6 m from A and 20 kN downward at 4 m from A.

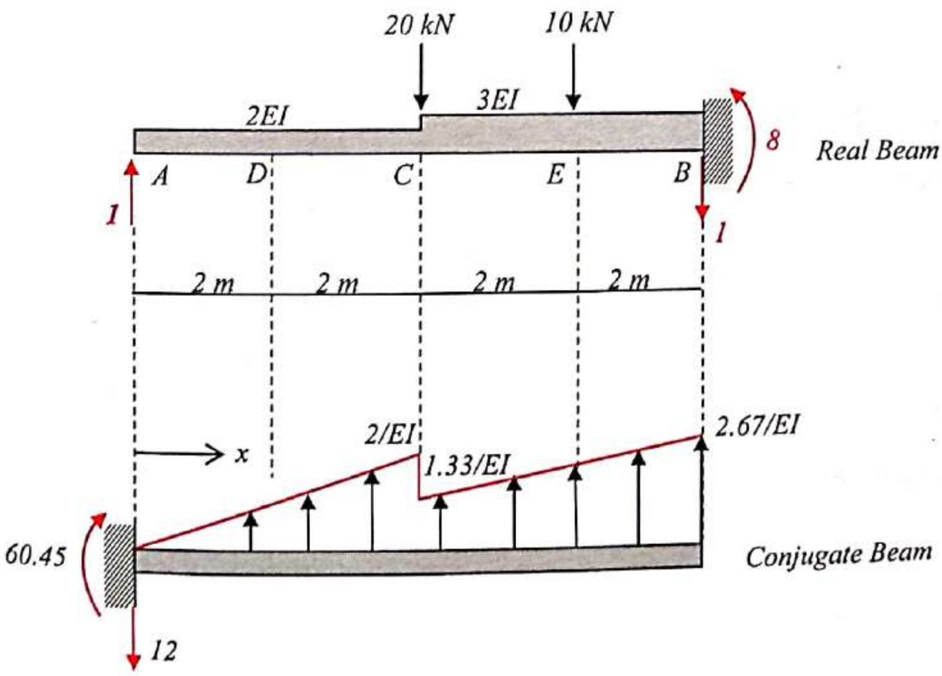


Solution

Influence line of R_A



Find fx_A by conjugate beam



For $x=0 \rightarrow 4, L_e=\frac{x}{2}$ $\quad$ [Assume $EI=1$]

$$V_e=\int L_e dx=\int \frac{x}{2} dx=\frac{x^2}{4}+C=\frac{x^2}{4}-12$$

$$M_e=\int V_e\,dx=\int\left(\frac{x^2}{4}-12\right) dx=\frac{x^3}{12}-12x+C=\frac{x^3}{12}-12x+60.45$$

$at\ x=0m \Rightarrow\ \delta_{AA}=+60.45$

$at\ x=2m \Rightarrow\ \delta_{DA}=+37.11$

$at\ x=4m \Rightarrow\ \delta_{CA}=+17.78$

For $x=4 \rightarrow 8, L_e=\frac{4}{3}+\frac{x}{3}$ $\quad$ [Assume $EI=1$ and origin of x at C]

$$V_e=\int L_e dx=\int\left(\frac{4}{3}+\frac{x}{3}\right) dx=\frac{4x}{3}+\frac{x^2}{6}+C=\frac{4x}{3}+\frac{x^2}{6}-8$$

$$M_e=\int V_e\,dx=\int\left(\frac{4x}{3}+\frac{x^2}{6}-8\right) dx=\frac{4x^2}{6}+\frac{x^3}{18}-8x+C=\frac{4x^2}{6}+\frac{x^3}{18}-8x+17.78$$

$at\ x=2m\ (6m$ from A$) \Rightarrow\ \delta_{EA}=+4.89$

$at\ x=4m\ (8m$ from A$) \Rightarrow\ \delta_{BA}=+0.00$

Point	*x (m)*	δ_{xA}	δ_{xA}/δ_{AA}	R_A	R_B	M_B	V_B	V_C	M_C
A	0	60.45	1	1	0	0	0	0	0
D	2	37.11	0.614	0.614	0.386	-1.088	-0.386	-0.386	0.456
C	4	17.78	0.294	0.294	0.706	-1.648	-0.706	-0.706 0.294	1.176
E	6	4.89	0.081	0.081	0.919	-1.352	-0.919	0.081	0.324
B	8	0	0	0	1	0	-1	0	0

