

Live MCQ™

৪৯তম স্পেশাল বিসিএস (শিক্ষা) বিষয়ভিত্তিক প্রস্তুতি

বিষয়: ফলিত গণিত (৫৬১)

Compendium PDF

(সিলেবাস অনুসারে সর্বাধিক গুরুত্বপূর্ণ টপিক ও এমসিকিউ-এর সমন্বয়ে রচিত)

Mentor:

মোঃ আবু তারেক

প্রভাষক (গণিত)

সাপাহার সরকারি কলেজ, নওগাঁ।

৪৩তম বিসিএস (সাধারণ শিক্ষা)

প্রিয় শিক্ষার্থী, ৪৯তম স্পেশাল বিসিএস (শিক্ষা)-এর বিষয়ভিত্তিক প্রস্তুতি হিসাববিজ্ঞান অংশের নির্ধারিত সিলেবাসের অন্তর্ভুক্ত সকল ক্লাস সম্পন্ন হয়েছে। আসন্ন চূড়ান্ত পরীক্ষাকে সামনে রেখে লাইভ এমসিকিউ একাডেমিক টিম আপনাদের প্রস্তুতিকে শাণিত করতে বিশেষ কমপেনডিয়াম পিডিএফ প্রদান করছে। এখানে যা যা থাকছেঃ

Contents

পিএসসি কর্তক নির্ধারিত সিলেবাস.....	3
রেফারেন্স.....	6
পরামর্শ.....	6
টপিক ভিত্তিক সংক্ষিপ্ত আলোচনা	7
200 Important Questions	64

পিএসসি কর্তক নির্ধারিত সিলেবাস

APPLIED MATHEMATICS (POST RELATED)

Subject Code: 561

Total Marks-100

Part-I

Marks: 50

Group-A: Statics

- (a) Forces acting in a plane, parallel forces, moments and couples.
- (b) Equilibrium of coplanar forces. Astatic, stable and unstable equilibrium.
- (c) Work, Virtual work.
- (d) Centre of gravity. Forces in three dimensions.

Group-B: Dynamics

- (a) Motion in a straight line, Simple harmonic motion.
- (b) Motion in a plane referred to Cartesian and polar coordinates. Radial and transverse velocities; central, tangential and normal acceleration. Central forces.
- (c) Motion in resisting medium.
- (d) Motion in three dimensions.

Group-C: Mathematical Methods

- (a) The Laplace Transform: Definition, existence and basic properties Differentiation and integration, Inverse Laplace transform and convolution, Solution of linear differential equations with constant coefficients.
- (b) Bessel's Equation: Solution, Generating function, Recurrence relation, Orthogonality.
- (c) Legendre's Equation: Solution, Generating function, Recurrence relation, Rodrigui's formula and orthogonality of Legendre Polynomial.
- (d) Fortier Series: Fortier Coefficients, Sine and Cosine series, Dirichlet's theorem, Properties and applications.

Part-II

Marks: 50

Group-A: Real Analysis

- (a) Metric Spaces: Definition and examples. Open and closed sets, Compact sets, perfect set and cantor set.
- (b) Sequence: Convergent sequence, bounded' sequence, subsequence, Cauchy sequence, and completeness of IR.
- (c) Differentiation: Continuous function. Derivative of a function. Rolle's Theorem, Meanvalue theorem, T aylor's theorem.

- (d) Functions of Several Variables: Limit and Continuity. Partial differentiation. Schwarz's theorem, Young's theorem.

Group-B: Complex Analysis

- (a) Complex Functions: Single and many valued functions. Limit, Continuity and differentiability of complex function.
- (b) Analytic Functions: Necessary and sufficient conditions. Harmonic functions. Mobius transformation and power series.
- (c) Complex Integration: Zeros of analytic functions, Cauchy's theorem, Morera's theorem, Cauchy's integral formula, Singularities, Classification of singularities.
- (d) Complex integration: The open mapping theorem, Taylor's and Laurent series. Fundamental theorem of algebra, Rouches theorem, The residue theorem, Contour integration.

Group-C: Numerical Analysis (a) Solution of algebraic and transcendental equation, Interpolation.

- (a) Numerical solution of linear and non-linear system of equations.
- (b) Numerical differentiation and integration.
- (c) Numerical solutions of ordinary differential equation.

Group-D: Hydrodynamics

- (a) Velocity and acceleration of fluid particles, Steady and unsteady flows, Uniform and non-uniform flows, Stream lines, path lines, vortex lines and velocity potential. Rotational and irrotational flows. Equation of continuity.
- (b) Euler's equation of motion, conservative field force. Lamb's equations of motion. Bernoulli's equation.
- (c) Motion in two-dimensions, stream function and its physical meaning, velocity in polar coordinates, relation between stream function and velocity.
- (d) Sources, sinks and doublets. Complex potential and complex velocity, stagnation points, Complex potential due to a source and a doublet. Circulation and vorticity, relation between circulation and vorticity. Kelvin's Circulation theorem.

Subject	Topic	Importance
Group-A: Statics	Forces acting in a plane, parallel forces, moments and couples, Equilibrium of coplanar forces. Astatic, stable and unstable equilibrium, Virtual work, Centre of gravity.	***
	Forces in three dimensions	**
	Work	*

Group-B: Dynamics	Simple harmonic motion, Motion in a plane referred to Cartesian and polar coordinates. Radial and transverse velocities; central, tangential and normal acceleration. Central forces,	***
	Motion in resisting medium, Motion in three dimensions	**
	Motion in a straight line	*
Group-C: Mathematical Methods	The Laplace Transform, Bessel's Equation, Legendre's Equation, Fourier Series, Dirichlet's theorem	***
	Solution of linear differential equations with constant coefficients.	**
Group-A: Real Analysis	Metric Spaces, Sequence, Differentiation: Continuous function. Derivative of a function. Rolle's Theorem, Meanvalue theorem, Functions of Several Variables: Limit and Continuity.	***
	Taylor's theorem, Partial differentiation. Schwarz's theorem, Young's theorem	**
Group-B: Complex Analysis	Complex Functions, Analytic Functions, Harmonic functions. Mobius transformation. Complex Integration: Zeros of analytic functions, Cauchy's theorem, Morera's theorem, Cauchy's integral formula, Singularities, Classification of singularities, Rouches theorem, The residue theorem,	***
	power series, The open mapping theorem, Taylor's and Laurent series. Fundamental theorem of algebra, Contour integration.	**
Group-C: Numerical Analysis	Solution of algebraic and transcendental equation, Interpolation. Numerical solution of linear and non-linear system of equations. Numerical differentiation and integration	***

	Numerical solutions of ordinary differential equation	**
Group-D: Hydrodynamics	Velocity and acceleration of fluid particles, Steady and unsteady flows, Uniform and non-uniform flows, Stream lines, path lines, vortex lines and velocity potential. Rotational and irrotational flows. Equation of continuity.	***

রেফারেন্স

- ১। তিতাস বুক পাবলিকেশন
- ২। সাইন'স ভিউ পাবলিকেশন

পরামর্শ

১. শেষ মুহূর্তে সময় কম থাকলে কঠিন টপিকসগুলো এড়িয়ে চলা যেতে পারে।
২. যদিও আশা করা যায় জেনারেল পার্টে সহজ হবে, তারপরও উভয় পার্টকে গুরুত্ব দেওয়া উচিত।
৩. আমরা যেই টপিক পড়বো সেই টপিক যেন সবকিছু কমপ্লিট করি। (বেসিক + ম্যাথ)।
৪. যেহেতু অনেক গুলো ম্যাথ করতে হবে সেহেতু সময় ব্যবস্থাপনা সম্পর্কে সচেতন হতে হবে। পরীক্ষার হলে এমন ম্যাথ আসবে যেগুলোর আমরা সহজ উপায়ে সমাধান জানি না, সেগুলো দাগ দিয়ে রেখে শেষে করা যেতে পারে।
৫. পরীক্ষার হলে প্রশ্ন কঠিন দেখে হতাশ হওয়ার কিছু নেই। প্রশ্ন সবার জন্যই তাহলে কঠিন হয়েছে। সেক্ষেত্রে নিজের স্ট্রং জোনকে সামনে রেখে প্রশ্নগুলোর উত্তর করা উচিত।
৬. অতি স্বল্প সময় করা নোট। টাইপিং মিসটেক থেকে বিভিন্ন ধরনের মিসটেক থেকে যেতে পারে সেক্ষেত্রে আমাদের উচিত মূল বই পাশে নিয়ে পড়া। কোন টপিকে বিরোধ দেখা দিলে বইটাকে গ্রহণ করতে হবে। নোটকে সহায়িকা হিসেবে ব্যবহার উচিত।
৭. এআই ব্যবহারের সতর্ক থাকতে হবে। কারণ এ আই অনেক সময় ভুল উত্তর দেয়।

টপিক ভিত্তিক সংক্ষিপ্ত আলোচনা

Forces acting in a plane:

When two or more forces act on a body in the same plane, they are called *coplanar forces*.

Types:

Concurrent forces: meet at a point.

Parallel forces: act parallel (same or opposite directions).

General coplanar forces: neither concurrent nor parallel.

Resultant: A single force that has the same effect as all forces combined (found by vector addition, parallelogram or triangle law).

☞ In short: Forces acting in a plane can be reduced to one resultant (or equilibrium if resultant = 0).

Parallel forces in a plane:

Forces whose lines of action are parallel.

- **Like parallel forces** → act in the same direction → Resultant = sum of magnitudes, direction same, line of action found by Varignon's theorem.
- **Unlike parallel forces** → act in opposite directions → Resultant = difference of magnitudes, direction of larger force.
- If equal and opposite & parallel but not collinear → form a **couple** (causes rotation).

☞ In short: Parallel forces combine into a single resultant force (or a couple if balanced but non-collinear).

Moments and Couples

- **Moment of a force:** Turning effect of a force about a point or axis.

$$M = F \cdot d$$

where F = force, d = perpendicular distance from the point/axis.

- **Clockwise vs. Anticlockwise moments:** Direction depends on sense of rotation.
- **Couple:** Two equal and opposite forces whose lines of action are parallel but not collinear.
 - Net force = 0
 - Net effect = pure rotation
 - **Moment of couple:**

$$M = F \times \text{distance between forces}$$

☞ In short: **Moment** measures turning effect of one force, while a **couple** is a special case where only rotation (no translation) occurs.

Q1. A force of 20 N acts at a perpendicular distance of 0.5 m from a pivot. Find the moment.

Solution:

$$M = F \times d = 20 \times 0.5 = 10\text{ Nm (clockwise/anticlockwise)}$$

Q2. Two equal and opposite forces of 50 N act parallel to each other, separated by 0.2 m . Find the moment of the couple.

Solution:

$$M = F \times d = 50 \times 0.2 = 10\text{ Nm (pure rotation)}$$

Equilibrium of coplanar forces

A body is in equilibrium under coplanar forces when:

1. $\Sigma F_x = 0$ → sum of all horizontal forces is zero.
2. $\Sigma F_y = 0$ → sum of all vertical forces is zero.
3. $\Sigma M = 0$ → sum of all moments about any point is zero.
 - If only two forces → they must be **equal, opposite, and collinear**.
 - If three forces → they must be **concurrent** (meet at a point) and satisfy triangle law of forces.

🔑 In short: **Equilibrium = no translation + no rotation.**

Q1. A body is acted on by two forces: 40 N to the right, and 40 N to the left. Is the body in equilibrium?

Solution:

$$\Sigma F_x = 40 - 40 = 0, \Sigma F_y = 0, \Sigma M = 0$$

Yes, equilibrium.

Q2. A body is in equilibrium under three forces of magnitudes 10 N , 15 N , 20 N . Can this happen?

Solution:

For 3 forces in equilibrium → must satisfy triangle law.

Check: $10 + 15 = 25 > 20$

and other conditions hold.

So they can form a closed triangle → equilibrium possible.

Types of Equilibrium

1. **Stable equilibrium**
 1. When body is slightly displaced, it returns to its original position.
 2. Center of gravity (C.G.) rises when displaced.
 3. Example: pendulum at lowest point, cone resting on base.

2. **Unstable equilibrium**

1. When body is slightly displaced, it moves farther away from original position.
2. C.G. lowers when displaced.
3. Example: pendulum at top point, cone balancing on tip.

3. **Neutral (Astatic) equilibrium**

1. When body is slightly displaced, it remains in the new position without returning or moving farther.
2. C.G. neither rises nor lowers.
3. Example: rolling ball on a horizontal surface, cube on a flat face.

☞ In short:

- **Stable** → returns,
- **Unstable** → falls away,
- **Neutral/Astatic** → stays where moved.
- **Neutral equilibrium:**
Body stays in its new position **without any restoring or disturbing tendency**.
Example: ball on a flat horizontal surface.
- **Astatic equilibrium:**
A special case of neutral, where **equilibrium exists in every position** of the body.
Example: a smooth cylinder lying on a flat surface → it can rest in any orientation.

☞ In short:

- **Neutral** → stays where moved, but not in all positions.
- **Astatic** → equilibrium possible in *any* position.
- ☞ In short: **All astatic equilibriums are neutral, but not all neutral equilibriums are astatic.**

Principle of Components

A force can be replaced by two or more components along given directions.
Sum of components along any direction = component of resultant along same direction.

Formula: $F_x = F \cos \theta$, $F_y = F \sin \theta$

Use: Resolving forces along any convenient direction → simplifies equilibrium calculations.

Example: If $F_1 = 10\text{ N}$, $F_2 = 15\text{ N}$ at a point, along x-axis:

$$F_{1x} + F_{2x} = R_x$$

Lami's Theorem

- For **three concurrent, non-collinear forces** in equilibrium.
- Statement: Each force $\propto$ sine of angle between the other two.
- Formula: $\frac{F_1}{\sin \alpha} = \frac{F_2}{\sin \beta} = \frac{F_3}{\sin \gamma}$
- Use: Solve three-force equilibrium without resolving components.

Two Like Parallel Forces

Magnitude of Resultant (R):

$$R = F_1 + F_2$$

Direction: Same as that of the forces (since they act in the same direction).

Point of Application (distance from F₁):

If distance between forces = d , then line of action of resultant measured from F_1 :

$$x = \frac{F_2 \cdot d}{F_1 + F_2}$$

Notes:

Resultant acts **between the two forces**, closer to the **larger force**.

Used in **beam and lever problems**.

Q: Two like parallel forces $F_1 = 30\text{ N}$ and $F_2 = 50\text{ N}$ act 2 m apart. Find **resultant magnitude** and **point of application from F₁**.

Solution:

- **Magnitude:**

$$R = F_1 + F_2 = 30 + 50 = 80\text{ N}$$

- **Point of application from F₁:**

$$x = \frac{F_2 \cdot d}{F_1 + F_2} = \frac{50 \cdot 2}{80} = 1.25\text{ m}$$

Resultant = 80 N, acts 1.25 m from F₁ toward F₂.

Two Unlike Parallel Forces

Magnitude of Resultant (R):

$$R = |F_1 - F_2|$$

(Direction of **larger force**)

Point of Application (distance from F₁):

$$x = \frac{F_2 \cdot d}{F_1 - F_2}$$

Notes:

If $F_1 = F_2$ and opposite $\rightarrow$ Resultant = 0 $\rightarrow$ forms a **couple**.

Used in **beam, lever, and couple problems**.

Q: Two parallel forces $F_1 = 60\text{ N}$ and $F_2 = 40\text{ N}$ act in opposite directions, 3 m apart. Find **resultant magnitude** and **point of application from F1**.

Solution:

- **Magnitude:**

$$R = F_1 - F_2 = 60 - 40 = 20\text{ N (in direction of F1)}$$

- **Point of application from F1:**

$$x = \frac{F_2 \cdot d}{F_1 - F_2} = \frac{40 \cdot 3}{20} = 6\text{ m}$$

Resultant = 20 N, acts 6 m from F1 toward F2.

Work

- **Definition:** Work done by a force = force $\times$ displacement in the direction of force.

$$W = F \cdot d \cdot \cos \theta$$

- **Positive work:** Force and displacement in same direction.
- **Negative work:** Force and displacement in opposite direction.

Virtual Work

- **Definition:** Work done by applied forces during a **small imagined (virtual) displacement** consistent with constraints.
- **Principle of Virtual Work:** A system in equilibrium $\rightarrow$ **total virtual work = 0** for any virtual displacement.

$$\delta W = 0$$

- **Use:** Solving equilibrium problems without resolving forces completely.

Centre of Gravity (C.G.)

- **Definition:** The point at which the **entire weight of a body** can be considered to act.
- **Properties:**
 - Acts **vertically downward**.
 - For **symmetrical bodies**, C.G. lies on the axis of symmetry.
 - Can lie **inside or outside** the body depending on shape.
- **C.G. of composite body:** Found using **weighted average of individual C.G.s:**

$$x_{cg} = \frac{\sum (x_i m_i)}{\sum m_i}, \quad y_{cg} = \frac{\sum (y_i m_i)}{\sum m_i}$$

Examples:

- Rod $\rightarrow$ midpoint
- Rectangle $\rightarrow$ intersection of diagonals
- Uniform sphere $\rightarrow$ center

Examples of Centre of Gravity:

Body	C.G. Location
Uniform rod	Midpoint of length
Uniform square/rectangle	Intersection of diagonals
Uniform triangle	At centroid ($1/3$ from base along median)
Uniform semicircle	Along symmetry axis, at $4r/3\pi$ from flat side
Uniform circular lamina	Centre of circle
Uniform sphere	Geometrical center
Uniform hemisphere	Along vertical axis, $3r/8$ from flat face
L-shaped lamina	Use composite body method to find weighted average
Hollow cylinder	Midpoint of axis of symmetry
Cone	Along axis, at $1/4$ height from base

Problem setup:

- Upper sphere radius = r
- Lower sphere radius = R
- Centre of upper sphere at height h from contact point
- Line joining centers is vertical at equilibrium
- Rough surfaces $\rightarrow$ no sliding

We need to find the condition for **stable** or **unstable** equilibrium when the upper sphere is slightly displaced.

Condition	Type of Equilibrium	Reason
$-1h > -1r + -1R$	Unstable	C.G. lowers on displacement $\rightarrow$ moves away
$-1h < -1r + -1R$	Stable	C.G. rises on displacement $\rightarrow$ restoring tendency

Common Catenary

- **Definition:** The curve assumed by a **uniform flexible chain or cable** hanging under its own weight, supported at its ends, and acted upon by gravity.

- **Equation:**

$$y = a \cosh \frac{x}{a} = a \frac{e^{x/a} + e^{-x/a}}{2}$$

where $a = \frac{T_0}{w}$,

T_0 =horizontal tension at lowest point,

w = weight per unit length,

x measured horizontally from lowest point.

- **Properties:**
 - Lowest point at $y = a$.
 - Tension is minimum at lowest point, increases towards supports.
 - Curve is symmetrical about vertical line through lowest point.
- **Applications:** Suspension bridges, overhead cables, power lines.

Catenary Relations:

1. **Arc length from lowest point:**

$$s = c \sinh \frac{x}{c}$$

2. **Relation between height and arc length:**

$$y^2 = s^2 + c^2$$

3. **Height in terms of slope angle ψ :**

$$y = c \sec \psi$$

4. **Horizontal distance in terms of slope angle ψ :**

5. **Horizontal distance in terms of height and arc length:**

$$x = c \ln \frac{s + y}{c}$$

6. **Tension at any point:**

$$T = wy$$

where w = weight per unit length, y = vertical distance from lowest point.

Summary: These formulas relate **horizontal distance, height, arc length, slope angle, and tension** for a hanging chain.

Q: A uniform chain of weight $w = 10 \text{ N/m}$ hangs between two supports. The horizontal tension at the lowest point is $T_0 = 200 \text{ N}$. Find the parameter a and the height y at horizontal distance $x = 3 \text{ m}$ from the lowest point.

Solution:

1. **Parameter a :**

$$a = \frac{T_0}{w} = \frac{200}{10} = 20 \text{ m}$$

2. **Height at $x = 3 \text{ m}$:**

$$y = a \cosh \frac{x}{a} = 20 \cosh \frac{3}{20}$$

Height at $x = 3 \text{ m}$ from lowest point $\approx 20.23 \text{ m}$.

Motion in a straight line (briefly):

- **Definition:** Movement of a body along a straight path.
- **Key Quantities:**
 - Displacement x
 - Velocity $v = \frac{dx}{dt}$
 - Acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$
- **Equations of Motion (constant a):**
 - $v = u + at$
 - $x = ut + \frac{1}{2}at^2$
 - $v^2 = u^2 + 2ax$
- **Types:**
 - Uniform motion ($a = 0$)
 - Uniformly accelerated motion ($a = \text{constant}$)

Simple Harmonic Motion (SHM) – Brief:

- **Definition:** Oscillatory motion where acceleration a is proportional and opposite to displacement x : $a = -\omega^2 x$
- **Equation:**
- **Key Terms:**
 - A = amplitude
 - ω = angular frequency
 - $T = \frac{2\pi}{\omega}$ = period
 - $f = \frac{1}{T}$ = frequency
- **Energy:** Total energy $E = \frac{1}{2}kA^2$ (constant)

Problem 1:

A particle executes SHM with amplitude 0.1 m and angular frequency 5 rad/s . Find its maximum velocity.

Solution:

$$v_{\max} = \omega A = 5 \times 0.1 = 0.5\text{ m/s}$$

Problem 2:

A mass on a spring has period 2 s . Find its angular frequency.

Solution:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi\text{ rad/s}$$

Problem 3:

SHM particle has displacement $x = 0.05\text{ m}$, amplitude 0.1 m , angular frequency 10 rad/s . Find acceleration.

Solution:

$$a = -\omega^2 x = -(10)^2 \cdot 0.05 = -5\text{ m/s}^2$$

A particle moves in a straight line AO starting from rest A and moving with an acceleration which is always directed towards O and varies as the distance from O.

Find the motion and Periodic Time.

Acceleration always towards O , proportional to displacement ($a \propto -x$).

Equation of motion:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Solution: $x = A \cos(\omega t + \phi)$

Periodic Time:

$$T = \frac{2\pi}{\omega}$$

A particle moves in a straight line AO starting from rest A and moving with an acceleration which is always directed towards O and varies as the distance from O.

Find the motion and Periodic Time.

The ODE is

$$\ddot{x} - \omega^2 x = 0$$

Hence $x(t) = A \cosh(\omega t)$, $\dot{x}(t) = A\omega \sinh(\omega t)$

Motion and periodic time: the motion is **non-oscillatory** (exponential growth away from O); the equilibrium at O is **unstable**. There is **no real period** (no periodic time).

The characteristic growth time is $\tau = \frac{1}{\omega}$

1. Damped Oscillation

Oscillation with resistive force (e.g., friction/air resistance).

Amplitude decreases with time.

Equation: $\ddot{x} + 2\beta\dot{x} + \omega_0^2x = 0$.

Types: underdamped, critically damped, overdamped.

Brief summary of damping cases:

1. Underdamped

- Damping small.
- Oscillation occurs with gradually decreasing amplitude.

• Solution: $x(t) = Ae^{-\beta t} \cos(\omega' t + \phi),$

• where $\omega' = \sqrt{\omega_0^2 - \beta^2}$.

2. Critically damped

- Damping just enough to prevent oscillation.
- Fastest return to equilibrium without overshoot.
- Solution: $x(t) = (C_1 + C_2 t)e^{-\omega_0 t}.$

3. Overdamped

- Damping very large.
- No oscillation, very slow return to equilibrium.
- Solution: $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ with $r_1, r_2 < 0$.

☞ Only the **underdamped case oscillates**; the other two do not.

Case	Condition	Nature of Motion	Example Solution Form	Behavior
Underdamped	$\beta < \omega_0$	Oscillatory with decreasing amplitude		Oscillates, gradually dies out
Critically damped	$\beta = \omega_0$	No oscillation, fastest return to equilibrium	$x(t) = (C_1 + C_2 t)e^{-\omega_0 t}$	Returns quickly, no overshoot
Overdamped	$\beta > \omega_0$	No oscillation, very slow return to equilibrium	$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$	Returns sluggishly

Problem 1 — Underdamped free oscillation

A mass $m = 1$ kg on a spring $k = 100$ N/m has damping coefficient $c = 4$ kg/s. Initial amplitude $A_0 = 0.10$ m.

(a) Find the damped angular frequency ω' .

(b) Find the period T_d .

(c) Amplitude after $t = 2$ s.

Solution (very brief):

$$\omega_0 = \sqrt{k/m} = \sqrt{100/1} = 10 \text{ rad/s.}$$

$$\beta = \frac{c}{2m} = \frac{4}{2 \cdot 1} = 2 \text{ s}^{-1}.$$

$$\omega' = \sqrt{\omega_0^2 - \beta^2} = \sqrt{100 - 4} = \sqrt{96} = 9.797958 (\approx 9.798) \text{ rad/s.}$$

$$T_d = \frac{2\pi}{\omega'} = \frac{2\pi}{9.797958} = 0.641 \text{ s} (\approx 0.641 \text{ s}).$$

$$\text{Amplitude at } t = 2: A(2) = A_0 e^{-\beta t} = 0.10 e^{-2 \cdot 2} = 0.10 e^{-4} = 0.10 \times 0.018315 = 0.00183 \text{ m.}$$

Problem 2 — (Undamped) Forced oscillation

A mass $m = 0.5$ kg on a spring $k = 8$ N/m is driven by (N). Find the steady-state amplitude A .

Solution (very brief):

$$\omega_0 = \sqrt{k/m} = \sqrt{8/0.5} = \sqrt{16} = 4 \text{ rad/s.}$$

Driving freq $\omega = 3$ rad/s.

$$\text{Drive per unit mass} = \frac{F_0}{m} = \frac{1}{0.5} = 2.$$

$$\text{Amplitude } A = \frac{F_0/m}{\omega_0^2 - \omega^2} = \frac{2}{16 - 9} = \frac{2}{7} = 0.2857 \text{ m.}$$

Problem 3 — Damped forced oscillation (near resonance)

Mass $m = 2$ kg, damping $c = 8$ kg/s, spring $k = 18$ N/m. Driven by . Find steady-state amplitude A at $\omega = 3$ rad/s.

Solution (very brief):

$$\beta = \frac{c}{2m} = \frac{8}{4} = 2 \text{ s}^{-1}. \omega_0 = \sqrt{k/m} = \sqrt{18/2} = \sqrt{9} = 3 \text{ rad/s.}$$

Here $\omega = \omega_0$ so $\omega_0^2 - \omega^2 = 0$.

$$\text{Denominator} = \sqrt{0^2 + (2\beta\omega)^2} = 2\beta\omega = 2 \cdot 2 \cdot 3 = 12.$$

$$F_0/m = 5/2 = 2.5. A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}} = \frac{2.5}{12} = 0.20833 \text{ m.}$$

Resonance (very brief)

- **Definition:** Large amplitude response when a system is driven near its natural frequency.
- **Condition:** Driving frequency ω close to natural frequency ω_0 . For light damping the peak is near ω_0 .
- **Driven damped oscillator amplitude:**

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}$$

where $\beta = \frac{c}{2m}$.

- **Resonant frequency (peak) with damping:**

$$\omega_r = \sqrt{\omega_0^2 - 2\beta^2} \text{ (for } \beta < \omega_0/\sqrt{2}\text{)}$$

)For small damping $\omega_r \approx \omega_0$.

- **Max amplitude at resonance (approx, small damping):**

$$A_{\max} \approx \frac{F_0/m}{2\beta\omega_0}$$

- **Quality factor and bandwidth:**

$$Q = \frac{\omega_0}{2\beta} = \frac{m\omega_0}{c}, \Delta\omega \approx \frac{\omega_0}{Q} = 2\beta$$

Higher $Q \Rightarrow$ sharper, larger resonance.

- **Physical consequence / examples:** bridge oscillations, microwave cavities, musical instruments, tuned circuits. Dangerous if uncontrolled (structural failure).

Quick numeric example (very short):

$m = 1, k = 100 \Rightarrow \omega_0 = 10, c = 2 \Rightarrow \beta = 1$. Driven with $F_0 = 1$ at $\omega = \omega_0$.

$$A(\omega_0) = \frac{1/1}{\sqrt{0 + (2 \cdot 1 \cdot 10)^2}} = \frac{1}{20} = 0.05.$$

Motion in Two Dimensions (brief):

- **Definition:** Motion of a particle in a plane (x-y).
- **Position vector:** $\vec{r} = x\hat{i} + y\hat{j}$.
- **Velocity:** $\vec{v} = \frac{d\vec{r}}{dt} = v_x\hat{i} + v_y\hat{j}$.
- **Acceleration:** $\vec{a} = \frac{d\vec{v}}{dt} = a_x\hat{i} + a_y\hat{j}$.
- **Equations of motion (constant a):**

$$x = u_x t + \frac{1}{2} a_x t^2$$

- $y = u_y t + \frac{1}{2} a_y t^2$

- **Path:** Eliminating t gives trajectory.
- **Example:** Projectile motion — parabolic path.

Let position be $\vec{r} = r(t) \hat{r}$ where $\hat{r}$ and $\hat{\theta}$ are the unit vectors.

Unit-vector time derivatives

$$\dot{\hat{r}} = \dot{\theta} \hat{\theta}, \quad \dot{\hat{\theta}} = -\dot{\theta} \hat{r}.$$

Velocity

$$\vec{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}.$$

Acceleration

$$\vec{a} = (\ddot{r} - r \dot{\theta}^2) \hat{r} + (r \ddot{\theta} + 2\dot{r} \dot{\theta}) \hat{\theta}.$$

Radial & transverse (θ) equations of motion for mass m under forces F_r and F_θ :

$$m(\ddot{r} - r \dot{\theta}^2) = F_r, \quad m(r \ddot{\theta} + 2\dot{r} \dot{\theta}) = F_\theta.$$

Useful results

- If $F_\theta = 0$ (no tangential force) then $r^2 \dot{\theta} = \text{constant}$ (conservation of angular momentum).
- Centripetal term: $-r \dot{\theta}^2$ (radial inward).
- Coriolis-like term: $2\dot{r}$ (transverse if radial speed $\neq 0$).

Apse (brief):

- In orbital/planetary motion (polar coordinates), an **apse** is a point where the orbiting body is at an **extreme distance** from the center of force.
- **Types:**
 - **Periapsis:** nearest point to the focus (e.g., perihelion for Sun, perigee for Earth).
 - **Apoapsis:** farthest point (e.g., aphelion, apogee).
- At an apse: **radial velocity = 0** ($\dot{r} = 0$), (only transverse velocity exists).

🔗 In short: **Apse = turning points of the radial coordinate in orbital motion.**

Kepler's Three Laws (brief):

Law of Orbits:

Each planet moves in an **ellipse** with the Sun at one focus.

Law of Areas:

A line joining a planet and the Sun sweeps out **equal areas in equal times**.

Implies conservation of angular momentum: $r^2 \dot{\theta} = \text{constant}$.

Law of Periods:

Square of orbital period T is proportional to cube of semi-major axis a :

$$T^2 \propto a^3$$

☞ These laws describe **planetary motion** and are valid for any body under **central inverse-square force**.

Problem 1 — Law of Periods

Earth's orbit: $a_1 = 1 \text{ AU}$, $T_1 = 1 \text{ yr}$. Another planet has $a_2 = 4 \text{ AU}$. Find its orbital period T_2 .

Solution:

$$\frac{T_2^2}{T_1^2} = \frac{a_2^3}{a_1^3} \Rightarrow T_2^2 = 4^3 = 64 \Rightarrow T_2 = 8 \text{ yr}.$$

Problem 2 — Law of Areas

A satellite moves around Earth in a circular orbit. At a point, its radial distance is $r = 7000 \text{ km}$ and transverse velocity $v_\theta = 7.5 \text{ km/s}$. Find areal velocity.

Solution:

$$\text{Areal velocity} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} r v_\theta = \frac{1}{2} \cdot 7000 \cdot 7.5 = 26250 \text{ km}^2/\text{s}.$$

Problem 3 — Apse and Orbital Speed

A planet moves in an elliptical orbit around the Sun. Its closest distance (perihelion) is $r_p = 1.0 \times 10^{11} \text{ m}$, farthest distance (aphelion) is $r_a = 1.5 \times 10^{11} \text{ m}$. If its speed at perihelion is $v_p = 3.0 \times 10^4 \text{ m/s}$, find its speed at aphelion v_a .

Solution (very brief):

Use **conservation of angular momentum**:

$$r_p v_p = r_a v_a \Rightarrow v_a = \frac{r_p v_p}{r_a} = \frac{1.0 \times 10^{11} \cdot 3.0 \times 10^4}{1.5 \times 10^{11}} = 2.0 \times 10^4 \text{ m/s}.$$

Motion in a Resisting Medium:

Definition:

Motion of a body through a medium where it experiences a resistive force (like air or water), usually proportional to velocity (linear) or square of velocity (quadratic).

Properties:

1. Resistive force always opposes motion.
2. Motion eventually reaches a **terminal velocity** if resistive force depends on velocity.
3. Acceleration decreases over time due to resistive force.
4. Energy is dissipated as heat in the medium.

Geometrical Meaning:

Velocity-time graph shows **exponential decay** towards terminal velocity;
displacement-time graph is **concave** and approaches linear behavior at large time.

Numerical Example (linear resistance):

Force: $F = -kv$, mass m , gravity g .

Equation: $m \frac{dv}{dt} = mg - kv$

Solution: $v(t) = \frac{mg}{k} \left(1 - e^{-\frac{k}{m}t} \right)$

Terminal velocity: $v_t = \frac{mg}{k}$

Stokes' Law (for spherical object in viscous medium):

$$F = 6\pi\eta r v$$

Where η = viscosity, r = radius, v = velocity.

Definition:

Stokes' Law gives the **resistive (drag) force** experienced by a small spherical object moving slowly through a viscous fluid.

Formula:

$$F = 6\pi\eta r v$$

Where:

- F = drag force (N)
- η = viscosity of fluid (Pa·s)
- r = radius of the sphere (m)
- v = velocity of the sphere (m/s)

Properties:

1. Force is proportional to **velocity** (linear drag).
2. Force is proportional to **radius** of the sphere.
3. Applicable for **laminar flow** (low Reynolds number, $Re < 1$).
4. Used to calculate **terminal velocity** of spheres in fluids.

A small sphere of radius $r = 0.003 \text{ m}$ moves through a viscous fluid with viscosity $\eta = 0.002 \text{ Pa}\cdot\text{s}$ at a velocity $v = 0.05 \text{ m/s}$. Calculate the **drag force** on the sphere.

Solution:

Stokes' Law:

$$F = 6\pi\eta r v$$

$$F = 6\pi(0.002)(0.003)(0.05)$$

$$0.002 \cdot 0.003 \cdot 0.05 = 3 \times 10^{-7}$$

$$F = 6\pi \cdot 3 \times 10^{-7} \approx 5.65 \times 10^{-6} \text{ N}$$

Terminal Velocity (brief):

Definition: Constant velocity when resistive force = driving force (no acceleration).

Properties:

- Acceleration = 0
- Depends on size, shape, density, and medium
- Velocity-time graph levels off

Equation:

- Linear resistance: $v_t = \frac{mg}{k}$
- Sphere in viscous fluid (Stokes' law): $v_t = \frac{2}{9} \frac{r^2(\rho - \rho_m)g}{\eta}$

Numerical Problem (Terminal Velocity):

A small sphere of radius $r = 0.005 \text{ m}$ and density $\rho = 1200 \text{ kg/m}^3$ falls in glycerin ($\eta = 1.5 \text{ Pa}\cdot\text{s}$, $\rho_m = 1000 \text{ kg/m}^3$). Find its **terminal velocity**.

Solution (using Stokes' law):

$$v_t = \frac{2}{9} \frac{r^2(\rho - \rho_m)g}{\eta}$$

$$v_t = \frac{2}{9} \frac{(0.005)^2(1200 - 1000)9.8}{1.5}$$

$$v_t = \frac{2}{9} \frac{0.000025 \cdot 200 \cdot 9.8}{1.5} = \frac{0.000049}{1.5} \approx 3.27 \times 10^{-5} \text{ m/s}$$

Answer: $v_t \approx 3.27 \times 10^{-5} \text{ m/s}$

Definition:

Reynolds number (Re) is a **dimensionless quantity** that predicts the type of fluid flow—laminar or turbulent—around an object.

Formula:

$$Re = \frac{\rho v L}{\eta} \text{ or } Re = \frac{v L}{\nu}$$

Where:

- ρ = fluid density (kg/m^3), v = velocity of fluid (m/s)
- L = characteristic length (m), η = dynamic viscosity ($\text{Pa}\cdot\text{s}$)
- $\nu = \eta/\rho$ = kinematic viscosity (m^2/s)

Properties:

- Laminar flow:** $Re < 2000$, smooth, orderly motion
- Transitional flow:** $2000 < Re < 4000$, unstable, mixed motion
- Turbulent flow:** $Re > 4000$, chaotic, eddy-dominated
- Determines **flow pattern** and **drag characteristics**.

Geometrical Meaning:

It's the ratio of **inertial forces to viscous forces** in a fluid.