In the above table, $R_B=1-R_A$

Moment at B

1 unit load is on beam,

$M_B = x - 8 + 8R_A$

Shear at B

$V_B = -R_B$

Shear at C

1 unit load is on left of C,

$V_C = -R_B$

1 unit load is on right of C,

$V_C = 1-R_B$

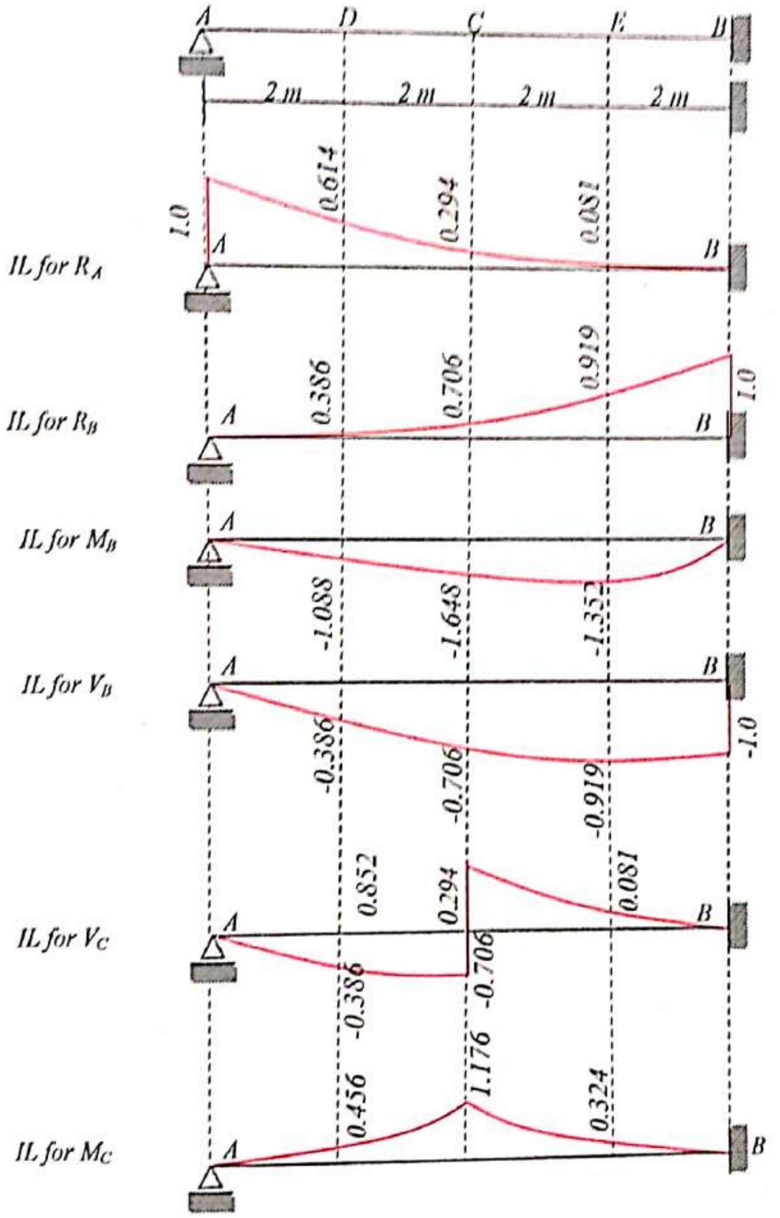
Moment at C

1 unit load is on left of C,

$M_C = x - 4 + 4R_A$

1 unit load is on right of C,

$M_C = 4R_A$



For $10\ kN$ load at $6m$ from A,

From the Influence Line diagram (or from the tabular value) above,

$R_A = 0.081 \times 10 = 0.81\ kN$

$R_B = 10 - R_A = 9.91\ kN$

For 10 kN load at $6m$ from A and $20\ kN$ load at $4m$ from A,

$R_A = 0.081\times10 + 0.294\times20 = 6.7\ kN$

$R_B = 30\text{-}R_A = 23.3\ kN$

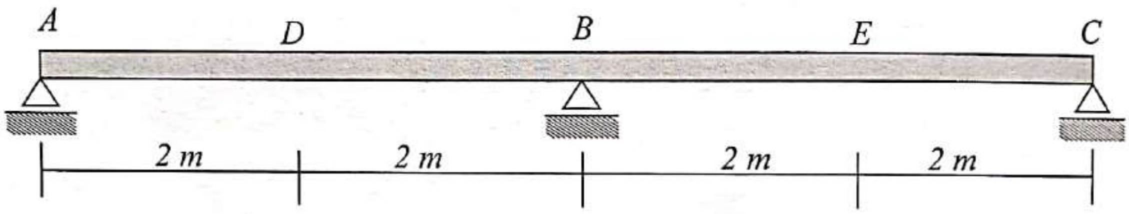
Find all other quantities by yourself from the IL and draw the shear force and bending moment diagram.

Exercise 1

For the beam shown below,

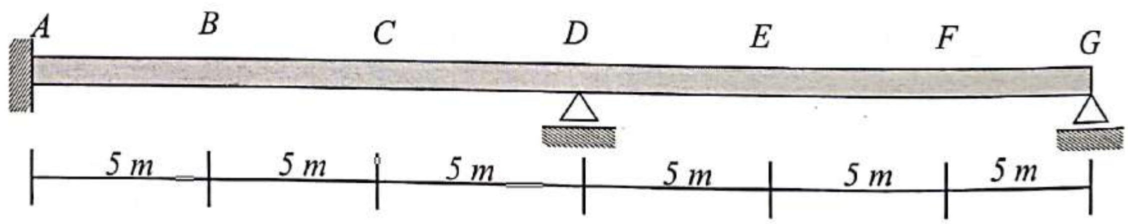
(a) Draw the influence line for the shear at D for the beam

(b) Draw the influence line for the bending moment at D for the beam EI is constant. Plot numerical values every $2\ m$.



Exercise 2

Draw the influence line for the reactions at supports for the beam shown in the figure below. EI is constant. Plot numerical values every *5 m*.