Numerical Example:

Water ($\rho = 1000 \text{ kg/m}^3$, $\eta = 0.001 \text{ Pa}\cdot\text{s}$) flows past a pipe of diameter 0.05 m at $v = 2 \text{ m/s}$:

$$Re = \frac{\rho v L}{\eta} = \frac{1000 \cdot 2 \cdot 0.05}{0.001} = 1 \times 10^5$$

Flow type: Turbulent.

Motion of a particle in space along **x, y, z axes**, described by **position vector** $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$.

Properties:

1. Velocity: $\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$
2. Acceleration: $\vec{a} = \frac{d\vec{v}}{dt} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$
3. Trajectory: The path in space traced by the particle.
4. Can be expressed in **Cartesian, cylindrical, or spherical coordinates**.

Cylindrical Coordinates (r, θ, z)

Position vector:

$$\vec{r} = r\hat{e}_r + z\hat{k}$$

Velocity:

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k}$$

Acceleration:

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta + \ddot{z}\hat{k}$$

Components meaning:

Radial: $\ddot{r} - r\dot{\theta}^2$

Transverse: $r\ddot{\theta} + 2\dot{r}\dot{\theta}$

Vertical: $\ddot{z}$

Spherical Coordinates (r, θ, ϕ)

Position vector:

$$\vec{r} = r\hat{e}_r$$

Velocity:

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\dot{\phi}\hat{e}_\phi$$

Acceleration:

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2)\hat{e}_\theta + (r\sin\theta\ddot{\phi} + 2\dot{r}\sin\theta\dot{\phi} + 2r\cos\theta\dot{\theta}\dot{\phi})\hat{e}_\phi$$

Components meaning:

- Radial: change along r
- Polar: change along θ
- Azimuthal: change along ϕ

Problem:

A particle moves as $r = 2t, \theta = 3t, z = t^2$. Find velocity and acceleration at $t = 1$ s.

Solution:

Velocity:

$$v_r = \dot{r} = 2, v_\theta = r\dot{\theta} = 2 \cdot 3 = 6, v_z = \dot{z} = 2t = 2$$

$$\vec{v} = 2\hat{e}_r + 6\hat{e}_\theta + 2\hat{k}$$

Acceleration:

$$\begin{aligned}a_r &= \ddot{r} - r\dot{\theta}^2 = 0 - 2 \cdot 9 = -18 \\a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 + 2 \cdot 2 \cdot 3 = 12 \\a_z &= \ddot{z} = 2 \\\vec{a} &= -18\hat{e}_r + 12\hat{e}_\theta + 2\hat{k}\end{aligned}$$

Laplace Transform

Definition:

The Laplace Transform of a function $F(t)$, defined for $t \geq 0$, is:

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t) dt$$

where s is a complex variable.

Existence:

The Laplace Transform exists if:

$F(t)$ is **piecewise continuous** for $t \geq 0$

$F(t)$ is of **exponential order**, i.e., there exist constants $M, \alpha > 0$ such that

$$|F(t)| \leq Me^{\alpha t} \text{ for all } t \geq 0$$

Basic Properties (with $F \leftrightarrow f$):

1. Linearity:

$$\mathcal{L}\{aF(t) + bG(t)\} = af(s) + bg(s)$$

2. s-Shifting (Multiplication by e^{at}):

$$\mathcal{L}\{e^{at}F(t)\} = f(s - a)$$

3. Time Shifting:

$$\mathcal{L}\{F(t - a)u(t - a)\} = e^{-as}f(s)$$

4. Differentiation in Time:

$$\mathcal{L}\{F'(t)\} = sf(s) - F(0), \quad \mathcal{L}\{F''(t)\} = s^2f(s) - sF(0) - F'(0)$$

5. Integration in Time:

$$\mathcal{L}\left\{\int_0^t F(\tau) d\tau\right\} = \frac{f(s)}{s}$$

6. Multiplication by t (Differentiation in s):

$$\mathcal{L}\{tF(t)\} = -\frac{df(s)}{ds}$$

$F(t)$	$f(s) = \mathcal{L}\{F(t)\}$	Property / Notes
1	$\frac{1}{s}$	Constant function
t	$\frac{1}{s^2}$	Power rule
t^2	$\frac{2}{s^3}$	Power rule: $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$	Exponential function
$e^{bt}t$	$\frac{1}{(s-b)^2}$	s-shifting property
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	Power + s-shifting
$\sin(at)$	$\frac{a}{s^2 + a^2}$	Trigonometric function
$\cos(at)$	$\frac{s}{s^2 + a^2}$	Trigonometric function
$e^{-bt} \sin(at)$	$\frac{a}{(s+b)^2 + a^2}$	Exponential decay $\times$ sin
$e^{-bt} \cos(at)$	$\frac{s+b}{(s+b)^2 + a^2}$	Exponential decay $\times$ cos
$u(t-a)$	$\frac{e^{-as}}{s}$	Heaviside / time-shift function

Definition

The **Inverse Laplace Transform** finds the original time function $F(t)$ from its Laplace transform $f(s)$:

$$F(t) = \mathcal{L}^{-1}\{f(s)\}$$

Properties

1. **Linearity:**

$$\mathcal{L}^{-1}\{af(s) + bg(s)\} = aF(t) + bG(t)$$

2. **s-Shifting:**

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$$

3. **Time Shifting:**

$$\mathcal{L}^{-1}\{e^{-as}f(s)\} = F(t-a)u(t-a)$$

4. **Differentiation in s:**

$$\mathcal{L}^{-1}\left\{\frac{df(s)}{ds}\right\} = -tF(t)$$

$f(s)$	$F(t) = \mathcal{L}^{-1}\{f(s)\}$
$\frac{1}{s}$	1
$\frac{1}{s^2}$	t
$\frac{1}{s-a}$	e^{at}
$\frac{s}{s^2+a^2}$	$\cos(at)$
$\frac{a}{s^2+a^2}$	$\sin(at)$
$\frac{1}{(s-a)^2}$	te^{at}

solving **linear differential equations with constant coefficients using Laplace**

Transform:

Steps:

1. Take **Laplace Transform** of both sides:

$$\mathcal{L}\{y^{(n)}\} = s^n Y(s) - s^{n-1}y(0) - \dots - y^{(n-1)}(0)$$

2. Substitute **initial conditions**.

3. Solve algebraically for $Y(s)$.

4. Take **Inverse Laplace Transform**: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Example

$$y'' - 3y' + 2y = 0, y(0) = 1, y'(0) = 0$$

Laplace:

$$(s^2Y - s) - 3(sY - 1) + 2Y = 0 \Rightarrow Y(s) = \frac{s-3}{(s-1)(s-2)}$$

Inverse Laplace:

$$Y(s) = \frac{-2}{s-1} + \frac{1}{s-2} \Rightarrow y(t) = e^{2t} - 2e^t$$

Definition (Convolution)

The **convolution** of two functions $F(t)$ and $G(t)$ is:

$$(F * G)(t) = \int_0^t F(\tau)G(t-\tau) d\tau$$

It gives a new function representing the “overlap” of F and G as one function slides over the other.

Bessel's Equation:

1. Definition

Bessel's equation of order n is:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0$$

- n = order of the Bessel function (can be integer or real)

2. Solutions

- Bessel functions of the first kind $J_n(x)$:**

$$y = J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n}$$

- Bessel functions of the second kind $Y_n(x)$ (Neumann function):**

$$y = Y_n(x) \text{ singular at } x = 0$$

- General solution:**

$$y(x) = AJ_n(x) + BY_n(x)$$

3. Properties

- Recurrence relations:**

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x), J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x)$$

- Orthogonality:**

$$\int_0^1 x J_n(\alpha_{n,m} x) J_n(\alpha_{n,k} x) dx = 0 \quad (m \neq k)$$

- Parity:**

$$J_{-n}(x) = (-1)^n J_n(x)$$

4. Example (Numerical)

Bessel equation of order 0:

$$x^2 y'' + xy' + x^2 y = 0$$

Solution:

$$y = AJ_0(x) + BY_0(x)$$

Values at $x = 0$

$$J_n(0) = \begin{cases} 1, & n = 0 \\ 0, & n = 1, 2, 3, \dots \end{cases}$$

(Generating Function)

The generating function for **Bessel functions of integer order $J_n(x)$** is:

$$e^{\frac{x}{2}(t - \frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

- Expands an **exponential** into a series of Bessel functions.

- Useful to derive **recurrence relations, addition formulas, and other properties**

Orthogonal

Definition

Bessel functions of the first kind $J_n(x)$ are **orthogonal** over $[0, 1]$ (or any interval scaled by roots) with weight x :

$$\int_0^1 x J_n(\alpha_{n,m}x) J_n(\alpha_{n,k}x) dx = 0 \quad (m \neq k)$$

- $\alpha_{n,m}$ = m -th zero of $J_n(x)$

2. Properties

1. Weighted orthogonality:

$$\int_0^a x J_n\left(\frac{\alpha_{n,m}x}{a}\right) J_n\left(\frac{\alpha_{n,k}x}{a}\right) dx = 0 \quad (m \neq k)$$

2. Normalization:

$$\int_0^a x \left[J_n\left(\frac{\alpha_{n,m}x}{a}\right) \right]^2 dx = \frac{a^2}{2} [J_{n+1}(\alpha_{n,m})]^2$$

- Used to **expand functions** in series of Bessel functions (like Fourier series).

3. Example (Brief)

For $J_0(x)$ and its first two zeros $\alpha_{0,1}, \alpha_{0,2}$:

$$\int_0^1 x J_0(\alpha_{0,1}x) J_0(\alpha_{0,2}x) dx = 0$$

- Shows **orthogonality between different modes**.

1. Legendre's Differential Equation

$$(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n + 1)y = 0$$

- $n = 0, 1, 2, \dots$
- Defined on $-1 \leq x \leq 1$

2. General Solution

1. For integer n :

$$y(x) = AP_n(x) + BQ_n(x)$$

- $P_n(x)$ = Legendre polynomial of order n (regular at $x = \pm 1$)
- $Q_n(x)$ = Legendre function of the second kind (singular at $x = \pm 1$)

2. If solution must be finite at $x = \pm 1$:

$$y(x) = AP_n(x) \text{ (discard } Q_n(x) \text{)}$$

3. Rodrigues' Formula for $P_n(x)$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

4. Series Form (Summation)

$$P_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r}{2^n} \frac{(2n-2r)!}{r! (n-r)! (n-2r)!} x^{n-2r}$$

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$

1. Generating Function

$$G(x, t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n, \quad |t| < 1$$

- Expands a single function into **all Legendre polynomials**.
- Useful for deriving **recurrence relations** and series expressions.

2. Rodrigues' Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

- Provides a **direct formula** for any order n .

3. Recurrence Relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

- Allows computing higher-order polynomials from **previous two orders**.

4. Orthogonality

$$\int_{-1}^1 P_m(x) P_n(x) dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

- Essential for **expanding functions in Legendre series**.

1. Fourier Series

A **periodic function** $f(x)$ with period $2L$ can be represented as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

- **Sine and cosine terms** represent the harmonic components of the function.

2. Fourier Coefficients

1. Constant term:

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

2. Cosine coefficients:

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, n = 1, 2, 3, \dots$$

3. Sine coefficients:

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, n = 1, 2, 3, \dots$$

Sine and Cosine Series

4. Even function ($f(-x) = f(x) \rightarrow$ (Cosine series only ($b_n = 0$))

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

5. Odd function ($f(-x) = -f(x) \rightarrow$ (Sine series only ($a_n = 0$))

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

4. Example (Brief)

- Function: $f(x) = x, -\pi < x < \pi$, odd function.
- Sine series:

$$\Rightarrow f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

You may find in period 2π .

1. Dirichlet's Theorem

For a function $f(x)$ with period 2π that is **piecewise continuous and has a finite number of maxima, minima, and discontinuities** in $[-\pi, \pi]$:

The Fourier series of $f(x)$ **converges** at each point x to:

$$S(x) = \frac{f(x^+) + f(x^-)}{2}$$

$f(x^+)$ = right-hand limit, $f(x^-)$ = left-hand limit

Remarks:

At **continuous points**, Fourier series converges to $f(x)$

At **discontinuities**, Fourier series converges to the **midpoint of the jump**

2. Properties of Fourier Series

1. Linearity:

$$\text{If } f(x) = \alpha g(x) + \beta h(x), \text{ then } F[f] = \alpha F[g] + \beta F[h]$$

2. Even/Odd Functions:

- Even $\rightarrow$ cosine series only
- Odd $\rightarrow$ sine series only

3. Parseval's Identity:

4. Periodicity:

Fourier series can represent **any piecewise continuous periodic function**.

3. Applications

1. **Signal processing:** Representing periodic signals as sums of sines and cosines.
2. **Heat conduction / Wave equation:** Solving PDEs using separation of variables.
3. **Approximation of functions:** Expanding functions into series for numerical computations.
4. **Electrical engineering:** Analysis of AC circuits, Fourier transforms.

Metric Spaces

1. Definition

A **metric space** is a pair (M, d) , where M is a set and $d: M \times M \rightarrow \mathbb{R}$ is a **distance function** (metric) satisfying, for all $x, y, z \in M$:

1. **Non-negativity:** $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$
2. **Symmetry:** $d(x, y) = d(y, x)$
3. **Triangle inequality:** $d(x, z) \leq d(x, y) + d(y, z)$

2. Properties

- Distance is never negative.
- Distance = 0 only if two points are identical.
- Distance is symmetric.
- Triangle inequality holds (direct path is shortest).

3. Geometric Meaning

A metric tells us **how to measure distance** between two points.

Different metrics = different geometry (circles may look round, diamond, or square).

4. Examples

i. Euclidean metric (usual distance):

(Unit circle = round circle).

ii. Taxi-Cab (Manhattan) metric: $d(x, y) = \sum_{i=1}^n |x_i - y_i|$

(Unit circle = diamond shape).

iii. Chebyshev metric: $d(x, y) = \max_i |x_i - y_i|$

(Unit circle = square in $\mathbb{R}^2$.)

iv. **Discrete metric:** $d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$

)All non-identical points are equally distant).

5. Numerical Example

Find distance between $A = (1, 2)$, $B = (4, 6)$.

- Euclidean: $d = \sqrt{(1-4)^2 + (2-6)^2} = \sqrt{9+16} = 5$
- Taxi-Cab: $d = |1-4| + |2-6| = 3+4 = 7$
- Chebyshev: $d = \max(|1-4|, |2-6|) = \max(3, 4) = 4$
- Discrete: $d = 1$ (since $A \neq B$)

Open Set

1. Definition

In a metric space (M, d) , a set $S \subseteq M$ is **open** if:

$$\forall x \in S, \exists r > 0 \text{ such that } B_r(x) \subseteq S$$

where

$$B_r(x) = \{y \in M : d(x, y) < r\}$$

is the **open ball** of radius r around x .

Meaning: around every point of S , you can draw a “small ball” completely inside S .

2. Properties

- Union of any collection of open sets is open.
- Finite intersection of open sets is open.
- Both the whole space M and the empty set $\emptyset$ are open.

3. Geometric Meaning

- **Open sets do not include boundary points.**
- Example: On the real line, $(0, 1)$ means you can move a bit left or right from any point inside and still stay inside.

4. Examples

i. In $\mathbb{R}$:

- $(0, 1)$ is open.
- Union $(0, 1) \cup (2, 3)$ is also open.

ii. In $\mathbb{R}^2$:

- Open disk $\{(x, y) : x^2 + y^2 < 1\}$ is open.

iii. Empty set $\emptyset$ and $\mathbb{R}$ are both open.

Closed Set

1. Definition

A set $S \subseteq M$ in a metric space (M, d) is **closed** if **its complement** $M \setminus S$ is open.

Equivalently:

S is closed $\Leftrightarrow$ it contains all its limit points.

Meaning: If a sequence of points in S converges, its **limit also lies in S** .

2. Properties

- Intersection of any collection of closed sets is closed.
- Finite union of closed sets is closed.
- The whole space M and the empty set $\emptyset$ are closed.
- A set can be **both open and closed** (clopen).

3. Geometric Meaning

- Closed sets **include their boundary points**.
- Example: On the real line, $[0, 1]$ contains 0 and 1, so it is closed.

4. Examples

i. In $\mathbb{R}$ (Euclidean metric):

- $[0, 1]$ is closed.
- $[0, 1] \cup [2, 3]$ is closed.

ii. In $\mathbb{R}^2$:

- Closed disk $\{(x, y): x^2 + y^2 \leq 1\}$ is closed.

iii. Whole space $\mathbb{R}$ and empty set $\emptyset$ are both closed.

iv. Half-open interval $(0, 1]$ is **not closed**, because 0 is a limit point not included in the set.

Clopen Set

1. Definition

A set $S \subseteq M$ in a metric space (M, d) is **clopen** if it is **both open and closed**.

- Open $\rightarrow$ contains a small neighborhood around each of its points.
- Closed $\rightarrow$ contains all its limit points.

So a **clopen set** satisfies both properties simultaneously.

2. Properties

- Complement of a clopen set is also clopen.
- Union and intersection of clopen sets are clopen.
- In **connected spaces** (like $\mathbb{R}$ with usual metric), the only clopen sets are M (whole space) and $\emptyset$ (empty set).
- In **discrete metric spaces**, every subset is clopen.

3. Geometric Meaning

- Clopen sets have **no “boundary” inside the space**.
- Moving a small distance from any point keeps you **inside the set**, and all limit points of the set are included.

4. Examples

- i. Whole space $M \rightarrow$ open and closed $\rightarrow$ clopen
- ii. Empty set $\emptyset \rightarrow$ open and closed $\rightarrow$ clopen
- iii. In **discrete metric space** $M = \{a, b, c\}$, any subset $\{a\}, \{a, b\}$ etc. $\rightarrow$ clopen

5. Quick Conceptual Check

- In $\mathbb{R}$:
 - $[0, 1] \rightarrow$ closed but not open $\rightarrow$ not clopen
 - $(0, 1) \rightarrow$ open but not closed $\rightarrow$ not clopen
 - $\mathbb{R} \rightarrow$ open and closed $\rightarrow$ clopen
 - $\emptyset \rightarrow$ open and closed $\rightarrow$ clopen

Compact Set

1. Definition

A set $K \subseteq M$ in a metric space (M, d) is **compact** if **every open cover of K has a finite subcover**.

Open cover: A collection of open sets $\{U_\alpha\}$ such that $K \subseteq \bigcup_\alpha U_\alpha$

Finite subcover: A finite number of these open sets still cover K .

Equivalent definition in $\mathbb{R}^n$:

A set is compact **iff it is closed and bounded** (Heine-Borel theorem).

2. Properties

Compact sets are **closed**.

Compact sets are **bounded** in $\mathbb{R}^n$.

Continuous functions on compact sets attain their **maximum and minimum** values.

Any sequence in a compact set has a **convergent subsequence** (Bolzano-Weierstrass property).

3. Geometric Meaning

- Compact sets are “small and complete” in the sense that they **fit inside a bounded region** and contain all their **limit points**.
- You can cover them with finitely many open sets — no “infinite spread.”

4. Examples

- i. In $\mathbb{R}$:
 - Closed interval $[0, 1] \rightarrow$ compact
 - Open interval $(0, 1) \rightarrow$ **not compact** (not closed)
- ii. In $\mathbb{R}^2$:
 - Closed disk $\{(x, y): x^2 + y^2 \leq 1\} \rightarrow$ compact
 - Open disk $\{(x, y): x^2 + y^2 < 1\} \rightarrow$ not compact
- iii. Finite sets:
 - Any finite set of points is compact.

5. Numerical / Conceptual Problem

Check whether $K = [0, 2]$ in $\mathbb{R}$ is compact.

- Closed? yes, contains all limit points.
- Bounded? yes, lies between 0 and 2.
- Hence, by Heine-Borel theorem, K is **compact**.
- Open interval $(0, 2) \rightarrow$ bounded but not closed $\rightarrow$ **not compact**

Perfect Set

1. Definition

A set $P \subseteq M$ in a metric space (M, d) is **perfect** if:

1. P is **closed**.
2. Every point of P is a **limit point** of P (no isolated points).

Meaning: A perfect set is **closed and dense in itself**—every point can be approached by other points of the set.

2. Properties

- Perfect sets are **closed**.
- Perfect sets have **no isolated points**.
- A non-empty perfect set in $\mathbb{R}$ is **uncountable** (like the Cantor set).
- Subsets of perfect sets may or may not be perfect.

3. Geometric Meaning

- Every point in the set is **“surrounded” by other points of the set**.
- There are no “lonely” points that stand apart from the rest of the set.

4. Examples

i. **Closed interval in $\mathbb{R}$:**

$[0, 1] \rightarrow$ perfect (every point is a limit point, and it is closed).

5. Numerical / Conceptual Problem

Check: $P = [0, 1]$

- Closed? yes
- Every point a limit point? yes (every point has neighbors in $[0, 1]$)
 $\rightarrow$ **Perfect**

Check: $Q = \{0, 1, 2\}$

- Closed? yes
- Every point a limit point? no (all points are isolated)
 $\rightarrow$ **Not perfect**

Cantor Set

1. Definition

The **Cantor set** is a subset of $[0, 1]$ constructed by repeatedly removing the **middle third** of each interval:

1. Start with $[0, 1]$.
2. Remove the middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$.
3. Repeat for each remaining interval infinitely.

The set of points **never removed** in this process forms the **Cantor set**.

2. Properties

- **Closed:** Cantor set contains all its limit points.
- **Perfect:** Every point is a limit point; no isolated points.
- **Uncountable:** There are as many points as in $[0, 1]$.
- **Measure zero:** Its total length = 0 (despite being uncountable).
- **Nowhere dense:** It contains no intervals.

3. Geometric Meaning

- The Cantor set is **totally disconnected**: between any two points there's a gap.
- Appears as an **infinitely "dusty" set** inside $[0, 1]$.
- Can be represented using **base-3 expansion**: all numbers with only 0s and 2s in base-3.

4. Examples / Representation

i. First few points left after iterations:

$$0, \frac{1}{3}, \frac{2}{3}, 1, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \dots$$

Sequence

1. Definition

A **sequence** is an ordered list of elements (numbers, vectors, or objects) indexed by natural numbers:

$$\{a_n\} = a_1, a_2, a_3, \dots$$

- Each a_n is called the **n-th term** of the sequence.
- A sequence can be **finite** or **infinite** (usually infinite in analysis).

2. Properties

Monotone:

Increasing: $a_{n+1} \geq a_n$

Decreasing: $a_{n+1} \leq a_n$

Bounded:

Bounded above: $a_n \leq M$

Bounded below: $a_n \geq m$

Bounded if both above and below.

Convergent: A sequence $\{a_n\}$ **converges** to L if

$$\lim_{n \rightarrow \infty} a_n = L$$

Divergent: If it does not converge.

Convergent Sequence

1. Definition

A sequence $\{a_n\}$ is **convergent** if there exists a real number L such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with:

$$|a_n - L| < \varepsilon \text{ for all } n > N$$

Meaning: The terms of the sequence get **arbitrarily close** to L as $n \rightarrow \infty$.

Here, L is called the **limit** of the sequence.

2. Properties

- **Unique Limit:** A convergent sequence has only one limit.
- **Boundedness:** Every convergent sequence is bounded.
- **Algebra of Limits:**
- **Monotone + bounded $\Rightarrow$ convergent.**

3. Geometric Meaning

- The terms of the sequence **approach a fixed point** L on the number line (or plane) as n increases.
- Think of “zooming in” on L ; the points cluster closer and closer.

4. Examples

i. $a_n = \frac{1}{n} \rightarrow$ converges to 0.

ii. $a_n = \frac{n}{n+1} = 1 - \frac{1}{n+1} \rightarrow$ converges to 1.

iii. $a_n = \frac{2n+1}{n+2} \rightarrow$ converges to 2.

iv. $a_n = (-1)^n \rightarrow$ **not convergent** (oscillates).

Bounded Sequence

1. Definition

A sequence $\{a_n\}$ is **bounded** if there exists a real number $M > 0$ such that

$$|a_n| \leq M \text{ for all } n \in \mathbb{N}.$$

Meaning: All the terms of the sequence lie within a **finite interval** $[-M, M]$.

2. Types

- **Bounded Above:** $a_n \leq M$ for all n .
- **Bounded Below:** $a_n \geq m$ for all n .
- **Bounded (both sides):** Exists m, M such that

$$m \leq a_n \leq M \quad \forall n.$$

Properties

- Every **convergent sequence** is bounded.
- A **bounded sequence** need not be convergent (e.g., $a_n = (-1)^n$).
- If a sequence is **monotone and bounded**, then it converges.

4. Geometric Meaning

- The sequence's terms stay **trapped inside a fixed interval**, never escaping to infinity.

5. Examples

- $a_n = \frac{1}{n}$: bounded between 0 and 1.
- $a_n = (-1)^n$: bounded between -1 and 1, but not convergent.
- $a_n = n$: bounded between -1 and 1, but does not converge.
- $a_n = n$: **not bounded** (grows to infinity).

Subsequence

Definition

A subsequence of $\{a_n\}$ is formed by choosing terms in order with increasing indices:

$$\{a_{n_k}\}, \quad n_1 < n_2 < n_3 < \dots$$

Properties

- Every subsequence of a convergent sequence $\rightarrow$ same limit.
- A bounded sequence always has a convergent subsequence (Bolzano–Weierstrass).
- Divergent sequences can have subsequences with different limits.

Examples

- $a_n = (-1)^n$: subsequences $a_{2k} = 1, a_{2k-1} = -1$.
- $a_n = \frac{1}{n}$: subsequence $a_{2k} = \frac{1}{2k} \rightarrow$ all go to 0.

Cauchy Sequence

Definition

A sequence $\{a_n\}$ is a **Cauchy sequence** if for every $\varepsilon > 0$, there exists N such that

$$|a_m - a_n| < \varepsilon \quad \forall m, n > N.$$

Meaning: The terms get arbitrarily close to each other as $n \rightarrow \infty$.

Properties

- Every **convergent sequence** is Cauchy.
- In $\mathbb{R}$ (and any complete metric space), every **Cauchy sequence is convergent**.
- Cauchy $\Rightarrow$ bounded.

Geometric Meaning

- Terms of the sequence **cluster together** as we go further, not spreading apart.

Examples

- $a_n = \frac{1}{n} \rightarrow$ Cauchy (converges to 0).
- $a_n = \sqrt{n} \rightarrow$ not Cauchy (terms keep spreading).

Completeness of $\mathbb{R}$

1. Definition

The real numbers $\mathbb{R}$ are said to be **complete** because:

Every Cauchy sequence in $\mathbb{R}$ converges to a limit in $\mathbb{R}$.

Equivalently:

- Every non-empty set of real numbers that is **bounded above** has a **least upper bound (supremum)** in $\mathbb{R}$.

2. Properties

- Completeness distinguishes $\mathbb{R}$ from $\mathbb{Q}$ (rationals).
- Guarantees existence of limits, suprema, and infima.
- Makes $\mathbb{R}$ a **complete metric space** under the usual distance $d(x, y) = |x - y|$.

3. Geometric Meaning

- The real line has **no “holes” or gaps**.
- Sequences that should converge (based on closeness) always land on a real number.

Continuous Function

- **Definition:** f is continuous at c if $\lim_{x \rightarrow c} f(x) = f(c)$.
- **Geometric:** Graph has no break at c .

- **Properties:** Sum/product/quotient (if denom $\neq 0$) of continuous functions are continuous.
- **Example:** $f(x) = x^2 \rightarrow$ continuous everywhere.

Derivative of a Function

- **Definition:**

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}.$$

- **Geometric:** Slope of tangent at $x = a$.
- **Properties:** Linearity, product rule, quotient rule, chain rule.
- **Example:** $f(x) = x^2 \Rightarrow f'(x) = 2x$.

Quick Problem: $f(x) = x^2 + 3x, f'(2) = 7$.

Necessary and Sufficient Condition for Differentiability

- **Necessary:** If f is differentiable at $x = a$, then f must be continuous at $x = a$.
(Differentiability $\Rightarrow$ Continuity).
- **Not sufficient:** Continuity at a does **not** guarantee differentiability at a .
Example: $f(x) = |x|$ is continuous at 0, but not differentiable there.
- **Sufficient condition:** If f is differentiable at a , then automatically it is continuous at a .

Rolle's Theorem

1. Statement / Definition

Let f be a function satisfying:

1. f is **continuous** on the closed interval $[a, b]$.
2. f is **differentiable** on the open interval (a, b) .
3. $f(a) = f(b)$.

Then, there exists at least one $c \in (a, b)$ such that:

$$f'(c) = 0$$

2. Geometric Meaning

- The curve starts and ends at the same height ($f(a) = f(b)$).
- Rolle's theorem guarantees **at least one horizontal tangent** somewhere in between.

3. Properties / Notes

- The theorem is a **special case of the Mean Value Theorem**.
- Can have **more than one** c satisfying $f'(c) = 0$.
- Only works if the function is **continuous and differentiable** as stated.

4. Examples

i. $f(x) = x^2 - 1$ on $[-1, 1]$

- $f(-1) = f(1) = 0$
- $f'(x) = 2x$
- $f'(0) = 0 \rightarrow c = 0$

Mean Value Theorem (MVT)

1. Statement / Definition

Let f be a function such that:

1. f is **continuous** on the closed interval $[a, b]$.
2. f is **differentiable** on the open interval (a, b) .

Then, there exists at least one $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Meaning: There exists a point where the **instantaneous rate of change (slope of tangent)** equals the **average rate of change** over $[a, b]$.

2. Geometric Meaning

- The tangent line at $x = c$ is **parallel to the secant line** connecting $(a, f(a))$ and $(b, f(b))$.
- Guarantees at least **one point with the same slope as the average slope**.

3. Properties / Notes

- **Rolle's Theorem** is a special case of MVT where $f(a) = f(b)$.
- Can have **more than one** c satisfying the theorem.
- Requires both **continuity on $[a, b]$** and **differentiability on (a, b)** .

4. Examples

i. $f(x) = x^2$ on $[1, 3]$

- Average slope: $\frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = 4$
- $f'(x) = 2x$
- Solve $2c = 4 \Rightarrow c = 2$

Taylor's Theorem

- **Definition:** For $f(x)$ n -times differentiable near a :

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x)$$

- **Remainder:** $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$, c between a and x .
- **Geometric:** Approximates $f(x)$ near $x = a$ with a polynomial.

- **Examples:**
 - $e^x = 1 + x + x^2/2! + \dots$
 - $\sin x = x - x^3/3! + \dots$
- **Numerical:** Approximate $e^{0.1}$ using $P_2(x) = 1 + x + x^2/2 = 1.105$.

Feature	Schwarz's Theorem	Young's Theorem
Scope	Applies in a region where mixed partials exist and are continuous.	Applies at a single point where mixed partials exist and are continuous.
Type	Global statement over a region.	Local statement at a point.
Requirement	f_{xy} and f_{yx} continuous throughout the region.	f_{xy} and f_{yx} continuous at that point only.
Conclusion	$f_{xy}(x, y) = f_{yx}(x, y)$ everywhere in the region.	$f_{xy} = f_{yx}$ at that specific point.
Example	$f(x, y) = x^2y + 3xy^2 \rightarrow$ holds for all (x, y) .	$f(x, y) = x^3y^2 + xy \rightarrow$ holds at a particular (x_0, y_0) .

Complex Analysis – Complex Functions

1. Single-Valued and Multi-Valued Functions

Definition

- **Complex Function:** $f: \mathbb{C} \rightarrow \mathbb{C}$, $z = x + iy \mapsto f(z) = u(x, y) + iv(x, y)$ where $u(x, y), v(x, y)$ are real-valued functions.
- A function $f(z)$ is **single-valued** if for each z in its domain, $f(z)$ has **exactly one value**.

Example:

- $f(z) = z^2, f(z) = e^z, f(z) = z + 1$

ii. Multi-Valued Function

- A function $f(z)$ is **multi-valued** if for some z , $f(z)$ can take **more than one value**.

Example:

- $f(z) = \sqrt{z} \rightarrow$ two values for each non-zero z .
- $f(z) = \log z \rightarrow$ infinitely many values, differing by $2\pi in, n \in \mathbb{Z}$.

2. Properties / Notes

- Multi-valued functions can be made **single-valued** using **branches** (branch cuts).
- Single-valued functions are easier to differentiate and integrate.
- Analyticity is defined for **single-valued functions**.

Complex Functions – Limit, Continuity, Differentiability

1. Limit

Definition:

$$\lim_{z \rightarrow z_0} f(z) = L$$

if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$$

Limit must be **same along all possible paths** approaching z_0 .

Example:

$$f(z) = z^2, \lim_{z \rightarrow 1+i} z^2 = (1+i)^2 = 2i$$

2. Continuity

Definition:

$f(z)$ is continuous at z_0 if

$$\lim_{z \rightarrow z_0} f(z) = f(z_0)$$

Example:

$f(z) = e^z$ is continuous everywhere in $\mathbb{C}$.

3. Differentiability

Definition:

f is differentiable at z_0 if

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and is independent of the direction of approach.

Geometric Meaning:

- **Limit:** value approached from **all paths**.
- **Continuity:** **no break** in mapping.
- **Differentiability:** tangent exists; function is smooth at that point.

4. Properties

- Differentiability $\Rightarrow$ Continuity (but not vice versa).
- If differentiable in a region, f is called **analytic** there.

5. Examples

i. $f(z) = z^2$

- Limit exists everywhere.
- Continuous everywhere.
- Differentiable everywhere with $f'(z) = 2z$.

ii. $f(z) = \bar{z}$

- Limit exists.
- Continuous everywhere.
- Not differentiable (since derivative depends on direction).

1. Analytic Functions: Necessary & Sufficient Conditions

- A function $f(z) = u(x, y) + iv(x, y)$ is **analytic** at a point z_0 if:
 - f is differentiable in a neighborhood of z_0 .
 - The **Cauchy–Riemann equations** hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- u, v have **continuous partial derivatives**.

Necessary: C–R must hold for differentiability.

Sufficient: If C–R + continuous partials, then f is analytic.

2. Harmonic Functions

- A real function $\phi(x, y)$ is **harmonic** if it satisfies **Laplace's equation**:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

- If $f(z) = u + iv$ is analytic, then both $u(x, y)$ and $v(x, y)$ are **harmonic functions**.
- They are called **harmonic conjugates**

Entire Function

Definition

A function $f(z)$ is called **entire** if it is **analytic at every point of the complex plane $\mathbb{C}$** .

Properties

1. Differentiable everywhere in $\mathbb{C}$.
2. Taylor series expansion valid for all $z \in \mathbb{C}$.

Examples

- Polynomials: $f(z) = z^n$.
- Exponential: $f(z) = e^z$.
- Trigonometric functions: $\sin z, \cos z$.
- Hyperbolic functions: $\sinh z, \cosh z$.

Non-Examples

- $\log z, \tan z, \left(\frac{1}{z}\right)$ not analytic everywhere).

Geometric Meaning: Entire functions are smooth mappings of the whole plane without singularities.

Möbius Transformation

Definition

A Möbius (linear fractional) transformation is

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

where $a, b, c, d \in \mathbb{C}$.

Properties

- Maps the **extended complex plane** $\mathbb{C} \cup \{\infty\}$ onto itself.
- Preserves **angles** (conformal map).
- Maps **lines and circles** into lines or circles.
- **Invertible**, and the inverse is also a Möbius transformation.

Geometric Meaning

Every Möbius map is a composition of **translation, scaling/rotation, and inversion**.

Example

$$f(z) = \frac{z - 1}{z + 1}$$

This maps the **right half-plane** $\Re(z) > 0$ to the **unit disk** $|w| < 1$.

Power Series

Definition

A power series about z_0 :

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Properties

1. Converges inside some radius R (radius of convergence).
2. Represents analytic functions within $|z - z_0| < R$.
3. Differentiation and integration term-by-term are valid within the radius.

Examples

- $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ (converges for all z).
- $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$

Zeros of Analytic Functions

Definition

If $f(z)$ is analytic in a domain D , then a point $z_0 \in D$ is called a **zero** of $f(z)$ if

$$f(z_0) = 0.$$

Order (Multiplicity) of a Zero

- If $f(z)$ can be written as

$$f(z) = (z - z_0)^m g(z) \quad g(z_0) \neq 0$$

then z_0 is a **zero of order m** .

- $m = 1$: **simple zero**.
- $m > 1$: **multiple zero**.

Properties

- Zeros of a non-constant analytic function are **isolated** (they do not accumulate inside the domain).
- Zeros play a key role in **contour integration** and **residue theory**.

Example

- $f(z) = \sin z$: zeros at $z = n\pi$, $n \in \mathbb{Z}$. Each is **simple zero**.
- $f(z) = (z - 1)^3 e^z$: zero at $z = 1$ of **order 3**.

Cauchy's Theorem

Statement

If $f(z)$ is **analytic** inside and on a **simple closed contour C** , then

$$\oint_C f(z) dz = 0$$

Example 1

$$f(z) = e^z, \quad C: |z| = 2$$

$f(z)$ analytic everywhere.

$$\Rightarrow \oint_C e^z dz = 0$$

Example 2

$$f(z) = \frac{1}{z - 2}, \quad C: |z| = 1$$

$f(z)$ **analytic inside** $|z| < 1$ (pole at $z = 2$ outside C).

$$\Rightarrow \oint_C \frac{1}{z - 2} dz = 0$$

Morera's Theorem

Statement

Let $f(z)$ be **continuous** in a domain D . If

$$\oint_C f(z) dz = 0$$

for **every closed contour** C in D , then $f(z)$ is **analytic** in D .