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Dr. Sohrabuddin Ahmed, Professor, BUET

2) **Indeterminate Structural Analysis –** C.K Wang

3) **Indeterminate Structurai Analysis –** J. S. Kinney

4) **Theory of Simple Structures-** T. C Shedd & J. Vawter

5) **Analysis of Statically Indeterminate Structures by the Direct Stiffness Method,** Version 2 CE IIT, Kharagpur

6) **Technical Guidance Note, The Structural Engineers,** Sep. 2012.

7) **Structural Analysis III, Moment distribution,** DSr. Colin Caprani, 2008.

8) **Mathematical Model of Influence Lines for Indeterminate Beams,** Dr. Moujalli Hourani, Dept. of CE, Manhattan College, Riverdale, NY 10471

Most Useful conversion

Length 1 in = 25.4 mm 1 m = 3.28 ft 1 mile = 1.61 km	**Pressure** 1 kN/m^2 = 20.885 psf 1 MPa = 1 N/mm^2 = 145 psi 1MPa = 10.197 kg/cm^2 1 tsf = 95.76 kN/m^2 (kPa) 1 bar = 14.5 psi
Area 1 m^2 = 10.76 sft 1 acre = 4840 sq yard 1 hactare = 10000 m^2 1 Dec = 435.6 sft 1 Katha = 720 sft	**Force** 1 kN = 224.8 lb 1 ton = 2000 lb = 2 kip
Mass 1 kg = 2.2 lb 1 lb = 453 gm 1 ounce = 28.35 gm	**Moment** 1 kN.m = 0.737 kip-ft
Volume 1 US gallon = 3.78 liter 1 m^3 = 100 liter 1 cft = 28.32 liter 1 m^3 = 35.31 cft	**Unit Weight** Steel 488 lb/ft^3 = 78 kN/m^3 =7850 kg/m^3 Concrete 150 lb/ft^3 = 24 kN/m^3 =2400 kg/m^3