Notes

Morera's theorem is a **converse of Cauchy's theorem**.

Useful to prove **analyticity** of functions defined by integrals.

Cauchy's Integral Formula (CIF)

Statement

If $f(z)$ is **analytic** inside and on a **simple closed contour** C , and z_0 is a point inside C , then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

Example 1

$$f(z) = e^z, \quad C: |z| = 2, \quad z_0 = 1$$

$$f(1) = \frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{z - 1} dz$$

The integral equals $e^1 = e$

Singularities of Complex Functions

Definition

A point z_0 is a **singularity** of a complex function $f(z)$ if $f(z)$ is **not analytic** at z_0 but is analytic at some neighborhood around z_0 (excluding z_0 itself).

Classification of Singularities

1. Removable Singularity

- $f(z)$ is undefined at z_0 but $\lim_{z \rightarrow z_0} f(z)$ exists.
- Example:** $f(z) = \frac{\sin z}{z}$ at $z = 0$.
- Can be **"removed"** by defining $f(0) = 1$.

Pole (of order m)

- $f(z)$ blows up at z_0 :

$$\lim_{z \rightarrow z_0} |f(z)| = \infty$$

- Order m :** $f(z) = \frac{g(z)}{(z - z_0)^m}$, $g(z_0) \neq 0$.
- Example:** $f(z) = \frac{1}{(z-1)^3}$, pole of order 3 at $z = 1$.

3. Essential Singularity

- Neither removable nor pole. Behavior near z_0 is **wild**, takes almost all values.
- Example:** $f(z) = e^{1/z}$ at $z = 0$.
- By **Casorati-Weierstrass theorem**, $f(z)$ near an essential singularity is dense in $\mathbb{C}$.

Geometric Meaning

- Removable:** hole in the surface.
- Pole:** function shoots to infinity.
- Essential:** function oscillates wildly; no limiting behavior.

1. Open Mapping Theorem

Statement

If $f(z)$ is **non-constant analytic** in a domain D , then f maps **open sets** in D to **open sets** in $\mathbb{C}$.

Geometric Meaning

Analytic functions “spread out” points — they cannot collapse an open region to a curve or point.

Example: $f(z) = z^2$ maps the open disk $|z| < 1$ to $|w| < 1$.

2. Taylor Series

Definition

If $f(z)$ is analytic at z_0 , it can be expanded as:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

Properties

Converges in **disk of radius R** (distance to nearest singularity).

Term-by-term differentiation/integration is valid.

Example: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

3. Laurent Series

Definition

If $f(z)$ is analytic in an **annulus** $r < |z - z_0| < R$, it can be expanded as:

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Notes

- The series has **positive and negative powers**.
- Negative powers represent **principal part**, crucial for **poles and residues**.

Example:

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1} = -\frac{1}{z} - \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

Fundamental Theorem of Algebra (FTA)

Statement

Every non-constant polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0, \quad a_n \neq 0$$

with complex coefficients has **at least one root** in $\mathbb{C}$.

Consequences / Properties

1. A polynomial of degree n has **exactly n roots** (counting multiplicities) in $\mathbb{C}$.
2. All polynomials can be **factored completely** over $\mathbb{C}$:

$$P(z) = a_n(z - z_1)(z - z_2) \dots (z - z_n)$$

3. Shows $\mathbb{C}$ is **algebraically closed**

Rouché's Theorem

Statement

Let $f(z)$ and $g(z)$ be analytic inside and on a **simple closed contour** C . If

$$|g(z)| < |f(z)| \text{ for all } z \in C,$$

then $f(z)$ and $f(z) + g(z)$ have **the same number of zeros** inside C (counting multiplicities).

Example 1

Determine the number of zeros of $P(z) = z^3 + 5z + 2$ inside $|z| = 1$.

- Let $f(z) = 5z$, $g(z) = z^3 + 2$.
- On $|z| = 1$: $|g(z)| \leq 1 + 2 = 3 < |f(z)| = 5$
- By Rouché's theorem, $P(z)$ has **same number of zeros as $f(z)$** inside $|z| = 1$, i.e., **1 zero**.

Example 2

$f(z) = z^5$, $g(z) = 3z^2 + 1$, $C: |z| = 2$

- On $|z| = 2$, $|g(z)| \leq 3 * 4 + 1 = 13$, $|f(z)| = 32$
- $|g(z)| < |f(z)|$, so $f + g$ has **5 zeros inside $|z|=2$** .

1. Residue Theorem

Statement

Let $f(z)$ be analytic in a domain except for **isolated singularities** $z_1, z_2, \dots, z_n$ inside a **simple closed contour** C . Then

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k)$$

where $\text{Res}(f, z_k)$ is the **residue** of f at z_k .

Residue at a Simple Pole

If $f(z)$ has a **simple pole** at z_0 :

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

Residue at Pole of Order m

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$$

Solution of Algebraic and Transcendental Equations (Numerical Methods)

Bisection Method

Definition

A numerical method to find a **real root** of $f(x) = 0$ in an interval $[a, b]$ where $f(a) \cdot f(b) < 0$.

Properties

Guaranteed convergence if $f(x)$ is continuous on $[a, b]$ and $f(a)f(b) < 0$.

Converges **linearly** (slower than Newton-Raphson).

Root lies **within the interval** at every step.

Simple and reliable; does **not require derivative**.

Newton-Raphson Method

Definition

An iterative numerical method to find a **real root** of $f(x) = 0$ using:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Properties

1. Requires **derivative** $f'(x)$.
2. **Converges quadratically** near the root (fast).
3. Sensitive to **initial guess**; may diverge if guess is poor.
4. Efficient for **single roots**.

Error Formula

If x^* is the exact root:

$$e_{n+1} \approx \frac{f''(x^*)}{2f'(x^*)} e_n^2$$

- Shows **quadratic convergence**: error roughly squares each iteration.

Secant Method

Definition

An iterative method to find a root of $f(x) = 0$ using **two initial approximations** x_0 and x_1 :

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

- Does **not require derivative** explicitly.

Properties

1. Uses **two previous approximations** instead of derivative.
2. **Faster than Bisection**, slower than Newton-Raphson.
3. Converges **superlinearly** (order ≈ 1.618).
4. Sensitive to initial guesses; may diverge if poorly chosen.

Error Formula (Approximate)

$$e_{n+1} \approx \frac{e_n e_{n-1}}{e_n - e_{n-1}} f''(\xi)/2$$

- Shows **superlinear convergence**.

Fixed Point Iteration (FPI)

Definition: A method to solve $f(x) = 0$ by rewriting it as:

$$x = g(x)$$

Then iterate:

$$x_{n+1} = g(x_n)$$

- Converges to a **fixed point** x^* where $x^* = g(x^*)$.

Properties

1. Convergence requires $|g'(x)| < 1$ near the root.
2. Simple to implement; derivative **not needed**.
3. Convergence may be **slow** compared to Newton-Raphson.
4. Choice of $g(x)$ affects convergence.

Error Formula (Approx.)

$$e_{n+1} \approx |g'(x^*)| e_n$$

- **Linear convergence** if $|g'(x^*)| < 1$.

Interpolation Methods (Numerical Analysis)

1. Lagrange Interpolation

- Uses formula:

$$P_n(x) = \sum_{i=0}^n y_i \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

- **Properties:** Passes through all points, derivative not needed.

2. Newton's Divided Difference Interpolation

- Polynomial:

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \dots$$

- Properties:** Easy to add new points, uses divided differences.

Forward Difference (Finite Difference) Interpolation

- Uses equally spaced points:

$$P_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \dots$$

- $p = \frac{x-x_0}{h}$, h = spacing.
- Properties:** Efficient for equally spaced data.

4. Backward Difference Interpolation

- Similar to forward, uses last data point:

$$P_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \dots$$

- $p = \frac{x-x_n}{h}$
- Use:** Interpolation near **end of table**.

5. Stirling's Formula

- Combines forward and backward differences for **central interpolation**.
- Best for **points near the middle** of the table.

6. Bessel's Formula

- Central interpolation for **even number of points**, more accurate than Stirling.

7. Cubic Spline Interpolation

- Piecewise cubic polynomials with **continuous first and second derivatives**.
- Best for smooth curves** and avoids Runge's phenomenon.

1. Linear Systems

Direct Methods

- Gauss Elimination** – Convert to upper triangular, back-substitute

Properties

Deterministic and gives exact solution (ignoring round-off errors).

Works for any **non-singular** matrix.

Can be modified for **partial or complete pivoting** to improve stability.

- Gauss-Jordan** – Reduce to diagonal form.

Steps / Formula:

Form augmented matrix $[A \mid b]$.

Use **row operations** to make **leading 1's** and zeros elsewhere in each column.

Read the solution directly from the final matrix.

Properties:

Direct method, no iteration.

Works for **any number of equations** if determinant $\neq 0$.

Produces **unique solution** if system is consistent.

3.LU Decomposition:

Properties

Efficient for multiple systems.

Direct method, no iteration.

Exists if matrix is nonsingular.

L lower, U upper triangular.

Reusable for different b .

iterative methods for solving linear systems:

1. Jacobi Method

- **Formula:** $x_i^{(k+1)} = \frac{1}{a_{ii}} (b_i - \sum_{j \neq i} a_{ij} x_j^{(k)})$
- **Property:** Uses previous iteration values; simple but slower.

2. Gauss-Seidel Method

- **Formula:** $x_i^{(k+1)} = \frac{1}{a_{ii}} (b_i - \sum_{j < i} a_{ij} x_j^{(k+1)} - \sum_{j > i} a_{ij} x_j^{(k)})$
- **Property:** Uses latest updates; converges faster than Jacobi.

3. Successive Over-Relaxation (SOR)

- **Type:** Iterative, acceleration of Gauss-Seidel
- **Formula:** $x_i^{(k+1)} = (1 - \omega)x_i^{(k)} + \frac{\omega}{a_{ii}} (b_i - \sum_{j < i} a_{ij} x_j^{(k+1)} - \sum_{j > i} a_{ij} x_j^{(k)})$
- **Property:** ω (relaxation factor) improves convergence speed.

4. Conjugate Gradient Method (for large symmetric positive-definite matrices)

- **Type:** Iterative, optimization-based
- **Property:** Fast convergence; avoids full matrix inversion

Common methods for non-linear systems:

1. Newton-Raphson Method (for systems)

- **Type:** Iterative
- **Formula:** $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - J^{-1}(\mathbf{x}^{(k)})\mathbf{F}(\mathbf{x}^{(k)})$
- **Idea:** Uses Jacobian to linearize the system at each iteration.
- **Property:** Fast convergence if initial guess is close; requires Jacobian.

2. Fixed-Point Iteration

- **Type:** Iterative
- **Formula:** Rewrite system as $x = g_1(x, y), y = g_2(x, y)$, iterate: $x_{n+1} = g_1(x_n, y_n), y_{n+1} = g_2(x_n, y_n)$

- **Property:** Convergence only if $|g'_i| < 1$ near solution.

3. Broyden's Method

- **Type:** Iterative, quasi-Newton
- **Idea:** Approximate Jacobian to reduce computation
- **Property:** Useful when Jacobian is hard to compute; slower than full Newton-Raphson but saves effort.

4. Homotopy / Continuation Method

- **Type:** Iterative / Global
- **Idea:** Transform a simple system continuously into target system, track solution path.
- **Property:** Can find multiple solutions; more complex.

Jacobian (for non-linear systems) [note: it is not a method, a useful tool]

Definition:

Matrix of **first-order partial derivatives** of vector function $\mathbf{F}(\mathbf{x})$, used in **Newton-Raphson for systems**.

Formula:

$$J(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Use:

Linearizes non-linear system: $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - J^{-1}(\mathbf{x}^{(k)})\mathbf{F}(\mathbf{x}^{(k)})$

Numerical Differentiation:

1. Definition:

Numerical differentiation is the **process of approximating the derivative of a function** using discrete data points or function values.

2. Properties:

- Provides **approximate derivatives**.
- Accuracy depends on **step size** and method.
- Can be **forward, backward, or central difference**.

3. Main Formulas:

A. Forward Difference:

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}$$

B. Backward Difference:

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{h}$$

C. Central Difference:

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1}))}{2h}$$

- More accurate than forward/backward.

D. Higher Derivatives:

$$f''(x_i) \approx \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1}))}{h^2}$$

Numerical Integration:

Numerical integration is the **approximation of definite integrals** when exact integration is difficult or impossible.

Trapezoidal Rule (brief):

Formula (single interval):

$$\int_a^b f(x) dx \approx \frac{h}{2} [f(a) + f(b)], h = b - a$$

Composite form (n subintervals):

$$\int_a^b f(x) dx \approx \frac{h}{2} [f_0 + 2(f_1 + f_2 + \dots + f_{n-1}) + f_n]$$

Properties:

Order of accuracy: $O(h^2)$

Easy but less accurate than Simpson's rule.

Geometric meaning: Approximates area under curve using trapezoids.

Example: $\int_0^1 x^2 dx, h = 1$:

$$\frac{1}{2} [f(0) + f(1)] = 0.5 \text{ (exact) } = 0.333$$

Error Term:

For $\int_a^b f(x) dx$:

$$E_T = -\frac{(b-a)^3}{12n^2} f''(\xi), \xi \in (a, b)$$

Order: $O(h^2)$ (depends on 2nd derivative).

Sign: Negative if $f''(x) > 0$ (convex curve), positive if $f''(x) < 0$

Simpson's 1/3 Rule (brief):

Definition: Approximates $\int_a^b f(x) dx$ using **parabolas** through 3 points.

Formula (single interval, $n = 2$):

$$\int_a^b f(x) dx \approx \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)], h = \frac{b-a}{2}$$

Composite form (even n):

$$\int_a^b f(x) dx \approx \frac{h}{3} [f_0 + 4(f_1 + f_3 + \dots + f_{n-1}) + 2(f_2 + f_4 + \dots + f_{n-2}) + f_n]$$

Error term:

$$E_S = -\frac{(b-a)}{180} h^4 f^{(4)}(\xi), \xi \in (a, b)$$

Order: $O(h^4)$ (much more accurate than trapezoidal).

Geometric meaning: Approximates curve by quadratic arcs (parabolas).

Properties of Simpson's 1/3 Rule (brief):

Type: Newton–Cotes formula (uses 3 equally spaced points).

Accuracy: Exact for all polynomials up to degree 3.

Error Order: $O(h^4)$, depends on the 4th derivative of $f(x)$.

Requirement: Number of subintervals n must be **even**.

Example: $\int_0^1 x^2 dx, h = 0.5$:

$$\frac{0.5}{3} [f(0) + 4f(0.5) + f(1)] = \frac{1}{6} (0 + 4(0.25) + 1) = 0.333$$

(exact).

Since, It is exact for all polynomial up to degree 3. In exam hall, it is not necessary to calculate simpson rule. Just do it in exact way.

Simpson's 3/8 Rule:

- **Formula (single interval, $n = 3$):**

$$\int_a^b f(x) dx \approx \frac{3h}{8} [f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)], h = \frac{b-a}{3}$$

- **Composite form (when n is multiple of 3):**

$$\int_a^b f(x) dx \approx \frac{3h}{8} [f_0 + f_n + 3 \sum f_{3k-2, 3k-1} + 2 \sum f_{3k}]$$

- **Error term:**

$$E_{3/8} = -\frac{3h^5}{80} f^{(4)}(\xi), \xi \in (a, b)$$

- **Order:** $O(h^4)$.

- **Properties:**

- Exact for polynomials up to degree 3.
- Requires number of subintervals n to be **a multiple of 3**.
- Useful when interval count is not even (where 1/3 rule cannot be applied).
- Approximates area with **cubic arcs**.

Euler's Method:

- **Definition:** Simplest numerical method to solve ODE $y' = f(x, y), y(x_0) = y_0$.

- **Formula:**

$$y_{n+1} = y_n + hf(x_n, y_n)$$

- **Properties:**
 - Single-step method.
 - Easy but low accuracy..
- **Error:**
 - Local truncation error: $O(h^2)$
 - Global error: $O(h)$
- **Example:** Solve $y' = y, y(0) = 1, h = 0.1$.

$$y_1 = 1 + 0.1(1) = 1.1$$

Euler's Method – Example (brief):

Problem: $y' = x + y, y(0) = 1, h = 0.1$

Formula: $y_{n+1} = y_n + hf(x_n, y_n)$

Step 1: $x_0 = 0, y_0 = 1$

$$y_1 = 1 + 0.1(0 + 1) = 1.1$$

Step 2: $x_1 = 0.1, y_1 = 1.1$

$$y_2 = 1.1 + 0.1(0.1 + 1.1) = 1.22$$

Step 3: $x_2 = 0.2, y_2 = 1.22$

$$y_3 = 1.22 + 0.1(0.2 + 1.22) = 1.362$$

Euler's approx at $x = 0.3 \rightarrow y_3 \approx 1.362$

Exact $y(0.3) \approx 1.395$

Improved Euler / Heun's Method :

Definition:

Improves Euler's method by averaging slopes at the beginning and end of the interval.

Formula:

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))]$$

Properties:

- Order: $O(h^2)$ more accurate than Euler).
- Single-step method.
- Geometric meaning: average slope $\rightarrow$ better approximation.

Example (brief):

Solve $y' = x + y, y(0) = 1, h = 0.1$

1. $f_0 = f(x_0, y_0) = 0 + 1 = 1$
2. Predictor: $y^* = y_0 + hf_0 = 1 + 0.1(1) = 1.1$
3. Corrector: $f_1 = f(x_1, y^*) = 0.1 + 1.1 = 1.2$
4. $y_1 = 1 + 0.1/2(1 + 1.2) = 1 + 0.11 = 1.11$

Runge-Kutta 4th Order Method (RK4) – Brief:

Definition:

Single-step method to solve ODE $y' = f(x, y)$, more accurate than Euler/Improved Euler by using weighted slopes.

Formula:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

with

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\ k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\ k_4 &= hf(x_n + h, y_n + k_3) \end{aligned}$$

Properties:

- Order: $O(h^4)$, very accurate.
- Single-step method.
- Geometric meaning: weighted average of slopes at several points in interval.

Problem: Solve

$$y' = 2, y(0) = 1, h = 0.1$$

Step 1: RK4 formula:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Step 2: Compute slopes:

$$\begin{aligned} k_1 &= hf(x_0, y_0) = 0.1 * 2 = 0.2 \\ k_2 &= hf(x_0 + h/2, y_0 + k_1/2) = 0.1 * 2 = 0.2 \\ k_3 &= hf(x_0 + h/2, y_0 + k_2/2) = 0.1 * 2 = 0.2 \\ k_4 &= hf(x_0 + h, y_0 + k_3) = 0.1 * 2 = 0.2 \end{aligned}$$

Step 3: Update:

$$y_1 = y_0 + \frac{1}{6}(0.2 + 2 * 0.2 + 2 * 0.2 + 0.2) = 1 + 0.2 = 1.2$$

For RK4 method, I think its enough to practice $y' = \text{constant}$.

Hydrodynamics

1. Velocity:

$$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}, q = \sqrt{u^2 + v^2 + w^2}$$

- Tangent to **streamline**.
- Rate of change of particle position.

2. Acceleration:

$$\mathbf{a} = \frac{D\mathbf{q}}{Dt} = \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q}$$

- **Local acceleration:** $\partial \mathbf{q} / \partial t$ (unsteady flow)
- **Convective acceleration:** $(\mathbf{q} \cdot \nabla)\mathbf{q}$ (motion through space)
- Tangent to **pathline**.

3. Steady and Unsteady Flows

- **Steady Flow:**
 - Velocity of a fluid particle at a point **does not change with time**.
 - $\frac{\partial \mathbf{q}}{\partial t} = 0$
 - Geometric meaning: Streamlines = Pathlines = Streaklines.
- **Unsteady Flow:**
 - Velocity **changes with time** at a point.
 - $\frac{\partial \mathbf{q}}{\partial t} \neq 0$
 - Streamlines may change with time.

4. Uniform and Non-Uniform Flows

- **Uniform Flow:**
 - Velocity **same at all points** in space at a given instant.
 - $\mathbf{q} = \text{constant in space}$
 - Example: Flow in a straight pipe with constant cross-section (ideal).
- **Non-Uniform Flow:**
 - Velocity **varies from point to point**.
 - $\mathbf{q} = \mathbf{q}(x, y, z)$
 - Example: Flow in a converging/diverging channel.

1. Streamlines

- **Definition:** Curve **tangent to velocity vector $\mathbf{q}$** at every point **at a given instant**.

3D Streamline Equation

Differential Form: $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

- **Properties:**
 - No fluid crosses a streamline.
 - For **steady flow**, streamlines = pathlines = streaklines.

2. Pathlines

- **Definition:** Actual **trajectory of a fluid particle** over time.
- **Properties:**
 - Describes particle motion.
 - For **steady flow**, pathlines = streamlines.

3. Vortex Lines

- **Definition:** Curve **tangent to vorticity vector** $\omega = \nabla \times \mathbf{q}$ at every point.
- **Properties:**
 - Show **rotation of fluid elements**.
 - Helps visualize vortices in the flow.

Quick Note:

- Streamlines $\rightarrow$ Instantaneous direction of flow.
- Pathlines $\rightarrow$ Particle path over time.
- Vortex lines $\rightarrow$ Rotation axes in fluid

Velocity Potential:

- **Definition:** Scalar function ϕ such that

$$\mathbf{q} = -\nabla\phi$$
- **Exists for:** Irrotational flow ($\nabla \times \mathbf{q} = 0$)
- **Incompressible flow:** $\nabla^2\phi = 0$
- **Geometric meaning:** Flow along **direction of maximum decrease of ϕ** ; equipotential surfaces $\perp$ streamlines.
- **Example:** $\phi = x^2 + y^2 \Rightarrow q_x = -2x, q_y = -2y$

1. Irrotational Flow

- **Definition:** Flow in which **vorticity is zero** everywhere.

$$\nabla \times \mathbf{q} = 0$$
- **Velocity potential exists:** $\mathbf{q} = \nabla\phi$
- **Property:** Streamlines are perpendicular to equipotential surfaces.
- **Example:** Flow due to a source or sink, uniform flow.

2. Rotational Flow

- **Definition:** Flow in which **vorticity is not zero**.

$$\nabla \times \mathbf{q} \neq 0$$
- **Velocity potential does not exist.**
- **Property:** Fluid particles rotate about their own axes.
- **Example:** Vortex flow, flow near rotating cylinders.

Quick Note:

- **Vorticity vector:** $\omega = \nabla \times \mathbf{q}$
- **Irrotational:** $\omega = 0$
- **Rotational:** $\omega \neq 0$

Equation of Continuity – Brief:

- **Definition:** Conservation of mass in fluid flow.
- **3D:** $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0$
- **Incompressible:** $\nabla \cdot \mathbf{q} = 0$ or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

- **1D pipe flow:** $A_1 q_1 = A_2 q_2$
- **Example:** $A_1 = 0.01, q_1 = 2, A_2 = 0.02 \Rightarrow q_2 = 1$
- **Meaning:** Mass conserved; flow speeds up in narrow sections, slows in wide sections.

Fluid Motion Equations

- **Euler's Equation:**

$$\frac{D\mathbf{q}}{Dt} = -\frac{1}{\rho} \nabla P + \mathbf{F} \text{ (inviscid flow)}$$

Where, $\mathbf{F}$ = Body force per unit mass (For gravity $\mathbf{F} = g$)

- **Conservative Force:**
 $\mathbf{F} = -\nabla \Omega$ (e.g., gravity potential)
- **Lamb's Equation:**

$$\frac{\partial \mathbf{q}}{\partial t} + \boldsymbol{\omega} \times \mathbf{q} + \nabla \frac{q^2}{2} = -\frac{1}{\rho} \nabla P + \mathbf{F}$$

- **Bernoulli's Equation (steady, incompressible, along streamline):**

$$\frac{P}{\rho} + \frac{q^2}{2} + \Omega = \text{constant}$$

where Ω is the potential corresponding to the body force $\mathbf{F}$ (e.g., $\Omega = gz$ for gravity).

Stream Function (ψ):

1. Definition:

- A **scalar function** used in **2D incompressible flow** such that:

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$$

where u, v are velocity components in x, y .

2. Physical Meaning:

1. **Streamlines:** $\psi = \text{constant}$ represents a **streamline**.
2. **Flow rate:** Difference in ψ between two streamlines = **volume flow rate per unit depth**.
3. **Continuity automatically satisfied:** $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Stream Function – Velocity Relation

Cartesian (2D):

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$$

Polar (2D):

$$v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, v_\theta = -\frac{\partial \psi}{\partial r}$$

1. Source

- **Definition:** Fluid **emanates radially outward**.
- **Velocity potential:**

$$\phi = m \ln r$$

- **Stream function:**

$$\psi = m\theta \quad \text{where, } m = \text{strength of source.}$$

2. Sink

- **Definition:** Fluid **converges radially inward**.
- **Velocity potential:**

$$\phi = -m \ln r$$

- **Stream function:**

$$\psi = -m\theta$$

3. Doublet (Dipole)

- **Definition:** Source + sink of equal strength **placed very close together**.
- **Velocity potential:**

$$\phi = \frac{\mu \cos \theta}{r}$$

- **Stream function:**

$$\psi = -\frac{\mu \sin \theta}{r}$$

- μ = strength of doublet.

Meaning:

- Source $\rightarrow$ fluid outflow, Sink $\rightarrow$ fluid inflow
- Doublet $\rightarrow$ models **flow around cylinders/obstacles**

1. Complex Potential $W(z)$

- **Definition:** A complex function representing **2D incompressible, irrotational flow**:

$$W(z) = \phi(x, y) + i\psi(x, y)$$

where:

- $\phi(x, y)$ = **velocity potential**
- $\psi(x, y)$ = **stream function**
- $z = x + iy$
- **Properties:**
 - $\phi = \text{constant}$ $\rightarrow$ equipotential lines
 - $\psi = \text{constant}$ $\rightarrow$ streamlines

2. Complex Velocity q_c

- **Definition:** Negative derivative of complex potential:

$$q_c = -\frac{dW}{dz} = u - iv$$

where u, v = velocity components along x, y .

- **Properties:**

- Magnitude: $q = |q_c| = \sqrt{u^2 + v^2}$
- Direction: tangent to streamlines

3. Example:

Source of strength m at origin: $W(z) = m \ln z$

$$q_c = -\frac{dW}{dz} = -\frac{m}{z} = -\frac{m}{r} e^{-i\theta}$$

Radial velocity: $v_r = \frac{m}{r}$, Tangential velocity: $v_\theta = 0$

1. Definition:

- **Stagnation Point:** Point in a flow where **fluid velocity is zero**.

$$\mathbf{q} = 0 \text{ or } u = v = 0$$

2. Properties:

1. Occur on **solid boundaries** or in **flow fields**.
2. **Pressure is maximum** at stagnation points (from Bernoulli's equation).
3. For **inviscid, incompressible flow**, $\phi = \text{constant}$ at stagnation point.

3. Example:

- Uniform flow $q = U_\infty$ combined with a **doublet** at origin:

$$W(z) = U_\infty \left(z + \frac{\mu}{z} \right)$$

- **Stagnation points** satisfy $\frac{dW}{dz} = 0 \Rightarrow U_\infty \left(1 - \frac{\mu}{z^2} \right) = 0 \Rightarrow z = \pm \sqrt{\mu/U_\infty}$
- **Meaning:** Fluid "stops" at these points, pressure is highest

Circulation and Vorticity :

- **Circulation:** $\Gamma = \oint_C \mathbf{q} \cdot d\mathbf{l}$, measures **rotational strength** around a closed curve.
- **Vorticity:** $\omega = \nabla \times \mathbf{q}$, measures **local rotation** of fluid particles.
- **Relation:** $\Gamma = \iint_S \omega \cdot \mathbf{n} dS \rightarrow \text{circulation} = \text{total vorticity through surface}$.

Kelvin's Circulation Theorem :

Statement: In an **inviscid, barotropic fluid** with **conservative forces**, the **circulation around a material contour** moving with the fluid is **constant**:

$$\frac{D\Gamma}{Dt} = 0$$

Implication: Circulation is **conserved**; irrotational flow stays irrotational

200 Important Questions

1. Two forces $F_1 = 3 \text{ N}$ and $F_2 = 4 \text{ N}$ act at 90° to each other. The resultant is:

- A) 5 N B) 7 N C) 1 N D) 12 N

Solution: $R = \sqrt{F_1^2 + F_2^2} = \sqrt{3^2 + 4^2} = 5 \text{ N}$ **A**

2. Two parallel forces of magnitude 10 N and 20 N, distance 2 m apart, form a couple. Moment of couple is:

- A) 10 Nm B) 20 Nm C) 30 Nm D) 40 Nm

Solution: $M = F \cdot d = 10 \cdot 2 = 20 \text{ Nm}$ **B**

3. A force of 50 N acts at 3 m from a pivot. The moment about the pivot is:

- A) 150 Nm B) 50 Nm C) 3 Nm D) 53 Nm

Solution: $M = F \cdot d = 50 \cdot 3 = 150 \text{ Nm}$ **A**

4. Two parallel forces of 5 N and 15 N act at points $x = 1 \text{ m}$ and $x = 3 \text{ m}$. Position of resultant:

- A) $x = 2.5 \text{ m}$ B) $x = 2 \text{ m}$ C) $x = 1.5 \text{ m}$ D) $x = 3 \text{ m}$

Solution: $x_R = \frac{5 \cdot 1 + 15 \cdot 3}{5 + 15} = \frac{50}{20} = 2.5 \text{ m}$ **A**

5. Which of the following is true for a couple?

- A) Produces translation only B) Produces rotation only
C) Produces both rotation and translation D) Has no effect

Solution: Moment of a couple $\rightarrow$ **pure rotation** **B**

6. Two forces $F_1 = 6 \text{ N}$ and $F_2 = 8 \text{ N}$ act at 60° . The resultant is:

- A) 10 N B) 12 N C) 14 N D) 8 N

Solution: $R = \sqrt{6^2 + 8^2 + 2 \cdot 6 \cdot 8 \cos 60^\circ} = \sqrt{148} \approx 12$ **B**

7. Two parallel forces of 12 N and 18 N act 4 m apart as a couple. Moment of the couple:

- A) 30 Nm B) 40 Nm C) 48 Nm D) 60 Nm

Solution: $M = F \cdot d = 12 \cdot 4 = 48 \text{ Nm}$ **C**

8. Two forces 5 N and 12 N along the same line. Resultant:

- A) 7 N B) 17 N C) 12 N D) 60 N

Solution: Add along the line: $R = 5 + 12 = 17 \text{ N}$ **B**

9. Moment of 20 N force at 0.5 m from pivot:

- A) 5 Nm B) 10 Nm C) 20 Nm D) 40 Nm

Solution: $M = F \cdot d = 20 \cdot 0.5 = 10 \text{ Nm}$ **B**

10. Two parallel forces 8 N and 12 N, 2 m apart. Resultant location from smaller force:

- A) 1 m B) 1.2 m C) 1.5 m D) 0.5 m

Solution: $x_R = \frac{8 \cdot 0 + 12 \cdot 2}{8 + 12} = 1.2 \text{ m}$ **B**

11. Moment of a couple is independent of:

- A) Force magnitude B) Distance between forces
C) Point of reference D) Direction of forces

Solution: Couples produce **pure rotation**, independent of point

C

12. Two perpendicular forces, resultant direction:

- A) Bisector of angle B) Along larger force
C) $\tan \theta = F_2/F_1$ D) Along smaller force

Solution: $\tan \alpha = F_2/F_1$

C

13. Position of resultant for three parallel forces along x-axis:

- A) Arithmetic mean B) Weighted mean $x_R = \frac{\sum F_i x_i}{\sum F_i}$
C) Midpoint D) Midpoint of largest force

Solution: Weighted mean

B

14. Two equal and opposite forces 1 m apart produce:

- A) Translation B) Rotation only
C) Zero effect D) Acceleration

Solution: Couple $\rightarrow$ rotation only

B

15. Two opposite forces along same line, 10 N and 15 N. Resultant:

- A) 25 N along larger force B) 5 N along smaller force
C) 5 N along larger force D) 0 N

Solution: Subtract $\rightarrow 15 - 10 = 5$ N along larger force

C

16. A body remains at rest under coplanar forces. Which condition is satisfied?

- A) $\sum F_x = 0$ only B) $\sum F_y = 0$ only
C) $\sum F = 0$ and $\sum M = 0$ D) $\sum M = 0$ only

Solution: Both force and moment conditions

C

17. For a body in stable equilibrium:

- A) Slight displacement $\rightarrow$ body moves further
B) Slight displacement $\rightarrow$ body returns to original position
C) Slight displacement $\rightarrow$ body remains at new position
D) Body rotates freely

Solution: Restoring tendency

B

18. A ball on a flat horizontal table exhibits:

- A) Stable equilibrium B) Unstable equilibrium
C) Neutral (Astatic) equilibrium D) No equilibrium

Solution: Remains in new position

C

19. A ball at the top of a hill exhibits:

- A) Stable equilibrium B) Unstable equilibrium
C) Neutral equilibrium D) Static equilibrium

Solution: Moves away if displaced

B

B) Only $\sum F_y = 0$

D) $\sum M = 0$

C

B) Only moments must balance

D) Neither need to balance

C

B) Ball on top of dome

D) Boat floating in calm water

B

B) Potential energy is maximum

D) Forces are unequal

C

B) Will rotate

D) Is in neutral equilibrium

B

B) Decreases on displacement

D) Is zero

A

D) 0 J

C

D) Zero

D

D) Unit: Watt

B

B) Random movement

D) Rotational motion

C

30. Principle of Virtual Work says:

- A) Net work = maximum
B) Net work = negative
C) Net work = applied load
D) Net virtual work = 0

Solution: Equilibrium condition: $\delta W = 0$.

D

31. Virtual work is useful in:

- A) Fluid mechanics
B) Thermodynamics
C) Dynamics
D) Statics

Solution: Used to test equilibrium of constrained systems.

D

32. If block is lifted and placed back at same height, work by gravity:

- A) Positive
B) Negative
C) Zero
D) Infinite

Solution: Gravity is conservative $\rightarrow$ net work = 0.

C

33. Example of virtual work application:

- A) Heat transfer
B) Satellite motion
C) Structural analysis
D) Fluid pressure

Solution: Used in frames, trusses, beams equilibrium.

C

34. The Centre of Gravity (C.G.) of a uniform rod lies at:

- A) One end
B) Midpoint
C) Two-thirds length
D) Variable point

Solution: Uniform rod $\rightarrow$ mass evenly distributed $\rightarrow$ midpoint.

B

35. For a semicircular thin plate of radius r , the C.G. lies at:

- A) At center
B) At diameter
C) At arc
D) $\frac{2r}{\pi}$ from the diameter

Solution: Known formula for semicircular plate.

D

36. The C.G. of a uniform sphere is located at:

- A) On surface
B) At center
C) At pole
D) At base

Solution: Symmetry $\rightarrow$ at geometric center.

B

37. In coordinates, C.G. of system of particles is given by:

- A) Arithmetic mean of coordinates
B) Weighted mean of coordinates
C) Geometric mean
D) Harmonic mean

Solution: $x_{cg} = \frac{\sum m_i x_i}{\sum m_i}$, etc.

B

38. If a body is balanced at its C.G., then:

- A) Only horizontal forces act
B) Only vertical forces act
C) It remains in rotational equilibrium
D) It becomes unstable

Solution: At C.G., weight acts without producing moment.

C

39. Resultant of 3D force system is given by:

- A) Sum of magnitudes
 B) Difference of magnitudes
 C) Cross product of forces
 D) Vector sum of components

Solution: $\vec{R} = \sum F_x \hat{i} + \sum F_y \hat{j} + \sum F_z \hat{k}$. D

40. Condition for 3D equilibrium of forces is:

- A) Only $\sum F = 0$
 B) Only $\sum M = 0$
 C) $\sum F = \text{constant}$
 D) $\sum F = 0$ and $\sum M = 0$

Solution: Both force and moment balance needed. D

41. A system of two particles: $m_1 = 2 \text{ kg}$ at $x = 0$, $m_2 = 4 \text{ kg}$ at $x = 6 \text{ m}$. Find C.G.

- A) 2 m
 B) 3 m
 C) 4 m
 D) 6 m

Solution:

$$x_{cg} = \frac{2(0) + 4(6)}{2+4} = \frac{24}{6} = 4 \text{ m}$$
C

42. Three equal masses at (0,0,0), (2,0,0), (0,2,0). Find C.G.

- A) (1,0,0)
 B) (0,1,0)
 C) (0,0,1)
 D) (2/3, 2/3, 0)

Solution:

$$x_{cg} = \frac{0+2+0}{3} = \frac{2}{3}, y_{cg} = \frac{0+0+2}{3} = \frac{2}{3}, z_{cg} = 0$$
D

43. A cube of side 2m. C.G. from one corner is at:

- A) (1,0,0)
 B) (0,1,1)
 C) (1,1,1)
 D) (2,2,2)

Solution: For uniform cube $\rightarrow$ center = half side in each direction. C

44. Work done by $\vec{F} = (2, 2, 1)$ in moving $\vec{d} = (1, 1, 1)$:

- A) 3
 B) 4
 C) 5
 D) 6

Solution: Dot product $W = 2(1) + 2(1) + 1(1) = 5$. C

45. C.G. of a hemisphere of radius r from base is:

- A) $r/2$
 B) $3r/8$
 C) $2r/3$
 D) r

Solution: Formula for solid hemisphere C.G. = $3r/8$. B

Motion in a Straight Line

46. A particle starts from rest and moves with acceleration 2 m/s^2 . Its velocity after 5 s is:

- A) 5 m/s
 B) 10 m/s
 C) 15 m/s
 D) 20 m/s

Solution: $v = u + at = 0 + 2(5) = 10$. B

47. A car travels 20 m in 2 s from rest. Find acceleration.

- A) 2 m/s^2
 B) 5 m/s^2
 C) 10 m/s^2
 D) 20 m/s^2

Solution: $s = ut + \frac{1}{2}at^2$. $20 = 0 + \frac{1}{2}a(4) \rightarrow a = 10/2 = 5$. B

48. In Simple Harmonic Motion (SHM), acceleration is proportional to:

- A) Velocity B) Displacement C) Mass D) Energy

Solution: $a = -\omega^2 x$.Acceleration $\propto$ displacement.

B

49. A particle executes SHM with amplitude 0.1 m and angular frequency $\pi\text{ rad/s}$.

Its time period is:

- A) 0.5 s B) 1 s C) 2 s D) $\pi\text{ s}$

Solution: $T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2.$

C

50. The total energy in SHM is proportional to:

- A) Displacement B) Amplitude² C) Velocity D) Mass only

Solution: $E = \frac{1}{2}m\omega^2 A^2$.Energy $\propto$ amplitude².

B

51. In Cartesian coordinates, velocity is expressed as:

- A) $\vec{v} = x\hat{i} + y\hat{j}$ B) $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$ C) $\vec{v} = \ddot{x}\hat{i} + \ddot{y}\hat{j}$ D) $\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

Solution: Velocity = time derivative of position vector.

B

52. In polar coordinates, the velocity of a particle is:

- A) $\dot{r}\hat{e}_r$ B) $r\dot{\theta}\hat{e}_\theta$ C) $\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$ D) $\ddot{r}\hat{e}_r + \ddot{\theta}\hat{e}_\theta$

Solution: Velocity has both radial and transverse components.

C

53. Radial component of velocity is:

- A) $r\dot{\theta}$ B) $\dot{r}$ C) $\ddot{r}$ D) $\dot{r} + r\dot{\theta}$

Solution: Rate of change of distance from origin = $\dot{r}$.

B

54. Transverse acceleration is given by:

- A) $r\dot{\theta}^2$ B) $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ C) $\ddot{r} - r\dot{\theta}^2$ D) $\dot{r}$

Solution: Tangential acceleration formula in polar coordinates.

B

55. Radial acceleration is:

- A) $\dot{r}$ B) $\ddot{r} - r\dot{\theta}^2$ C) $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ D) $r\dot{\theta}$

Solution: Radial acceleration has centripetal term $-r\dot{\theta}^2$.

B

56. A particle moves such that $r = 2t\text{ m}$, $\theta = t\text{ rad/s}$. Find radial velocity at $t = 2\text{ s}$:

- A) 1 m/s B) 2 m/s C) 4 m/s D) 6 m/s

Solution: $v_r = \dot{r} = \frac{d}{dt}(2t) = 2\text{ m/s}$

B

57. For the same particle ($r = 2t, \theta = t$), transverse velocity at $t = 2\text{ s}$ is:

- A) 2 m/s B) 4 m/s C) 6 m/s D) 8 m/s

Solution: $v_\theta = r\dot{\theta} = (2t)(1) = 2 * 2 = 4\text{ m/s}$

B

58. Particle moves in plane with $r = 3t^2$, $\theta = t$. Radial acceleration at $t = 1\text{ s}$ is:

- A) 0 m/s^2 B) 4 m/s^2 C) 2 m/s^2 D) 3 m/s^2

Solution: $\dot{r} = 6t \Rightarrow \ddot{r} = 6$

$r\dot{\theta}^2 = 3 * 1^2 = 3$

$a_r = \ddot{r} - r\dot{\theta}^2 = 6 - 3 = 3\text{ m/s}^2$

D

59. For same particle, transverse acceleration at $t = 1s$:

- A) 8 m/s^2 B) 9 m/s^2 C) 10 m/s^2 D) 12 m/s^2

Solution: $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 3 * 0 + 2 * 6 * 1 = 12 \text{ m/s}^2$

D

60. Particle moves with $r = t$, $\theta = 2t$. Find magnitude of velocity at $t = 1s$:

- A) 2 m/s B) $\sqrt{5} \text{ m/s}$ C) 3 m/s D) $\sqrt{10} \text{ m/s}$

B

61. The Laplace transform of a function $f(t)$ is defined as:

- A) $F(s) = \int_0^\infty f(t) dt$ B) $F(s) = \int_0^\infty f(t)e^{-st} dt$
C) $F(s) = \int_{-\infty}^\infty f(t) dt$ D) $F(s) = \int_0^\infty e^{st} dt$

Solution: Definition: $\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st} dt$.

B

62. For which function the Laplace transform exists?

- A) Any function B) Function of exponential order
C) Only polynomial D) Only trigonometric

Solution: LT exists if $|f(t)| \leq Me^{ct}$ as $t \rightarrow \infty$.

B

63. Linearity property of Laplace transform:

- A) $\mathcal{L}[f + g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$ B) $\mathcal{L}[af + bg] = a\mathcal{L}[f] + b\mathcal{L}[g]$
C) $\mathcal{L}[af] = a$ D) $\mathcal{L}[f - g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$

Solution: Linearity: constants factor out and sum splits.

B

64. Laplace transform of 1 is:

- A) $1/s^2$ B) $1/s$ C) s D) e^{-s}

Solution: $\mathcal{L}[1] = \int_0^\infty e^{-st} dt = 1/s$

B

65. Laplace transform of e^{at} is:

- A) $1/(s - a)$ B) $1/(s + a)$ C) $s/(s - a)$ D) $s/(s + a)$

Solution: $\mathcal{L}[e^{at}] = \int_0^\infty e^{at} e^{-st} dt = 1/(s - a), s > a$

A

66. Laplace transform of t^n ($n \geq 0$) is:

- A) $n!/s^{n+1}$ B) $s^n/n!$ C) $1/s^n$ D) $s^{n+1}/n!$

Solution: Standard formula: $\mathcal{L}[t^n] = n!/s^{n+1}$.

A

67 Time shift property: $\mathcal{L}[f(t - a)u(t - a)] = ?$

- A) $F(s)e^{as}$ B) $F(s)e^{-as}$ C) $F(s - a)$ D) $e^{-s}F(s - a)$

Solution: Shifting t by a multiplies transform by e^{-as} .

B

68. Derivative property: $\mathcal{L}[f'(t)] = ?$

- A) $sF(s) + f(0)$ B) $sF(s) - f(0)$ C) $F(s)/s$ D) $F'(s)$

Solution: $\mathcal{L}[f'] = s\mathcal{L}[f] - f(0)$

B

69. If $f(t)$ is of exponential order, Laplace transform:

- A) Always diverges B) Always converges for some $s > \alpha$
C) Exists only for polynomial functions D) Does not exist

Solution: Exponential order guarantees convergence for large enough s .

B

70. Linearity implies:

- A) $\mathcal{L}[f(t) - g(t)] = \mathcal{L}[f] - \mathcal{L}[g]$ B) $\mathcal{L}[f \cdot g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$
 C) $\mathcal{L}[af] = F(s)/a$ D) $\mathcal{L}[f + g] = \mathcal{L}[f]/\mathcal{L}[g]$

Solution: Linearity works for addition/subtraction.

B

71. Laplace transform of $\sin(at)$ is:

- A) $\frac{a}{(s^2+a^2)}$ B) $\frac{s}{(s^2+a^2)}$ C) $\frac{1}{(s^2+a^2)}$ D) $\frac{s^2}{(s^2+a^2)}$

Solution: .

A

72. Laplace transform of $\cos(at)$ is:

- A) $\frac{a}{(s^2+a^2)}$ B) $\frac{s}{(s^2+a^2)}$ C) $\frac{1}{(s^2+a^2)}$ D) $\frac{s^2}{(s^2+a^2)}$

Solution: Standard formula: .

B

73. Laplace transform of $e^{2t} \sin(3t)$ is:

- A) $3/[(s-2)^2 + 9]$ B) $2/[(s+2)^2 + 9]$
 C) $3/(s^2 + 9)$ D) $2/(s^2 + 9)$

Solution: LT of .

A

74. Laplace transform of te^{2t} is:

- A) $1/(s-2)^2$ B) $1/(s+2)^2$ C) $2/(s-2)^2$ D) $2/(s+2)^2$

Solution: $\mathcal{L}[te^{at}] = 1/(s-a)^2$.

A

75. Laplace transform of $u(t-3)$ (Heaviside function) is:

- A) $\frac{e^{-3s}}{s}$ B) $\frac{e^{3s}}{s}$ C) $\frac{1}{s}$ D) se^{-3s}

Solution: Time-shift property: $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$.

A

76. Laplace transform of $\cosh(at)$ is:

- A) $\frac{s}{(s^2-a^2)}$ B) $\frac{a}{(s^2-a^2)}$ C) $\frac{s}{(s^2+a^2)}$ D) $\frac{a}{(s^2+a^2)}$

Solution: Standard formula: .

A

77. Laplace transform of $\sinh(at)$ is:

- A) $\frac{a}{(s^2-a^2)}$ B) $\frac{s}{(s^2-a^2)}$ C) $\frac{s}{(s^2+a^2)}$ D) $\frac{a}{(s^2+a^2)}$

Solution: Standard formula: .

A

78. Laplace transform of t^2 is:

- A) $\frac{2}{s^3}$ B) $\frac{1}{s^2}$ C) $\frac{1}{s^3}$ D) $\frac{2}{s^2}$

Solution: $\mathcal{L}[t^n] = n!/s^{n+1}$. For t^2 , $2!/s^3 = 2/s^3$.

A

79. If $\mathcal{L}[f(t)] = F(s)$, then $\mathcal{L}[e^{3t}f(t)] = ?$

- A) $F(s+3)$ B) $F(s-3)$ C) $e^{3s}F(s)$ D) $e^{-3s}F(s)$

Solution: LT property: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

B

80. Standard form of Bessel's equation (order n):

- A) $x^2 y'' + xy' + (x^2 - n^2)y = 0$ B) $y'' + xy' + (n^2 - x^2)y = 0$
C) $y'' + y = 0$ D) $xy'' + y' + n^2 y = 0$

Solution: Standard definition.

A

81. Which solution is regular at $x = 0$?

- A) $Y_n(x)$ B) $J_n(x)$ C) Both D) Neither

Solution: J_n is finite at $x = 0$, Y_n is singular.

B

82. Series form of $J_n(x)$ involves:

- A) Factorials only B) Factorials and Gamma function
C) Only binomial expansion D) No series expansion

Solution: Formula uses $m!$ and $\Gamma(m + n + 1)$.

B

83. Generating function of $J_n(x)$:

- A) e^x B) $e^{\frac{x}{2}(t-1/t)}$ C) $e^{\frac{x}{2}(t+1/t)}$ D) $e^{\frac{x}{2}(t-1/2t)}$

Solution: Correct identity: $\sum J_n(x)t^n$

B

84. Recurrence relation:

Which is correct?

- A) $J_{n-1} + J_{n+1} = 2J'_n$ B) $J_{n-1} - J_{n+1} = 2J'_n$
C) $J_{n-1} + J_{n+1} = 0$ D) $J_{n+1} - J_{n-1} = 2J_n$

Solution: Correct: $J_{n-1} - J_{n+1} = 2J'_n$

B

85. Orthogonality property holds when:

- A) Same zeros of J_n B) Different zeros of J_n
C) Different orders of Bessel functions D) No condition

Solution: $\int_0^1 r J_n(\alpha_{np}r) J_n(\alpha_{nq}r) dr = 0$ for $p \neq q$.

B

86. The first two terms of $J_0(x)$ are:

- A) $1 - \frac{x^2}{4} + \dots$ B) $1 + \frac{x^2}{4} + \dots$ C) $x - \frac{x^3}{6} + \dots$ D) $\frac{x}{2} - \frac{x^3}{16} + \dots$

Solution: Series expansion gives alternating signs.

A

87. Value of $J_0(0)$:

- A) 0 B) 1 C) ∞ D) -1

Solution: From series, $J_0(0) = 1$.

B

88. Value of $J_1(0)$:

- A) 1 B) 0 C) ∞ D) -1

Solution: First term of $J_1(x)$ is proportional to x , so at $x = 0$, it vanishes.

B

89. The Legendre differential equation is:

- A) $(1 - x^2)y'' + 2xy' + n(n + 1)y = 0$ B) $(1 + x^2)y'' - 2xy' + n(n + 1)y = 0$
C) $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ D) $y'' + y = 0$

Answer: C

Correct form: $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$.

90. The polynomial solution for $n = 2$ is:

A) $P_2(x) = x^2 + 1$

B) $P_2(x) = \frac{1}{2}(3x^2 - 1)$

C) $P_2(x) = 2x^2 - 1$

D) $P_2(x) = 3x^2 - 2$

Answer: B

Formula gives $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

91. The generating function of Legendre polynomials is:

A) $(1 - xt)^{-1} = \sum P_n(x)t^n$

B) $\frac{1}{\sqrt{1-2xt+t^2}} = \sum P_n(x)t^n$

C) $e^{xt} = \sum P_n(x)t^n$

D) $\frac{1}{1-xt} = \sum P_n(x)t^n$

Answer: B

Generating function: $\frac{1}{\sqrt{1-2xt+t^2}}$.

92. Rodrigues' formula is:

A) $P_n(x) = \frac{d^n}{dx^n}(x^n)$

B) $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$

C) $P_n(x) = \sum_{k=0}^n \binom{n}{k} x^k$

D) $P_n(x) = (x - 1)^n$

Answer: B

Rodrigues' formula defines Legendre polynomials.

93. Orthogonality condition of Legendre polynomials is:

A) $\int_0^1 P_m(x)P_n(x) dx = 0, m \neq n$

B) $\int_{-1}^1 P_m(x)P_n(x) dx = 0, m \neq n$

C) $\int_{-1}^1 P_m(x)P_n(x) dx = \frac{2}{2n+1} \delta_{mn}$

D) Both B and C

Answer: D

✓ Orthogonal on $[-1, 1]$, with norm $\frac{2}{2n+1}$.

94. The recurrence relation is:

A) $(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)$

B) $P_{n+1}(x) = P_n(x) + P_{n-1}(x)$

C) $(n + 1)P_{n+1}(x) = (n + 1)xP_n(x) - nP_{n-1}(x)$

D) $xP_n(x) = P_{n+1}(x)$

Answer: A ✓ Main recurrence relation for Legendre polynomials.

95. Which of the following is correct for $P_1(x)$?

A) $P_1(x) = 1$

B) $P_1(x) = x$

C) $P_1(x) = \frac{1}{2}(3x^2 - 1)$

D) $P_1(x) = \frac{1}{2}(5x^3 - 3x)$

Answer: B

$P_1(x) = x$.

96. The integral $\int_{-1}^1 P_0(x)P_2(x) dx$ equals:

- A) 1 B) 0 C) $\frac{2}{3}$ D) -1

Answer: B

Orthogonality: $m \neq n \Rightarrow \int_{-1}^1 P_m P_n dx = 0$.

97. A function $f(x)$ with period 2π can be expressed as:

- A) $f(x) = \sum_{n=0}^{\infty} c_n e^{inx}$ B) $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos\left(\frac{\pi nx}{L}\right) + b_n \sin\left(\frac{\pi nx}{L}\right))$
C) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n$ D) Both A and B

Answer: D

Fourier series has both exponential and sine-cosine forms.

98. The Fourier coefficient a_0 is given by:

- A) $\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$ B) $\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$
C) $\int_0^{2\pi} f(x) dx$ D) $\frac{1}{\pi} \int_0^{\pi} f(x) dx$

Answer: A

Correct definition: $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$.

99. If $f(x)$ is an even function, then which coefficients vanish?

- A) All a_n B) All b_n
C) Both a_n and b_n D) None

Answer: B

Even $\rightarrow$ only cosines $\rightarrow$ all sine coefficients $b_n = 0$.

100. If $f(x)$ is an odd function, then:

- A) Only cosine terms appear B) Only sine terms appear
C) Both sine and cosine appear D) Constant term only

Answer: B

Odd $\rightarrow$ only sine terms.

101. A metric space is defined as a set X with a function d satisfying:

- A) Positivity, Symmetry, Triangle inequality B) Positivity only
C) Symmetry only D) None

Answer: A

These 3 properties define a metric.

102. The discrete metric on any set X is given by:

- A) $d(x, y) = |x - y|$ B) $d(x, y) = 0$ if $x = y$, 1 if $x \neq y$
C) $d(x, y) = \sqrt{(x - y)^2}$ D) None of these

Answer: B

Discrete metric assigns 0 to identical points, 1 otherwise.

103. Which of the following is an open set in $\mathbb{R}$?

- A) $[0, 1]$ B) $(0, 1)$ C) $\{0\}$ D) $[0, 1)$

Answer: B

Interval $(0, 1)$ is open; $[0, 1]$ is closed.

104. A set is closed if:

- A) It has no limit points B) It contains all its limit points
C) Its complement is closed D) It is bounded

Answer: B

Closed set $\Leftrightarrow$ contains all limit points.

105. In $\mathbb{R}^n$, a set is compact iff:

- A) Closed only B) Bounded only
C) Closed and bounded D) Open and bounded

Answer: C

By Heine–Borel theorem: compact $\Leftrightarrow$ closed & bounded.

106. A perfect set must be:

- A) Open only B) Closed with no limit points
C) Closed and every point is a limit point D) Countable and bounded

Answer: C

Perfect = closed + all points are limit points.

107. The Cantor set is:

- A) Countable, closed, and dense
B) Uncountable, perfect, nowhere dense, measure zero
C) Finite, closed, open D) Countable, bounded, open

Answer: B

Cantor set has these classical properties.

108. Which of the following is NOT compact in $\mathbb{R}$?

- A) $[0, 1]$ B) $[0, 1] \cup [2, 3]$ C) $(0, 1)$ D) A finite set of points

Answer: C

$(0, 1)$ is bounded but not closed $\rightarrow$ not compact.

109. The closure of an open set is always:

- A) Open B) Closed C) Compact D) Perfect

Answer: B

Closure of any set is always closed.

110. A sequence (x_n) converges to L if:

- A) $|x_n - L| < 1$ for all n B) $|x_n| \rightarrow \infty$
C) $|x_n - L| \rightarrow 0$ as $n \rightarrow \infty$ D) $|x_n| \leq L$ for all n

Answer: C

Convergence means the distance from x_n to L tends to 0.

D) $x_n = \ln n$

D) Oscillatory

D) Must be constant

D) $x_n = \sin n$

D) Every divergent sequence is unbounded

D) $\mathbb{R}^2$

D) All of the above

Sequence oscillates $\rightarrow$ bounded but not convergent, not Cauchy.

118. A subsequence of a convergent sequence:

- A) Always convergent B) Always divergent
C) Can be unbounded D) None of the above

Answer: A

Any subsequence converges to the same limit as the original sequence.

119. A function $f(x)$ is continuous at $x = a$ if:

- A) $\lim_{x \rightarrow a} f(x) \neq f(a)$ B) $f'(a)$ exists
C) $\lim_{x \rightarrow a} f(x) = f(a)$ D) $f(x)$ is differentiable

Answer: C

Continuity means the limit at a equals the function value.

120. The derivative of $f(x) = x^3$ is:

- A) x^2 B) $3x^2$ C) $2x$ D) x^3

Answer: B

Using the power rule: $\frac{d}{dx} x^n = nx^{n-1}$.

121. Rolle's Theorem applies if:

- A) f is continuous on $[a, b]$ B) f is differentiable on (a, b)
C) $f(a) = f(b)$ D) All of the above

Answer: D

All three conditions are required; then $\exists c \in (a, b)$ s.t. $f'(c) = 0$.

122. If $f(x) = x^2 - 4x + 3$ on $[1, 3]$, then Rolle's Theorem gives $c =$

- A) 1 B) 2 C) 3 D) 4

Answer: B

$f(1) = 0, f(3) = 0$; derivative $f'(x) = 2x - 4 = 0 \Rightarrow x = 2$.

123. The Mean Value Theorem states that:

- A) $\exists c \in (a, b)$ s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}$ B) $f'(x) = 0$
C) $f(a) = f(b)$ D) $f(x)$ is continuous only

Answer: A

MVT gives a point where derivative equals average rate of change.

124. For $f(x) = x^3$ on $[0, 2]$, MVT gives $c =$

- A) 0 B) $2/\sqrt{3}$ C) 2 D) 1

Answer: B

Average rate: $(f(2) - f(0))/(2 - 0) = 4$; $f'(x) = 3x^2 = 4 \Rightarrow x = c = 2/\sqrt{3}$

125. Taylor series of $f(x) = e^x$ about $x = 0$ is:

- A) $1 + x + x^2/2! + x^3/3! + \dots$ B) $1 + x - x^2/2! + x^3/3! - \dots$
C) $x + x^2/2! + x^3/3! + \dots$ D) $1 - x + x^2 - x^3 + \dots$

Answer: A

Standard Taylor expansion: $e^x = 1 + x + x^2/2! + x^3/3! + \dots$

126. Young's theorem states that:

- A) $f_{xy} = f_{yx}$ if partial derivatives are continuous
- B) Only $f_x = f_y$
- C) Only applies to functions of one variable
- D) Partial derivatives do not exist

Answer: A

Young's theorem ensures equality of mixed partials under continuity.

127. The function $f(x, y) = \frac{xy}{x^2+y^2}$ at $(0, 0)$ is:

- A) Continuous
- B) Discontinuous
- C) Differentiable
- D) Constant

Answer: B

limit does not exist $\rightarrow$ discontinuous.

128. A complex function $f(z)$ is said to be continuous at z_0 if:

- A) $\lim_{z \rightarrow z_0} f(z)$ exists and equals $f(z_0)$
- B) $f'(z_0)$ exists
- C) $f(z)$ is bounded near z_0
- D) $f(z)$ is differentiable

Answer: A

Continuity means the limit at z_0 equals the function value at that point.

129. The function $f(z) = \sqrt{z}$ is:

- A) Single-valued
- B) Multi-valued
- C) Entire
- D) Analytic everywhere

Answer: B

Square root in the complex plane has two values $\rightarrow$ multi-valued function.

130. The derivative of $f(z) = z^2$ is:

- A) $2\bar{z}$
- B) $2z$
- C) z^2
- D) $\bar{z}^2$

Answer: B

Using standard complex differentiation: $\frac{d}{dz} z^n = nz^{n-1}$.

131. A function $f(z) = u(x, y) + iv(x, y)$ is analytic if it satisfies:

- A) $u_x = v_y, u_y = -v_x$
- B) $u_x = -v_y, u_y = v_x$
- C) $u_x = u_y, v_x = v_y$
- D) $u = v$

Answer: A

These are the **Cauchy-Riemann equations**, necessary and sufficient for differentiability.

132. The function $f(z) = \bar{z}$ is:

- A) Analytic
- B) Entire
- C) Non-analytic
- D) Multi-valued

Answer: C

$\bar{z}$ does not satisfy Cauchy-Riemann equations $\rightarrow$ not analytic.

133. The function $f(z) = \ln z$ is:

- A) Single-valued
- B) Multi-valued
- C) Entire
- D) Analytic at $z = 0$

Answer: B

Complex logarithm is multi-valued due to $2\pi i$ ambiguity.

134. A function $f(z)$ is differentiable at z_0 implies:

- A) $f(z)$ is continuous at z_0
- B) $f(z)$ is bounded
- C) $f(z)$ is constant
- D) $f(z)$ is multi-valued

Answer: A

Differentiability in complex analysis implies continuity.

135. The function $f(z) = e^z$ is:

- A) Analytic everywhere
- B) Analytic nowhere
- C) Analytic only on real axis
- D) Multi-valued

Answer: A

Exponential function is entire $\rightarrow$ analytic for all $z \in \mathbb{C}$.

136. The limit $\lim_{z \rightarrow 0} \frac{|z|}{z}$ exists?

- A) Yes
- B) No
- C) Only on real axis
- D) Its Uncertain

Answer: B

Limit depends on direction $\rightarrow$ does not exist in complex plane.

137. A complex function $f(z) = u(x, y) + iv(x, y)$ is analytic if:

- A) Only u is continuous
- B) Only v is continuous
- C) u and v satisfy Cauchy-Riemann equations and have continuous first derivatives
- D) $u = v$

Answer: C

Necessary and sufficient condition for analyticity: Cauchy-Riemann equations with continuous partial derivatives.

138. Which of the following is a harmonic function?

- A) $u(x, y) = x^2 - y^2$
- B) $u(x, y) = e^{x+y}$
- C) $u(x, y) = \ln(x^2 + y^2)$
- D) All of the above

Answer: D

Harmonic functions satisfy Laplace's equation: $u_{xx} + u_{yy} = 0$.

139. The real and imaginary parts of an analytic function are:

- A) Independent
- B) Harmonic and conjugate
- C) Constant
- D) Non-differentiable

Answer: B

Analytic function $\rightarrow$ real and imaginary parts are harmonic conjugates.

140. Möbius transformation maps:

- A) Circle to circle or line B) Circle to ellipse
C) Line to parabola D) Hyperbola to ellipse

Answer: A

General form $w = \frac{az+b}{cz+d}$ maps circles/lines to circles/lines.

141. The Möbius transformation $w = \frac{z-i}{z+i}$ maps $z = i$ to:

- A) 0 B) 1 C) ∞ D) -1

Answer: A

Substitute $z = i$: $w = (i - i)/(i + i) = 0$.

142. The power series $\sum_{n=0}^{\infty} z^n$ converges for:

- A) $|z| < 1$ B) $|z| > 1$ C) $|z| = 1$ D) All complex numbers

Answer: A

Geometric series converges inside the unit circle $|z| < 1$.

143. A function $f(z)$ has a zero at z_0 if:

- A) $f'(z_0) = 0$ B) $f(z_0) = 0$ C) $f(z_0) \neq 0$ D) $f''(z_0) = 0$

Answer: B

A zero of a function is a point where the function value itself is zero.

144. Cauchy's theorem states that:

- A) Integral of an analytic function over a closed contour is zero
B) Integral of any function over a closed contour is zero
C) Function must be real
D) Function must be singular

Answer: A

For a function analytic inside and on a closed contour, $\oint f(z)dz = 0$.

145. Morera's theorem is used to:

- A) Test if a function is analytic B) Find residues
C) Classify singularities D) Solve real integrals

Answer: A

If a continuous function has zero integral over every closed contour, it is analytic.

146. Cauchy's integral formula for $f(z_0)$ is:

- A) $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz$ B) $f(z_0) = \oint_C f(z)dz$
C) $f(z_0) = f'(z_0)$ D) $f(z_0) = \int_0^{z_0} f(z)dz$

Answer: A

It expresses the value of an analytic function inside a contour in terms of a contour integral.

147. A singularity of a function is:

- A) A point where the function is zero B) A point where the function is not analytic
C) A point where the derivative is zero D) A point where the function is bounded

Answer: B

Singularities are points where a function fails to be analytic.

148. Classification of isolated singularities includes:

- A) Removable, pole, essential B) pole, essential, imaginary
C) Removable, pole, bounded D) Single-valued, multi-valued, pole

Answer: A

Isolated singularities are classified as removable, poles, or essential singularities.

149. The open mapping theorem states that:

- A) Analytic non-constant functions map open sets to open sets
B) Analytic functions map closed sets to closed sets
C) Analytic functions map points to circles
D) Analytic functions preserve distances

Answer: A

Non-constant analytic functions map open sets in the domain to open sets in the codomain.

150. The fundamental theorem of algebra states that:

- A) Every polynomial has at least one zero in $\mathbb{C}$
B) Every function is analytic
C) Every series converges
D) Polynomials have no zeros in $\mathbb{C}$

Answer: A

Any non-constant polynomial with complex coefficients has at least one complex root.

151. Rouché's theorem is used to:

- A) Count zeros of analytic functions inside a contour B) Compute integrals
C) Solve differential equations D) Classify singularities

Answer: A

If $|f(z)| > |g(z)|$ on a closed contour, f and $f + g$ have the same number of zeros inside.

152. The Bisection method is used to:

- A) Solve differential equations B) Solve algebraic and transcendental equations
C) Interpolate data D) Compute definite integrals

Answer: B

Bisection method finds roots of equations by repeatedly halving an interval where the function changes sign.

153. For the equation $x^3 - 4x + 1 = 0$,which interval contains a root using Bisection method?

- A) [0,1] B) [1,2] C) [2,3] D) [-1,0]

Answer: B

$f(1) = -2$, $f(2) = 1$;function changes sign $\rightarrow$ root exists in [1,2].

154. The Newton-Raphson method requires:

- A) Only function values B) Function and its derivative
C) Two initial guesses D) Integral of the function

Answer: B

Newton-Raphson formula: $x_{n+1} = x_n - f(x_n)/f'(x_n)$.

155. The False Position (Regula Falsi) method is similar to:

- A) Bisection method B) Newton-Raphson method
C) Secant method D) Lagrange interpolation

Answer: A

It also brackets the root like Bisection, but uses a straight line to estimate the root.

156. Lagrange interpolation formula is used to:

- A) Find roots B) Approximate integrals
C) Construct a polynomial passing through given points
D) Solve differential equations

Answer: C

Lagrange interpolation constructs a polynomial $P(x)$ that exactly passes through the given data points.

157. Newton's forward interpolation is suitable for:

- A) Unevenly spaced data B) Evenly spaced data
C) Solving equations D) Integrating functions

Answer: B

Forward differences require equally spaced points.

158. In Newton-Raphson method, if $f'(x_n) = 0$,then:

- A) Iteration fails B) Root is found
C) Converges faster D) Bisection is used

Answer: A

Division by zero occurs $\rightarrow$ iteration cannot proceed.

159. The Gauss Elimination method is used to:

- A) Solve nonlinear equations B) Solve linear systems of equations
C) Interpolate data D) Integrate functions

Answer: B

It transforms the system into an upper triangular form to solve by back-substitution.

01701377322

174. Acceleration of a fluid particle is given by:

- A) $\frac{\partial \vec{q}}{\partial t}$ B) $\frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q}$ C) $\nabla \cdot \vec{q}$ D) $\nabla \times \vec{q}$

Answer: B

Total acceleration = local acceleration + convective acceleration.

175. Steady flow is characterized by:

- A) Velocity varies with time at a point B) Velocity does not vary with time at a point
C) Pressure is zero D) Density is zero

Answer: B

In steady flow, $\frac{\partial \vec{q}}{\partial t} = 0$.

176. Uniform flow means:

- A) Velocity is same at all points in the flow field
B) Velocity varies with time
C) Pressure is uniform
D) Flow is irrotational

Answer: A

Uniform flow: velocity vector is constant throughout the flow field.

177. Streamlines are:

- A) Paths traced by fluid particles over time
B) Curves tangent to velocity at a given instant
C) Curves of constant acceleration
D) Curves of constant pressure

Answer: B

Streamlines show instantaneous direction of fluid velocity.

178. Pathlines are:

- A) Curves tangent to velocity at a given instant
B) Actual paths traced by individual fluid particles over time
C) Curves of zero vorticity
D) Curves of constant density

Answer: B

Pathline = trajectory of a particle over time.

179. Vortex lines represent:

- A) Curves tangent to velocity vectors B) Curves tangent to vorticity vectors
C) Curves of constant pressure D) Streamlines

Answer: B

Vortex lines indicate direction of local rotation of fluid.

180. Velocity potential ϕ exists if:

- A) Flow is compressible B) Flow is irrotational
C) Flow is rotational D) Flow is viscous

Answer: B

Irrotational flow $\rightarrow \vec{q} = \nabla\phi$.

181. A flow is irrotational if:

- A) $\nabla \cdot \vec{q} = 0$ B) $\nabla \times \vec{q} = 0$
C) $\vec{q} = 0$ D) Velocity is constant

Answer: B

Irrotational flow $\rightarrow$ vorticity = curl of velocity = 0.

182. Equation of continuity for incompressible flow is:

- A) $\nabla \cdot \vec{q} = 0$ B) $\nabla \times \vec{q} = 0$ C) $\frac{\partial \vec{q}}{\partial t} = 0$ D) $\vec{q} = \nabla\phi$

Answer: A Continuity equation ensures mass conservation: divergence of velocity = 0 for incompressible flow.

183. Euler's equation of motion for an inviscid fluid is:

- A) $\frac{D\vec{q}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{F}$ B) $\nabla \cdot \vec{q} = 0$
C) $\vec{q} = \nabla\phi$ D) $\frac{dp}{dx} = 0$

Answer: A

Euler's equation expresses acceleration of a fluid particle in terms of pressure gradient and body force $\vec{F}$.

184. A conservative body force is characterized by:

- A) $\vec{F} = 0$ B) $\vec{F} = -\nabla\Omega$ C) $\nabla \times \vec{F} \neq 0$ D) $\vec{F} \cdot \vec{q} = 0$

Answer: B

Conservative forces derive from a potential Ω , e.g., gravity: $\vec{F} = -\nabla\Omega$.

188. The stream function $\psi(x, y)$ represents:

- A) Potential energy of fluid B) Curves tangent to velocity
C) Flow pattern, with streamlines as $\psi = \text{constant}$
D) Pressure distribution

Answer: C

Lines of constant ψ are streamlines; they represent the flow pattern.

189. For incompressible 2D flow, the velocity components in terms of stream function are:

- A) $q_x = \frac{\partial \psi}{\partial x}, q_y = \frac{\partial \psi}{\partial y}$ B) $q_x = \frac{\partial \psi}{\partial y}, q_y = -\frac{\partial \psi}{\partial x}$
C) $q_x = -\frac{\partial \psi}{\partial x}, q_y = -\frac{\partial \psi}{\partial y}$ D) $q_x = q_y = \psi$

Answer: B

Definition of 2D stream function ensures $\nabla \cdot \vec{q} = 0$ automatically.

190. The physical meaning of stream function is:

- A) Measures pressure
- B) Difference in ψ between two streamlines = volume flow rate per unit depth
- C) Measures density
- D) Measures velocity magnitude

Answer: B

$\Delta\psi$ between streamlines equals the volumetric flow rate per unit depth in 2D flow

191. The relation between stream function ψ and polar velocity components is:

- A) $q_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, q_\theta = -\frac{\partial \psi}{\partial r}$
- B) $q_r = \frac{\partial \psi}{\partial r}, q_\theta = \frac{\partial \psi}{\partial \theta}$
- C) $q_r = -\frac{\partial \psi}{\partial \theta}, q_\theta = \frac{\partial \psi}{\partial r}$
- D) $q_r = q_\theta = \psi$

Answer: A

This ensures continuity equation $\nabla \cdot \vec{q} = 0$ in polar coordinates.

192. The complex potential of a source of strength m at the origin is:

- A) $W(z) = m \ln z$
- B) $W(z) = \frac{m}{z}$
- C) $W(z) = \frac{1}{2} m z^2$
- D) $W(z) = -m \ln z$

Answer: A

Source potential in 2D: $W(z) = m \ln z$. For sink, sign is negative.

193. The streamlines of a source are:

- A) Concentric circles
- B) Radial lines from the origin
- C) Ellipses
- D) Parabolas

Answer: B

For a source, fluid flows radially outward $\rightarrow$ streamlines are straight lines through the origin.

194. The complex velocity is related to complex potential as:

- A) $q = \frac{dW}{dz}$
- B) $q = -\frac{dW}{dz}$
- C) $q = \nabla W$
- D) $q = \frac{1}{W}$

Answer: B

By definition, complex velocity = negative derivative of potential.

195. A stagnation point is defined as:

- A) A point of maximum velocity
- B) A point where velocity is zero
- C) A point of maximum pressure only
- D) A point where streamlines cross

Answer: B

At stagnation points, $q = 0$, so fluid comes to rest.

B) $\Gamma = \oint \vec{q} \cdot d\vec{s}$

C) $\Gamma = \nabla \cdot \vec{q}$

D) $\Gamma = \int \vec{F} \cdot d\vec{r}$

A) $\vec{\omega} = \nabla \cdot \vec{q}$

B) $\vec{\omega} = \nabla \times \vec{q}$

C) $\vec{\omega} = \oint \vec{q} \cdot d\vec{s}$

D) $\vec{\omega} = \frac{dq}{dz}$

$$A) \Gamma = \int \vec{\omega} \, dA$$

B) $\Gamma = \nabla \cdot \vec{\omega}$

C) $\Gamma = \frac{d\omega}{dt}$

D) $\Gamma = \omega Z$